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TECHNIQUES IN MACHINE FUNCTION INTEGRAL CALCULATION*

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A. Introduction

1.) This note is a summary of machine function integral expressions the author has accumulated in several years' work on accelerator physics. It is not of theoretical importance, but it can help much in practical calculation. Many accelerator physicists have noticed that to express such integrals by functions at some special points and parameters of the magnet in question has an advantage over step-by-step summation, owing to less time elapsed and better accuracy obtained. However, most of the formulae they present in papers or programs still have much room for simplification. To express the integrals as simply as possible has the following benefits: it saves more time; it exhibits conclusions in better clarity so as to reduce chances of error; it can help set some parameters as "function of goodness" or "fit function" in searching for an ideal lattice configuration, though the parameters are usually considered too complicated. For example, it is possible to make the non-coupling emittance as well as some other functions minimized in designing a synchrotron radiation source, and it may be found easy to fit the momentum compaction factor to a given goal value for choosing a very short bunch length lattice. Both of these were realized in the author's work on the Hefei 800 MeV Storage Ring.

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2.) Let 1 and 2 denote the entrance and the exit of a magnet respectively. The effective length of the magnet is $L = z_2 - z_1$. Supose P is a z-dependent machine function. Let average symbol [ ] and difference symbol Δ be defined as below:

$$[P] = \frac{1}{L}\int_{z_1}^{z_2} P(z)\, dz \tag{1}$$

$$\Delta P = P_2 - P_1 = P(z_2) - P(z_1) \tag{2}$$

The problem of function integral evaluation is how to express [P] by known parameters.

Suppose Q is another z-dependent function and A is piecewisely constant, namely, A doesn't change between z_1 and z_2. Obviously the following relations can be established:

$$[A] = A \;;\quad [A\,P] = A\,[P] \;;\quad [P + Q] = [P] + [Q] \;;$$

$$[P'] = \Delta P/L \;;\quad [P\,Q'] = \Delta(P\,Q)/L - [P'Q] \tag{3}$$

3.) No mathematical approximation is made in any equations throughout this note. It is assumed that magnetic field is constant within a magnet. The particle motion is described in a natural orthogonal x-y-z coordinate system, with y-axis fixed vertically, which implies no vertical bending. Then the first order motion equation of a particle without energy deviation reads

$$u'' + F_u\, u = 0 \tag{4}$$

where u may be x or y, and

$$F_x = K + \frac{1}{\rho^2} \;;\qquad F_y = -K \tag{5}$$

ρ is the curvature radius of the ideal orbit in a bending magnet where magnetic field $B_y = (B\rho)_0/\rho$, with $(B\rho)_0$ the particle magnetic rigidity. K is quadrupole component defined as $K = (\partial B_y/\partial x)/(B\rho)_0$. F_x, F_y and ρ are all piecewise constants.

As well known, the behavior of particles in a machine can be described by Courant-Snyder[1] beta function β_x and β_y, energy dispersion function η, and some functions associated with β_u such as α_u, γ_u and phase advance ψ_u. Usually a computer program evaluates all these functions at any magnet edges, after L, K and $1/\rho$ of all the elements in the machine are given.

4.) A summary of the functions whose integrals over a magnet one may be interested in is as follows.

$[\eta]$, the essential part in calculating machine integrals I_1 and I_4, which will in turn determine the momentum compaction factor and the damping partition numbers respectively. See Refs.2) and 3) for explanation of this statement as well as of what follows.

$[\mathcal{H}]$, where function $\mathcal{H}$ is defined as

$$\mathcal{H} = (\eta^2 + (\alpha_x \eta + \beta_x \eta')^2) / \beta_x \tag{6}$$

From $[\mathcal{H}]$, the non-coupling emittance and consequently the equilibrium beam size can be found.

$[\beta_u]$, the dominant term in calculating the natural chromaticities and an important parameter in estimating either the dependence of the tunes on magnetic gradient errors or the dependence of closed orbit distortion rms values on magnet misalignments[4]. It also plays a role in obtaining beam size rms values in a magnet, since

$$[\sigma_x]_{rms} = (\varepsilon_x[\beta_x] + (\frac{\sigma_E}{E_0})^2[\eta^2])^{1/2} ; \quad [\sigma_y]_{rms} = (\varepsilon_y[\beta_y])^{1/2} ;$$

$$[\sigma_x']_{rms} = (\varepsilon_x[\gamma_x] + (\frac{\sigma_E}{E_0})^2[\eta'^2])^{1/2} ; \quad [\sigma_y']_{rms} = (\varepsilon_y[\gamma_y])^{1/2} \tag{7}$$

where ε_u is the emittance on u plane and the explanation for the other symbols can be found in Ref.2). The beam size rms values (sigmas) are useful in calculations related to Touschek lifetime and instabilities and in featuring synchrotron light sources.

$[\gamma_u]$, $[\eta^2]$ and $[\eta'^2]$, all are needed in evaluating Eq.(7).

$[\beta_u^2]$, used to estimate tune shift rms values and β function distortion due to magnetic errors.[4)]

$[\beta_u \eta]$ in sextupoles has to be calculated for chromaticity correction. And $[\beta_x \eta]$ in nonzero-gradient bending magnets is needed for natural chromaticity calculation. The formula of $[\beta_y \eta]$ in bending magnets will not be presented, both because there is no need for it in chromaticity calculation[5)], and because no simple expression can be found for it under the most general condition in which neither K nor $1/\rho$ is zero. But some formulae in the Appendix can help those really interested in $[\beta_y \eta]$.

5.) All the formulae of integrals introduced later will be grouped in two sets. In the first set the integrals are expressed by functions at both edges, while in the second set by functions at the midpoint. One is free in choosing that formula he feels more convenient. Generally speaking, the first set is more suitable for handling quadrupoles and, if some special conditions such as "separate function" are given, for bending magnets also. The second set can serve better if bending magnets under general conditions are treated.

The Appendix presents a detailed description of a few special functions named as $C_u(z)$, $S_u(z)$ and $D_u(z)$. Their properties profit the author very much in almost every piece of work concerning accelerator physics, so their use is not limited in integral claculations.

B. Integrals Expressed by Function Values at Magnet Edges

1.) The following relations are well known[1)]

$$\beta_u' = -2\,\alpha_u ; \qquad \alpha_u' = F_u \beta_u - \gamma_u$$

$$\gamma_u' = 2\,F_u\,\alpha_u ; \qquad \gamma_u = (1+\alpha_u^2)/\beta_u \tag{8}$$

which hold on the condition that the particle motion is described by Eq.(4).

One can make a fuller use of them if he defines $A_{\beta u}$ as

$$A_{\beta u} = F_u \cdot \beta_u + \gamma_u \tag{9}$$

and finds that $A_{\beta u}$ is a piecewise constant, since $A'_{\beta u} = 0$ when F_u remains unchanged.

The special case in which $F_u = 0$ will be discussed in the last part of this section. So suppose $F_u \neq 0$, and one can easily obtain

$$[\beta_u] = [\frac{1}{2F_u}(A_{\beta u} + \alpha_u')] = \frac{1}{2F_u}(A_{\beta u} + \Delta\alpha_u / L) \tag{10}$$

$$[\gamma_u] = \frac{1}{2}(A_{\beta u} - \Delta\alpha_u / L) \tag{11}$$

and, incidentally,

$$[\alpha_u] = -\frac{1}{2}\Delta\beta_u / L$$

Here $A_{\beta u}$ as well as other piecewise constants to be defined later can be evaluated at any point in the magnet. Sometimes an index i is attached to the functions involved to denote this arbitrariness, with the understanding that i stands for either 1, 2 or other point indices. For example, Eq.(10) can be written as

$$[\beta_u] = \frac{1}{2F_u}(F_u \beta_{ui} + \gamma_{ui} + (\alpha_{u2} - \alpha_{u1}) / L)$$

If $F_u = 0$, then $\gamma_{ui} + \Delta\alpha_u / L = 0$ and Eq.(10) will be indefinite. This will also happen to the other equations where F_u appears in the denominator. But, when one is looking for natural chromaticities or for the tune shifts due to a relative gradient error ($\Delta K/K$), if the magnet in question is a quadrupole, the term he has to calculate will fortunately be $\int_{z_1}^{z_2} F_u \beta_u dz$. So Eq.(10) can be rewritten as

$$[F_u \beta_u] = \frac{1}{2}(F_u \beta_{ui} + \gamma_{ui} + \Delta\alpha_u / L) \tag{12}$$

In this case Eq.(12) always works, no matter how much F_u is.

