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Existence and stability of periodic orbits of periodic difference equations with delays *

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Abstract

In this paper, we investigate the existence and stability of periodic orbits of the p-periodic difference equation with delays $x_n = f(n-1, x_{n-k})$. We show that the periodic orbits of this equation depend on the periodic orbits of p autonomous equations when p divides k. When p is not a divisor of k, the periodic orbits depend on the periodic orbits of $gcd(p,k)$ nonautonomous $\frac{p}{gcd(p,k)}$-periodic difference equations. We give formulas for calculating the number of different periodic orbits under certain conditions. In addition, when p and k are relatively prime integers, we introduce what we call the pk-Sharkovsky's ordering of the positive integers, and extend Sharkovsky's theorem to periodic difference equations with delays. Finally, we characterize global stability and show that the period of a globally asymptotically stable orbit must divide p.

1 Introduction

Autonomous (time-invariant) difference equations with delays of the general form

$$x_n = f(x_{n-1}, x_{n-2}, \cdots, x_{n-k}) \tag{1.1}$$

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have shown up prominently in the books of Kocic and Ladas [19] and Elaydi [10]. Such equations have been used to model biological populations in which year classes may develop independently. In general, delay equations are heavily used as epidemic models, neural networks model, ecological models, economic models and then systems with memory [10], [19], [17].

Recently, there have been a surge of research activities focusing on a special case of equation (1.1), namely, the equation

$$x_n = f(x_{n-k}). \tag{1.2}$$

A prototype of equation (1.2) is the popular fish model, the Beverton-Holt delay equation [20, 10, 3, 4, 5, 11, 12]

$$x_n = \frac{\mu K x_{n-k}}{K + (\mu - 1)x_{n-k}}, \mu > 0,\ K > 0,\ n, k \in \mathbb{N}. \tag{1.3}$$

Here, μ is the intrinsic growth rate, K is the carrying capacity, and k is the delay time period. One group of researchers, that includes, Balibrea and Linero[1] [2]. Liang [21], Der Heiden and Liang[6], Diekman and Van Gill [7], focused their attention on the combinatorial structure of the periodic orbits of equation (1.2) and the extension of Sharkovsky's theorem. In another direction, several researchers turned their attention to nonautonomous periodic difference equations of the form

$$x_{n+1} = f_n(x_n),\ n \in \mathbb{N}, \tag{1.4}$$

where $f_{n+p} = f_n$ for all $n \in \mathbb{N}$ and some integer $p \geq 2$ (see Franke and Selgrade [14], Franke and Yakubu [15], Selgrade and Roberds [22], and Henson [16]). In those papers, the overriding consideration was to investigate populations with periodically fluctuating habitat. However, the study of periodic difference equations received great inputs by the publication of two conjectures by Cushing and Henson [5, 4]. In a series of papers, Elaydi and Sacker [11, 12] not only proved Cushing-Henson conjectures, but also laid the necessary machinery to study periodic difference equations. This was followed by the papers of Kocic [18] and Kon [17] who addresses the question of whether periodically fluctuating habitat would enhance the growth of the population (resonance) or would have an adverse effect on its growth (attenuance).

In [1] AlSharawi et al focused, among other things, on the extension of Sharkovsky's theorem to the periodic difference equation (1.4). Moreover, they were able to describe the combinatorial structure of the periodic orbits of equation (1.4). In this paper, we extend our work in [1] to periodic difference equations with delays of the form

$$x_n = f_{n-1}(x_{n-k}),\ f_{n+p} = f_n \text{ and } x_n \in \mathbb{X},\ \text{for all } n \in \mathbb{N}, \tag{1.5}$$

[1]The authors received the 2004 prize of best paper on difference equations by the International Society of Difference Equations (ISDE)

where $\mathbb{X}$ is a metric space, which will be restricted as needed, $k > 1, p > 1$, and $k, p \in \mathbb{N}$. In Section 2, we study the case where the period p of the system divides the delay k. Using the Möbius inversion formula we provide a formula for counting the number of different periodic orbits of a given minimal period. In Section 3, we extend the results of Section 2 to the case where p does not divide k. In Section 4 we provide an extension of Sharkovsky's theorem [23], [8], [9] to equation (1.5). Finally, in Section 5, we investigate the stability of the periodic orbits of equation (1.5).

In the sequel, we use the following notations: $\mathbb{Z}^+$ denotes the set of positive integers and $\mathbb{N} := \mathbb{Z}^+ \cup \{0\}$. $gcd(p,k)$ and $lcm(p,k)$ denote the greatest common divisor and the least common multiple between p and k, respectively. A difference equation $x_{n+1} = f(n, x_n) = f_n(x_n)$ is called p-periodic if p is the minimal positive integer for which $f_{n+p} = f_n$ for all $n \in \mathbb{N}$. Similarly, a periodic orbit (geometric cycle) $\{c_0, c_1, \cdots, c_{r-1}\}$ is called r-cycle if r is the minimal period. For a function $f(x)$, $f^2(x) = f(f(x))$ and inductively, $f^m(x) = f(f^{m-1}(x))$. Finally, for our convenience, we write $f_{m \bmod p}(x)$ simply as $f_m(x)$.

2 The periodic orbits when p divides k

Throughout this section we assume $p|k$, i.e., $k = mp$ for some $m \in \mathbb{N}$. The orbit of (1.5) will be denoted by

$$\mathcal{O}^+ (x_{-k+1}, x_{-k+2}, \ldots, x_0) = \{x_{-k+1}, x_{-k+2}, \ldots, x_0, x_1, x_2, \ldots\},$$

and since $p|k$ it can be partitioned into the k suborbits

$$\mathcal{O}_i^+ (x_{-k+i}) = \left\{x_{i+k(j-1)} : j \in \mathbb{N}\right\}, \quad 1 \le i \le k,$$

where $\mathcal{O}_i^+$ is the orbit associated with the autonomous difference equation

$$x_{i+kn} = f_{i-1}(x_{i+k(n-1)}), \quad n \in \mathbb{N}. \tag{2.1}$$

This shows that the periodic orbits of equation (1.5) depend on periodic orbits of the p autonomous equations

$$x_{n+1} = f_j(x_n), \quad n \in \mathbb{N}, \quad 0 \le j \le p-1.$$

A way to visualize these orbits is provided by the following array, where the initial elements $x_{-k+1}, \ldots, x_0$ form the first row and subsequent rows are found by applying the maps f_j, $0 \le j \le p-1$.

$$
\begin{array}{ccccccc}
\mathcal{O}_0^+(x_{-k+1}) & \cdots & \mathcal{O}_p^+(x_{-k+1+p}) & \cdots & \mathcal{O}_{2p}^+(x_{-k+1+2p}) & \cdots & \mathcal{O}_{k-1}^+(x_0) \\
\hline
x_{-k+1} & \cdots & x_{-k+1+p} & \cdots & x_{-k+1+2p} & \cdots & x_0 \\
\downarrow & \cdots & \downarrow & \cdots & \downarrow & \cdots & \downarrow \\
f_0(x_{-k+1}) & \cdots & f_{p-1}(x_{-k+1+p}) & \cdots & f_{p-1}(x_{-k+1+2p}) & \cdots & f_{p-1}(x_0) \\
\downarrow & \cdots & \downarrow & \cdots & \downarrow & \cdots & \downarrow \\
f_0^2(x_{-k+1}) & \cdots & f_{p-1}^2(x_{-k+1+p}) & \cdots & f_{p-1}^2(x_{-k+1+2p}) & \cdots & f_{p-1}^2(x_0) \\
\downarrow & \cdots & \downarrow & \cdots & \downarrow & \cdots & \downarrow \\
f_0^3(x_{-k+1}) & \cdots & f_{p-1}^3(x_{-k+1+p}) & \cdots & f_{p-1}^3(x_{-k+1+2p}) & \cdots & f_{p-1}^3(x_0) \\
\downarrow & \cdots & \downarrow & \cdots & \downarrow & \cdots & \downarrow
\end{array}
$$

The following two lemmas describe when equation (1.5) has a periodic orbit as well as structural properties of the periodic orbit.

Lemma 2.1. *Let $k = mp$. Then each of the following holds true.*

(i) Equation (1.5) has a periodic orbit if and only if each autonomous equation $x_{n+1} = f_j(x_n)$, $n \in \mathbb{N}$, $0 \le j \le p-1$, has a periodic orbit.

