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SYSTEM CHARACTERIZATION IN NONLINEAR RANDOM VIBRATION

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Linear structural models are frequently used for structural system characterization and analysis. In most situations they can provide satisfactory results, but under some circumstances they are insufficient for system definition. The present investigation proposes a model for nonlinear structure characterization, and demonstrates how the functions describing the model can be identified using a random vibration experiment. Further, it is shown that the model is sufficient to completely characterize the stationary random vibration response of a structure that has a harmonic frequency generating form of nonlinearity. An analytical example is presented to demonstrate the plausibility of the model.

Introduction

Experimental identification of structural systems usually employs a linear model for the structure. The frequency response function of a linear system can be identified using either a deterministic analysis or a probabilistic analysis with random excitation. When the physical system being tested is truly linear then use of the linear model and analysis are appropriate. Further, when the system is slightly nonlinear a reasonable representation of system behavior can, in some senses, be established with the linear model. If the identified model is used for prediction of response or for the computation of the excitation that causes a specific response, then the analysis may, remain satisfactory as long as the nonlinearity effects are negligible.

The procedures commonly used for the identification of the frequency response functions of linear systems involve averaging operations. For example, the stationary random vibration procedure for estimation of the frequency response function (FRF) requires the generation and measurement of a random excitation This excites structural response which is then measured at points of interest. The measured excitation is used to estimate the auto spectral density of the excitation; the excitation and responses are then used to estimate the cross spectral densities between the excitation and responses. Each cross spectral density is ratioed with the excitation auto spectral density to establish an estimate for the structural FRFs at the points of interest. Details of the procedures described above are given in References 1, 2 and 3.

Modeling and identification of nonlinear systems, however, is not as straightforward as the procedure outlined in the previous paragrph. The literature contains many models for specific types of nonlinear structural systems and dèscribes approaches for computing their responses when the excitation is defined. See, for example, References 4 and 5. In some cases, experimental techniques useful in the identification of system parameters are described. However, the difficulty with using such models in general applicatons is that it is not usually easy to ascertain that a structural system has a nonlinearity that is appropriately modeled with a specific parametric form, and it is usually not clear what error is introduced when one nonlinear model is used to simulate a system with a different form of nonlinearity. This problem has been avoided by the use of the Volterra model for nonlinear systems. This is a nonparametric model that characterizes nonlinear systems using higher order impulse response functions and their Fourier transforms. This type of model and its identification is described, for example, in References 6, 7 and 8. The shortcoming of this model appears to be its inability to model frequency generating forms of nonlinearity.

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The harmonic frequency generating form of nonlinearity is one that is commonly seen in practice. This form of nonlinearity is related to harmonic distortion of motion at frequencies where motion is substantial, such as modal frequencies. Response in a nonlinear, harmonic generating system often shows substantial power at a particular fundamental frequency, and some fractional level of that power at odd multiples of the fundamental, whether or not structural excitation power is applied at the higher frequencies. The frequencies where a nonlinear system shows signal content in the response are related to the shapes of the displacement and velocity restoring force functions. The harmonic frequency generating form of nonlinearity appears often in random vibration tests and causes difficulty in system characterization and test control.

The present investigation establishes a model for nonlinear, harmonic generating systems. It is shown first that the model can be identified using a random vibration approach similar to that used in linear system analysis. Second, it is shown that once the system is identified the model can be used to establish the response characteristics for random excitation. An example demonstrates the use of the model.

The Nonlinear Model and Its Identification through Random Vibration Tests

The model established in this investigation is for a nonlinear, harmonic generating system. It is a model that can be used to describe the behavior of a structure which, for a mechanical reason, when excited at a particular frequency, executes response not only at the excitation frequency, but also at harmonics of that frequency. Many real structures display this characteristic. The model is established first and discussed briefly. Then the method for identification of the functions in the model using a random vibration test is established.