Let us stick to the supposition that $F_u \neq 0$. Then

$$[\alpha_u' \beta_u] = \Delta(\alpha_u \beta_u)/L - [\alpha_u \beta_u'] = \Delta(\alpha_u \beta_u)/L - 2 + 2[\beta_u \gamma_u]$$

$$= \Delta(\alpha_u \beta_u)/L - 2 + A_{\beta u}[\beta_u] - [\alpha_u' \beta_u]$$

Therefore, $[\alpha_u' \beta_u] = \frac{1}{2} A_{\beta u}[\beta_u] + \frac{1}{2L}\Delta(\alpha_u \beta_u) - 1$.

So one arrives at

$$[\beta_u^2] = \frac{1}{2F_u}[A_{\beta u}\beta_u + \alpha_u' \beta_u]$$

$$= \frac{1}{4F_u}(3A_{\beta u}[\beta_u] + \Delta(\alpha_u \beta_u)/L - 2) \tag{13}$$

And, at the same time, some more equations are obtained such as

$$[\beta_u \gamma_u] = \frac{1}{4} A_{\beta u}[\beta_u] - \frac{1}{4L}\Delta(\alpha_u \beta_u) + \frac{1}{2}$$

$$[\alpha_u^2] = \frac{1}{4} A_{\beta u}[\beta_u] - \frac{1}{4L}\Delta(\alpha_u \beta_u) - \frac{1}{2}$$

2.) η function is the periodic solution of equation

$$\eta'' + F_x \eta = \frac{1}{\rho} \tag{14}$$

If $F_x = K + \frac{1}{\rho^2} \neq 0$, it is easy to give

$$[\eta] = [\frac{1}{\rho} - \eta'']/F_x = (\frac{1}{\rho} - \Delta\eta'/L)/F_x \tag{15}$$

A special example is, for separate function bending magnets where $K = 0$, one consequently has

$$[\eta] = \rho(1 - \Delta\eta'/\theta_B)$$

where θ_B is the bending angle.

In order to find $[\eta^2]$ and $[\eta'^2]$, one can make use of another piecewise constant, which is defined as

$$A_\eta = (\eta_i - \frac{1}{F_x \rho})^2 + \frac{1}{F_x}\eta_i'^2 \tag{16}$$

It is obvious that $A_\eta' = 0$, and that

$$[(\eta - \frac{1}{F_x\rho})^2] = -\frac{1}{F_x L}\Delta(\eta'(\eta - \frac{1}{F_x\rho})) + \frac{1}{F_x}[\eta'^2]$$

Therefore,

$$[\eta'^2] = \frac{1}{2}F_x A_\eta + \frac{1}{2L}\Delta(\eta'(\eta - \frac{1}{F_x\rho})) \tag{17}$$

$$[\eta^2] = \frac{1}{2}A_\eta - \frac{1}{2F_x L}\Delta(\eta'(\eta - \frac{1}{F_x\rho})) + \frac{1}{(F_x\rho)^2} - \frac{2\Delta\eta'}{F_x^2\rho L} \tag{18}$$

3.) It seemed more difficult at first thought to find relatively simple expressions for integrals of $\beta-\eta$ combined functions, such as the $\mathcal{H}$ function defined by Eq.(6). Because $\mathcal{H}' = (2/\rho)(\alpha_x\eta + \beta_x\eta')$, $\mathcal{H}$ itself is a piecewise constant if $1/\rho = 0$. But this doesn't help anything since one is only interested in calculating $[\mathcal{H}]$ of bending magnets where $1/\rho$ must be non-zero. However, this idea encourages attempts to find another function which is similar to $\mathcal{H}$ but is piecewisely constant even if $1/\rho \neq 0$.

This is done by defining several functions:

$$U(z) = \alpha_x(\eta - \frac{1}{F_x\rho}) + \beta_x\eta' \quad ; \quad V(z) = \gamma_x(\eta - \frac{1}{F_x\rho}) + \alpha_x\eta' \quad ;$$

$$A_H = (\eta - \frac{1}{F_x\rho})V + \eta' U = ((\eta - \frac{1}{F_x\rho})^2 + U^2)/\beta_x \tag{19}$$

And one can use Eqs.(8) and (14) to prove the following equations:

$$U' = -V \quad ; \qquad V' = F_x U \quad ; \qquad A_H' = 0 \tag{20}$$

An interesting conclusion can be drawn from Eq.(20) that V and U are a pair of independent solutions to equation $u'' + F_x u = 0$. A_H is a new piecewise constant which is nothing but $\mathcal{H}$ function with η replaced by $(\eta - \frac{1}{F_x\rho})$.

Therefore,

$$[\mathcal{H}] = [\ A_H + \frac{2}{F_x\rho} V + \frac{1}{(F_x\rho)^2}\gamma_x\ ] = A_H - \frac{2}{F_x\rho}\Delta U/\ L + \frac{1}{(F_x\rho)^2}[\ \gamma_x\ ] \tag{21}$$

where A_H and U are evaluated by Eq.(19) and $[\ \gamma_x]$ by Eq.(11). For example, in a separate function machine, one can give

$$[\mathcal{H}] = \frac{1}{\beta_{xi}}((\eta_i - \rho)^2 + U_i^2) + \frac{2\rho}{L}(U_1 - U_2) + \frac{1}{2}(\beta_{xi} + \rho^2\gamma_{xi}) + \frac{\rho^2}{2L}(\alpha_{x1} - \alpha_{x2})$$

where $U = \alpha_x(\eta - \rho) + \beta_x\eta'$.

Functions U and V also help get the expression for $[\beta_x\eta]$ in the way shown below. Since

$$F_x[\beta_x\eta] = [(A_{\beta x} - \gamma_x)\eta] = A_{\beta x}[\eta] - [\gamma_x(\eta - \frac{1}{F_x\rho})] - \frac{1}{F_x\rho}[\gamma_x]$$

and $2\cdot F_x[\beta_x\eta] = [(A_{\beta x} + \alpha_x')\eta] = A_{\beta x}[\eta] + \Delta(\alpha_x\eta)/L - [\alpha_x\eta']$,

one comes to

$$[\beta_x\eta] = \frac{1}{3F_x}(\ 2A_{\beta x}[\eta] - \frac{1}{F_x\rho}[\gamma_x] + \Delta(\ U + \alpha_x\eta)/\ L\ )$$

$$= \frac{1}{3F_x}(\ 2A_{\beta x}[\eta] - \frac{1}{\rho}[\beta_x] + \Delta(\ 2\ \alpha_x\eta + \beta_x\eta'\ )/\ L\ ) \tag{22}$$

All the integrals mentioned in the first section have been expressed by functions at magnet edges through Eqs.(10),(11),(13),(15),(17),(18),(21) and (22) as long as $F_u \neq 0$.

4.) If $F_u = 0$, the integrals can be directly obtained by using the following expressions which are valid in this case

$$\beta_u = \beta_{u1} - 2\alpha_{u1}(z - z_1) + \gamma_{u1}(z - z_1)^2\ ;$$

$$\alpha_u = \alpha_{u1} - \gamma_{u1}(z - z_1)\ ;\ \gamma_u = \gamma_{u1}\qquad (\text{ constant })$$

$$\eta = \eta_1 + \eta_1'(z - z_1) + \frac{1}{2\rho}(z - z_1)^2\ ;\ \eta' = \eta_1' + \frac{1}{\rho}(z - z_1) \tag{23}$$

and using equation $[(z - z_1)^n] = L^n/(n+1)$. Therefore,

$$[\beta_u] = \beta_{u1} - \alpha_{u1}L + \frac{1}{3}\gamma_{u1} L^2 = \frac{1}{2}(\beta_{u1} + \beta_{u2}) - \frac{1}{6}\gamma_{ui} L^2 \; ;$$

$$[\gamma_u] = \gamma_{ui} \; ;$$

$$[\beta_u^2] = \beta_{u1}^2 - 2\alpha_{u1}\beta_{u1}L + \frac{2}{3}(1 + 3\alpha_{u1}^2)L^2 - \alpha_{u1}\gamma_{u1}L^3 + \frac{1}{5}\gamma_{u1}^2 L^4$$

$$= [\beta_u]^2 + \frac{1}{3}L^2(\alpha_{u1}\alpha_{u2} + \frac{4}{15}\gamma_{ui}^2 L^2) \; ;$$

$$[\eta] = \eta_1 + \frac{1}{2}\eta_1' L + \frac{1}{6\rho}L^2 = \frac{1}{2}(\eta_1 + \eta_2) - \frac{1}{12\rho}L^2 \; ;$$

$$[\eta^2] = [\eta]^2 + \frac{1}{12}L^2(\eta_1'\eta_2' + \frac{4}{15}(L/\rho)^2) \; ;$$

$$[\eta'^2] = \eta_1'\eta_2' + \frac{1}{3}(L/\rho)^2 \; ;$$

$$[\mathcal{H}] = \frac{1}{2}(\mathcal{H}_1 + \mathcal{H}_2) - \frac{1}{6\rho}L^2(\frac{1}{\rho}\beta_{x1} - \gamma_{xi}\eta_1 - \alpha_{x1}\eta_1') + \frac{1}{4\rho^2}L^3\alpha_{x1} - \frac{3}{40\rho^2}L^4\gamma_{xi} \; ;$$

$$[\beta_x\eta] = [\beta_x]\cdot[\eta] - \frac{1}{12}L^2(\alpha_{x1}\eta_2' + \alpha_{x2}\eta_1' - \frac{8}{15\rho}L^2\gamma_{xi}) \qquad (24)$$

and, if $\frac{1}{\rho} = K = 0$,

$$[\beta_u\eta] = \frac{1}{2}(\eta_1 + \eta_2)[\beta_u] - \frac{1}{12}L^2(\alpha_{u1}\eta_2' + \alpha_{u2}\eta_1')$$