(ii) Suppose each autonomous equation $x_{n+1} = f_j(x_n)$, $n \in \mathbb{N}$, $0 \le j \le p-1$, has m periodic orbits $S_j, S_{j+p}, \ldots, S_{j+(m-1)p}$ (not necessarily distinct) of minimal periods $p_j, p_{j+p}, \ldots, p_{j+(m-1)p}$, respectively. Then the initial conditions $(x_{-k+1}, x_{-k+2}, \ldots, x_0) \in S_0 \times S_1 \times \cdots \times S_{k-1}$, provide a periodic orbit of equation (1.5) of period $\operatorname{lcm}(p_0, p_1, \ldots, p_{k-1}) \cdot k$, not necessarily minimal.

Proof. Trivial. □

Lemma 2.2. *Suppose $C_r = \{c_0, c_1, \ldots, c_{r-1}\}$ is an r-cycle of equation (1.5). Then each of the following holds true.*

(i) If $r|k$, then for each $0 \le i \le r-1$ the maps $f_i, f_{i+r}, \ldots, f_{i+k-r}$ have the same fixed point c_i.

(ii) If $r < k$ and $d := \gcd(r, k) \ne r$, then for each $0 \le j \le d-1$,

$$S_j = \left\{c_j, c_{j+k}, c_{j+2k}, \ldots, c_{j+(\frac{r}{d}-1)k}\right\}$$

is a cycle of period $\frac{r}{d}$ (not necessarily minimal) to each of the maps $f_j, f_{j+d}, \ldots, f_{j+k-d}$.

(iii) If $r > k$ and $d = \gcd(r, k) \ne k$, then for each $0 \le j \le d-1$,

$$S_j = \left\{c_j, c_{j+k \bmod r}, c_{j+2k \bmod r}, \ldots, c_{j+\left(\frac{r}{d}-1\right)k \bmod r}\right\}$$

is a cycle of period $\frac{r}{d}$ (not necessarily minimal) to each of the maps $f_j, f_{j+d}, \ldots, f_{j+k-d}$.

(iv) If $r = mk$, $m > 1$ then for each $0 \le j \le k-1$, $S_j = \{c_j, c_{j+k}, \ldots, c_{j+(m-1)k}\}$ is a cycle of period m to the map f_j. Furthermore, if the minimal period of S_j is p_j, then $m = \operatorname{lcm}(p_0, p_1, \ldots, p_{k-1})$.

Proof. The proof of (i) is trivial. To prove (ii) and (iii), observe that $\frac{r}{d} \cdot k = 0 \bmod r$, then track the orbits $\mathcal{O}^+(c_0, \ldots, c_{k-1 \bmod r})$ and $\mathcal{O}^+(c_0, \ldots, c_{k-1})$ respectively. For (iv), track the orbit $\mathcal{O}^+(c_0, c_1, \ldots, c_{k-1})$ and use Lemma 2.1. $\square$

We observe from Lemma 2.1 that periodic orbits of equation (1.5) are determined by the cycles of each of the maps f_j, $0 \le j \le p-1$. Therefore, suppose $S_j = \{c_{0,j}, c_{1,j}, \ldots, c_{p_j-1,j}\}$, $0 \le j \le k-1$ is a p_j-cycle of the autonomous equation $x_{n+1} = f_j(x_n)$, $n \in \mathbb{N}$. We assume throughout this section the initial conditions $(x_{-k+1}, x_{-k+2}, \ldots, x_0) \in S_0 \times S_1 \times \cdots \times S_{k-1}$. It is convenient to observe that the first $k \cdot \operatorname{lcm}(p_0, p_1, \ldots, p_{k-1})$ points in the orbit $\mathcal{O}^+(x_{-k+1}, x_{-k+1}, \ldots, x_0)$ of equation (1.5) can be viewed as rows of $q \times k$ matrix, where $q = \operatorname{lcm}(p_0, p_1, \ldots, p_{k-1})$. To see this we write the iterates x_n as

$$x_{ik+j} = f_{j-1}^{i+1}(x_{-k+j}) = c_{i+1,j-1}, \quad 1 \le j \le k, \quad -1 \le i \le q-2,$$

and so the matrix is

$$\mathcal{O}_{qk} = \begin{bmatrix} c_{0,0} & c_{0,1} & \cdots & c_{0,k} \\ c_{1,0} & c_{1,1} & \cdots & c_{1,k} \\ \vdots & \vdots & & \vdots \\ c_{p_0-1,0} & c_{p_1-1,1} & \cdots & c_{p_k-1,k} \end{bmatrix}.$$

Two questions now arise in understanding the nature of the periodic orbits of equation (1.5).

- What is the relation between the minimal periods p_j and the minimal periods of the associated cycles of equation (1.5)?

- What is the relation between the numbers p_j, $0 \le j \le k-1$, and the total number of associated distinct cycles of equation (1.5)?

We end this section with partial answers to these questions. The next lemma aids in answering the first question. We make use of the following sets used in [1].

$$\mathcal{A}_{k,q} = \{n \in \mathbb{Z}^+ : \operatorname{lcm}(n, q) = kq\}.$$

Lemma 2.3. *For each $0 \le i \le k-1$, suppose $S_i = \{c_{0,i}, c_{1,i}, \ldots, c_{0,p_i-1}\}$ is a p_i-cycle of the map f_i. Let $q = \operatorname{lcm}(p_0, p_1, \ldots, p_{k-1})$. Then each initial condition $(x_{-k+1}, x_{-k+2}, \ldots, x_0) \in \prod_{k=0}^{k-1} S_i$ produces an r-cycle, $r \in \mathcal{A}_{k,q}$.*

Proof. By Lemma 2.1, $(x_{-k+1}, x_{-k+2}, \ldots, x_0) \in \prod_{k=0}^{k-1} S_i$ produces a periodic orbit of equation (1.5) with minimal period r with $r | qk$. We consider three cases.

(i) If r is a divisor of k, then Lemma 2.2 implies that each cycle S_i is a fixed point. Therefore $q = 1$ and since $\mathcal{A}_{k,1}$ contains the divisors of k, $r \in \mathcal{A}_{k,q}$.

(ii) If r is a multiple of k, then Lemma 2.2 implies $r = qk \in \mathcal{A}_{k,q}$.

(iii) If $\gcd(r, k) = d \notin \{r, k\}$, then Lemma 2.2 implies that each cycle S_i has period equals to $\frac{r}{d}$. This tells us that $p_i | \frac{r}{d}$ for all $0 \le i \le k-1$. Therefore $q | \frac{r}{d}$, but $\frac{r}{d} = \frac{\operatorname{lcm}(r,k)}{k}$ and consequently kq divides $\operatorname{lcm}(r, k)$. However, by Lemma 2.1, $r | kq$, so $\operatorname{lcm}(r, k) | kq$. Hence $\operatorname{lcm}(r, k) = kq$, i.e., $r \in \mathcal{A}_{k,q}$.

□

Note that this lemma does not assure that $r \in \mathcal{A}_{k,q}$ is a minimal period. The following lemma will be needed to clarify the relations between the elements of the sets $\mathcal{A}_{k,q}$, as well as, to determine when r is a minimal period.

Lemma 2.4. *Let $r, r^* \in \mathcal{A}_{k,q}$. Denote $d^* = \gcd(r^*, k)$, $d = \gcd(r, k)$, and $q = \operatorname{lcm}(p_0, \ldots, p_{k-1})$ for some positive integers $p_0, \ldots, p_{k-1}$. Then each of the following holds true.*

(i) r^* divides r if and only if d^* divides d.

(ii) $r^* < r$ if and only if $d^* < d$.

(iii) $d = \frac{r}{q}$, and $\frac{kq}{r} = \frac{k}{d}$.

(iv) There exists a positive integer h such that $\frac{k}{d} h \equiv 1 \mod q$.

(v) $\left(\frac{k}{d} \mod q\right) h \equiv 1 \mod p_i$, for all $0 \le i \le d-1$.

