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The Convergence of Difference Boxes

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Abstract

We consider an elementary mathematical puzzle known as a "difference box" in terms of a discrete map from R^4 to R^4 or, canonically, from a subset of the first quadrant of R^2 into itself. We find the map's unique canonical fixed point and answer the general question of how many iterations a given "difference box" takes to reach zero.

1 Introduction

"Difference boxes", also known as "diffy boxes", are a simple mathematical puzzle which provides elementary grades students with subtraction practice. The idea's original author is unknown, although among many Texas teachers it can be traced back to Prof. Juanita Copley of the University of Houston, who introduced it as a problem-solving activity in professional development sessions about twenty years ago (and who in turn cites her grandmother). Diffy boxes have been used in numerous places as an example for elementary teachers of a way to practice arithmetic without the tedium of drill (its use as a problem-solving activity will be addressed in the final section of this paper).

One creates a difference box as follows:

1. Draw a (large) square, and label each vertex with a (real) number.
2. On the midpoint of each side, write the (unsigned) difference between the two numbers at the endpoints.
3. Inscribe a new square in the old one, using these new numbers as vertices.
4. Repeat this process, and continue inscribing new boxes until one has all four vertices labeled with 0.

It is perhaps surprising that diffy boxes always tend to "converge" rather quickly, that is, it usually takes no more than a handful of iterations to get a box with all zeroes. Figure 1 shows a simple example which converges to all zeroes after four iterations (on the fifth box).

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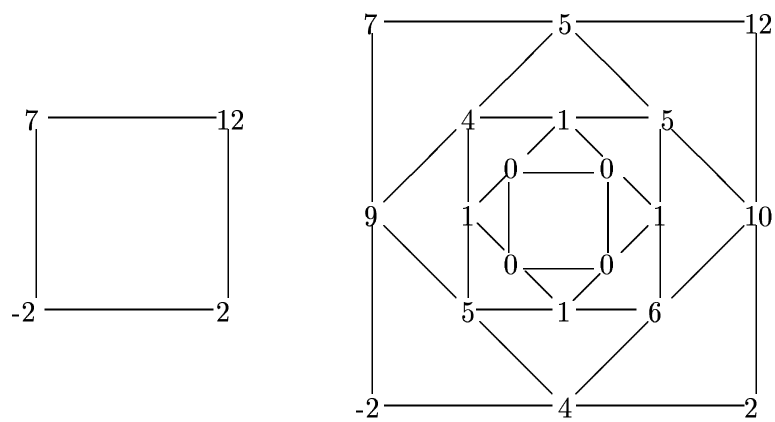

Figure 1: A simple diffy box (left) and its "descendants" (right)

The question we want to investigate is whether all diffy boxes really do converge to the zero box, and, if so, how quickly? We will approach the problem by considering the diffy box process as a map from the set of all possible 4-tuples, $\mathbf{R}^4$, into itself.

2 Definitions

We begin our analysis by defining some notation. We will denote by $[a\ b\ c\ d]$ the diffy box which contains the numbers a, b, c and d on its upper left, upper right, lower right, and lower left corners, respectively (which makes the first box in Figure 1 $[7\ 12\ 2\ -2]$).

Next we define notation for the diffy box iteration process, and some terms related to convergence:

> DEFINITION 1. The *child* of a given box $B = [a\ b\ c\ d]$ is $C(B) = [|b-a|\ |c-b|\ |d-c|\ |a-d|]$. B is a *parent* of $C(B)$. (We shall see that any given box has many parents.) The parent-child relation shall be denoted $B \triangleright C(B)$, and we shall denote $C(C(B))$ by $C^2(B)$, etc.
>
> DEFINITION 2. A given box B *converges in n generations* if and only if $C^n(B) = [0\ 0\ 0\ 0]$ but $C^{n-1}(B) \neq [0\ 0\ 0\ 0]$. B can then be said to have *longevity n*.
>
> DEFINITION 3. Let $|B|$ be the largest difference between two (not necessarily adjacent) vertices of B.
>
> DEFINITION 4. A box B is *monotone* if and only if its vertices are distinct and occur around the square in numerical order from least to greatest.

Note that we shall consider boxes such as $[2\ 1\ 4\ 3]$ monotone, since the definition allows us to start with any vertex and proceed clockwise or counterclockwise from it. A couple of relatively quick results may help the reader begin to develop some intuition as to how and why diffy boxes tend to converge.

PROPOSITION 1. $|C(B)| \leq |B|$, and the inequality is strict if the four numbers in B are distinct.

Proof. Let $w \geq x \geq y \geq z$ be the numbers in B (not necessarily in order of appearance). We have $|B| = w - z$. The numbers in $C(B)$ all fall between 0 and $w-z$; hence $|C(B)| \leq (w-z) - 0 = |B|$. Furthermore, if w, x, y, z are distinct then $C(B)$ does not contain zero, and hence $|C(B)| < |B|$. □

PROPOSITION 2. Any nonmonotone diffy box converges in 6 or fewer generations.

Proof. Table 1 details the longevity of all diffy boxes whose vertices are nonmonotone. The proof is technical, by cases (a simpler proof follows from Figure 3 in Section 3). To put a given diffy box into a form listed in the table, reorder the vertices (using reflection and/or rotation, isometry properties which we shall discuss in Section 4) so that the smallest vertex is listed first, followed by the smaller of the two vertices adjacent to it. (If two copies of the smallest number occupy adjacent vertices, put them first and second, followed by the smaller of the two vertices adjacent to them.) This reordering does not affect longevity. □

Longevity	*Isometric form of diffy box*
1	$[a\ a\ a\ a]$, $a \neq 0$
2	$[a\ b\ a\ b]$
	$[a\ b\ c\ b]$, $b = \frac{a+c}{2}$
3	$[a\ a\ b\ b]$
	$[a\ b\ b\ c]$, $b = \frac{a+c}{2}$
	$[a\ b\ d\ c]$, $a - b = c - d$
	$[a\ c\ b\ d]$, $a - b = c - d$
4	$[a\ a\ a\ b]$
	$[a\ a\ b\ c]$, b at least as close to c as to a
	$[a\ b\ a\ c]$
	$[a\ b\ b\ b]$
	$[a\ b\ b\ c]$, $b \neq \frac{a+c}{2}$
	$[a\ b\ c\ b]$, $b \neq \frac{a+c}{2}$
	$[a\ b\ c\ c]$, b at least as close to a as to c
	$[a\ b\ d\ c]$, $a - b \neq c - d$, $(a+d)/2$ between b, c
	$[a\ c\ b\ c]$
	$[a\ c\ b\ d]$, $a - b \neq c - d$
6	$[a\ a\ b\ c]$, b closer to a than to c
	$[a\ b\ c\ c]$, b closer to c than to a
	$[a\ b\ d\ c]$, b and c on the same side of $(a+d)/2$

Table 1: Longevities for all nonmonotone diffy boxes (here $a < b < c < d$)

Note that Proposition 2 includes any box whose vertex numbers are not all distinct. We will observe in Section 4 that diffy boxes $[a\ b\ c\ d]$ whose vertices *are* monotone (with $a < b < c < d$ or $a > b > c > d$) have longevity 5 or greater.