Consider a nonlinear, harmonic generating system where the Fourier transform of the response is represented

$$Z(\omega_i) = \sum_{j=1}^{M}\sum_{k=0}^{n} H_j(\omega_k,\omega_i)\ (X(\omega_k))^j \qquad i=0,\ldots,n \quad (1)$$

$Z(\omega_i)$ is the Fourier transform of the response at frequency ω_i. $X(\omega_k)$ is the Fourier transform of the excitation at frequency ω_k. $H_j(\omega_k,\omega_i)$ is an element in the sequence of coefficients that characterize the structure; the coefficients are deterministic and independent of the excitation, and might be thought of as forming a harmonic generating transfer function. This model generates a response component, characterized by magnitude and phase, at frequency ω_i as a complex valued, algebraic, power function of excitation components at frequencies ω_k, $k=0,\ldots,n$. The $H_j(\omega_k, \omega_i)$ are coefficients of the power function. A special case of this is the linear excitation - response case. This occurs when N=1 and the $H_j(\omega_k, \omega_i)$ are zero except when $\omega_k=\omega_i$. After the $H_j(\omega_k, \omega_i)$ are established using measured excitation and response, the representation (1) can be used to predict the response of the structure to random excitation. Further, the coefficients serve as a descriptor of structural behavior.

A method for establishing the $H_j(\omega_k,\omega_i)$ is now developed. Let the excitation be a zero mean, stationary, normal random process, $\{X(t),-\infty<t<\infty\}$. Let $X(\omega_k)$ represent the discrete Fourier transform (DFT) of a segment of the excitation whose duration is T seconds. Let the excitation be defined such that all its frequency components are uncorrelated. Multiply both sides of (1) by $(X^*(\omega_\ell))^m$, the mth power of the complex conjugate of $X(\omega_\ell)$, and then take the expected value on both sides of the equation. The result is

$$E(Z(\omega_i)(X^*(\omega_\ell))^m) = \sum_{j=1}^{M}\sum_{k=0}^{n} H_j(\omega_k,\omega_i)\ E((X(\omega_k))^j(X^*(\omega_\ell))^m) \qquad i=0,\ldots,n;\ \ell=0,\ldots,n \quad (2)$$

This expression can be simplified. Because the components of the stationary excitation are uncorrelated, the moment $E[(X(\omega_k))^j(X^*(\omega_\ell))^m]$ is zero except when $\omega_k=\omega_\ell$. (The specific reason for this is shown in Appendix 1.) Therefore,

$$E(Z(\omega_i)(X^*(\omega_\ell))^m) = \sum_{j=1}^{M} H_j(\omega_\ell,\omega_i)\ E((X(\omega_\ell))^j(X^*(\omega_\ell))^m) \qquad i=0,\ldots,n;\ \ell=0,\ldots,n \quad (3)$$

Further, the only situation where the expectation on the right hand side is nonzero occurs when j=m. (The reason for this is also shown in Appendix 1.) Because of this

$$E(Z(\omega_i)(X^*(\omega_\ell))^m) = H_m(\omega_\ell,\omega_i)\ E(|X(\omega_\ell)|^{2m}) \qquad \begin{matrix} i=0,\dots,n \\ \ell=0,\dots,n \end{matrix} \quad (4)$$

At this point it is possible to conduct a stationary random vibration experiment. First, we would estimate the moments $E[Z(\omega_i)(X^*(\omega_\ell))^m]$ and $E[|X(\omega_\ell)|^{2m}]$ using standard statistical techniques, and then we would ratio the results to obtain an estimate for the coefficient function $H_m(\omega_\ell, \omega_i)$. However, it is useful to establish the relation between the excitation spectral density and the moment $E[|X(\omega_\ell)|^{2m}]$, and to write a special expression for the moment $E[Z(\omega_i)(X^*(\omega_\ell))^m]$ before proceeding to estimate the $H_m(\omega_\ell, \omega_i)$. As mentioned previously, the excitation random process is a zero mean, stationary, normal random process. Let $S_{XX}(\omega)$ denote the spectral density of the excitation, {X(t)}. Then it can be shown (See Appendix 2.) that

$$E(|X(\omega_\ell)|^{2m}) = m!\ T^m\ S^m_{XX}(\omega_\ell) \qquad \begin{matrix} m=1,\dots,M \\ \ell=0,\dots,n \end{matrix} \quad (5)$$

when $\omega T>>1$, and where T is the time associated with the DFT's (and later, with the statistical analyses). Note that $E[|X(\omega_\ell)|^{2m}]$ is a function of the DFT time interval because $S_{XX}(\omega)$ is time independent.