5.) Most equations introduced in this section exhibit a symmetric appearance of the functions at the two edges so that the contributions from the two halves of the magnet will be the same if the function in question is mirror symmetric in the magnet. This may explain why the expressions using functions at the two edges are simpler than those using functions at only one edge, say at the entrance. In the case where $F_u \neq 0$, it is interesting that all the expressions proved in this section don't depend on which mathematical functions are used to describe β_u or η in the magnet. In fact, even no consideration was given to such descriptions.

C. Integrals Expressed by Function Values at Magnet Midpoint

1.) One can make use of the symmetry of the integrand functions in an alternative way, that is, by expressing them with functions evaluated at the midpoint of the magnet. The Appendix attached describes three functions, with the aid of which the expressions required can be much simplified. The functions are defined as

$$C_u(z) = \sum_{n=0}^{\infty} (-F_u)^n \, z^{2n} / (2n)! \quad = \begin{cases} \cos(\sqrt{F_u}\, z) \,, & \text{if } F_u > 0 \,, \\ 1 \,, & \text{if } F_u = 0 \,, \\ \cosh(\sqrt{-F_u}\, z) \,, & \text{if } F_u < 0 \,; \end{cases}$$

$$S_u(z) = \sum_{n=0}^{\infty} (-F_u)^n \, z^{2n+1} / (2n+1)! = \begin{cases} \sin(\sqrt{F_u}\, z)/\sqrt{F_u} \,, & \text{if } F_u > 0 \,, \\ z \,, & \text{if } F_u = 0 \,, \\ \sinh(\sqrt{-F_u}\, z)/\sqrt{-F_u} \,, & \text{if } F_u < 0 \,; \end{cases}$$

$$D_u(z) = \sum_{n=0}^{\infty} (-F_u)^n \, z^{2n+2} / (2n+2)! = \begin{cases} (1 - C_u(z))/ F_u \,, & \text{if } F_u \neq 0 \,, \\ \frac{1}{2} z^2 \,, & \text{if } F_u = 0 \end{cases} \qquad (25)$$

Their properties are given in the Appendix in much detail.

Let m denote the midpoint of the magnet. The main machine functions are given in terms of the functions defined by Eq.(25) as

$$\beta_u(z) = \beta_{um} C_u^2(z - z_m) + \gamma_{um} S_u^2(z - z_m) - 2\, \alpha_{um} C_u(z - z_m) S_u(z - z_m) \;;$$

$$\alpha_u(z) = \alpha_{um} C_u^2(z - z_m) - F_u \alpha_{um} S_u^2(z - z_m) + (F_u \beta_{um} - \gamma_{um}) C_u(z - z_m) S_u(z - z_m) \;;$$

$$\gamma_u(z) = \gamma_{um} C_u^2(z - z_m) + F_u^2 \beta_{um} S_u^2(z - z_m) + 2\, F_u \alpha_{um} C_u(z - z_m) S_u(z - z_m) \;;$$

$$\eta(z) = \eta_m \, C_x(z - z_m) + \eta'_m \, S_x(z - z_m) + \frac{1}{\rho} D_x(z - z_m) \;;$$

$$\eta'(z) = \eta'_m \, C_x(z - z_m) + \left(\frac{1}{\rho} - F_x \eta_m \right) S_x(z - z_m) \qquad (26)$$

It is seen that the use of functions C_u, S_u and D_u makes function expressions independent on the sign of F_u. For example, if $F_u = 0$, Eq.(26) will automatically read the same as Eq.(23).

It is obvious that, if $f(z - z_m)$ is an odd function, $[\, f(z - z_m) \,] = 0$. This reduces the number of the terms one has to calculate almost to its half, since $S_u(z)$ is an odd function while both $C_u(z)$ and $D_u(z)$ are even functions.

2.) The terms involved in the integrals are treated one by one as follows. The details can be found in the Appendix. For brevity, the variable of the functions in the following expressions will be omitted if it is $(z - z_m)$. Some terms are named as P_i, $(i = 1, 2, \ldots 6)$ to keep the succeeding expressions independent on whether F_u is 0.

$$[\, C_u \,] = \frac{2}{L} S_u(L/2) \quad ; \qquad [\, C_u^2 \,] = \frac{1}{2} \left(1 + S_u(L)/L \right) \quad ;$$

$$[\, C_u S_u^2 \,] = \frac{2}{3L} S_u^3(L/2) \quad ; \qquad [\, C_u^3 \,] = \frac{2}{L} S_u(L/2) - \frac{2}{3L} F_u S_u^3(L/2) \quad ;$$

$$[\, S_u^2 \,] = P_1 = \begin{cases} \frac{1}{2F_u} \left(1 - S_u(L)/L \right) , & \text{if } F_u \neq 0 , \\ \frac{1}{12} L^2 , & \text{if } F_u = 0 \ ; \end{cases}$$

$$[\, D_u \,] = P_2 = \begin{cases} \frac{1}{F_u} \left(1 - \frac{2}{L} S_u(L/2) \right) , & \text{if } F_u \neq 0 , \\ \frac{1}{24} L^2 , & \text{if } F_u = 0 \ ; \end{cases}$$

$$[\, S_u^2 D_u \,] = P_3 = \begin{cases} \frac{1}{F_u} \left(P_1 - \frac{2}{3L} S_u^3(L/2) \right) , & \text{if } F_u \neq 0 , \\ \frac{1}{160} L^4 , & \text{if } F_u = 0 \ ; \end{cases}$$

$$[\, C_u D_u \,] = P_1 - P_2 \quad ; \qquad [\, C_u^2 D_u \,] = P_2 - F_u P_3 = P_2 - P_1 + \frac{2}{3L} S_u^3(L/2) \quad ;$$

$$[\, D_u^2 \,] = P_4 = \begin{cases} \frac{1}{F_u} \left(2 P_2 - P_1 \right) , & \text{if } F_u \neq 0 , \\ \frac{1}{320} L^4 , & \text{if } F_u = 0 \ ; \end{cases}$$

$$[\, C_u^2 S_u^2 \,] = P_5 = \begin{cases} \frac{1}{8F_u} \left(1 - \frac{1}{2L} S_u(2L) \right) , & \text{if } F_u \neq 0 , \\ \frac{1}{12} L^2 , & \text{if } F_u = 0 \ ; \end{cases}$$

$$[\, C_u^4 \,] = \frac{1}{2} \left(1 + S_u(L)/L \right) - F_u P_5 \quad ;$$

$$[\, S_u^4 \,] = P_6 = \begin{cases} \frac{1}{F_u} \left(P_1 - P_5 \right) , & \text{if } F_u \neq 0 , \\ \frac{1}{80} L^4 , & \text{if } F_u = 0 \end{cases} \qquad (27)$$

It is convenient in writing programs to have some more parameters defined as

$$Q_1 = [\, C_u \,] \; ; \qquad Q_2 = [\, C_u^{\,2} \,] \; ; \qquad Q_3 = [\, C_u \, S_u^{\,2} \,] \tag{28}$$

If $F_u = 0$, the definition of $S_u(z)$, Eq.(25), gives that $Q_1 = Q_2 = 1$, $Q_3 = \frac{1}{12} L^2$.