Since, following Proposition 1, $|C(B)| = |B|$ only when B's vertices are not all distinct, we can bound the longevity of any box that has *all-integer* vertices by observing that the range $|B|$ of any such box must decrease by at least 1 per iteration, until we reach a box C with at least one pair of identical vertices. At this point we compare in Table 1 the possible longevities of C (2, 3, 4 or 6) with the corresponding minimum possible range (1, 1, 1 or 3, respectively) and note the greatest difference. We therefore have the following result.

COROLLARY. If B consists of integers, then the longevity of B is less than or equal to $|B| + 3$.

3 Canonical form

A little experimentation with different sets of numbers quickly leads to the observation that there are different boxes which behave the same way as regards the iteration process which leads to zero. For example, adding the same number to each of the vertices of a box B ($B + k,\ k \in \mathbf{R}$), or changing the signs of all the vertices ($-B$), will produce other parents of $C(B)$, since each iteration only records differences between successive vertices; thus the C map is many-to-one. Furthermore, since our real interest is this history of families rather than of individuals, we observe three other types of changes which produce family histories parallel to the original: Multiplying each vertex of a box B by a positive constant k will produce a box whose child is that same multiple of $C(B)$: $kB \triangleright kC(B)$; B and kB should therefore take the same number of iterations to reach the all-zero box. We may think of B and kB as "cousins" with the same family histories. Finally, rotating or reflecting the numbers on the vertices of a box B (call this $r(B)$) will create another cousin, a box whose child is the rotated or reflected version of $C(B)$, $r(B) \triangleright r(C(B))$, and so on through successive iterations. These changes are merely cosmetic, since the numbers retain their positions relative to each other.

To simplify our analysis below, we will therefore define a set of equivalence classes which will reduce the number of distinct boxes we must consider (and thereby reduce the dimension of the problem considerably, as we shall see).

DEFINITION 5. The equivalence class of a box $B = [a\ b\ c\ d]$ is given by all combinations of the following five properties:

(1) translation: $\forall \alpha \in \mathbf{R}\ \ [a\ b\ c\ d] \sim [(a+\alpha)\ (b+\alpha)\ (c+\alpha)\ (d+\alpha)]$;

(2) negation: $[a\ b\ c\ d] \sim [-a\ \ -b\ \ -c\ \ -d]$;

(3) positive scaling: $\forall \alpha \in \mathbf{R}^+\ \ [a\ b\ c\ d] \sim [(\alpha a)\ (\alpha b)\ (\alpha c)\ (\alpha d)]$;

(4) rotation: $[a\ b\ c\ d] \sim [b\ c\ d\ a] \sim [c\ d\ a\ b] \sim [d\ a\ b\ c]$;

(5) reflection: $[a\ b\ c\ d] \sim [d\ c\ b\ a]$.

We can consider the first three properties as field properties, and the last two as isometry properties. It can quickly be verified that this definition is indeed an equivalence relation, i.e., is reflexive, symmetric, and transitive. We can also observe, following the same arguments given informally above, that if $B_1 \sim B_2$, then $C(B_1) \sim C(B_2)$, and B_1 and B_2 have the same longevity unless one of them is the zero box and the other is $[a\ a\ a\ a]$ ($a \neq 0$).

We would now like to find a way to select a unique member of each equivalence class so that we can concentrate our remaining analysis on a reduced domain. Toward this end, we shall define the canonical form for an equivalence class.

DEFINITION 6. The canonical form for a given diffy box equivalence class shall be one of the following: (i) $[0\ 0\ 0\ 0]$ for the class containing this (zero) box; (ii) $[0\ 0\ 1\ 1]$ for the class containing this box; or (iii) the unique member of the form $[0\ 1\ x\ y]$ which has $(x, y) \in S = \{x \geq 0,\ y \geq 1,\ x - 1 \leq y \leq x + 1\}$, otherwise.

Equivalence class (i) consists of all boxes $[a\ a\ a\ a]$ ($a \in \mathbf{R}$), which converge to the zero box in one iteration (if $a \neq 0$). Equivalence class (ii) consists of all boxes $[a\ a\ b\ b]$ and $[a\ b\ b\ a]$ ($a, b \in \mathbf{R}$), and can be seen to converge in three iterations ($[0\ 0\ 1\ 1] \triangleright [0\ 1\ 0\ 1] \triangleright [1\ 1\ 1\ 1] \triangleright [0\ 0\ 0\ 0]$). Identifying the canonical form for all other classes besides these two requires the notion of an *extreme* element, that is, an element a of a box which is either maximal (at least as big as each of the other elements) or minimal (at least as small as each of the other elements). Note that at least two of the four elements of each box must be extreme. The following result gives the procedure for determining a type (iii) canonical form, as well as a justification of its uniqueness.

THEOREM 1. Any diffy box equivalence class of type (iii) (i.e., not of the form $[a\ a\ a\ a]$ or $[a\ a\ b\ b]$ $(a, b \in \mathbf{R})$) has a unique representative $[0\ 1\ x\ y]$ with $(x, y) \in S = \{x \geq 0,\ y \geq 1,\ x-1 \leq y \leq x+1\}$.

Proof. We first demonstrate an algorithm for finding the representative, using our equivalence relations. We begin with an arbitrary diffy box $[a\ b\ c\ d]$.

1. Rotate (property (4)) until $|d-a|$ is maximal among $|a-b|$, $|b-c|$, $|c-d|$, $|d-a|$.
2. (a) Observe that either a or d must be extreme. If necessary, reflect (property (5)) to ensure that a is extreme.