If the response were a normal random process and the correlation between the frequency components of the excitation and the frequency components of the response were known, then it would be possible to obtain an expression similar to (5) for the moment $E[Z(\omega_i)(X^*(\omega_\ell))^m]$, and this expression would reveal a dependence of that moment on $T^{(m+1)/2}$. However, because of the nonlinearity of the excitation - response relation, the response is not usually normally distributed, and the expression cannot be obtained. Nevertheless, it probably remains a fact that the moment $E[Z(\omega_i)(X^*(\omega_\ell))^m]$ is a function of $T^{(m+1)/2}$, and this is so assumed. Specifically, it is assumed that

$$E(Z(\omega_i)(X^*(\omega_\ell))^m) = T^{(m+1)/2}\ S_{ZX^{*m}}(\omega_i,\omega_\ell) \qquad \begin{matrix} i=0,\dots,n \\ \ell=0,\dots,n \end{matrix} \quad (6)$$

where $S_{ZX^{*m}}(\omega_i, \omega_\ell)$ is the spectral function that relates $Z(\omega_i)$ and $(X^*(\omega_\ell))^m$ in the frequency domain, and, as before, T is the time over which the DFTs are taken. $S_{ZX^{*m}}(\omega_i, \omega_\ell)$ is assumed time independent.

Based on (4), (5) and (6), $H_m(\omega_\ell, \omega_i)$ can be used to write a time independent, harmonic generating transfer function. This is

$$F_m(\omega_\ell,\omega_i) = T^{(m-1)/2}\ H_m(\omega_\ell,\omega_i) = \frac{S_{ZX^{*m}}(\omega_i,\omega_\ell)}{m!\ S^m_{XX}(\omega_\ell)} \qquad \begin{matrix} m=1,\dots,M \\ i=0,\dots,n \\ \ell=0,\dots,n \end{matrix} \quad (7)$$

This function describes the harmonic generating character of a structural system.

In order to establish a numerical estimate of (7), statistical estimates of the moments $E[|X(\omega_\ell)|^{2m}]$ and $E[Z(\omega_i)(X^*(\omega_\ell))^m]$ are required. These can be obtained using standard statistical procedures. The approach and formulas required to obtain the statistical estimate for $E[Z(\omega_i)(X^*(\omega_\ell))^m]$ is given in Appendix 3.

The functions established in (7) contain a substantial amount of information that includes, but goes far beyond, the information in a linear FRF. In fact, the harmonic generating transfer functions defined in (7) could be used to describe how response is generated at every frequency given excitation at every frequency. When the response is characterized by Fourier components at n frequencies, the harmonic generating transfer function defined for each value of m contains n^2 points, therefore, when m takes the values 1 through M, Mn^2 items of information can be used to define the discrete functions $F_m(\omega_\ell, \omega_i)$. For realistic values of M and (especially) n it is not realistic to assume that Mn^2 values could be stored. Note, however, that in realistic situations, it is not anticipated that the functions $F_m(\omega_\ell, \omega_i)$ will

When the excitation is

$$x(t) = \sum_k X_k \exp(i\omega_k t) \qquad (13)$$

it can be shown that the first order approximation to the response is

$$z(t) = \sum_k H(\omega_k)\, X_k \exp(i\omega_k t) + \sum_j \sum_k \sum_\ell H(\omega_j+\omega_k+\omega_\ell)\, H(\omega_j)\, H(\omega_k)\, H(\omega_\ell) * X_j X_k X_\ell \exp(i(\omega_j+\omega_k+\omega_\ell)t) \qquad (14)$$

where $H(\omega)$ is the FRF function of a linear single-degree-of-freedom system. This formula includes more terms than (1), therefore (1) can only represent the frequency domain response in an approximate, limited sense, in the general case. However, for stationary, random vibration analysis, the terms in (14) where $j\neq k\neq \ell$, and $j\neq \ell$ are unimportant and