Proof. The proofs of (i)–(iii) are trivial. (iv) follows from the fact that $\frac{k}{d}$ is in the group of units, $U(q)$, of the ring $\mathbb{Z}_q$. The proof of (v) follows from the following string of equivalences. Let $b = \frac{k}{d} \mod q$. Then

$$
\begin{aligned}
\tfrac{k}{d} &\equiv b \bmod q \\
\tfrac{k}{d}h &\equiv bh \bmod q \\
\tfrac{kh}{d} &\equiv 1 \bmod q \\
0 &\equiv bh - 1 \bmod q \\
bh &\equiv 1 \bmod q \\
bh &\equiv 1 \bmod p_i, \quad 0 \le i \le d-1 \\
\left(\tfrac{k}{d} \bmod q\right) h &\equiv 1 \bmod p_i, \quad 0 \le i \le d-1.
\end{aligned}
$$

□

The next theorem provides a necessary and sufficient condition for the period $r \in \mathcal{A}_{k,q}$ to be minimal.

Theorem 2.1. *Assume the conditions of Lemma 2.3 hold, and let* $r \in \mathcal{A}_{k,q}$, $q > 1$ *and* $d = \gcd(r,k)$. *The initial vector* $(x_{-k+1}, x_{-k+2}, \dots, x_0) \in \prod_{i=0}^{k-1} S_i$ *produces an* r*-cycle if and only if* $S_i = S_{i+d} = S_{i+2d} = \cdots = S_{i+\left(\frac{k}{d}-1\right)d}$ *for all* $0 \le i \le d-1$.

Proof. Suppose $S_i = S_{i+d} = S_{i+2d} = \cdots = S_{i+\left(\frac{k}{d}-1\right)d}$ for all $0 \le i < d$. The proof will be complete by constructing an r-cycle. From (i) in Lemma 2.4, we have

$$\mathbb{Z}_q = \{jh \bmod q : 0 \le j \le q-1\} = \{0, 1, 2, \dots, q-1\}.$$

Assume that for any $r^* \in \mathcal{A}_{k,q}$, $r^* < r$ and $d^* = \gcd(r^*, k)$, the condition $S_i = S_{i+d^*} = \cdots = S_{i+k-d^*}$ is not satisfied. Since $r = qd$, the construction is divided into two cases.

(i) $r \le k$, i.e. $q \le \frac{k}{d}$.

Let $y = h < q$ be the unique solution of $\frac{k}{d} \cdot y = 1 \bmod q$. Define the initial vector $(x_{-k+1}, \dots, x_0) \in \prod_{i=0}^{k-1} S_k$ to be

$$x_{-k+1+id+j} = \begin{cases} c_{ih \bmod p_j, j}, & 0 \le i \le q-1, \\ c_{(i \bmod q)h \bmod p_j, j}, & q \le i < \frac{k}{d}, \end{cases}$$

for each $0 \le j \le d-1$. This initial condition provides an r-cycle. To see this, let equation (1.5) act on the first d components of the given initial vector. Obviously, this implies $f_i(c_{0,i}) = c_{1,k}$, $0 \le i \le d-1$, which is the $\frac{k}{d}$th d components of the orbit. On the other hand, the $\frac{k}{d}$th d components under periodic assumption are

$$x_{1+j} = c_{\left(\frac{k}{d} \bmod q\right)h \bmod p_j, j}, \quad 0 \le j \le d-1.$$

But by (ii) in Lemma 2.4,

$$x_{1+j} = c_{\left(\frac{k}{d} \bmod q\right)h \bmod p_j, j} = c_{1,j}, \quad 0 \le j \le d-1.$$

(ii) $r > k$.

Consider the initial vector $(x_{-k+1}, \dots, x_0) \in \prod_{i=0}^{k-1} S_i$ to be, $x_{-k+id+j} = c_{ih \bmod p_j, j}$, $0 \le j \le d-1$ and $0 \le i < \frac{k}{d}$. As in part (i), all that is necessary is to check that this initial condition preserves the orbit structure as in the matrix $\mathcal{O}_{qk}$. From the fact that $\frac{k}{d} \in U(q)$, the initial vector provides a periodic orbit of minimal period $r = qd$.

Second, assume there exists $r^* \in \mathcal{A}_{kq}$, $r^* < r$, $r^*|r$, and $d = \gcd(r^*, k)$ such that the condition $S_i = S_{i+d^*} = \cdots = S_{i+k-d^*}$, is satisfied and suppose that r^* is the smallest such element. Then by Lemma 2.1, $d^*|d$. Now define the initial vector by taking the first d^* components to be $x_{-k+1+j} = c_{0,j}$, $0 \le j \le d^* - 1$ and consider that as a block. Then then replicate this block $\frac{d}{d^*}$ times to obtain the first d components, $x_{-k+1+id^*+j} = c_{0,j}$, $0 \le j \le d^* - 1$ and $0 \le i < \frac{d}{d^*}$. The proof is completed in the same fashion as the first case.

The converse is a direct result of the given assumptions and Lemma 2.2. $\square$

The following corollary is a direct consequence of Theorem 2.1.

Corollary 2.1. *For each $0 \le i \le k-1$, suppose S_i is a p_i-cycle of the map $f_{i \bmod p}$, $0 \le i \le k-1$. Let $q = \operatorname{lcm}(p_0, p_1, \ldots, p_{k-1})$, $s_1, s_2, \ldots, s_m$ be distinct elements of $\mathcal{A}_{k,q} \backslash \{kq\}$, and define $d_i = \gcd(s_i, k)$, $1 \le i \le m$. If $S_i \ne S_{d_i}$, for all $1 \le i \le m$, then each initial condition $(x_{-k+1}, x_{-k+2}, \ldots, x_0) \in \prod_{i=0}^{k-1} S_i$ produces a qk-cycle of equation (1.5). Furthermore, the total number of different periodic orbits provided by the given initial conditions is $\dfrac{pp_0p_1 \cdots p_{k-1}}{qk}$.*

Proof. Since $S_i \not\equiv S_{d_i}$ for all $1 \le i \le m$, then by Theorem 2.1, all produced periodic orbits are of minimal period qk. Now the j^{th} component of the initial vector $(x_{-k+1}, x_{-k+2}, ..., x_0) \in \prod_{i=0}^{k-1} S_i$ can be occupied by p_j choices; however, since $S_i \ne S_{d_i}$ then for each given cycle there are exactly $\frac{qk}{p}$ phase shifts. Thus the total number of different periodic orbits provided by the given initial conditions is $\frac{pp_0p_1 \ldots p_{k-1}}{qk}$. $\square$

After the existence of r-cycles, $r \in \mathcal{A}_{k,q}$, is assured by Theorem 2.1, it remains to decide the number of different r-cycles generated by the given initial conditions. In the case where $S := S_i = S_j$, for all $0 \le i, j \le k-1$, even if some of the functions f_i, $0 \le i \le k-1$ are different, the restrictions on S can be treated as an autonomous system. This has been extensively studied in [6, 7, 21]. We focus on the general case where the sets S_i, $0 \le i \le k-1$, are not all equal.

Let $r \in \mathcal{A}_{k,q}$ and define

$$\mathcal{B}(r) := \{r^* \in \mathcal{A}_{k,q} : \ r^* | r\}, \ \mathcal{B}^*(r) := \mathcal{B}(r) \backslash \{r\}. \text{ Then } \mathcal{B}(r) = \mathcal{A}_{\frac{r}{q},q}.$$

Also, denote by $P(r)$ the number of distinct r-cycles provided by the initial conditions $(x_{-k+1}, x_{-k+2}, \ldots, x_0) \in \prod_{i=0}^{k} S_i$. If the condition $S_i = S_{i+d} = \cdots S_{i+k-d}$, $d = gcd(r,k)$, $\forall i, 0 \le i \le d-1$, is not satisfied, then $P(r) = 0$. Otherwise we a give a recurrence formula of $P(r)$ in the following theorem.

Theorem 2.2. *For each $0 \le i \le k-1$, suppose S_i is a p_i-cycle of the map f_i. Let $q = lcm(p_0, p_1, ..., p_{k-1})$, $r \in \mathcal{A}_{k,q}$, and $d := gcd(r,k)$. If $S_i = S_{i+d} = ... = S_{i+k-d}$, $0 \le i < d$, then $(x_{-k+1}, x_{-k+2}, ..., x_0) \in \prod_{i=0}^{k-1} S_i$ provide $P(r)$ different r-cycles of equation (1.5), where*

$$P(r) = \frac{1}{r} \left(\min\{p, \tilde{d}\} p_0 p_1 \ldots p_{d-1} - \sum_{j \in \mathcal{B}^*(r)} j P(j) \right), \tag{2.2}$$

and $\tilde{d}$ is the smallest divisor of k for which $S_i = S_{i+\tilde{d}} = ... = S_{i+k-\tilde{d}}$, $0 \le i < \tilde{d}$ holds true.