   (b) If a and d are both extreme, it's possible that $|a-b| < |c-d|$. If necessary, reflect (property (5)) to make $|a-b| \geq |c-d|$.
3. If a is maximal, use negation (property (2)) to make a minimal.
4. Use translation (property (1)) to make $a = 0$.
5. Use positive scaling (property (3)) to make $b = 1$.

Observe that the properties created in each step are preserved in subsequent steps. The last step is always possible since steps 2(b) and 4 together imply that $b = 0 \Rightarrow c = d$, and the only two such cases are equivalence classes (i) $(c = d = 0)$ and (ii) $(c = d \neq 0)$. Otherwise $b > 0$ (from steps 3 and 4), so positive scaling can be used to normalize b.

We now have $[0\ 1\ c\ d]$ (from steps 4 and 5). From steps 3 and 4, $c, d \geq 0$. From steps 1, 4 and 5, $d \geq 1$. From step 2(b), $|c-d| \leq 1$, so $c-1 \leq d \leq c+1$.

It remains to show that the canonical form (iii) is unique, that is, given $[0\ 1\ a\ b] \sim [0\ 1\ c\ d]$ where $(a, b), (c, d) \in S$ as given in Definition 6, $(a, b) = (c, d)$. This can be seen first by observing that 0 is a unique minimal number, i.e., $a, b, c, d > 0$ (except for $[0\ 1\ 0\ 1]$, which has no other $[0\ 1\ c\ d]$ representation), and second on a case-by-case basis by taking a or b as maximal, and showing that each of the seven transformations that would translate 0, 1, a or b to 0 and normalize either of the numbers adjacent to it, results in a $(c, d) \notin S$. As the details are simple but technical, we leave them as an exercise for the reader (likewise the proof that the equivalence classes containing $[0\ 0\ 0\ 0]$ and $[0\ 0\ 1\ 1]$ have no type (iii) canonical form representation). □

4 A two-dimensional map

We can now focus our remaining analysis on what happens to diffy box equivalence classes of the form $[0\ 1\ x\ y]$ with $(x, y) \in S$. We first observe that in the special case $y = 1$, the children will be of type (i) or (ii) (since $[0\ 1\ x\ 1] \triangleright [1\ |x-1|\ |x-1|\ 1]$), and hence will converge in 2 $(x = 0, 2)$ or 4 generations.

We shall now consider the diffy box process as a continuous map from $S \backslash \{y = 1\}$ into S, which calculations show is given by

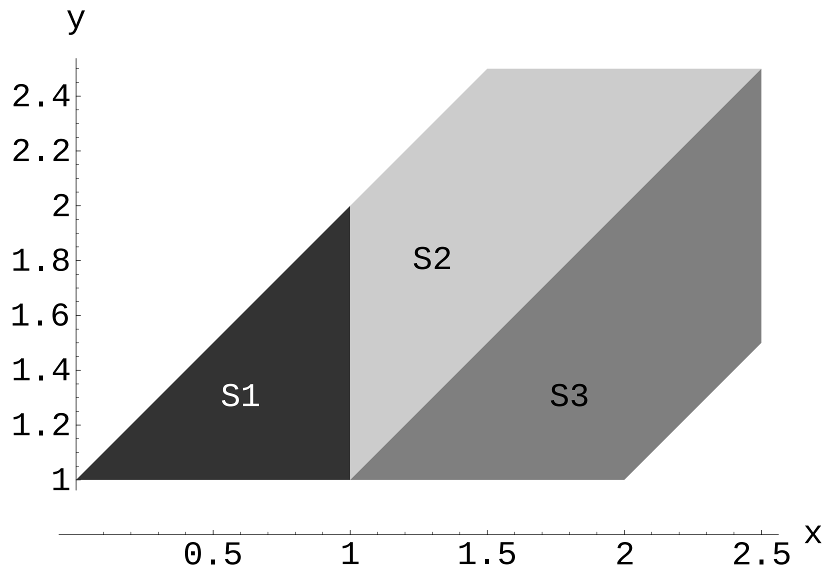

Figure 2: Subdivision of S into regions corresponding to the three branches of f

$$f(x,y)=\begin{cases}\left(1+\dfrac{x}{y-1},\dfrac{x}{y-1}\right), & 0\le x\le 1,\ 1<y\le x+1;\ (S_1)\\[2ex] \left(1+\dfrac{2-x}{y-1},\dfrac{x}{y-1}\right), & x>1,\ x\le y\le x+1;\ (S_2)\\[2ex] \left(1+\dfrac{2-x}{y-1},2+\dfrac{2-x}{y-1}\right), & x,y>1,\ x-1\le y\le x.\ (S_3)\end{cases}$$

Figure 2 illustrates the three domains S_1, S_2, S_3 into which this definition decomposes S.

By inspection we can see that the first and third branches of this function are infinite-to-one mappings: the first branch sends all points to the line $v=u-1$ (with $u\ge 2$ since $y\le x+1$ implies $\frac{x}{y-1}\ge 1$), and the third branch sends all points to the line $v=u+1$ (with $u\ge 0$ since $y\ge x-1$ implies $\frac{2-x}{y-1}\ge -1$). On the first branch (where $x\le 1$), $f(x_1,y_1)=f(x_2,y_2)$ implies $\frac{y_1-1}{x_1-0}=\frac{y_2-1}{x_2-0}$, that is, (x_1,y_1) and (x_2,y_2) lie on the same line segment from (0,1). Likewise, on the third branch (where $y\le x$), $f(x_1,y_1)=f(x_2,y_2)$ implies $\frac{y_1-1}{x_1-2}=\frac{y_2-1}{x_2-2}$, that is, (x_1,y_1) and (x_2,y_2) lie on the same line segment from (2,1). On these branches, f compresses two-dimensional regions into rays on the boundary of S.

The second branch, however, is one-to-one: for (x_1,y_1) and (x_2,y_2) both in S_2, $f(x_1,y_1)=f(x_2,y_2)$ implies $(x_1,y_1)=(x_2,y_2)$ (we can see this by rewriting $\frac{x_1}{y_1-1}=\frac{x_2}{y_2-1}$ as $y_2-1=\left(\frac{y_1-1}{x_1}\right)x_2$ and substituting into $1+\frac{2-x_1}{y_1-1}=1+\frac{2-x_2}{y_2-1}$).