3.) Using Eq.(26), one can express any machine functions he is interested in by C_u, S_u and D_u, such as:

$$\beta_u^{\,2}(z) = \beta_{um}^2 C_u^{\,4} + \gamma_{um}^2 S_u^{\,4} + 2(1+3\alpha_{um}^2) C_u^{\,2} S_u^{\,2} - 4\alpha_{um}(\beta_{um} C_u^{\,2} + \gamma_{um} S_u^{\,2}) C_u S_u \; ;$$

$$\eta^2(z) = \eta_m^2 C_x^{\,2} + \eta_m'^2 S_x^{\,2} + \frac{1}{\rho^2} D_x^{\,2} + \frac{2}{\rho}\eta_m C_x D_x + \frac{2}{\rho}\eta'_m S_x D_x + 2\eta_m \eta'_m C_x S_x \; ;$$

$$\eta'^2(z) = \eta_m'^2 C_x^{\,2} + \left(\frac{1}{\rho} - F_x \eta_m\right)^2 S_x^{\,2} + 2\eta'_m\left(\frac{1}{\rho} - F_x \eta_m\right) C_x S_x \; ;$$

$$\beta_x \eta = \beta_{xm}\eta_m C_x^{\,3} + (\gamma_{xm}\eta_m - 2\alpha_{xm}\eta'_m) C_x S_x^{\,2} + \frac{1}{\rho}\beta_{xm} C_x^{\,2} D_x + \frac{1}{\rho}\gamma_{xm} S_x^{\,2} D_x$$
$$+ \gamma_{xm}\eta'_m S_x^{\,3} + (\beta_{xm}\eta'_m - 2\alpha_{xm}\eta_m) C_x^{\,2} S_x - \frac{2}{\rho}\alpha_{xm} C_x S_x D_x \tag{29}$$

It may be a surprise that the formula for $\mathcal{H}$ is relatively very simple. The results are:

$$\alpha_x \eta + \beta_x \eta' = (\alpha_{xm}\eta_m + \beta_{xm}\eta'_m) C_x - (\alpha_{xm}\eta'_m + \gamma_{xm}\eta_m) S_x$$
$$+ \frac{1}{\rho}\beta_{xm} C_x S_x + \frac{1}{\rho}\gamma_{xm} S_x D_x + \frac{1}{\rho}\alpha_{xm}(D_x - 2 S_x^{\,2}) \; ;$$

$$\alpha_x \eta' + \gamma_x \eta = (\alpha_{xm}\eta'_m + \gamma_{xm}\eta_m) C_x - (\alpha_{xm}\eta_m + \beta_{xm}\eta'_m) F_x S_x$$
$$+ \frac{1}{\rho}\beta_{xm} F_x S_x^{\,2} - \frac{1}{\rho}\gamma_{xm} C_x D_x + \frac{1}{\rho}\alpha_{xm} S_x (2 C_x - 1) \; ;$$

$$\mathcal{H} = \eta(\alpha_x \eta' + \gamma_x \eta) + \eta'(\alpha_x \eta + \beta_x \eta')$$
$$= \mathcal{H}_m + \frac{2}{\rho}(\beta_{xm}\eta'_m + \alpha_{xm}\eta_m) S_x - \frac{2}{\rho}(\alpha_{xm}\eta'_m + \gamma_{xm}\eta_m) D_x$$
$$+ \frac{1}{\rho^2}(\beta_{xm} S_x^{\,2} + \gamma_{xm} D_x^{\,2} - 2\alpha_{xm} S_x D_x) \tag{30}$$

where $\mathcal{H}_m = \gamma_{xm} \eta_m^2 + 2\, \alpha_{xm} \eta_m \eta'_m + \beta_{xm} \eta'^2_m$.

Now there is no difficulty for one to arrive at

$$[\beta_u] = \beta_{um} Q_2 + \gamma_{um} P_1 \quad ;$$

$$[\gamma_u] = \gamma_{um} Q_2 + F_u^2 \beta_{um} P_1 \quad ;$$

$$[\eta] = \eta_m Q_1 + \frac{1}{\rho} P_2 \quad ;$$

$$[\beta_x \eta] = \beta_{xm} \eta_m (Q_1 - F_x Q_3) + (\gamma_{xm} \eta_m - 2\, \alpha_{xm} \eta'_m) Q_3 + \frac{1}{\rho} \beta_{xm} (P_2 - F_x P_3) + \frac{1}{\rho} \gamma_{xm} P_3 \quad ;$$

$$[\mathcal{H}] = \mathcal{H}_m - \frac{2}{\rho} (\alpha_{xm} \eta'_m + \gamma_{xm} \eta_m) P_2 + \frac{1}{\rho^2} (\beta_{xm} P_1 + \gamma_{xm} P_4) \quad ;$$

$$[\eta^2] = \eta_m^2 Q_2 + \eta'^2_m P_1 + \frac{1}{\rho^2} P_4 + \frac{2}{\rho} \eta_m (P_1 - P_2) \quad ;$$

$$[\eta'^2] = \eta'^2_m Q_2 + (\frac{1}{\rho} - F_x \eta_m)^2 P_1 \quad ;$$

$$[\beta_u^2] = \beta_{um}^2 (Q_2 - F_u P_5) + \gamma_{um}^2 P_6 + 2(1 + 3\alpha_{um}^2) P_5 \qquad (31)$$

For clarity, the parameters $Q_i (i=1,2,3)$ and $P_i (i=1,2...6)$ are given again by

$$Q_1 = \frac{2}{L} S_u(L/2) \quad ; \quad Q_2 = \frac{1}{2} (1 + S_u(L)/L) \quad ; \quad Q_3 = \frac{2}{3L} S_u^3(L/2) \quad ;$$

$$P_1 = \begin{cases} \frac{1}{F_u} (1 - Q_2) , & \text{if } F_u \neq 0 , \\ Q_3 , & \text{if } F_u = 0 ; \end{cases}$$

$$P_2 = \begin{cases} \frac{1}{F_u} (1 - Q_1) , & \text{if } F_u \neq 0 , \\ \frac{1}{2} Q_3 , & \text{if } F_u = 0 ; \end{cases}$$

$$P_3 = \begin{cases} \frac{1}{F_u} (P_1 - Q_3) , & \text{if } F_u \neq 0 , \\ 0.15\, L^2 P_2 , & \text{if } F_u = 0 ; \end{cases}$$

$$P_4 = \begin{cases} \frac{1}{F_u} (2 P_2 - P_1) , & \text{if } F_u \neq 0 , \\ \frac{1}{2} P_3 , & \text{if } F_u = 0 ; \end{cases}$$

$$P_5 = \begin{cases} \frac{1}{8F_u}(1 - S_u(2L)/(2L)) , & \text{if } F_u \neq 0 , \\ Q_3 , & \text{if } F_u = 0 ; \end{cases}$$

$$P_6 = \begin{cases} \frac{1}{F_u}(P_1 - P_5) , & \text{if } F_u \neq 0 , \\ 2 P_3 , & \text{if } F_u = 0 \end{cases} \qquad (32)$$

where P_5 and P_6 are only used in calculating $[\beta_u^2]$. All these formulae, Eqs. (31) and (32), can be carried out by a program very easily. Readers who check them will find that, after the functions at magnet edges are evaluated with Eq.(26), all these formulae are well equivalent to those introduced in the last section. An advantage of Eqs.(31) and (32) is that they are general enough to cover all commonly used magnet types. The sign of F_u only influences how to evaluate Q_i's and P_i's.

For a rough estimate, one may expand Q_i's and P_i's as power series in L and use the first several terms only. The series read

$$Q_1 = 1 - \frac{1}{24} F_u L^2 + \frac{1}{1920} F_u^2 L^4 - \ldots ;$$

$$Q_2 = 1 - \frac{1}{12} F_u L^2 + \frac{1}{240} F_u^2 L^4 - \ldots ; \qquad Q_3 = \frac{1}{12} L^2 (1 - \frac{1}{8} F_u L^2 + \ldots) ;$$

$$P_1 = \frac{1}{12} L^2 (1 - \frac{1}{20} F_u L^2 + \ldots) ; \qquad P_2 = \frac{1}{24} L^2 (1 - \frac{1}{80} F_u L^2 + \ldots) ;$$

$$P_3 = \frac{1}{160} L^4 - \ldots ; \qquad P_4 = \frac{1}{320} L^4 - \ldots ;$$

$$P_5 = \frac{1}{12} L^2 (1 - \frac{1}{5} F_u L^2 + \ldots) ; \qquad P_6 = \frac{1}{80} L^4 - \ldots \qquad (33)$$