$$z(t) = \sum_k H(\omega_k)\, X_k \exp(i\omega_k t) - \epsilon \sum_k H(3\omega_k)\, H^3(\omega_k)\, X_k^3 \exp(i3\omega_k t) \qquad (15)$$

represents the response with all the terms necessary for a first order analysis. In view of this, the harmonic generating transfer functions for the first approximation to the Duffing oscillator are

$$H_j(\omega_k,\omega_i) = \begin{cases} H(\omega_i), & j=1,\ k=i \\ -\epsilon H(\omega_i)H^3(\omega_i/3), & j=3,\ k=i/3 \\ 0, & \text{otherwise} \end{cases} \qquad (16)$$

The first expression simply establishes the nature of the linear part of the response. The second term transfers excitation at frequency $\omega_i/3$ to response at frequency ω_i. The moduli of these functions (a normalized form, in the second case) are plotted in Figures 2 and 3 for the case where $\omega_n=1$, $\zeta=0.05$. The frequency generating nature of $H_3(\omega_i/3, \omega_i)$ is apparent in Figure 3.

Similar analyses are possible for higher approximations to the Duffing oscillator response and for other nonlinear systems.

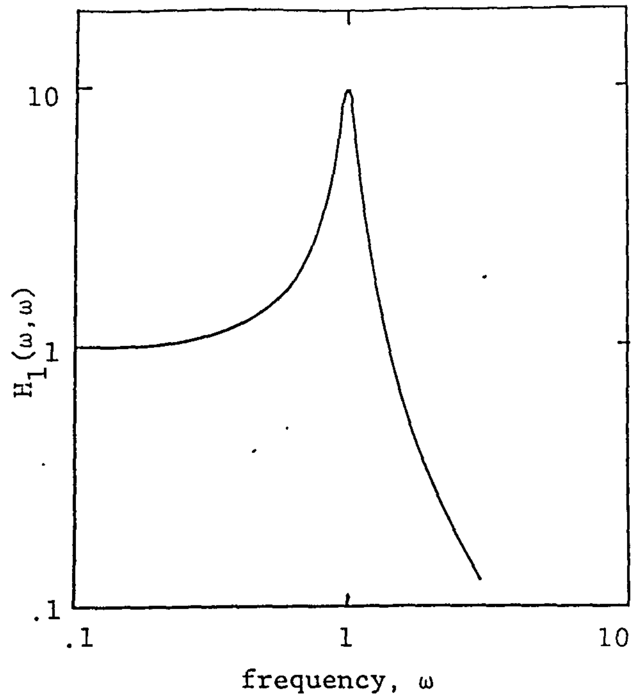


Figure 2. The first expression in (16). FRF of a linear single-degree-of-freedom system.