We can also see that the first and third branches have no fixed points inside their respective domains: the first branch sends all points to the line $v=u-1$, which isn't in S_1, and branch 3 sends all points to the line $v=u+1$, which isn't in S_3. (In fact, there is no possible periodicity in these regions, either, since for (x,y) in S_1 or S_3, $(c,d)=f(f(x,y))$ has $d=1$.) Therefore any fixed points must lie in S_2. Indeed, straightforward calculations show that S_2 contains the unique fixed point $(q(q-1),q)\approx(1.5437,1.8393)$, where

$$q = \frac{1}{3}\left(1 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}}\right)$$

is the unique real solution to $q^3 - q^2 - q - 1 = 0$. Here $C([0\ 1\ q(q-1)\ q]) \sim [0\ 1\ q(q-1)\ q]$, i.e., the child is in the same equivalence class as the parent (and therefore takes as long to converge — in other words, it has infinite longevity). The only other equivalence class for which this is true is $[0\ 0\ 0\ 0]$ (but properly speaking this class has no longevity).

We should note, however, that this does *not* imply that $B = C(B)$ for members of the class $[0\ 1\ q(q-1)\ q]$. In fact, in an $\mathbf{R}^4$-norm sense, as well as that of Definition 3, members of this class do also approach the zero box under iteration (cf. Prop. 1) — they just take infinitely long to get there. For example, calculation of a few iterations of the diffy box process beginning with $[0\ 1\ q(q-1)\ q]$ show that after the first step, the four entries gradually diminish in size (by a factor of $q-1$) as they cycle around counterclockwise, in keeping with Proposition 1.

We now determine whether the fixed point $(q(q-1), q)$ is stable. To determine the stability of a fixed point of a complex map, one looks at the Jacobian matrix. This is derived from the linearization of the map, and consists of the map's partial derivatives, evaluated at a given fixed point. For $f = (f_1(x,y), f_2(x,y))$, the Jacobian is $\begin{bmatrix} \partial f_1/\partial x & \partial f_1/\partial y \\ \partial f_2/\partial x & \partial f_2/\partial y \end{bmatrix}$. This (second) branch of f has

$$J(x,y) = \begin{bmatrix} -\frac{1}{y-1} & \frac{x-2}{(y-1)^2} \\ \frac{1}{y-1} & -\frac{x}{(y-1)^2} \end{bmatrix} \text{ and } J(q(q-1), q) \approx \begin{bmatrix} -1.1915 & -0.6478 \\ 1.1915 & -2.1915 \end{bmatrix}.$$

This matrix has eigenvalues $\lambda \approx -1.6915 \pm 0.7224i$. Because the eigenvalues have magnitude greater than 1, the fixed point is unstable. Because the imaginary components are nonzero, we see that points near $(q(q-1), q)$ spiral away from it under (repeated) application of f. To read more about stability analysis of fixed points, see [2, 3]. Because this is the only fixed point, one might expect that further applications of f will eventually move any other point to the boundary of the domain, and then out of it entirely. Therefore, we might expect that those diffy boxes that take longest to converge correspond to those points in S closest to the fixed point $(q(q-1), q)$. As we shall see below, these intuitive notions turn out to be correct.

If we begin a case-by-case analysis of the successive applications of f in S, we find the domain subdividing into regions beginning along the boundaries and working in toward the fixed point.

Example 1. Any box $[0\ 1\ x\ y]$ with $x \leq 1$ converges within 4 generations (3 if $y = x+1$, 2 if $(x,y) = (0,1)$). We calculate

$$[0\ 1\ x\ y] \triangleright [1\ (1-x)\ (y-x)\ y] \triangleright [x\ (y-1)\ x\ (y-1)] \triangleright [|y-x-1|\ |y-x-1|\ |y-x-1|\ |y-x-1|] \triangleright [0\ 0\ 0\ 0]$$

and observe convergence to the zero box one or two generations sooner in the aforementioned special cases. (This corresponds to region S_1 in Figure 2.)

Example 2. Any box $[0\ 1\ x\ y]$ with $x \geq y$, $x \geq 2$ converges within 4 generations (3 if $y = x-1$, 2 if $(x,y) = (2,1)$). We calculate

$$\begin{aligned} [0\ 1\ x\ y]\ &\triangleright [1\ (x-1)\ (x-y)\ y] \triangleright [(x-2)\ (y-1)\ (2y-x)\ (y-1)] \\ &\triangleright [(y-x+1)\ (y-x+1)\ (y-x+1)\ (y-x+1)] \triangleright [0\ 0\ 0\ 0], \end{aligned}$$

again observing the quicker convergence for the special cases.

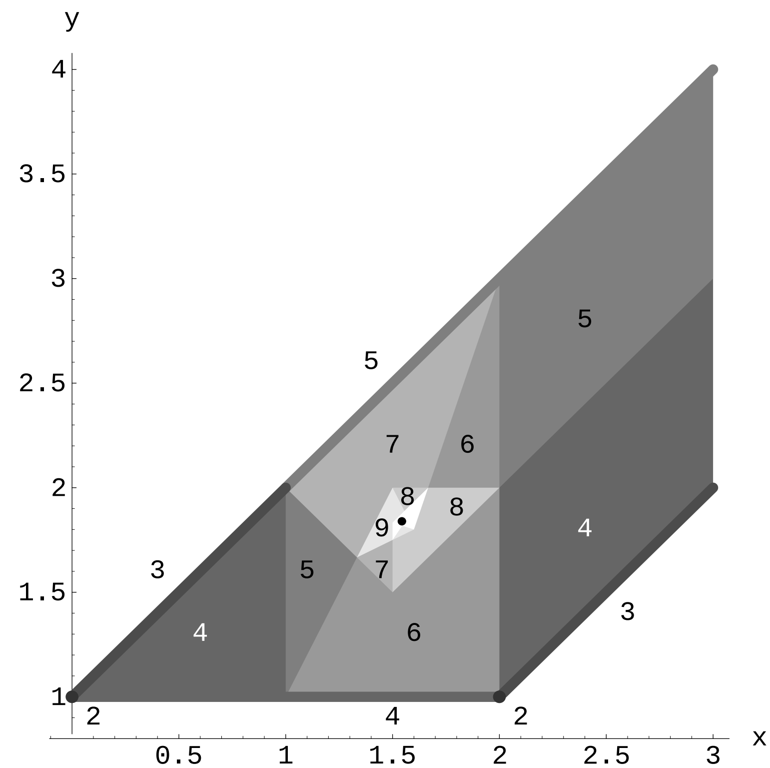

Figure 3: Subdivision of S into regions colored by longevity

EXAMPLE 3. Any box $[0\ 1\ x\ y]$ with $1 < y \le x < 2$ converges within 6 generations. We calculate

$$\begin{aligned}[0\ 1\ x\ y]\ \ &\rhd [1\ (x-1)\ (x-y)\ 2] \rhd [(2-x)\ (y-1)\ (2y-x)\ (y-1)] \\ &\rhd [|x+y-3|\ (y-x+1)\ (y-x+1)\ |x+y-3|] \rhd [p\ 0\ p\ 0] \\ &\rhd [p\ p\ p\ p] \rhd [0\ 0\ 0\ 0]\end{aligned}$$

where $p = ||x+y-3| - y + x - 1|$. (Examples 2 and 3 together comprise region S_3 in Figure 2.)