Proof. By Theorem 2.1 there exists an r-cycle of equation (1.5) provided by the given initial conditions. Consider r^* to be the smallest divisor of r in $\mathcal{A}_{k,q}$, in which $d^* := gcd(r^*, k)$ satisfy $S_i = S_{i+d^*} = ... = S_{i+k-d^*}$, then $\tilde{d}$ divides d^*. By Lemma 2.4, d^* divides d. If $j \in \mathcal{B}(r)$ and for all $0 \le i < d_j$, $S_i = S_{i+d_j} = ... = S_{i+k-d_j}$, where $d_j := gcd(j, k)$, then Theorem 2.1 assures the existence of j-cycle, while if $j \in \mathcal{B}(r)$ and $S_i = S_{i+d_j} = ... = S_{i+k-d_j}$ is not satisfied then the given initial condition does not produce any j-cycles, i.e. $P(j) = 0$. Now the total number of periodic solutions of minimal periods in $\mathcal{B}(r)$ is given by the total choices of fixing the first d components of the initial vector $(x_{-k+1}, ..., x_0) \in \prod_{i=0}^{k-1} S_i$, which can be done in $p_0p_1...p_{d-1}$ choices. On the other hand, each j-cycle, $j \in \mathcal{B}(r)$, has $\frac{j}{\min\{p,\tilde{d}\}}$ phase shifts. Thus

$$p_0p_1...p_{d-1} = \frac{r}{\min\{p,\tilde{d}\}}P(r) + \sum_{j\in\mathcal{B}^*(r)} \frac{j}{\min\{p,\tilde{d}\}}P(j)$$

implies

$$P(r) = \frac{1}{r}\left(\min\{p,\tilde{d}\}p_0p_1...p_{d-1} - \sum_{j\in\mathcal{B}^*(r)} jP(j)\right).$$

□

The following corollary is immediate from Theorem 2.2.

Corollary 2.2. *For each $0 \le i \le k-1$, suppose S_i is a p_i-cycle of the map f_i. Let $q = \operatorname{lcm}(p_0, p_1, ..., p_{k-1})$, $r \in \mathcal{A}_{k,q}$, and $d := \gcd(r, k)$. If $j = d$ is the smallest divisor of k so that $S_i = S_{i+j} = ... = S_{i+k-j}$, $0 \le i < j$, then $(x_{-k+1}, x_{-k+2}, ..., x_0) \in \prod_{i=0}^{k-1} S_i$ provide $\frac{p_0p_1...p_{d-1}}{\max\{q,\frac{r}{p}\}}$ different r-cycles of equation (1.5).*

To give a more friendly version of formula (2.2), we need the Möbius μ-function and a special version of the Möbius inversion formula [24, 7].

Lemma 2.5. *Define $\mathcal{A}^*_{k,q} = \{\frac{r}{q} : r \in \mathcal{A}_{k,q}\}$. Let G and g be two functions defined on $\mathbb{Z}^+$ for which*

$$G(k) = \sum_{j\in\mathcal{A}^*_{k,q}} g(j).$$

Then

$$g(k) = \sum_{j\in\mathcal{A}^*_{k,q}} \mu\left(\frac{k}{j}\right)G(j),$$

where $\mu(k)$ is the Möbius μ-function.

Proof. Observe that $\mathcal{A}^*_{k,q} = \{j|k : gcd(\frac{k}{j}, q) = 1\}$, and refer to Lemma 3.4 in [7]. □

Now, we give the friendly version of formula (2.2) in the following corollary.

Corollary 2.3. *Formula (2.2) in Theorem 2.2 can be written as*

$$P(r) = \frac{\min\{p, \tilde{d}\}}{r} \sum_{\substack{j \in \mathcal{A}_{\frac{r}{q},q} \\ \tilde{d} | r}} \mu\left(\frac{r}{j}\right) p_0 p_1 \cdots p_{\frac{j}{q}-1}. \tag{2.3}$$

Proof. Formula (2.2) is equivalent to

$$p_0 p_1 \cdots p_{d-1} = \sum_{j \in \mathcal{A}_{\frac{r}{q},q}} \frac{j}{\min\{p, \tilde{d}\}} P(j) = \sum_{j \in \mathcal{A}^*_{\frac{r}{q},q}} \frac{qj}{\min\{p, \tilde{d}\}} P(qj).$$

Recall that $P(r) = 0$ when the condition $S_i = S_{i+d} = \cdots = S_{i+k-d}$, $d = gcd(r, k)$, $\forall i, 0 \leq i \leq d-1$, is not satisfied. Take that into consideration and invoke Lemma 2.5 to obtain

$$P(r) = \frac{\min\{p, \tilde{d}\}}{r} \sum_{\substack{j \in \mathcal{A}^*_{\frac{r}{q},q} \\ \tilde{d} | r}} \mu\left(\frac{r/q}{j}\right) p_0 p_1 \cdots p_{j-1} = \frac{\min\{p, \tilde{d}\}}{r} \sum_{\substack{j \in \mathcal{A}_{\frac{r}{q},q} \\ \tilde{d} | r}} \mu\left(\frac{r}{j}\right) p_0 p_1 \cdots p_{\frac{j}{q}-1}.$$

□

To clarify our developed theory, we give the following example:

Example 2.1. Suppose $k = 360 = 2^3 3^2 5$, and define

$$\begin{aligned} f_{18j}(x) &= -\frac{1}{2}(3x+1)(x-1) + j\sum_{j=0}^{3}(x-i), \quad 0 \leq j < 20 \\ f_{18j+i}(x) &= (2-x)^i + j(x-1), \quad 1 \leq i < 18,\ 0 \leq j < 20. \end{aligned}$$

Observe the 3-cycles $S_{18j} = \{0, 1, 2\}$, for all j, $0 \leq j < 20$, and the 1-cycles $S_{18j+i} = \{1\}$, $1 \leq i < 18$, $0 \leq j < 20$. Thus $S_i = S_{i+18} = \cdots = S_{i+342}$, $0 \leq i \leq 17$ and $q = \operatorname{lcm}(3, 1) = 3$. In this case

$$\mathcal{A}_{360,3} = 3 \cdot 3^2 \cdot \{1, 2, 4, 5, 8, 10, 20, 40\} = \{27, 54, 108, 135, 216, 270, 540, 1080\},$$

and $\tilde{d} = 18$ divides $r = 54, 108, 216, 270, 540, 1080$. Therefore

$$\begin{aligned} \mathcal{A}_{18,3} &= 3 \cdot 3^2 \cdot \{1, 2\} \\ \mathcal{A}_{36,3} &= 3 \cdot 3^2 \cdot \{1, 2, 4\} \\ \mathcal{A}_{72,3} &= 3 \cdot 3^2 \cdot \{1, 2, 4, 8\} \\ \mathcal{A}_{90,3} &= 3 \cdot 3^2 \cdot \{1, 2, 5, 10\} \\ \mathcal{A}_{180,3} &= 3 \cdot 3^2 \cdot \{1, 2, 4, 5, 10, 20\}. \end{aligned}$$

Now, we calculate the values of $P(r)$.

$$\begin{aligned} P(54) &= \frac{18}{54} \sum_{\substack{j\in\mathcal{A}_{18,3}\\ \tilde{d}|j}} \mu\left(\frac{54}{j}\right) p_0p_1\cdots p_{\frac{j}{3}-1} \\ &= \frac{1}{3}\left(\mu(1)p_0\cdots p_{17}\right) = 1. \end{aligned}$$

$$\begin{aligned} P(108) &= \frac{18}{108} \sum_{\substack{j\in\mathcal{A}_{36,3}\\ \tilde{d}|j}} \mu\left(\frac{108}{j}\right) p_0p_1\cdots p_{\frac{j}{3}-1} \\ &= \frac{1}{6}\left(\mu(2)p_0\cdots p_{17} + \mu(1)p_0\cdots p_{35}\right) \\ &= \frac{1}{6}(-3+3^2) = 1. \end{aligned}$$

$$\begin{aligned} P(216) &= \frac{18}{216} \sum_{\substack{j\in\mathcal{A}_{72,3}\\ \tilde{d}|j}} \mu\left(\frac{216}{j}\right) p_0p_1\cdots p_{\frac{j}{3}-1} \\ &= \frac{1}{12}\left(\mu(2^2)p_0\cdots p_{17} + \mu(2)p_0\cdots p_{35} + \mu(1)p_0\cdots p_{71}\right) \\ &= \frac{1}{12}(0-3^2+3^4) = 6. \end{aligned}$$