4.) The functions at the midpoit as well as $S_u(L/2)$ can be found by making use of a half-element transfer matrix. Usually this is only needed for each bending magnet. A display of the function values at all the bending magnet **midpoints may be considered worth doing, especially if the machine is to be a synchrotron radiation source. If this is not preferred, Eq.(26) can be used to give the relations between the functions at the midpoint and those at the two edges, the latter are usually calculated by every program. Since the whole-element transfer matrix must have been known, one can get $C_u(L) = M_{11}$, $S_u(L) = M_{12}$ on either x or y plane. Then the required functions are given by**

$$C_u(L/2) = \left(\frac{1}{2}\,(1 + C_u(L))\right)^{1/2} \quad ; \quad S_u(L/2) = S_u(L)/(2\, C_u(L/2)) \quad ;$$

$$D_x(L/2) = S_x^{\,2}(L/2)/(1 + C_x(L/2)) \quad \left(\text{or} \begin{cases} \frac{1}{F_x}\,(1 - C_x(L/2)) \;, & \text{if } F_x \neq 0 \;, \\ \frac{1}{8}\; L^2 \;, & \text{if } F_x = 0 \end{cases} \right) ;$$

$$S_u(2\,L) = 2\, S_u(L)\; C_u(L) \quad ;$$

$$\beta_{um} = \frac{1}{2}\,(\beta_{u1} + \beta_{u2} + S_u(L/2)\,(\alpha_{u2} - \alpha_{u1})/C_u(L/2)) \quad ;$$

$$\alpha_{um} = (\beta_{u1} - \beta_{u2})/(2\, S_u(L)) \quad \left(\text{or} \quad (\alpha_{u1} + \alpha_{u2})/(2\, C_u(L)) \right) \quad ;$$

$$\eta_m = \frac{1}{2}\,(\eta_1 + \eta_2 - \frac{2}{\rho}\, D_x(L/2))/C_x(L/2) \quad ;$$

$$\eta'_m = (\eta_2 - \eta_1)/(2\, S_x(L/2)) \quad \left(\text{or} \quad (\eta'_1 + \eta'_2)/(2\, C_x(L/2)) \right) \qquad (34)$$

5.) Separate function type is perhaps most commonly adopted nowadays in machine design. More attention is therefore paid to this special case in which, for all the bending magnets, K = 0 and consequently $F_x = 1/\rho^2$, $F_y = 0$. The following formulae can be used in a program specially made for this case:

$$[\eta] = \rho + (\eta_m - \rho)\, \sin(\theta_B/2)/(\theta_B/2) \quad ;$$

$$[\mathcal{H}] = \mathcal{H}_m - 2\rho\,(\alpha_{xm}\,\eta'_m + \gamma_{xm}(\eta_m - \rho))\,(1 - \sin(\theta_B/2)/(\theta_B/2))$$
$$+ \frac{1}{2}\,(\beta_{xm} - \rho^2\,\gamma_{xm})\,(1 - \sin\theta_B/\theta_B) \quad ;$$

$$[\beta_x] = \frac{1}{2}\,\beta_{xm}(1 + \sin\theta_B/\theta_B) + \frac{1}{2}\,\rho^2\,\gamma_{xm}(1 - \sin\theta_B/\theta_B) \quad ;$$

$$[\gamma_x] = \frac{1}{2}\,\gamma_{xm}(1 + \sin\theta_B/\theta_B) + \frac{1}{2\rho^2}\,\beta_{xm}(1 - \sin\theta_B/\theta_B) \quad ;$$

$$[\beta_x^{\,2}] = \frac{1}{2}\,\beta_{xm}^2(1 + \sin\theta_B/\theta_B) + \frac{1}{2}\,\rho^4\,\gamma_{xm}^2(1 - \sin\theta_B/\theta_B)$$
$$+ \frac{1}{8}\,\rho^2(2 + 6\,\alpha_{xm}^2 - \frac{1}{\rho^2}\beta_{xm}^2 - \rho^2\,\gamma_{xm}^2)\,(1 - \sin 2\theta_B/(2\,\theta_B)) \quad ;$$

$$[\eta^2]= \frac{1}{2}(\eta_m-\rho)^2(1+\sin\theta_B/\theta_B)+\frac{1}{2}\rho^2\eta_m'^2(1-\sin\theta_B/\theta_B)$$
$$+\rho^2+2\rho(\eta_m-\rho)\sin(\theta_B/2)/(\theta_B/2) \quad ;$$
$$[\eta'^2]= \frac{1}{2}\eta_m'^2(1+\sin\theta_B/\theta_B)+\frac{1}{2\rho^2}(\eta_m-\rho)^2(1-\sin\theta_B/\theta_B) \quad ;$$
$$[\beta_x\eta]= \beta_{xm}(\eta_m-\rho)(1-\frac{1}{3}\sin^2(\theta_B/2))\sin(\theta_B/2)/(\theta_B/2)+\frac{\rho}{2}\beta_{xm}(1+\sin\theta_B/\theta_B)$$
$$+\frac{1}{2}\rho^2\gamma_{xm}(1-\sin\theta_B/\theta_B)+\frac{1}{3}\rho^2(\gamma_{xm}(\eta_m-\rho)-2\alpha_{xm}\eta_m')\sin^3(\theta_B/2)/(\theta_B/2);$$
$$[\beta_y]= \beta_{ym}+\frac{1}{12}\rho^2\gamma_{ym}\theta_B^{\,2} \quad ; \qquad [\gamma_y]= \gamma_{ym} \quad ;$$
$$[\beta_y^{\,2}]= \beta_{ym}^2+\frac{1}{6}\rho^2(1+3\alpha_{ym}^2)\theta_B^{\,2}+\frac{1}{80}\rho^4\gamma_{ym}^2\theta_B^{\,4} \qquad (35)$$

where $\theta_B = L/\rho$ is the bending angle. Usually $[\beta_x\eta]$ is not needed in this case.

The first two of Eq.(35) are much more significant than the rest. In the procedure of machine design, ρ and θ_B of every bending magnet are usually decided before lattice optimization. So, during lattice optimization, $[\eta]$ is determined by η_m alone and, therefore, the momentum compaction factor is linearly dependent on η_m of every bending magnet and can be made a "fit function" of the program. With all the θ_B-dependent coefficients precalculated, $[\mathcal{H}]$ is determined so fast that its minimization can also be set as a criterion of optimization.

The partition numbers J_x, J_y and J_E are also related to $[\eta]$. The formulae are[2), 3)]

$$J_x = 1-\mathcal{D} \; ; \qquad J_y = 1 \; ; \qquad J_E = 2+\mathcal{D} \; ; \qquad \mathcal{D} = I_4 / I_2$$

where the machine integrals I_4 and I_2 are given by

$$I_4 = \sum_B \frac{1}{\rho}\left(\frac{1}{\rho^2}+2K\right)\int\eta\, dz - \sum_e \frac{1}{\rho^2}\eta_e \tan\theta_e \quad ; \quad I_2 = \sum_B \int \frac{1}{\rho^2}dz \qquad (36)$$

$\sum_B$ and $\sum_e$ denote summations for all the bending magnets and all the bending magnet edges, respectively.

Suppose $F = 0$ and edge angles $\theta_1 = \theta_2 = \theta_e$ in every bending magnet. Then the contribution from a bending magnet and its edges to I_4 is

$$I_{4(B)} = \frac{1}{\rho}(\theta_B - 2\tan\theta_e) + \frac{2}{\rho^2}(\eta_m - \rho)(\sin(\theta_B/2) - \cos(\theta_B/2)\tan\theta_e)$$

I_4 and thus the partition numbers are all determined by η_m alone. Especially, if the bending magnet is rectangular, that is, $\theta_e = \theta_B/2$, then

$$I_4 = \sum_B \frac{1}{\rho}(\theta_B - 2\tan(\theta_B/2))$$

is entirely independent on lattice configurations, provided that ρ and θ_B are chosen already. Furthermore, if ρ is identical for all the bending magnets, then

$$\mathcal{D} = 1 - \frac{1}{\pi}\sum_B \tan(\theta_B/2) \qquad (37$$

This means the partition numbers are determined by θ_B alone. If all the bending magnets are wholly identical, flat (no gradient) and rectangular, then

$$J_x = \tan(\theta_B/2)/(\theta_B/2) = \tan\theta_e/\theta_e \quad ; \quad J_y = 1 \quad ; \quad J_E = 3 - J_x \qquad (38$$

J_x is greater than 1 but very close to 1.

