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Julio R. Hasfura-Buenaga  
Trinity University, [jhasfura@trinity.edu](mailto:jhasfura@trinity.edu)

Phillip Lynch

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# PERIODIC POINTS OF THE FAMILY OF TENT MAPS

ROBERTO HASFURA-B. AND PHILLIP LYNCH

**1. INTRODUCTION.** Of interest in this article is the dynamical behavior of the one-parameter family of maps  $T_\omega(x) = \omega(1/2 - |x - 1/2|)$ ,  $x \in \mathbb{R}$ ,  $\omega > 0$ . For each such  $\omega$ , the map  $T_\omega$ , which can be described piecewise by

$$T_\omega(x) = \begin{cases} \omega x & \text{if } x \leq 1/2 \\ \omega(1 - x) & \text{if } x > 1/2 \end{cases} ,$$

is continuous, linear on each of the intervals  $(-\infty, 1/2]$  and  $[1/2, \infty)$  (with respective slopes  $\omega$  and  $-\omega$ ), and has the points  $(0, 0)$  and  $(1, 0)$  in its graph. The figure below, which shows the graph of the restriction of  $T_2$  to the unit interval  $I = [0, 1]$ , illustrates why the maps of the family  $T_\omega$  are referred to as tent maps.

If  $x \in \mathbb{R}$ , the set  $\{T_\omega^n(x)\}_{n=0}^\infty$ , where  $T_\omega^n$  denotes the  $n^{\text{th}}$  iterate of  $T_\omega$ , is called the *orbit* of  $x$ . If  $T_\omega(x) = x$  then  $x$  is called a fixed point, and if  $T_\omega^k(x) = x$  for some positive integer  $k$ , then  $x$  (and its orbit) is called *periodic* or *cyclic*. If in addition  $T_\omega^n(x) \neq x$ , for  $1 \leq n < k$ ,  $x$  is said to have (prime) period  $k$ . The dynamical properties of a map  $T_\omega$  are to some extent determined by the nature of the orbits of points in its domain. For example, that the dynamical behavior of  $T_\omega$  when  $0 < \omega < 1$  is docile can be inferred from the following facts concerning the orbits of  $T_\omega$ :

- (i)  $T_\omega(0) = 0$ , so 0 is a fixed point of  $T_\omega$ ,
- (ii)  $T_\omega(1) = 0$ , so the point 1 is eventually fixed, and
- (iii) for  $x \neq 0, 1$ ,  $T_\omega^n(x) \rightarrow 0$  as  $n \rightarrow \infty$ .

(The dynamical behavior of  $T_1$  is even easier to understand: if a point is not already fixed it is sent to one by an application of  $T_1$ .)

The behavior of  $T_\omega$  for all other values of  $\omega$  is far more complicated as we now see. First, let's assume that  $\omega > 2$ . In this case, by imitating the essential steps of the proof of Thm. ?? in [?], we can find a Cantor set  $C_\omega \in [0, 1]$ , invariant under  $T_\omega$  (that is, for which  $T_\omega(C_\omega) = C_\omega$ ) with these properties:

- (1) If  $x \notin C_\omega$  then  $T_\omega^n(x) \rightarrow -\infty$  as  $n \rightarrow \infty$ ;
- (2) The restriction  $T_\omega|_{C_\omega}$  of  $T_\omega$  to  $C_\omega$  is chaotic.

The term chaotic will be used throughout this paper in the sense of Devaney. Specifically, a continuous dynamical system  $f : M \rightarrow M$  on the metric space  $(M, d)$  is called *chaotic* if:

- C1** It is topologically transitive, i.e. given nonempty open sets  $U, V \subset M$  there is an  $n > 0$  for which  $f^n(U) \cap V$  is not empty;
- C2** The periodic points are dense in  $M$ ; and
- C3**  $f$  has sensitive dependence on initial conditions, i.e.

That  $T_\omega$  is chaotic on  $C_\omega$  when  $\omega > 2$  is established by finding a symbolic model of  $T_\omega|_{C_\omega}$  on which the three conditions above are easily verified (see [?].) As we will be concerned with periodic points, we mention here that this symbolic model also informs us that  $T_\omega$  has periodic points of all periods and allows us to count their number.