Similarly,

$$P(270) = 16,\ P(540) = 1960,\ \text{and}\ P(1080) = 58112088.$$

3 The periodic orbits when p does not divide k

In this section we assume p is not a divisor of the delay k, and we let $\hat{d} := \gcd(k,p)$. In this case, each orbit $\mathcal{O}^+(x_{-k+1}, x_{-k+2}, \ldots, x_0) = \{x_{-k+1}, \ldots, x_0, x_1, x_2, \ldots\}$, of equation (1.5) can be partitioned into k-suborbits

$$\mathcal{O}_j^+(x_j) = \{x_j, x_{j+k}, x_{j+2k}, \ldots\}, \quad -k+1 \le j \le 0,$$

where $\mathcal{O}_j^+(x_j)$ is associated with the nonautonomous $\frac{p}{\hat{d}}$-periodic difference equation

$$x_{k(n+1)+j} = f_{k(n+1)+j-1}(x_{kn+j}), \quad n \in \mathbb{N}. \tag{3.1}$$

Denote by G_f the set of functions $\{f_0, f_1, \ldots, f_{p-1}\}$, with the operation $\star$ defined as

$$f_i \star f_j = f_{i+j}, \quad 0 \le i, j < p.$$

Then $(G_f, \star)$ is a group that is isomorphic to $(Z_p, +)$ (integers $\bmod\ p$ under addition). For $j = -k+1$, the maps in equation (3.1) are $H_f := \{f_0, f_k, f_{2k}, ..., f_{\frac{p}{\hat{d}}k}\}$, and $(H_f, \star)$ is a cyclic subgroup of $(G_f, \star)$. Thus at $j = -k+2$, the maps in equation (3.1) contribute to the coset $f_1 \star H_f$. At $j = -k+3$, the maps contribute to the coset $f_2 \star H_f$, and so forth. Since the cyclic subgroup H_f has $\frac{p}{\hat{d}}$ elements, then by Lagrange's theorem, the quotient group G_f/H_f has $\hat{d}$ elements. Hence, in equations (3.1), the first $\hat{d}$ equations are different, i.e. $x_{kn+j+1} = f_{kn+j}(x_{k(n-1)+j+1}),\ n \in \mathbb{N},\ j = 0, 1, ..., \hat{d}-1$, are different, while the next $k - \hat{d}$ equations are time shifts. In fact we can consider the last $k - \hat{d}$ equations repetitions of the first $\hat{d}$ equations.

As in the previous section, the orbit $\mathcal{O}^+(x_{-k+1}, x_{-k+2}, \ldots, x_0)$, the suborbits $\mathcal{O}^+(x_j)$, and the k equations in (3.1) are visualized using the following diagram.

$$\begin{array}{ccccc}
x_{-k+1} & x_{-k+2} & \cdots & x_{-1} & x_0 \\
\downarrow & \downarrow & \cdots & \downarrow & \downarrow \\
f_0(x_{-k+1}) & f_1(x_{-k+2}) & \cdots & f_{k-2}(x_{-1}) & f_{k-1}(x_0) \\
\downarrow & \downarrow & \cdots & \downarrow & \downarrow \\
f_k f_0(x_{-k+1}) & f_{k+1} f_1(x_{-k+2}) & \cdots & f_{2k-2} f_{k-2}(x_{-1}) & f_{2k-1} f_{k-1}(x_0) \\
\downarrow & \downarrow & \cdots & \downarrow & \downarrow \\
\vdots & \vdots & \cdots & \vdots & \vdots,
\end{array} \tag{3.2}$$

From (3.2) we have the following lemma that is analogous to Lemma 2.1. The proof is similar.

Lemma 3.1. *Each of the following holds true:*

(i) *Equation (1.5) has a periodic orbit if and only if for each $j = 0, 1, ..., k-1$, the $\frac{p}{\hat{d}}$-periodic difference equation $x_{kn+j+1} = f_{kn+j}(x_{k(n-1)+j+1}),\ n \in \mathbb{N}$, has a periodic orbit.*

(ii) *Suppose each equation $x_{kn+j+1} = f_{kn+j}(x_{k(n-1)+j+1}),\ n \in \mathbb{N},\ 0 \le j < k$, has a p_j-cycle S_j. Then the initial condition $(x_{-k+1}, x_{-k+2}, ..., x_0) \in \prod_{i=0}^{k-1} S_i$ of equation (1.5) provides either no periodic orbit, or a periodic orbit of minimal period $r \in \mathcal{A}_{k,q}$, where $q = lcm(p_0, p_1, ..., p_{k-1})$.*

Next, assume each equation $x_{kn+j+1} = f_{kn+j}(x_{k(n-1)+j+1}),\ n \in \mathbb{N},\ 0 \le j < k$, has a p_j-cycle $S_j := \{c_{0,j}, c_{1,j}, ..., c_{p_j-1,j}\}$, and let $d_j = gcd(p_j, \frac{p}{\hat{d}})$. Here, we stress that we are considering the value of j as the reference time for the jth equation. From the combinatorial structure of geometric cycles [1], we define the non-ordered sets

$$S_j^* := \{c_{0,j}, c_{d_j,j}, c_{2d_j,j}, ..., c_{p_j-d_j-1,j}\},\ 0 \le j < k, \tag{3.3}$$

and hence, the initial conditions

$$(x_{-k+1}, x_{-k+2}, ..., x_0) \in \prod_{i=0}^{k-1} S_i^* \tag{3.4}$$

are the right candidates to provide periodic orbits of equation (1.5). Thus, we focus on these initial conditions. As in the previous section, we find the total number of periodic orbits and the minimal periods. The next theorem gives conditions about the existence of periodic orbits; although, not necessarily minimal. The proof of which is similar to that of Theorem 2.1.

Theorem 3.1. *For each $0 \leq j < k$, suppose $S_j := \{c_{0,j}, c_{1,j}, ..., c_{p_j-1,j}\}$ is a p_j-cycle of the $\frac{p}{d}$-periodic difference equation $x_{kn+j+1} = f_{kn+j}(x_{k(n-1)+j+1})$. Let $q = \operatorname{lcm}(p_0, p_1, ..., p_{k-1})$, S_j^*, $0 \leq j < k$ be defined as in equation (3.3), $r \in \mathcal{A}_{k,q}$, $d = gcd(r, k)$, and $h \leq q$ the unique solution of $\frac{k}{d}h = 1 \bmod q$. Then the initial condition $(x_{-k+1}, ..., x_0) \in \prod_{i=0}^{k-1} S_i^*$ provides an r-cycle if and only if*

$$c_{0,id+j} = c_{ih \bmod q, j},\ 1 \leq i < \frac{k}{d},\ 0 \leq j \leq d-1. \tag{3.5}$$

There are two concerns raised by this theorem. First, when do the conditions hold and second, when is the period minimal? In the case of fixed points, i.e., $f_i(x^*) = x^*$, for all $0 \leq i \leq p-1$. A periodic orbit $C_r = \{c_0, c_1, \ldots, c_{r-1}\}$ of minimal period $r \in \mathcal{A}_{k,1}$ exists if and only if $f_{\frac{k}{r}i+jk}(c_i) = c_i$, for all $0 \leq i \leq r-1$ and $0 \leq j \leq \frac{p}{d} - 1$. We stress this case in the following remark.

Remark 3.1. *Suppose $S_0 = S_1 = ... = S_{k-1} = \{x^*\}$, then $(x_{-k+1}, ..., x_0) \in \prod_{i=0}^{k-1} S_i^*$ provides one fixed point of equation (1.5). If $S_0, ..., S_{k-1}$ are allowed to take any one of two fixed points $\{x^*\}, \{y^*\}$, then equation (1.5) has r-cycles for all $r \in \mathcal{A}_{k,1}$.*

Next, assume $q = lcm(p_0, p_1, \ldots, p_{k-1}) > 1$. Then condition (3.5) is a very strong condition; nevertheless, we can weaken this condition by restricting the relation between the period p and the delay k.