In the calculation of I_4, effects of bending magnets and their edges are combined and it seems that the formula can be simplified to the greatest extent when the magnets are flat and rectangular. Similar attempts are made for first order chromaticity calculation, in which a similar combination takes place. But the results are not very satisfactory, giving a relatively simple formula for ξ_x and a complicated one for ξ_y.[5)]

6.) Two more integrals are sometimes useful in solving problems and their evaluations also benefit from the properties of C_u, S_u and D_u. They are

$$\int_{z_1}^{z_2}\sqrt{\beta_u}\sin(\psi_u - \psi_{u1})\,dz \quad \text{and} \quad \int_{z_1}^{z_2}\sqrt{\beta_u}\cos(\psi_u - \psi_{u1})\,dz$$

where $\psi_u - \psi_{u1} = \int_{z_1}^{z} (1/\beta_u(\bar{z}))\, d\bar{z}$ is the phase advance from z_1 to another point in the magnet, indicated by z. The relation between transfer matrix elements and β function gives[4]

$$C_u(z-z_1) = \sqrt{\beta_u(z)/\beta_{u1}}\,(\cos(\psi_u(z)-\psi_{u1}) + \alpha_{u1}\sin(\psi_u(z)-\psi_{u1}))\quad ;$$

$$S_u(z-z_1) = \sqrt{\beta_u(z)\cdot\beta_{u1}}\,\sin(\psi_u(z)-\psi_{u1}) \qquad (39)$$

Since $S_u(z) = \int C_u(z)dz$, $D_u(z) = \int S_u(z)dz$, one can soon obtain

$$\int_{z_1}^{z_2}\sqrt{\beta_u}\,\sin(\psi_u-\psi_{u1})\,dz = D_u(L)/\sqrt{\beta_{u1}}\quad ;$$

$$\int_{z_1}^{z_2}\sqrt{\beta_u}\,\cos(\psi_u-\psi_{u1})\,dz = (\beta_{u1}\,S_u(L) - \alpha_{u1}\,D_u(L))/\sqrt{\beta_{u1}} \qquad (40)$$

Of course, these two integrals can also be expressed by functions at z_2 or z_m. If the phase advance is written as $\psi_u - \psi_{um} = \int_{z_m}^{z}(1/\beta_u)\,d\bar{z}$, one gets

$$\int_{z_1}^{z_2}\sqrt{\beta_u}\sin(\psi_u-\psi_{um})dz = 0\quad ;\quad \int_{z_1}^{z_2}\sqrt{\beta_u}\cos(\psi_u-\psi_{um})dz = 2\sqrt{\beta_{um}}S_u(L/2) \qquad (41)$$

Eq.(41) looks much simpler than but is equivalent to Eq.(40).

* * * * *

All the equations introduced above have been carefully checked to assure their mathematical correctness. Most of them have been used in programs and they gave exactly the same results as obtained from other programs, though the formulae adopted by the latter are more complicated.

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Attached to "TECHNIQUES IN MACHINE FUNCTION INTEGRAL CALCULATIONS"

APPENDIX

Functions $C_u(z)$, $S_u(z)$ and $D_u(z)$

1.) This appendix describes three functions and presents a summary of their valuable properties. The functions are dependent both on a parameter F_u, that is the focusing strength on u plane, and on a variable z, that is usually the azimuthal coordinate. u is understood to be x or y, corresponding to horizontal or vertical plane respectively. If expressed by these functions, most formulae commonly used in accelerator physics will give a uniform appearance.

The functions are defined as

$$C_u(z) = \sum_{n=0}^{\infty} (-F_u)^n \, z^{2n} / (2n)! = \begin{cases} \cos(\sqrt{F_u}\, z) , & \text{if } F_u > 0 , \\ 1 , & \text{if } F_u = 0 , \\ \cosh(\sqrt{-F_u}\, z) , & \text{if } F_u < 0 \end{cases} \qquad \text{(A1)}$$

$$S_u(z) = \sum_{n=0}^{\infty} (-F_u)^n \, z^{2n+1} / (2n+1)! = \begin{cases} \sin(\sqrt{F_u}\, z)/\sqrt{F_u} , & \text{if } F_u > 0 , \\ z , & \text{if } F_u = 0 , \\ \sinh(\sqrt{-F_u}\, z)/\sqrt{-F_u} , & \text{if } F_u < 0 \end{cases} \qquad \text{(A2)}$$

$$D_u(z) = \sum_{n=0}^{\infty} (-F_u)^n \, z^{2n+2} / (2n+2)! = \begin{cases} (1 - C_u(z))/ F_u , & \text{if } F_u \neq 0 , \\ \frac{1}{2} z^2 , & \text{if } F_u = 0 \end{cases} \qquad \text{(A3)}$$

All of them are continuous either with respect to z or with respect to F_u, even in the vicinity of $F_u = 0$.

They may be named as cosine-like function, sine-like function and dispersion-arising function respectively.

2.) The fundamental properties of these functions are as follows:

Let $'$ denote d/dz . $C_u(z)$ is the cosine-like solution of the differential equation $u'' + F_u\ u = 0$, where F_u is a constant, no matter whether positive, zero or negative. $S_u(z)$ is the sine-like solution of the equation. $D_u(z)$ is the particular solution of equation $u'' + F_u\ u = 1$, with initial value and initial first derivative both equal to zero. Expressed by formulae, that is

$$C_u'' + F_u\ C_u = 0\ ;\quad C_u(0) = 1\ ;\quad C_u'(0) = 0\ ;$$

$$S_u'' + F_u\ S_u = 0\ ;\quad S_u(0) = 0\ ;\quad S_u'(0) = 1\ ;$$

$$D_u'' + F_u\ D_u = 1\ ;\quad D_u(0) = 0\ ;\quad D_u'(0) = 0 \qquad \text{(A4)}$$

So, if magnet length is measured in meters, F_u is in m^{-2} and C_u in unit, S_u in m, D_u in m^2. If one tries to solve Eq.(A4) by series, the results will be just the definition equations(A1), (A2) and (A3).

3.) In a sense these functions are pseudo-trigonometric functions, among which C_u and D_u are even functions while S_u is odd. One can give

$$C_u(-z) = C_u(z)\ ;\quad S_u(-z) = -S_u(z)\ ;\quad D_u(-z) = D_u(z) \qquad \text{(A5)}$$

and $C_u(z_1 + z_2) = C_u(z_1)\cdot C_u(z_2) - F_u\cdot S_u(z_1)\cdot S_u(z_2)$;

$$S_u(z_1 + z_2) = S_u(z_1)\cdot C_u(z_2) + C_u(z_1)\cdot S_u(z_2) \qquad \text{(A6)}$$

Combination of Eqs.(A5) and (A6) makes almost all the trigonometrical invariant equations still valid for S_u and C_u after necessary modification. For example,

$$C_u^2(z) + F_u\ S_u^2(z) = 1 \qquad \text{(A7)}$$

$$C_u(2z) = C_u^2(z) - F_u S_u^2(z) = 2\,C_u^2(z) - 1 = 1 - 2\,F_u S_u^2(z) \quad ;$$

$$S_u(2z) = 2\,S_u(z)\,C_u(z) \quad ;$$

$$\frac{F_u\,S_u(z/2)}{C_u(z/2)} = \frac{F_u\,S_u(z)}{1 + C_u(z)} = \frac{1 - C_u(z)}{S_u(z)} \qquad \text{(A8)}$$

From Eq.(A3), one gets

$$C_u(z) + F_u\,D_u(z) = 1 \qquad \text{(A9)}$$

Therefore, the relation among D_u, S_u and C_u is

$$D_u(z) = S_u^2(z)/(1 + C_u(z)) = 2\,S_u^2(z/2) \qquad \text{(A10)}$$

or $$S_u^2(z) - C_u(z)\,D_u(z) = D_u(z) \qquad \text{(A11)}$$

Eqs.(A7), (A9) and (A11) are the three invariant equations used most frequently in formula simplification.

4.) The derivatives of the functions with respect to z are

$$C_u'(z) = -F_u\,S_u(z) \quad ; \quad S_u'(z) = C_u(z) \quad ; \quad D_u'(z) = S_u(z) \qquad \text{(A12)}$$

So $D_u(z)$ can also be defined as $\int_0^z S_u(\bar{z})\,d\bar{z}$.