Note that Examples 1, 2 and 3, together with the case $y = 1$ discussed earlier and the type (i) and (ii) classes, cover all nonmonotone classes of diffy boxes[1]. (Monotone classes have $0 < 1 < x < y$.) Further calculations show monotone classes have longevity at least 5. Figure 3 shows how S is subdivided into regions of different longevities (the lighter the region, the greater the longevity). The only two equivalence classes not depicted are (i) and (ii); here (ii) can be considered as the point at infinity (by which is meant the one-point compactification of $\mathbf{R}^2$). The black dot in the center is the fixed point, and the white region around it contains all equivalence classes of longevity 10 or more generations. Note that where two regions of different longevity meet, the boundary between them "belongs" to the region of lower longevity.

To determine the longevity of monotone classes in full detail, we shall change our approach from the sort of increasingly detailed calculations in the above examples to a consideration of pre-images under f. We will therefore need the following result regarding the invertibility of the map f.

THEOREM 2. The map f has an inverse b which is well-defined on the interior of S (and $\{(x, 1) : 0 < x < 2\}$), which maps the interior of S into the interior of S, and which preserves line segments.

[1]and thus provide an alternate proof of Proposition 2

Proof. We have already seen that of the three branches given in the definition of f at the beginning of this section, only the second is one-to-one. Since the images of the first and third branches lie on the left and right boundaries of S, the inverse map $b = f^{-1}$ should be well-defined in the interior of S (and $y = 1$). We can invert the expression for f on the second branch to find

$$b(x,y) = \left(\frac{2y}{x+y-1}, \frac{x+y+1}{x+y-1}\right)$$

for (x, y) in the interior of S (and $\{(x, 1) : 0 < x < 2\}$). We observe that b maps the interior of S into the interior of S: for $b(x, y) = (u, v)$, $x > 0$ and $y > 1$ together imply that $u > 0$, $v > 1$, and $u-1 < v < u+1$. Of course, we can also see that, as mentioned above, $f^{-1}(\text{int}\, S)$ must fall within the second region in the definition of f, $S_2 = \{(x, y) : 1 < x \le y \le x + 1\}$.

We can also observe, by calculation, that b preserves line segments in the interior of S: If we have $y = k_1x + k_2$ in this region, then we find

$$b(x, k_1x+k_2) = (u,v) = \left(\frac{2(k_1x+k_2)}{(k_1+1)x+k_2-1}, \frac{(k_1+1)x+k_2+1}{(k_1+1)x+k_2-1}\right),$$

which obeys

$$v = \left(\frac{k_1+1}{k_1+k_2}\right)u + \left(\frac{k_2-k_1}{k_1+k_2}\right).$$

Likewise f preserves line segments in region 2:

$$\begin{aligned} f(x, k_1x+k_2) &= (u,v) = \left(1 + \frac{2-x}{k_1x+k_2-1}, \frac{x}{k_1x+k_2-1}\right) \\ &\Rightarrow v = \left(\frac{1-k_2}{k_1+k_2+1}\right)u + \left(\frac{2+k_2}{k_1+k_2+1}\right). \ \square \end{aligned}$$

We shall now define a sequence of sets T_n $(n > 1)$ inductively, as follows. Let $T_2 = S$, and for $n > 1$ let $T_{n+1} = f^{-1}(T_n)$. Because the backward map b is only well-defined in the interior of S, we shall consider the first few examples individually, until we have a $T_n \subset \text{int}\, S$. We shall also need to consider the type (ii) class (the point at infinity), as for $0 < x < 2$, $[0\ 1\ x\ 1] \rhd [1\ |x-1|\ |x-1|\ 1] \sim [0\ 0\ 1\ 1] \rhd [0\ 1\ 0\ 1]$, that is, the diffy box process sends points on the boundary $y = 1$ $(0 < x < 2)$ to the type (ii) class, and then sends the type (ii) class to the point $(0, 1) \in S$. Excluding this point from S, we find $T_3 = f^{-1}(S) = (S\backslash\{y = 1\}) \cup \infty$ (see Figure 4), since the only points in S which do not have images in S are on the lower boundary $y = 1$, and in addition the type (ii) class at ∞ maps to (0,1).

Next we find that $T_4 = f^{-1}(T_3) = S\backslash\{(x, x \pm 1)\}$, as $f^{-1}(\{y = 1\}) = \{(x, x \pm 1)\}$ (i.e., $f(x, y) = (k, 1) \Rightarrow y = x \pm 1$), and $f^{-1}(\infty) = \{(x, y) : 0 < x < 2,\ y = 1\}$. Following this, $T_5 = f^{-1}(T_4) = \{(x, y) : 1 < x < y\}$, since $f^{-1}(0, 1) = \{(x, x-1) : x > 2\}$, $f^{-1}(2, 1) = \{(x, x+1) : 0 < x \le 1\}$, $f^{-1}(x, x-1) : x > 2\} = \{(x, y) : 0 < x \le 1,\ 1 < y < x+1\}$, $f^{-1}(x, x+1) : 0 < x \le 1\} = \{(x, y) : x \ge 2,\ x-1 < y < x\}$, and $f^{-1}(x, x+1) : x > 1\} = \{(x, y) : 1 < x < 2,\ 1 < y < x\}$ (and, again, $f^{-1}(\infty) = \{(x, y) : 0 < x < 2,\ y = 1\}$). Finally, we find that $T_6 = f^{-1}(T_5)$ is the interior of the triangle with vertices (1,1), (2,1), and (2,3), again by excluding the preimages of the parts of S excluded from T_5 (see Figure 4 for sketches of all these). Since $T_6 \subset \text{int}\, S$, we can use the one-to-one backward map b to determine T_n for $n > 6$.