Theorem 3.2. *Suppose p and k are relatively prime, and let $S_0 := \{c_0, c_1, ..., c_{q-1}\}$ be a q-cycle of minimal period $q \notin \mathcal{A}_{p,1}$, of the p-periodic difference equation $x_{kn+1} = f_{kn}(x_{k(n-1)+1})$, $n \in \mathbb{N}$. For each $0 < i < k$, define $S_i := \{c_i, c_{i+1}, ..., c_0, .., c_{i-1}\}$. Then for all $r \in \mathcal{A}_{k,q}$, the initial conditions $(x_{-k+1}, ..., x_0) \in \prod_{i=0}^{k-1} S_i^*$ provide an r-cycle. Furthermore, the number of different r-cycles provided by those initial conditions is given by*

$$P(r) = \frac{1}{r} \sum_{j \in \mathcal{B}(r)} \mu\left(\frac{r}{j}\right) d^{*gcd(j,k)+1},\ \textit{where } d^* = gcd(p, q).$$

Proof. Let $r \in \mathcal{A}_{k,q}$, and $d := gcd(r,k)$. We show condition (3.5) of Theorem 3.1 is satisfied. First, observe from the structure of S_j's that for each $0 \le j \le k-1$, S_j is a q-cycle of the j^{th} equation in (3.1). Next, divide the initial condition $(x_{-k+1}, ..., x_0)$ into $\frac{k}{d}$ blocks, and occupy the first block by $(c_i, c_{1+i}, \ldots, c_{d-1+i \bmod q})$ for some $i = 0, d^*, ..., q-d^*$. Without loss of generality, say $(c_0, c_1, ..., c_{d-1 \bmod q})$. To have condition (3.5) satisfied, we need the second block to be $(c_h, c_{h+1}, ..., c_{d-1+h \bmod q})$; however, we also need this block to be in $S^*_d \times S^*_{d+1} \times \ldots \times S^*_{2d-1}$. Define the non-ordered set $G := \{c_0, c_1, ..., c_{q-1}\}$, and define the operation $\star$ on G as follows

$$c_i \star c_j = c_{i+j \bmod q}, \quad 0 \le i, j < q.$$

Then $(G, \star)$ is a group, isomorphic to $(Z_q, +)$. $(S^*_0, \star)$ is a cyclic subgroup of $(G, \star)$, while $S^*_0, S^*_1, ..., S^*_{d^*-1}$ are the d^* different cosets (equivalence classes). Also, recall the group of maps $(G_f, \star)$, and consider the cyclic subgroup $H_f := \{f_0, f_{d^*}, ..., f_{p-d^*}\}$. Each element of the coset $f_j \star H_f$, $0 \le j < d^*$ maps the coset S^*_j onto the coset $S^*_{j+1 \mod d^*}$. Since

$$\begin{aligned}
f_{hk} \star H_f &= f_{(h\frac{k}{d}d)} \star H_f \\
&= f_{(h\frac{k}{d}-1+1)d} \star H_f, && \tfrac{k}{d}h = 1 \bmod q \\
&= f_{(m_1 q+1)d} \star H_f, && \text{write } \tfrac{hk}{d} - 1 = m_1 q \\
&= (f_{m_1 q d} \star H_f) \star (f_d \star H_f) \\
&= H_f \star (f_d \star H_f) && m_1 q d \text{ is a multiple of } d^* \\
&= f_d \star H_f,
\end{aligned}$$

then f_{hk} and f_d are in the same coset. From the fact G/S^*_0 and G_f/H_f are isomorphic, f_{hk} and f_d are in the same coset if and only if

$$c_h \in S^*_d \Leftrightarrow c_{h+j \bmod q} \in S^*_{d+j \bmod d^*} \Leftrightarrow c_{ih+j \bmod q} \in S^*_{id+j \bmod d^*}.$$

Now, we show the initial condition can be chosen so that r is a minimal period. If $r \ne \min \mathcal{A}_{k,q}$, write all elements of $\mathcal{A}_{k,q}$ that divide r in ascending order as $r_0, r_1, ..., r_m = r$, and define $d_i = gcd(r_i, k)$, $0 \le i \le m$. The above argument assures the existence of an r_0-cycle. Fix such a cycle and consider

$$(\overbrace{x_{-k+1}, \ldots, x_{-k+d_0}}^{d_0 \text{ block}}, \overbrace{x_{-k+d_0+1}, \ldots, x_{-k+2d_0}}^{d_0 \text{ block}}, \ldots, \overbrace{x_{-d_0}, \ldots, x_0}^{d_0 \text{ block}}).$$

as the associated initial condition. Next, since $r_0 < r_1$, then by Lemma 2.4, $d_0 < d_1$. Keep the first $d_1 - 1$ components $x_{-k+1}, ..., x_{-k+d_1-1}$ fixed as they are, and change the x_{-k+d_1} component by another component from $S^*_{d_1-1} \setminus \{x_{-k+d_1}\}$. This is always possible, since $q \notin \mathcal{A}_{p,1}$, and consequently $S^*_{d_1-1}$ has more than one element. Then choose the next $k - d_1$ components so that condition (3.5) of Theorem 3.1 is satisfied. This constructed initial condition provides an r_1-cycle. Similarly, we construct initial conditions that provide r_i-cycles, $i = 2, 3, \cdots, m$.

Finally, we are ready to prove the count formula. Since S_0^* contributes to the creation of d^* cosets, then each existed r-cycle has $\frac{r}{d^*}$ phase shifts. Also, if r_1 divides r then an r_1-cycle is of period r, and the initial condition can be occupied in $d^{*gcd(r,k)}$ choices. Therefore,

$$d^{*gcd(r,k)} = \sum_{j\in\mathcal{B}(r)} \frac{j}{d^*}P(j).$$

Apply Möbius inversion formula to obtain

$$\frac{r}{d^*}P(r) = \sum_{j\in\mathcal{B}(r)} \mu\left(\frac{r}{j}\right) d^{*gcd(j,k)},$$

and hence

$$P(r) = \frac{1}{r}\sum_{j\in\mathcal{B}(r)} \mu\left(\frac{r}{j}\right) d^{*gcd(j,k)+1}.$$

□

4 Sharkovsky's theorem for periodic difference equations with delays

An extension of Sharkovsky's theorem to p-periodic difference equations is given in [1]. In this section, we generalize that to periodic difference equations with delays, particularly, when the period p and the delay k are relatively prime. To achieve this objective, we introduce what we call the pk-Sharkovsky's ordering of the positive integers, which in fact depends on the p-Sharkovsky's ordering given in [1]. The pk-Sharkovsky's ordering is given by

$$\begin{array}{l}
\mathcal{A}_{pk,3} \rhd \mathcal{A}_{pk,5} \rhd \mathcal{A}_{pk,7} \rhd \cdots \\
\mathcal{A}_{pk,2\cdot 3} \rhd \mathcal{A}_{pk,2\cdot 5} \rhd \mathcal{A}_{pk,2\cdot 7} \rhd \cdots \\
\vdots \\
\mathcal{A}_{pk,2^n\cdot 3} \rhd \mathcal{A}_{pk,2^n\cdot 5} \rhd \mathcal{A}_{pk,2^n\cdot 7} \rhd \cdots \\
\vdots \\
\rhd \mathcal{A}_{pk,2^n} \rhd \cdots \rhd \mathcal{A}_{pk,2^2} \rhd \mathcal{A}_{pk,2} \rhd \mathcal{A}_{pk,1}.
\end{array}$$

It is easy to check that this ordering is well defined.

Before we give the natural extension of Sharkovsky's theorem to periodic difference equations with delays, the following lemma is needed.