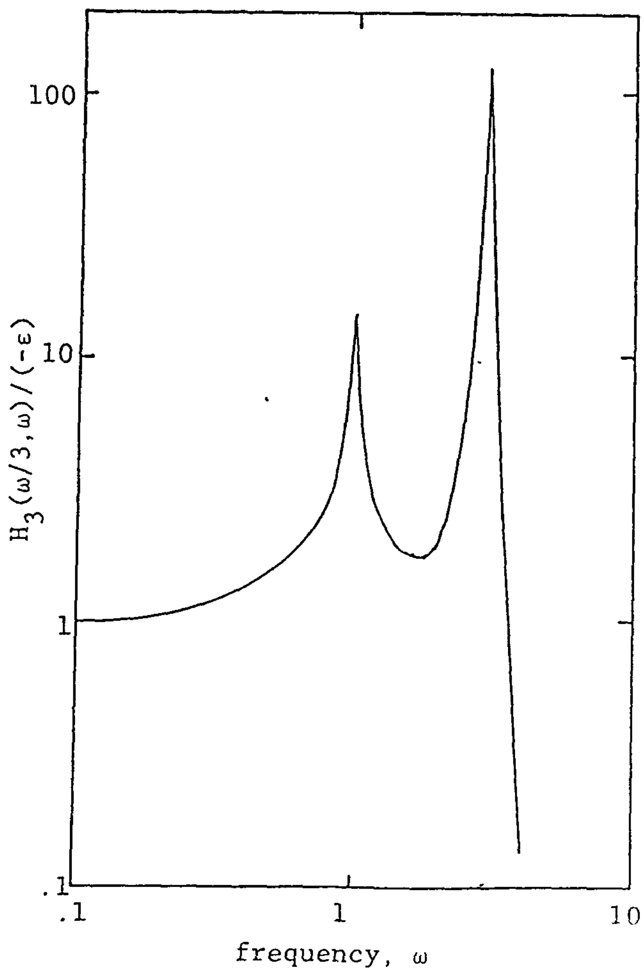


Figure3. The second expression in (16) Third harmonic generating transfer function of a Duffing oscillator.

have substantial values at all frequency pairs (ω_ℓ, ω_i). In most situations one would expect substantial values when $\omega_i = \omega_\ell, 3\omega_\ell, 5\omega_\ell$, etc., and possibly when $\omega_i = 2\omega_\ell, 4\omega_\ell$, etc. For example, if knowledge of those elements in $F_m(\omega_\ell, \omega_i)$ which create response at $\omega_i = \omega_\ell$ and $3\omega_\ell$ are desired for m=1 and m=3, then 4n items of information need to be established to characterize $F_m(\omega_\ell, \omega_i)$. Thus, most practical situations will require a reasonably accomodated amount of data storage.

Random Vibration Analysis Using the Harmonic Generating System Model

Aside from basic system characterization, the fundamental reason for establishing the mathematical model defined in (1) is to provide the capability for random vibration analysis of the harmonic generating system. Given the coefficients $H_j(\omega_k, \omega_i)$, a random vibration analysis can be easily executed. To do this, the complex conjugate of (1) is taken and multiplied times (1). Then the expected value is taken on both sides; the result is

$$E(|Z(\omega_i)|^2) = \sum_j \sum_k \sum_\ell \sum_m H_j(\omega_k, \omega_i)\, H_\ell^*(\omega_m, \omega_i) * E((X(\omega_k))^j (X^*(\omega_m))^\ell) \qquad i=0,\ldots,n \quad (8)$$

Recall that the expected value on the right hand side is zero except when m=k and ℓ=j; therefore, the expression simplifies to

$$E(|Z(\omega_i)|^2) = \sum_j \sum_k |H_j(\omega_k, \omega_i)|^2 E(|X(\omega_k)|^{2j}) \qquad i=0,\ldots,n \quad (9)$$

Now (6) can be used to simplify the left hand side (using $Z(\omega_i)$ in place of $X(\omega_\ell)$ and m=1) and the right hand side, and (7) can be used to establish an expression for $H_j(\omega_k, \omega_i)$ that can be used above. The result is

$$S_{zz}(\omega_i) = \sum_j \sum_k j!\, |F_j(\omega_k, \omega_i)|^2\, S^j_{xx}(\omega_k) \qquad i=0,\ldots,n \quad (10)$$

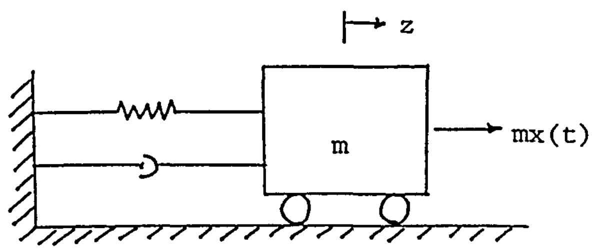
This formula establishes a means for computing the spectral density of structural response to stationary random vibration excitation. The formula is complete in the sense that it includes all the terms necessary for characterization of the response of a nonlinear, harmonic generating system to stationary, random vibration excitation. The reason for this completeness is the fact that different frequency components of a stationary random process are uncorrelated.

Example

This section presents an example that demonstrates the use of the formulas developed in the previous sections. The system to be considered is *a simple* Duffing oscillator. Figure 1 is a schematic display of the system. It is governed by the nonlinear, ordinary differential equation

$$\ddot{z} + 2\zeta\omega_n\dot{z} + \omega_n^2 z + \epsilon z^3 = x \qquad (11)$$

where ω_n is the natural frequency of the associated linear system, ζ is the system damping factor, ϵ is a small positive constant, x denotes the excitation, z denotes the displacement response, and dots denote differentiation with respect to time.