If we continue on, we will see that T_n for $n > 6$ are also interiors of triangles; the second half of Theorem 2 shows that b preserves triangles in the interior of S, and for $n \ge 8$ the vertices of these triangles are also in the interior of S, which allows us to keep track of the T_n via their

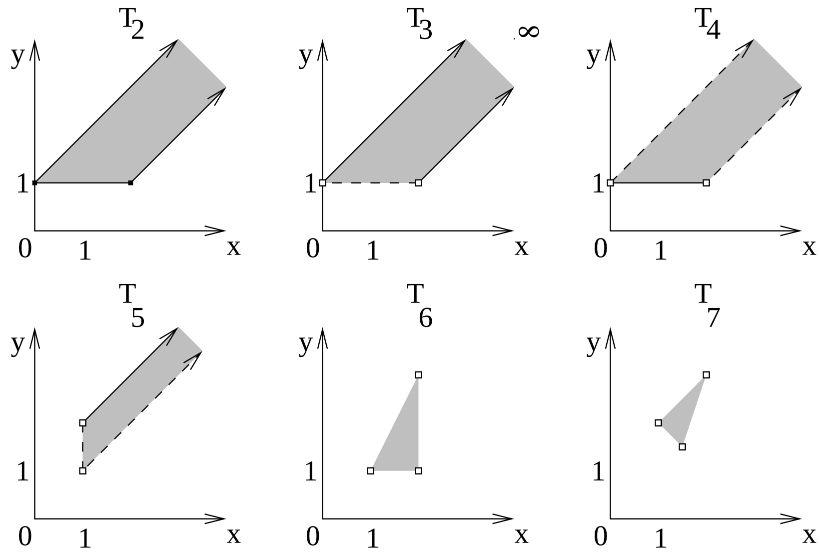

Figure 4: The set T_n for $2 \le n \le 7$

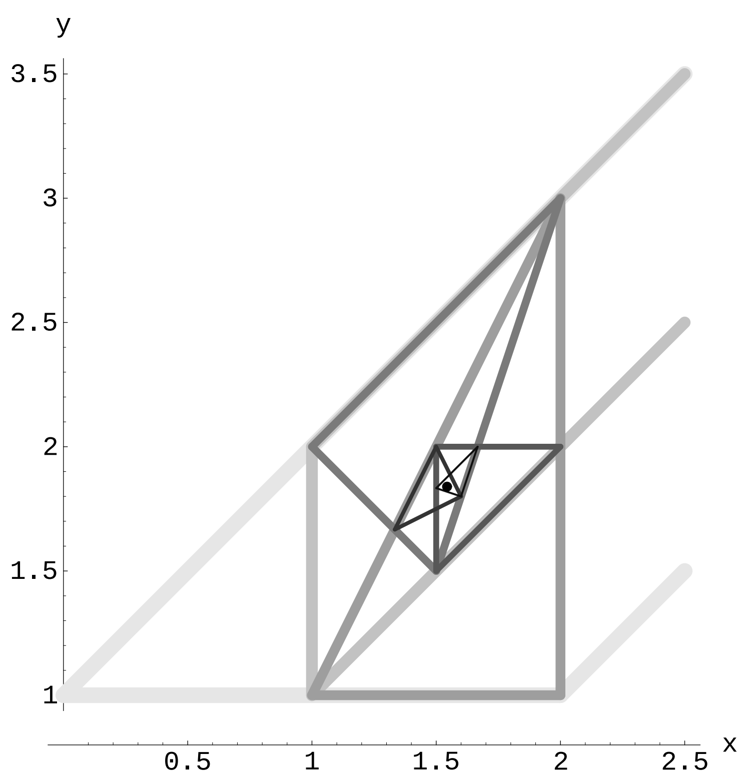

Figure 5: T_n $(n \le 10)$ superimposed upon each other

n	Vertex 1	Vertex 2	Vertex 3
6	(1,1)	(2,1)	(2,3)
7	(2,3)	(1,2)	$(\frac{3}{2},\frac{3}{2})$
8	$(\frac{3}{2},\frac{3}{2})$	(2,2)	$(\frac{3}{2},2)$
9	$(\frac{3}{2},2)$	$(\frac{4}{3},\frac{5}{3})$	$(\frac{8}{5},\frac{9}{5})$
10	$(\frac{8}{5},\frac{9}{5})$	$(\frac{5}{3},2)$	$(\frac{3}{2},\frac{11}{6})$

Table 2: Vertices of T_n for $6 \leq n \leq 10$

vertices (Table 2 provides a partial list, and Figure 5 shows boundaries of some of the T_n ($n \leq 10$) superimposed upon each other).

The utility of the T_n follows from the fact that all equivalence classes of longevity n ($n > 1$) are contained in T_n. It is simple enough to check this for the first few examples; thereafter the result follows by induction. It is also worth noting that although T_n does not contain all classes of longevity $n+1$, it does contain all classes of longevity $n+2$ or greater. Furthermore, we observe (again by induction, starting with $n = 2$) that $T_{n+2} \subset T_n$, so that $T_{n+3} \subset T_{n+1}$, and we can classify the set of all equivalence classes of longevity n ($n > 1$) as precisely $T_n \backslash \{T_{n+1}, T_{n+2}\}$. That is, an equivalence class has longevity n if and only if it is in T_n but not T_{n+1} or T_{n+2}.

At this point the question arises of how to test where a given point (i.e., equivalence class) falls relative to the sequence of triangles T_n ($n \geq 6$). Arguably the simplest is just to plot it on a graph containing (enough of) the T_n. There are also several simple algebraic approaches, however, to test whether a point falls within a given triangle. One is to write the points involved in vector form. First let the interior of each triangle be written as the set of points whose coordinates are a weighted average of the coordinates of the three vertices V_1, V_2 and V_3: $\hat{T} = \{(x, y) : (x, y) = rV_1 + sV_2 + (1 - r - s)V_3$ for some $r, s > 0,\ r + s < 1\}$. Now, to test whether a point P is inside $\hat{T}$, calculate the corresponding "coordinates" r and s:

$$\begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} \overrightarrow{V_3V_1} & \overrightarrow{V_3V_2} \end{bmatrix}^{-1} \overrightarrow{V_3P},$$

and see if $r, s > 0,\ r + s < 1$. For example, $T_6 = \{(x, y) : (x, y) = r(1, 1) + s(2, 1) + (1 - r - s)(2, 3)$ for some $r, s > 0,\ r + s < 1\}$, and the fixed point $P = (q(q - 1), q) \approx (1.5437, 1.8393)$, making $\overrightarrow{V_3V_1} = (-1, -2)$, $\overrightarrow{V_3V_2} = (0, -2)$, $\overrightarrow{V_3P} \approx (-0.4563, -1.1607)$, and $\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right] = \left[\begin{smallmatrix} -1 & 0 \\ -2 & -2 \end{smallmatrix}\right]^{-1} \left[\begin{smallmatrix} -0.4563 \\ -1.1607 \end{smallmatrix}\right] = (0.4563, 0.124)$, verifying that $P \in T_6$.