Because these functions keep continuous when F_u varies, one can get their derivatives with respect to F_u, which also present a uniform appearance well independent on the sign of F_u.

$$\partial C_u(z)/\partial F_u = -\frac{1}{2}\,z\,S_u(z) \quad ;$$

$$\partial S_u(z)/\partial F_u = \begin{cases} \frac{1}{2F_u}(z\,C_u(z) - S_u(z)) \;, & \text{if } F_u \neq 0 \;, \\ -\frac{1}{6}\,z^3 \;, & \text{if } F_u = 0 \;; \end{cases}$$

$$\partial D_u(z)/\partial F_u = \begin{cases} -\frac{1}{F_u}\left(D_u(z) - \frac{1}{2} z\, S_u(z)\right) , & \text{if } F_u \neq 0 , \\ -\frac{1}{24} z^4 , & \text{if } F_u = 0 ; \end{cases}$$

$$\partial(-F_u S_u(z))/\partial F_u = -\frac{1}{2}(z\, C_u(z) + S_u(z)) \qquad \text{(A13)}$$

For the relation among the derivatives one has

$$\partial D_u(z)/\partial F_u = 4\cdot S_u(z/2)\cdot \partial(S_u(z/2))/\partial F_u \; ; \quad (\partial D_u/\partial F_u)' = \partial S_u/\partial F_u \; ;$$

$$(\partial S_u/\partial F_u)' = \partial C_u/\partial F_u \; ; \quad (\partial C_u/\partial F_u)' = \partial(-F_u S_u)/\partial F_u \qquad \text{(A14)}$$

Let W_u represent either C_u, S_u or D_u. Function $\partial W_u/\partial F_u$ satisfies

$$(\partial W_u/\partial F_u)'' + F_u(\partial W_u/\partial F_u) = -W_u \; ;$$

$$(\partial W_u/\partial F_u)|_{z=0} = (\partial W_u/\partial F_u)'|_{z=0} = 0 \qquad \text{(A15)}$$

The differential equation can be directly obtained by deriving the equation $W_u'' + F_u W_u = 0$ or 1 with respect to F_u. Functions $(\partial W_u/\partial F_u)$ are useful in finding the linear dependence of a transfer matrix on the focusing strength.

For the linear dependence of a transfer matrix on the coupling strength from the other transverse plane, another group of functions can help. They are defined as:

$$\Delta W_u/\Delta F_u = \begin{cases} (W_x - W_y)/(F_x - F_y) , & \text{if } F_x \neq F_y , \\ \partial W_x/\partial F_x , & \text{if } F_x = F_y \end{cases} \qquad \text{(A16)}$$

where W_u may be C_u, S_u, D_u or $-F_u S_u$. This group of functions satisfies

$$(\Delta D_u/\Delta F_u)' = \Delta S_u/\Delta F_u \; ; \quad (\Delta S_u/\Delta F_u)' = \Delta C_u/\Delta F_u \; ;$$

$$(\Delta C_u/\Delta F_u)' = \Delta(-F_u S_u)/\Delta F_u \; ; \quad (\Delta W_u/\Delta F_u)|_{z=0} = 0 \; ;$$

$$(\Delta W_u/\Delta F_u)'' + F_x\,(\Delta W_u/\Delta F_u) = -W_y \; ;$$

$$(\Delta W_u/\Delta F_u)'' + F_y\,(\Delta W_u/\Delta F_u) = -W_x \qquad \text{(A17)}$$

5.) The standard form of the first order particle motion equation in a magnet is

$$u'' + F_u\, u = \frac{\delta}{\rho_u} \qquad \text{(A18)}$$

where u is x or y, δ is energy deviation, ρ_u is the curvature radius of the ideal central orbit on u plane. ρ_u and F_u are constant within a magnet, and they are related with magnetic field components by

$$1/\rho_x = B_y/(B\rho)_0 \;; \qquad 1/\rho_y = -B_x/(B\rho)_0 \;;$$

$$F_x = (\partial B_y/\partial x)/(B\rho)_0 + (1/\rho_x)^2 \;; \qquad F_y = -(\partial B_y/\partial x)/(B\rho)_0 + (1/\rho_y)^2$$

where $(B\rho)_0$ is the particle rigidity.

Let u_o and u'_o denote $u|_{z=z_o}$ and $u'|_{z=z_o}$ respectively. The solution of Eq.(A18) in the magnet is

$$u(z) = u_o\, C_u(z-z_o) + u'_o\, S_u(z-z_o) + \frac{\delta}{\rho_u} D_u(z-z_o) \;;$$

$$u'(z) = u'_o\, C_u(z-z_o) + \left(\frac{\delta}{\rho_u} - F_u\, u_o\right) S_u(z-z_o) \qquad \text{(A19)}$$

Therefore, in the theory of transfer matrices, the matrix of an L-meter-long magnet reads

$$M_u(L) = \begin{pmatrix} C_u(L) & S_u(L) & D_u(L)/\rho_u \\ -F_u \cdot S_u(L) & C_u(L) & S_u(L)/\rho_u \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{(A20)}$$

Some computer programs need the derivatives of the transfer matrix with respect to the focusing strength or the length of the magnet in order to get the linear dependence of machine parameters. The deriatives can be expressed by

$$\frac{\partial M_u}{\partial L} = \begin{pmatrix} -F_u \cdot S_u(L) & C_u(L) & S_u(L)/\rho_u \\ -F_u \cdot C_u(L) & -F_u \cdot S_u(L) & C_u(L)/\rho_u \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{(A21)}$$

$$\frac{\partial M_u}{\partial F_u} = \begin{pmatrix} -\frac{1}{2} L \cdot S_u(L) & \partial S_u(L)/\partial F_u & (\partial D_u(L)/\partial F_u)/\rho_u \\ -\frac{1}{2}(L \cdot C_u(L) + S_u(L)) & -\frac{1}{2} L \cdot S_u(L) & (\partial S_u(L)/\partial F_u)/\rho_u \\ 0 & 0 & 0 \end{pmatrix} \quad \text{(A22)}$$

where $\partial S_u(L)/\partial F_u$ and $\partial D_u(L)/\partial F_u$ are evaluated by Eq.(A13) with $z = L$.

Whatever value F_u is, Eqs.(A20), (A21) and (A22) as well as all the other equations introduced in this appendix keep correct. This helps to make a universal subroutine program for calculating all the elements of either a transfer matrix or its derivative matrices. The subroutine is as short as about 50 lines but able to cover almost all the cases one usually meets with (except the matrices for magnet edges). Input information is 4 arguments: F_u, $1/\rho_u$, L and an integer number indicating which are wanted as output — the elements of the transfer matrix of the magnet, or of the derivative matrix with respect to F_u or of the derivative matrix with respect to L. Here what the word "magnet" means is a quadrupole, a bending magnet or a drift. The matrix may represent the motion on either x or y plane. The only condition is that F_u and $1/\rho_u$ remain unchanged within the length L. An explanation for the sign of the parameters is as follows.

Focusing strength F_u is positive for focusing magnets, negative for defocusing magnets, or zero for non-focusing elements such as a drift. Magnetic field $1/\rho_u$ is positive for normally (inward) bending magnets, negative for reversely (outward) bending magnets, or zero for non-bending elements. For example, $1/\rho_y$ is always zero in a machine with only horizontal bending. Effective length L is usually positive. If L is negative, output will be the inverse transfer matrix, in other words,

$$M_u(-L) = (M_u(L))^{-1} \quad \text{or} \quad M_u(-L) \cdot M_u(L) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

If $L = 0$, M_u will be the unit matrix and $\partial M_u/\partial F_u$ will be the zero matrix, whatever F_u and $1/\rho_u$ are.

Not only the matrix elements, but also all the widely used machine functions can be given a uniform, simple description. Let index o denote function value at point z_o and suppose F_u and $1/\rho_u$ are constant between z_o and z.

As a solution to equation $\eta'' + F_x \eta = \frac{1}{\rho_x}$,

$$\eta(z) = \eta_o C_x(z-z_o) + \eta'_o S_x(z-z_o) + \frac{1}{\rho_x} D_x(z-z_o) \quad ;$$

$$\eta'(z) = \eta'_o C_x(z-z_o) + \left(\frac{1}{\rho_x} - F_x \eta_o\right) S_x(z-z_o) \qquad \text{(A23)}$$

β function is a solution to equation $\beta_u''' + 4 F_u \beta_u' = 0$, which is obtained from the relations $\beta_u' = -2\alpha_u$, $\alpha_u' = F_u \beta_u - \gamma_u$ and $\gamma_u' = 2 F_u \alpha_u$ on the condition that $F_u' = 0$. Therefore,

$$\beta_u(z) = \beta_{uo} C_u^2(z-z_o) + \gamma_{uo} S_u^2(z-z_o) - 2\,\alpha_{uo} C_u(z-z_o) S_u(z-z_o) \quad ;$$

$$\alpha_u(z) = \alpha_{uo} C_u^2(z-z_o) - F_u \alpha_{uo} S_u^2(z-z_o) + (F_u \beta_{uo} - \gamma_{uo}) C_u(z-z_o) S_u(z-z_o) \quad ;$$

$$\gamma_u(z) = \gamma_{uo} C_u^2(z-z_o) + F_u^2 \beta_{uo} S_u^2(z-z_o) + 2 F_u \alpha_{uo} C_u(z-z_o) S_u(z-z_o) \qquad \text{(A24)}$$