SPRING RESTORING FORCE = $m\omega_n^2 z + m\epsilon z^3$

DAMPER RESTORING FORCE = $2m\zeta\omega_n\dot{z}$

MASS NORMALIZED FORCE = x(t)

Figure 1. Duffing oscillator.

An approximate expression for the response can be developed using the perturbation approach. (See Reference 4.) With this approach it is assumed that the response can be expressed as an expansion in the small term ϵ. That is

$$z(t) = z_0(t) + \epsilon z_1(t) + \ldots \qquad (12)$$

(12) is used in (11), and linear equations governing z_0, z_1, etc., are developed by grouping terms by coefficients of ϵ^0, ϵ^1, etc., and noting that the coefficients must equal zero if the components z_0, z_1, etc., are to be independent and arbitrary.

Conclusions

The response of a nonlinear, harmonic generating structure to stationary, random vibration excitation can be represented using an expression that is a series of power series in the Fourier transform of the excitation. The coefficients of the power series describe the character of the structure. An analytical example shows that it can be easy to establish the coefficients. Further, it is demonstrated that the coefficient functions required for the nonlinear representation can be obtained experimentally. The magnitudes of the coefficients of the harmonic generating terms can be used to assess the degree of nonlinearity of a structure tested in the laboratory.

Future investigations must demonstrate the usefulness of this model with experimental data, and must consider the use of this model when the excitation is not a stationary random process.

Appendix 1

This appendix considers $\{X(t), 0<t<T\}$, a segment of mean zero, stationary, normal random process with Fourier representation

$$X(t) = \sum_{k=0}^{n-1} X_k \exp(i2\pi tk/n\Delta t), \qquad 0<t<T \tag{A1}$$

where

$$X_k = C_k \exp(i\phi_k), \qquad k=0,\dots,n-1 \tag{A2}$$

In this expression $C_0=0$, $C_{n-k}=C_k$, $k=1,\dots,n/2$, are deterministic constants related to the random process spectral density, and $\phi_0=0$, $\phi_{n-k}=\phi_k$, $k=1,\dots,n/2$, are uniformly distributed random variables on $(-\pi,\pi)$, where ϕ_j and ϕ_k are independent for $j\neq k$. Reference 9 establishes (A1) as a valid representation for a stationary random process.

It will be shown that $E[X_k^j (X_\ell^*)^m]$ is zero except when $k=\ell$ and $j=m$. Based on (A2)

$$E(X_k^j (X_\ell^*)^m) = C_k^j C_\ell^m \, E(\exp(i(j\phi_k - m\phi_\ell)) \tag{A3}$$

Because the ϕ_k, $k=0,\dots,n/2$, are independent

$$E(\exp(i(j\phi_k - m\phi_\ell)) = \begin{cases} E(\exp(i(j-m)\phi_\ell), & k=\ell \\ E(\exp(ij\phi_k))\, E(\exp(-im\phi_\ell)), & k\neq\ell \end{cases} \tag{A4}$$

Because $E[e^{ir\phi_s}]=0$ for all s and all $r\neq 0$, all the moments in (A4) are zero except when $k=\ell$ and $j=m$, and when $j=m=0$. Therefore, (A3) is nonzero only when $k=\ell$ and $j=m$.

Appendix 2

This appendix considers a random process $\{X(t), 0<t<T\}$, a segment of a mean zero, stationary, normal random process with autocorrelation function $R_{XX}(\tau)$ and spectral density $S_{XX}(\omega)$. The moment $E[|X(\omega_\ell)|^{2m}]$ will be evaluated where

$$X(\omega_\ell) = \int_0^T X(t) \exp(-i\omega_\ell t)\, dt \tag{A5}$$

The real and imaginary parts of $X(\omega_\ell)$ are normal random variables given by

$$X_R(\omega_\ell) = \int_0^T X(t) \cos(\omega_\ell t)\, dt \tag{A6}$$

$$X_I(\omega_\ell) = \int_0^T X(t) \sin(\omega_\ell t)\, dt \tag{A7}$$

The random variables $X_R(\omega_\ell)$ and $X_I(\omega_\ell)$ have zero means. The variance of $X_R(\omega_\ell)$ is

$$E(X_R^2(\omega_\ell)) = \int_0^T dt \int_0^T ds\, R_{xx}(s-t) \cos(\omega_\ell t) \cos(\omega_\ell s) \tag{A8}$$