The only drawback to an algebraic approach is that it is inescapably recursive, and the number of calculations required to continue testing whether a given point falls inside each T_n is comparable to the number of calculations required simply to take the diffy box process toward its eventual end. A graphical approach merely requires plotting a sufficient number of T_n so that the point falls outside two consecutive triangles.

We close this section with one more way to look at the domain S. Figure 6 divides S into three disjoint invariant regions by shades of gray: that is, each shade (light, medium or dark) represents a sequence of images and pre-images under f, jumping around and toward the fixed point. The fading of the colors near the fixed point indicates increasing longevity.

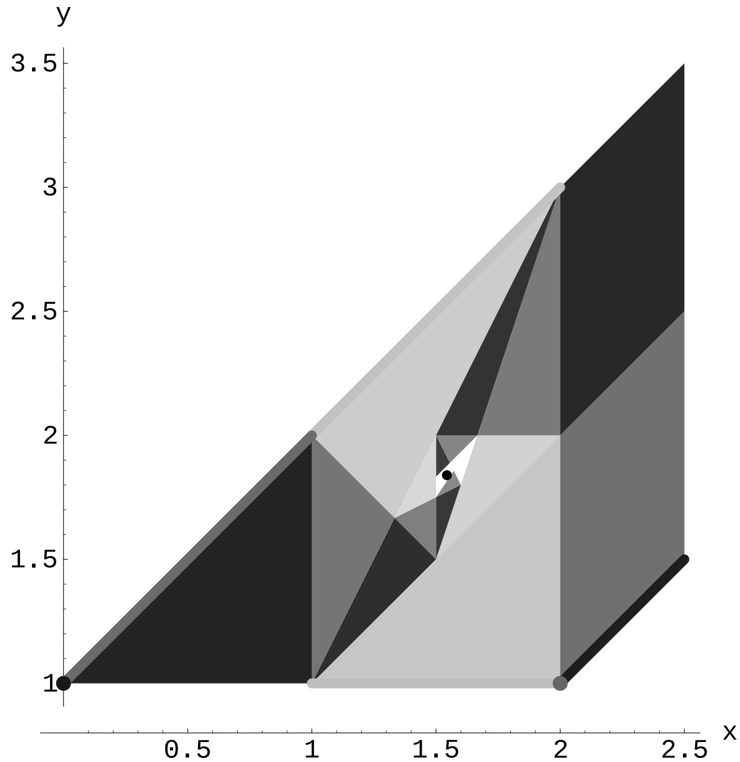

Figure 6: Subdivision of S into 3 invariant regions (by color)

5 Conclusions, applications and extensions

We now return to our original question: Does every diffy box converge to the zero box, and, if so, how many generations will it take? We can now reinterpret the results of our analysis on equivalence classes in terms of boxes as 4-tuples. We might first make the observation that (as seen by the regions into which S is subdivided in Figure 3) the use of "complicated" numbers such as radicals or transcendentals does not really prolong convergence much, as within a couple of generations the differences have propagated through the four vertices and get subtracted out. The answer to our original question is yes, and for any longevity you specify, there are some classes of diffy box that take that long to converge. However, there is one class of diffy box (the fixed point of f) which takes infinitely long to converge; any diffy box in this class has entries (vertex numbers) which become smaller and smaller but never actually reach zero. Nonmonotone boxes converge quickly (in no more than 6 generations), while to determine the longevity of a monotone diffy box, it is simplest to put the box in canonical form and compare its coordinates (x, y) with a graph of the regions of various longevities identified in the previous section.

As mentioned in the introduction, diffy boxes can be used as problem-solving contexts for elementary grades students (e.g., [1]). After working through several diffy boxes, children can group them according to longevity and begin to observe some patterns in the forms of boxes which converge in 1, 2, 3, and possibly 4 generations. They can also observe the properties we used in Definition 5 to define equivalence, as well as the effects (or lack thereof) of using numbers other than whole numbers.

Although we have classified boxes by how many generations they take to converge to the zero box, it is an open question what number appears on all the vertices of the penultimate box, i.e., the last box in the sequence before [0 0 0 0].

A natural extension of this problem which we leave to the reader is the generalization from squares to other polygons. For example, a quick investigation of "diffy triangles" reveals a peculiar chasing pattern and the surprising(?) result that *no* "diffy triangles" ever converge to the all-zero triangle, except for those with all three numbers the same (to convince yourself of this, try to construct the parent of such a diffy triangle). From this observation one might try to classify the types of possible behavior of diffy triangles, or else move to a larger scale and perhaps consider convergence of "diffy polygons" with even numbers of edges vs. odd numbers of edges. (If we place "diffy polygons" in the context of graph theory, we see that any simple generalization to a more general class of graphs is prevented by the fact that only the cycle graphs C_n, i.e., polygons, have line graphs isomorphic to themselves. Considering polygons as cycle graphs, however, does allow us to include the trivial example C_2, the two-sided polygon, which converges in two steps for any two starting values.)

Another possible extension is a change in the distance function used to calculate the vertices of a given box's child. We have used the symmetric "one-dimensional" norm $f(a,b) = |a-b|$, but we might instead have used a "two-dimensional" norm $f(a,b) = \sqrt{a^2 - b^2}$, or an asymmetric one-norm $f(a,b) = |Aa - Bb|$, for fixed weights $A + B = 2$ which place more emphasis on one vertex than the other (in this case we would clearly have to identify vertices by orientation, e.g., b is clockwise from a).