6.) Some integrals are useful in parameter calculation. Here is a list of the indefinite integrals possibly involved:

$$\int C_u \, dz = S_u(z)$$

$$\int S_u \, dz = D_u(z)$$

$$\int D_u \, dz = \begin{cases} \frac{1}{F_u}(z - S_u(z)) , & \text{if } F_u \neq 0 , \\ \frac{1}{6} z^3 , & \text{if } F_u = 0 \end{cases}$$

$$\int C_u^2 \, dz = \frac{1}{2}(z + C_u(z) S_u(z)) = \frac{1}{2}\left(z + \frac{1}{2} S_u(2z)\right)$$

$$\int C_u S_u \, dz = \frac{1}{2} S_u^2(z) = \frac{1}{4} D_u(2z)$$

$$\int S_u^2 \, dz = \begin{cases} \frac{1}{2F_u}(z - C_u(z) S_u(z)) , & \text{if } F_u \neq 0 , \\ \frac{1}{3} z^3 , & \text{if } F_u = 0 \end{cases}$$

$$\left(\text{or} \int S_u^2 \, dz = \frac{1}{2}\left(S_u(z) D_u(z) + \int D_u \, dz \right) \right)$$

$$\int S_u D_u \, dz = \frac{1}{2} D_u^2(z)$$

$$\int C_u D_u \, dz = \int S_u^2 \, dz - \int D_u \, dz$$

$$\int D_u^2 \, dz = \begin{cases} \frac{1}{F_u} \left(2\int D_u \, dz - \int S_u^2 \, dz \right) , & \text{if } F_u \neq 0 , \\ \frac{1}{20} z^5 , & \text{if } F_u = 0 \end{cases}$$

$$\int C_u S_u^2 \, dz = \frac{1}{3} S_u^3(z)$$

$$\int C_u^3 \, dz = S_u(z) - \frac{1}{3} F_u \, S_u^3(z)$$

$$\int S_u^3 \, dz = D_u^2(z) \left(1 - \frac{1}{3} F_u \, D_u(z) \right)$$

$$\int C_u^2 S_u \, dz = D_u(z) \left(C_u(z) + \frac{1}{3} F_u^2 \, D_u^2(z) \right)$$

$$\int S_u^2 D_u \, dz = \begin{cases} \frac{1}{F_u} \left(\int S_u^2 \, dz - \frac{1}{3} S_u^3(z) \right) , & \text{if } F_u \neq 0 , \\ \frac{1}{10} z^5 , & \text{if } F_u = 0 \end{cases}$$

$$\int C_u^2 D_u \, dz = \int D_u \, dz - F_u \int S_u^2 D_u \, dz$$

$$\int C_u S_u D_u \, dz = \frac{1}{2} D_u^2(z) - \frac{1}{3} F_u \, D_u^3(z)$$

$$\int S_u D_u^2 \, dz = \frac{1}{3} D_u^3(z)$$

$$\int C_u D_u^2 \, dz = S_u(z) \, D_u^2(z) - 2\int S_u^2 D_u \, dz$$

$$\int C_u^2 S_u^2 \, dz = \begin{cases} \frac{1}{8 F_u} \left(z - \frac{1}{4} S_u(4z) \right) , & \text{if } F_u \neq 0 , \\ \frac{1}{3} z^3 , & \text{if } F_u = 0 \end{cases}$$

$$\int C_u^4 \, dz = \frac{1}{2} \left(z + \frac{1}{2} S_u(2z) \right) - F_u \int C_u^2 S_u^2 \, dz$$

$$\int S_u^4 \, dz = \begin{cases} \frac{1}{F_u} \left(\int S_u^2 \, dz - \int C_u^2 S_u^2 \, dz \right) , & \text{if } F_u \neq 0 , \\ \frac{1}{5} z^5 , & \text{if } F_u = 0 \end{cases}$$

More complicated integrals can also be worked out but are less useful. It is easy to convert these equations into expressions of averaged functions.

Sometimes the integrands one has to deal with are combinations of functions on the two transverse planes, for example, in calculating $\int \beta_y \eta \, dz$. Some indefinite integrals of this kind are presented below. Note that the indices x and y can be exchanged, that is, they are not fixed to a certain plane.

Suppose $F_x \neq F_y$. Otherwise one can make $W_y = W_x$, and find the results in the preceding list.

$$\int C_x C_y \, dz = (F_x S_x(z) C_y(z) - F_y C_x(z) S_y(z))/(F_x - F_y)$$

$$\int S_x S_y \, dz = (S_x(z) C_y(z) - C_x(z) S_y(z))/(F_x - F_y)$$

$$\int C_x S_y \, dz = (F_x S_x(z) S_y(z) + C_x(z) C_y(z))/(F_x - F_y)$$

$$\int C_x D_y \, dz = S_x(z) D_y(z) - \int S_x S_y \, dz$$

$$\int S_x D_y \, dz = (S_x(z) S_y(z) - C_x(z) D_y(z) - D_x(z))/(F_x - F_y)$$

$$\int D_x D_y \, dz = \begin{cases} (z - S_x(z) - S_y(z) + \int C_x C_y \, dz)/(F_x F_y) , & \text{if } F_x \neq 0,\ F_y \neq 0, \\ \frac{1}{6F_x} z^2 (z - 3 S_x(z)) + \frac{1}{F_x^2} (S_x(z) - z C_x(z)), & \text{if } F_y = 0 \end{cases}$$

And one can get expressions of $\int S_x z \, dz$, $\int C_x z \, dz$, $\int C_x z^2 \, dz$, etc. by transformation of the above equations on the supposition that F_x or $F_y = 0$.

Suppose $F_x \neq 4 F_y$. Otherwise, one can relate $W_x(z)$ to $W_y(2z)$ and find the results in the preceding list.

$$\int S_x C_y S_y \, dz = (S_x(z) C_y(2z) - \frac{1}{2} C_x(z) S_y(2z))/(F_x - 4 F_y)$$

$$\int C_x C_y S_y \, dz = (C_x(z) C_y(2z) + \frac{1}{2} F_x S_x(z) S_y(2z))/(F_x \text{[illegible]}$$

$$\int D_x C_y S_y \, dz = (\frac{1}{2} (S_y^2(z) - S_x(z) S_y(2z)) + D_x(z) C_y(2z))/(F_x - 4 F_y)$$

$$\int C_x C_y^2 \, dz = S_x(z) C_y^2(z) + 2 F_y \int S_x C_y S_y \, dz$$

$$\int C_x S_y^2 \, dz = S_x(z) S_y^2(z) - 2 \int S_x C_y S_y \, dz$$

$$\int S_x C_y^2\, dz = D_x(z)\, C_y^2(z) + 2\, F_y \int D_x C_y S_y\, dz$$

$$\int S_x S_y^2\, dz = D_x(z)\, S_y^2(z) - 2 \int D_x C_y S_y\, dz$$

$$\int D_x C_y^2\, dz = \begin{cases} \left(\frac{1}{2} z + \frac{1}{4} S_y(2z) - \int C_x C_y^2\, dz \right)/F_x\ , & \text{if } F_x \neq 0\ , \\ \left(z\, C_y(2z) + \frac{2}{3} F_y z^3 - \left(\frac{1}{2} - F_y z^2\right) S_y(2z)\right)/(8F_y)\ , & \text{if } F_x = 0 \end{cases}$$

$$\int D_x S_y^2\, dz = \begin{cases} \left(\int D_x\, dz - \int D_x C_y^2\, dz \right)/\ F_y\ , & \text{if } F_y \neq 0\ , \\ 2 \int D_x D_y\, dz & \text{if } F_y = 0 \end{cases}$$

$$\int W_x C_y D_y\, dz = \int W_x S_y^2\, dz - \int W_x D_y\, dz \qquad (W_x \text{ is } C_x,\ S_x \text{ or } D_x)$$

$$\int C_x S_y D_y\, dz = S_x(z) S_y(z) D_y(z) - \int S_x C_y D_y\, dz - \int S_x S_y^2\, dz$$

$$\int S_x S_y D_y\, dz = D_x(z) S_y(z) D_y(z) - \int D_x C_y D_y\, dz - \int D_x S_y^2\, dz$$

This appendix has summarized almost all possibly useful information about C_u, S_u and D_u so as to make them very convenient tools in accelerator physics calculations.