Lemma 4.1. *Given relatively prime positive integers p and k, then*

$$\cup_{q\in\mathcal{A}_{p,r}}\mathcal{A}_{k,q} = \mathcal{A}_{pk,r}.$$

Proof. Let p and k be relatively prime positive integers, and fix $r \in \mathbb{Z}^+$. Factor p, k and r as

$$k = k_0^{\alpha_0} k_1^{\alpha_1} \cdots k_{m_1}^{\alpha_{m_1}},\ p = p_0^{\beta_0} p_1^{\beta_1} \cdots p_{m_2}^{\beta_{m_2}},\ r = k_0^{\alpha_0^*} \cdots k_{t_1}^{\alpha_{t_1}^*} p_0^{\beta_0^*} \cdots p_{t_2}^{\beta_{t_2}^*} r_0^{\gamma_0} r_1^{\gamma_1} \cdots r_{m_3}^{\gamma_{m_3}},$$

where $k_0, \ldots, k_{m_1}$ are the distinct prime factors of k, $p_0, \ldots, p_{m_2}$ are the distinct prime factors of p, and $r_0, \ldots, r_{m_3}$ are the distinct prime factors of r that are not in common with neither k nor p. Observer that

$$\mathcal{A}_{pk,r} = r k_0^{\alpha_0} k_1^{\alpha_1} \cdots k_{t_1}^{\alpha_{t_1}} p_0^{\beta_0} \cdots p_{t_2}^{\beta_{t_2}} \left\{ M_1 M_2 :\ M_1 | k_{t_1+1}^{\alpha_{t_1+1}} \cdots k_{m_1}^{\alpha_{m_1}},\ M_2 | p_{t_2+1}^{\beta_{t_2+1}} \cdots p_{m_2}^{\beta_{m_2}} \right\}$$

and

$$\mathcal{A}_{p,r} = r p_0^{\beta_0} \cdots p_{t_2}^{\beta_{t_2}} \left\{ M_2 :\ M_2 | p_{t_2+1}^{\beta_{t_2+1}} \cdots p_{m_2}^{\beta_{m_2}} \right\}.$$

Let $q \in \mathcal{A}_{p,r}$, then $q = r p_0^{\beta_0} \cdots p_{t_2}^{\beta_{t_2}} M_2$ for some fixed M_2. Now

$$\mathcal{A}_{k,q} = r k_0^{\alpha_0} k_1^{\alpha_1} \cdots k_{t_1}^{\alpha_{t_1}} p_0^{\beta_0} \cdots p_{t_2}^{\beta_{t_2}} \left\{ M_1 M_2 :\ M_1 | k_{t_1+1}^{\alpha_{t_1+1}} k_{t_1+2}^{\alpha_{t_1+2}} \cdots k_{m_1}^{\alpha_{m_1}} \right\}.$$

Hence

$$\cup_{q \in \mathcal{A}_{p,r}} \mathcal{A}_{k,q} = \mathcal{A}_{pk,r}.$$

□

Theorem 4.1 (Sharkovsky's theorem for periodic difference equations with delays). *Let p and k relatively prime positive integers. Suppose $f_i : I \to I$ are continuous on a closed interval I, and suppose the p-periodic difference equation $x_{k(n+1)} = f_{k(n+1)-1}(x_{kn})$, $n \in \mathbb{N}$, has an r-cycle. Let $\ell := \frac{lcm(p,r)}{p}$. Then each set $\mathcal{A}_{kp,q}$, such that $\mathcal{A}_{kp,\ell} \rhd \mathcal{A}_{kp,q}$, $q \neq 1$, contains a subset $\mathcal{A}_{k,qm}$ for some m, $m|p$, and each element $r^* \in \mathcal{A}_{k,qm}$ is a period of a cycle of $x_n = f_{n-1}(x_{n-k})$. Furthermore, if $x_{k(n+1)} = f_{k(n+1)-1}(x_{kn})$ has two fixed points, then the previous statement holds true for all $\mathcal{A}_{kp,q} \rhd \mathcal{A}_{kp,\ell}$.*

Proof. Suppose $x_{k(n+1)} = f_{k(n+1)-1}(x_{nk})$ has an r-cycle. By Sharkovsky's theorem for periodic difference equations, each set $\mathcal{A}_{p,q}$, such that $\mathcal{A}_{p,\ell} \trianglerighteq \mathcal{A}_{p,q}$, contains a period of some geometric cycle of $x_{k(n+1)} = f_{k(n+1)-1}(x_{kn})$, say $q^* \in \mathcal{A}_{p,q}$; moreover, $q^* = qm$ for some positive integer m, $m|p$. By Theorem 3.2, this q^*-cycle assures the existence of r-cycles of equation (1.5) for all $r \in \mathcal{A}_{k,q^*}$, whenever $q^* \neq 1$. To this end, we have verified that each collection

$$\cup_{\hat{q} \in \mathcal{A}_{p,q}} \mathcal{A}_{k,\hat{q}}, \quad \mathcal{A}_{p,\ell} \trianglerighteq \mathcal{A}_{p,q}$$

contains a set $\mathcal{A}_{k,q^*}$, $q^* = qm \neq 1$, and $m|p$ such that each element $r^* \in \mathcal{A}_{k,q^*}$ is the minimal period of a cycle of $x_n = f_{n-1 \bmod p}(x_{n-k})$. But by Lemma 4.1, $\cup_{q \in \mathcal{A}_{p,r}} \mathcal{A}_{k,q} = \mathcal{A}_{pk,r}$. Finally, the last statement follows from Remark 3.1 □

Observe that when $k = 1$, Theorem 4.1 reduces to the extension in [1], and when $p = 1$, it reduces to the extension given in [6] . Finally if $p = k = 1$ then it reduces to the original Sharkovsky's Theorem. Using Theorem 20 in [1] we obtain the converse of Sharkovsky's theorem for periodic difference equations with delays as given in the next corollary.

Corollary 4.1. *Given positive integers r, k, and p, such that* $\gcd(k,p) = 1$. *Define* $\ell := \frac{lcm(r,p)}{p}$. *There exists a periodic difference equation $x_{n+1} = f_{n \bmod p}(x_{n-k})$ that has an r-cycle, but has no r^*-cycles for all $r^* \in \mathcal{A}_{kp,q} \rhd \mathcal{A}_{kp,\ell}$.*

5 Stability Analysis

Again in this section, we divide the analysis into two parts according to the divisibility of the delay k by the periodicity p.

5.1 The period divides the delay

Assume the period p of equation (1.5) divides the delay k, and let $C_r := \{c_0, c_1, \ldots, c_{r-1}\}$ be an r-cycle of equation (1.5). Then it follows from our analysis in Section 2 that each map f_i, $0 \le i \le k-1$, has the periodic orbit

$$S_r^{p_i} := \{c_{i \mod r}, c_{(i+k) \mod r}, \ldots, c_{(i+(q-1)k) \mod r}\},$$

of minimal period p_i that divides $q := \mathrm{lcm}(r,k)/k$. Thus $S_r^{p_i}$ has a period q (not necessarily minimal) under the map f_i.

Definition 5.1. *The r-cycle $C_r = \{c_0, c_1, \cdots, c_{r-1}\}$ is stable {asymptotically stable} {globally asymptotically stable (GAS)} if for each i, $0 \le i \le k-1$, the cycle $S_r^{p_i}$ is stable {asymptotically stable } {GAS} under the map f_i. The r-cycle C_r, is unstable if for some i, $0 \le i \le k-1$, $S_r^{p_i}$ is unstable under the map f_i.*

Lemma 5.1. *Suppose the maps f_i of equation (1.5) are differentiable on an interval X. Then the following statements hold true. The r-cycle $C_r = \{c_0, c_1, \cdots, c_{r-1}\}$ is*

(i) asymptotically stable if for each i, $0 \le i \le k-1$,

$$J_i' = |f_i'(c_{i \mod r})f_i'(c_{(i+k) \mod r}) \ldots f_i'(c_{(i+(q-1)k) \mod r})| < 1.$$

(ii) unstable if for some i,

$$J_i' > 1.$$

The circumstances under which global stability is assured is the subject of the following theorem.

Theorem 5.1. *Suppose each map $f_j : X_j \to X_j,\ 0 \le j \le p-1$ is continuous on a connected metric space X_j. If $C_r = \{c_0, ..., c_{r-1}\}$ is a GAS r-cycle of the p-periodic difference equation $x_n = f_{n-1}(x_{n-k})$, then $r \in \mathcal{A}_{p,1}$. Furthermore, for each $0 \le i \le r-1$, the maps $f_i, f_{i+r}, \ldots, f_{i+k-r}$ have the same fixed point c_i.*

Proof. Suppose $C_r := \{c_0, c_1, ..., c_{r-1}\}$ is an r-cycle of equation (1.5). By Lemma 2.3, for each $i \in \{0, 1, ..., k-1\}, c_i \in S_i$, where S_i is a GAS cycle of f_i. If the phase space X_i is connected then S_i must be of period one under f_i [13]. Furthermore, since a map can not have more than one GAS fixed point, then $r|p$. Finally, the last statement comes directly from Lemma 2.2. □

We close this case with the following example:

Example 5.1. It is well known that the logistic map $f(x) = \mu x(1-x)$, where $\mu \in (0,1]$ and $x \in [0,1]$ has the GAS equilibrium point $x^* = 0$. Also, when $\mu \in (1,3)$ and $x \in (0,1)$, it has the GAS equilibrium point $x^* = \frac{\mu-1}{\mu}$. Now take these facts into consideration, and consider $p = k = 8$, we construct examples of GAS r-cycles of the equation

$$x_n = f_{n-1 \bmod 8}(x_{n-8}),$$

for all $r \in \mathcal{A}_{8,1}$.