It is also interesting to note that the irrational number q involved in the fixed-point diffy box class is also associated with sequences of numbers called Tribonacci numbers. Similar to the notion of Fibonacci numbers, Tribonacci numbers are a sequence of numbers t_n which obey the recursive equation $t_n = t_{n-1} + t_{n-2} + t_{n-3}$. (The sequence typically begins with $t_1 = 1$, $t_2 = 1$, $t_3 = 2$.) Like Fibonacci numbers, any sequence of Tribonacci numbers tends toward a geometric increase, with the ratio of any two successive numbers in the sequence approaching a fixed constant. For Tribonacci numbers, that constant is q. In fact, beginning as above, $t_n \to q^n$ as $n \to \infty$. (If we look for geometric solutions $t_n = a^n$ to the recursive relation above, we see that we must have $a^n = a^{n-1} + a^{n-2} + a^{n-3}$ or $a^3 = a^2 + a + 1$, the same equation we solved to obtain q.)

It is remarkable how mathematically rich such a simple notion can be, and we invite the reader to explore further.

CMKZ thanks George Christ for introducing him to diffy boxes.

References

[1] Charles A. Dana Center, *TEXTEAMS Rethinking Elementary School Mathematics, Part I*, Austin: University of Texas, 2002.

[2] John Guckenheimer and Philip Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, New York: Springer-Verlag, 1983.

[3] Stephen H. Strogatz, *Nonlinear Dynamics and Chaos*, New York: Addison-Wesley, 1994.

Referee's appendix

Proof of Proposition 2 (Section 2). By cases. Let $a > b > c > d$.

$[a\ a\ a\ a] \triangleright [0\ 0\ 0\ 0]$.

$[a\ b\ a\ b] \triangleright [(a-b)\ (a-b)\ (a-b)\ (a-b)] \triangleright [0\ 0\ 0\ 0]$.

$[a\ a\ b\ b] \triangleright [0\ (a-b)\ 0\ (a-b)] \triangleright [(a-b)\ (a-b)\ (a-b)\ (a-b)] \triangleright [0\ 0\ 0\ 0]$.

$[a\ a\ a\ b] \triangleright [0\ 0\ (a-b)\ (a-b)] \triangleright [0\ (a-b)\ 0\ (a-b)] \triangleright [(a-b)\ (a-b)\ (a-b)\ (a-b)] \triangleright [0\ 0\ 0\ 0]$.

$[a\ b\ a\ c] \triangleright [(a-b)\ (a-b)\ (a-c)\ (a-c)] \triangleright [0\ (b-c)\ 0\ (b-c)] \triangleright [(b-c)\ (b-c)\ (b-c)\ (b-c)] \triangleright [0\ 0\ 0\ 0]$.

$[a\ b\ b\ b] \triangleright [(a-b)\ 0\ 0\ (a-b)] \triangleright [(a-b)\ 0\ (a-b)\ 0] \triangleright [(a-b)\ (a-b)\ (a-b)\ (a-b)] \triangleright [0\ 0\ 0\ 0]$.

$[a\ b\ b\ c] \triangleright [(a-b)\ 0\ (b-c)\ (a-c)] \triangleright [(a-b)\ (b-c)\ (a-b)\ (b-c)] \triangleright [|a+c-2b|\ |a+c-2b|\ |a+c-2b|\ |a+c-2b|] \triangleright [0\ 0\ 0\ 0]$ (converges after 3 generations if $b = \frac{a+c}{2}$).

$[a\ b\ c\ b] \triangleright [(a-b)\ (b-c)\ (b-c)\ (a-b)] \triangleright [|a+c-2b|\ 0\ |a+c-2b|\ 0] \triangleright [|a+c-2b|\ |a+c-2b|\ |a+c-2b|\ |a+c-2b|] \triangleright [0\ 0\ 0\ 0]$ (converges after 2 generations if $b = \frac{a+c}{2}$).

$[a\ c\ b\ c] \triangleright [(a-c)\ (b-c)\ (b-c)\ (a-c)] \triangleright [(a-b)\ 0\ (a-b)\ 0] \triangleright [(a-b)\ (a-b)\ (a-b)\ (a-b)] \triangleright [0\ 0\ 0\ 0]$.

$[a\ c\ b\ d] \triangleright [(a-c)\ (b-c)\ (b-d)\ (a-d)] \triangleright [(a-b)\ (c-d)\ (a-b)\ (c-d)]$ which converges within 2 generations by Lemma 1 (within 1 if $a+d = b+c$).

$[a\ a\ b\ c] \triangleright [0\ (a-b)\ (b-c)\ (a-c)] \triangleright [(a-b)\ |a+c-2b|\ (a-b)\ (a-c)]$ which is of the form $[a\ b\ a\ c]$ and converges within 4 generations (see above), or 2 if $b \leq \frac{a+c}{2}$.

$[a\ b\ c\ c] \triangleright [(a-b)\ (b-c)\ 0\ (a-c)] \triangleright [|a+c-2b|\ (b-c)\ (a-c)\ (b-c)]$ which is of the form $[a\ b\ a\ c]$ and converges within 4 generations (see above), or 2 if $b \geq \frac{a+c}{2}$.

$[a\ b\ d\ c] \triangleright [(a-b)\ (b-d)\ (c-d)\ (a-c)] \triangleright [|a+d-2b|\ (b-c)\ |a+d-2c|\ (b-c)]$ which converges within 4 generations by Lemma 1 (within 1 if $a+d = b+c$, and 2 if $c \leq \frac{a+d}{2} \leq b$). □

Proof of Corollary (Section 2).

If the all-integer-valued B has at least one pair of identical vertices, then note simply that the greatest discrepancy between B's possible longevity and its corresponding smallest possible range is 3. Otherwise, let n be the smallest number such that $C^m(B)$ have all vertices distinct for $0 \leq m \leq n-1$, but $C^n(B)$ has at least one pair of repeated vertices. Then (from the first half of the definition of n) $|C^n(B)| \leq |B| - n$. (Obviously $n < |B|$, or else we have a contradiction.)

If we denote the longevity of B by $L(B)$, we have $L(B) = L(C^n(B)) + n$. Consulting Table 1, we see that

(i) if $L(C^n(B)) = 2$, then $|C^n(B)| \geq 1$, so that $L(B) \leq |B| + 1$.

(ii) if $L(C^n(B)) = 3$, then $|C^n(B)| \geq 1$, so that $L(B) \leq |B| + 2$.

(iii) if $L(C^n(B)) = 4$, then $|C^n(B)| \geq 1$, so that $L(B) \leq |B| + 3$.

(iv) if $L(C^n(B)) = 6$, then $|C^n(B)| \geq 3$, so that $L(B) \leq |B| + 1$. □