Define the change of variables $\tau=s-t$, $\Upsilon=s+t$, and allow τ to cover the interval $(-\infty, \infty)$ to establish an approximation. Then

$$E(X_R^2(\omega_\ell)) \cong \frac{T}{2} S_{xx}(\omega_\ell) + \frac{1}{4\omega_\ell} S_{xx}(0) \sin(2\omega_\ell T) \tag{A9}$$

When $\omega_\ell T >> 1$ and $S_{XX}(0)$ is near in value to $S_{XX}(\omega_\ell)$, this is approximately

$$E(X_R^2(\omega_\ell)) \cong \frac{T}{2} S_{xx}(\omega_\ell) \tag{A10}$$

Similarly, it can be shown that

$$E(X_I^2(\omega_\ell)) \cong \frac{T}{2} S_{xx}(\omega_\ell)$$
$$E(X_R(\omega_\ell) X_I(\omega_\ell)) \cong 0 \tag{A11}$$

Note that

$$|X(\omega_\ell)|^2 = X_R^2(\omega_\ell) + X_I^2(\omega_\ell) \qquad (A12)$$

Therefore

$$E(|X(\omega_\ell)|^{2m}) = \sum_{r=0}^{m} \binom{m}{r} E((X_R^2(\omega_\ell))^{m-r})\ E((X_I^2(\omega_\ell))^r) \qquad (A13)$$

Because of (A10) and (A11) and the fact that $X_R(\omega_\ell)$ and $X_I(\omega_\ell)$ are normal random variables

$$E((X_R(\omega_\ell))^{2n}) = E((X_I(\omega_\ell))^{2n}) = \frac{(2n)!}{2^n\ n!}\left[\frac{T}{2} S_{xx}(\omega_\ell)\right]^n \qquad (A14)$$

Use of this expression in (A13) yields

$$E(|X(\omega_\ell)|^{2m}) = m!\ (TS_{xx}(\omega_\ell))^m \qquad (A15)$$

for $\omega_\ell T >> 1$, which is the desired result.

Appendix 3

This appendix shows how the moment $E[Z(\omega_i)(X^*(\omega_\ell))^m] = S$ can be statistically estimated. Let $\{Z(t)\}$ and $\{X(t)\}$ be stationary and ergodic, mean zero random processes. Assume that measured realizations of the random processes are available; denote these z_j and x_j, $j=1,\ldots,n$. Divide each time series into M blocks of equal length N, such that MN=n. Denote the jth elements of the kth blocks z_{jk} and x_{jk}, $j=0,\ldots,N-1$, $k=1,\ldots,M$. Multiply each data block by an amplitude adjusted data window, w_j, $j=1,\ldots,N$, (if desired) to obtain

$$z'_{jk} = z_{jk}\, w_j, \qquad j=0,\ldots,N-1 \quad (A16)$$

$$x'_{jk} = x_{jk}\, w_j, \qquad k=1,\ldots,M \quad (A17)$$

Fourier transform the time series (A16) and (A17) to obtain the DFTs.

$$Z_{\ell k} = \Delta t \sum_{j=0}^{N-1} z'_{jk} \exp(-i2\pi j\ell/N) \qquad (A18)$$

$$X_{\ell k} = \Delta t \sum_{j=0}^{N-1} x'_{jk} \exp(-i2\pi j\ell/N) \qquad (A19)$$

$$\ell=0,\ldots,N-1$$
$$k=1,\ldots,M$$

The Δt is the sampling period of the measured data. Form products like $Z_{ik}(X^*_{\ell k})^m$ and average these over all blocks.

$$\hat{S} = \frac{1}{M}\sum_{k=1}^{M} Z_{ik}(X^*_{\ell k})^m \qquad (A20)$$

This is an estimate of the moment that appears in (4) and (6).

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