(1) Define $f_i(x) = \frac{1}{1+i}x(1-x),\ 0 \le i \le 7,\ x \in [0,1]$. Then $x^* = 0$ is a GAS 1-cycle.

(2) Define

$$f_0(x) = f_2(x) = f_4(x) = f_6(x) = 2x(1-x),\ x \in (0,1),$$

and

$$f_j(x) = \frac{1}{j+1}x(1-x),\ j = 1,3,5,7,\ x \in [0,1].$$

Then $\{\frac{1}{2}, 0\}$ is a GAS 2-cycle.

(3) Define

$$\begin{aligned} f_0(x) &= \tfrac{1}{2}x(1-x),\ x \in [0,1], \\ f_4(x) &= \tfrac{1}{4}x(1-x),\ x \in [0,1], \\ f_1(x) &= f_5(x) = (1+\tfrac{1}{5})x(1-x),\ x \in (0,1), \\ f_2(x) &= f_6(x) = (1+\tfrac{1}{6})x(1-x),\ x \in (0,1), \\ f_3(x) &= f_7(x) = (1+\tfrac{1}{7})x(1-x),\ x \in (0,1). \end{aligned}$$

Then $\{0, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}\}$ is a GAS 4-cycle.

(4) Define

$$f_j(x) = (1 + \frac{1}{1+i})x(1-x),\ 0 \le i \le 7,\ x \in (0,1).$$

Then $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}\}$ is a GAS 8-cycle.

Observe that the method used above can be generalized to construct an equation of the form $x_n = \mu_{n-1 \bmod p} x_{n-k}(1 - x_{n-k})$ with a GAS r-cycle for any $r \in \mathcal{A}_{p,1}$.

5.2 The period does not divide the delay

In this section we adapt Definition 7 in AlSharawi et al [1] with some obvious modifications. We define the operator

$$\Phi_n^k(f_{n_0}) := f_{(n_0+(n-1)k)} \circ \cdots \circ f_{n_0+k} \circ f_{n_0},$$

for $0 \le n_0 \le p-1$, and $n \in \mathbb{Z}^+$.

Let $C_r = \{c_0, c_1, \ldots, c_{r-1}\}$ be an r-cycle of equation (1.5). Let $q = \dfrac{\mathrm{lcm}(r,k)}{k}$, $s = \dfrac{\mathrm{lcm}(p,k)}{k}$, and $l = \mathrm{lcm}(q,s)$. Then we have the following adaptation of Definition 7 in Alsharawi et al [1].

Definition 5.2. *The r-cycle C_r is*

(i) uniformly stable if given $\varepsilon > 0$, there exists $\delta > 0$ such that for any $n_0 = 0, 1, \ldots, k-1$, and $x \in X$, $|x - c_{n_0 \bmod r}| < \delta$ implies $|\Phi_n^k(f_{n_0})x - \Phi_n^k(f_{n_0})c_{n_0 \bmod r}| < \varepsilon$, for all $n \in \mathbb{Z}^+$

(ii) uniformly attracting if there exists $\eta > 0$ such that for any $n_0 = 0, 1, \ldots, k-1$, and $x \in X$, $|x - c_{n_0 \bmod r}| < \eta$ implies $\lim_{n\to\infty} \Phi_{nl}(f_{n_0})x = c_{n_0 \bmod r}$, where $l = \mathrm{lcm}(q,s)$ as defined above.

(iii) uniformly asymptotically stable (UAS) if it is both uniformly stable and uniformly attracting.

(iv) GAS if it is UAS and $\eta = \infty$.

Now, it is straight forward to prove the following stability criteria.

Corollary 5.1. *Suppose the maps f_i, $0 \le i \le p-1$, are differentiable on an interval X. Let $C_r = \{c_0, c_1, ..., c_{r-1}\}$ be an r-cycle of equation (1.5). Let $d = \gcd(r,k)$, $\hat{d} = \gcd(p,k)$, and $l = \mathrm{lcm}\left(\frac{p}{\hat{d}}, \frac{r}{d}\right)$. Then*

(i) *C_r is UAS if*

$$\left|\prod_{i=0}^{l-1} f'_{ik+j}(c_{ik+j \bmod r})\right| < 1, \ \forall\, j = 0, ..., k-1.$$

(ii) *C_r is unstable if*

$$\left|\prod_{i=0}^{l-1} f'_{ik+j}(c_{ik+j \bmod r})\right| > 1, \text{ for some } j = 0, ..., k-1.$$

This leads to the following generalization of a theorem due to Elaydi and Sacker, [11].

Theorem 5.2. *Let p and k be positive integers in which p is not a divisor of k,, and define $\hat{d} := gcd(p,k)$. Suppose each map $f_{ik+j} : X_j \to X_j$, $0 \leq j < \hat{d}$, $0 \leq i < \frac{p}{\hat{d}}$ is continuous on a connected metric space X_j. If $C_r := \{c_0, c_1, ..., c_r\}$ is a GAS r-cycle of the p-periodic difference equation with delays $x_n = f_{n-1}(x_{n-k})$, then $r \in \mathcal{A}_{p,1}$.*

Proof. For each $0 \leq j \leq k-1$, c_j is the start of a p_j-cycle of the $\frac{p}{\hat{d}}$-periodic difference equation

$$x_{kn+j+1} = f_{kn+j}(x_{k(n-1)+j+1}),\ n \in \mathbb{N}.$$

Say $S_j = \{c_{j \bmod r}, c_{j+k \bmod r}, \cdots, c_{j+(p_j-1)k \bmod r}\}$. Furthermore, this S_j cycle is GAS, consequently, it is the unique cycle of the jth equation. By Elaydi-Sacker theorem, $p_j | \frac{p}{\hat{d}}$, and by the discussion provided after equation (3.1), we obtain $S_j = S_{j+\hat{d}} = S_{j+2\hat{d}} = \cdots = S_{j+k-\hat{d}}$ for all $0 \leq j \leq \hat{d}-1$. Now, $q := lcm(p_0, \cdots, p_{k-1}) = lcm(p_0, \cdots, p_{\hat{d}-1})$ and $q | \frac{p}{\hat{d}}$. From the facts we provided in equation (3.3), the points $c_{\hat{d}}, c_{\hat{d}+1}, \cdots, c_{k-1 \bmod r}$ are determined uniquely by the structure of $S_0, S_1, \cdots, S_{\hat{d}-1}$, and visa versa, which implies $gcd(r,k) | \hat{d}$. Hence, $r = q \cdot gcd(r,k) | p$. $\square$

The next, example clarifies the notion of global stability when p is not a divisor of k.

Example 5.2. Let p and k be positive integers in which p is not a divisor of k, and let $r \in \mathcal{A}_{p,1}$. Define $\hat{d} := gcd(p,k)$ and $d := gcd(r,k)$, then $lcm(r,k)/k$ divides $lcm(p,k)/k$, which implies r/d divides $p/\hat{d}$. Furthermore, since $\frac{k}{d}$ and $\frac{r}{d}$ are relatively prime, then there exists a unique positive integer h, $1 \leq h < \frac{r}{d}$, such that $\frac{k}{d} \cdot h = 1 \bmod \frac{r}{d}$. Now, define $m := \frac{p}{\hat{d}}$ and $m^* = \frac{r}{d}$, and for $0 \leq j \leq m-1$, $0 \leq i \leq k-1$, $0 \leq s \leq \frac{d}{d} - 1$, define the maps

$$f_{jk+i}(x) := \frac{\lfloor j/m^* \rfloor}{m+1}(x - (sh + j \bmod m^*)) + (j + sh + 1 \bmod m^*),\ \text{if } sd \leq i < (s+1)d,$$

where $\lfloor\cdot\rfloor$ is the greatest integer function. Then the initial condition $(x_{-k+1}, \ldots, x_0) :=$

$$(\overbrace{0,...,0}^{d\ times}, \overbrace{h \bmod m^*, ..., h \bmod m^*}^{d\ times}, ..., \overbrace{(\frac{k}{d}-1)h \bmod m^*, ..., (\frac{k}{d}-1)h \bmod m^*}^{d\ times})$$

generates a GAS r-cycle.

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