Now assume, as we will for the rest of this paper, that  $1 < \omega \leq 2$ . We begin with the simple observations that  $T_\omega(0) = 0$ ,  $T_\omega(1) = 0$ , and that if  $x < 0$  or  $x > 1$  then  $T_\omega^n(x) \rightarrow -\infty$  (monotonically) as  $n \rightarrow \infty$ . Consequently, outside the interval  $(0, 1)$  the dynamical behavior of  $T_\omega$  is understood (and very simple.) Also worthy of notice is the fact that the interval  $I_\omega = [\omega - \omega^2/2, \omega/2]$  remains invariant under the action of  $T_\omega$ . Indeed, if  $\omega - \omega^2/2 \leq x \leq 1/2$ , then  $\omega - \omega^2/2 \leq x < T_\omega(x) = \omega x \leq \omega/2$  because  $\omega > 1$ ; the condition  $1/2 < x \leq \omega/2$  forces  $\omega - \omega^2/2 \leq T_\omega(x) = \omega(1-x) \leq \omega/2$ . Furthermore, the points of  $(0, 1) \setminus I_\omega$  are eventually mapped into  $I_\omega$ : For  $0 < x < \omega - \omega^2/2$ , let  $n_0$  be the smallest nonnegative integer for which  $\omega^{n_0+1}x \geq \omega - \omega^2/2$ . Then  $\omega^{n_0}x < \omega - \omega^2/2 < 1/2$  implies that  $\omega - \omega^2/2 \leq T_\omega^{n_0+1}(x) < \omega/2$ . On the other hand, if  $\omega/2 < x < 1$  then  $0 < T_\omega(x) = \omega(1-x) < \omega - \omega^2/2$ . It follows from these considerations that to get a more complete picture of the dynamics of  $T_\omega$  for  $1 < \omega \leq 2$  it is necessary to shed light on the behavior of the restriction  $T_\omega|_{I_\omega}$ . To this task we devote the rest of this paper.

**2.  $1 < \omega \leq 2$ .** From now on we confine our attention to  $T_\omega|_{I_\omega}$ ,  $1 < \omega \leq 2$  which we will simply denote by  $T_\omega$ . We will further simplify our writing by finding models of these  $T_\omega$ 's in a common interval as follows: Let  $I$  denote  $[0, 1]$ . We define the map  $\bar{T}_\omega : I \rightarrow I$  as  $\bar{T}_\omega = h_\omega^{-1} \circ T_\omega \circ h_\omega$ , where  $h_\omega : I \rightarrow I_\omega$  is given by

$$h_\omega(x) = \left(\omega - \frac{\omega^2}{2}\right)(1-x) + \frac{\omega}{2}x.$$

Because  $h_\omega$  is a homeomorphism, the relation between the maps  $T_\omega$  and  $\bar{T}_\omega$  is that of *conjugacy*. We justify the introduction of the  $\bar{T}_\omega$ 's by reminding the reader that conjugate maps share dynamical properties. In particular,  $T_\omega$  is chaotic if and only if  $\bar{T}_\omega$  is. The graph of a typical  $\bar{T}_\omega$  is shown in figure 2.

We now claim that  $\bar{T}_\omega$  (and thus  $T_\omega$ ) is chaotic if  $\sqrt{2} < \omega \leq 2$ . To verify this claim all we need to check is that corresponding to any nondegenerate interval  $J \subset I$  there is an integer  $n$  for which  $\bar{T}_\omega^n(J) = I$ , for this condition obviously implies that  $\bar{T}_\omega$  is topologically transitive and, as shown in [?], on intervals **C1** implies both **C2** and **C3**. Thus, we just prove the following:

**Lemma 1.** *If  $\sqrt{2} < \omega \leq 2$  and  $J \subset I$  is a non degenerate interval then there is a positive integer  $n$  for which  $\bar{T}_\omega^n(J) = I$ .*

*Proof.* Set  $a = \omega/(\omega + 1)$ , the fixed point of  $\bar{T}_\omega$ . We leave to the reader the verification that if  $J$  contains  $a$  in its interior then  $J$  is eventually mapped onto  $I$ .

If  $a \notin \text{int}(J)$  there are several possibilities. First, we could have  $J = [a, b]$ ,  $a < b$ . Then  $\bar{T}_\omega(J) = [c, a]$ ,  $c < a$ , and if  $(\omega - 1)/\omega \notin [c, a]$  then  $|[c, a]| = \omega|[a, b]|$ . (Here,  $|\cdot|$  denotes length.) Further applications of  $\bar{T}_\omega$  eventually yield  $m$  such that  $\bar{T}_\omega^m(J) = [d, a]$  and  $(\omega - 1)/\omega \in [d, a]$ . Therefore,  $\bar{T}_\omega^{m+1}(J) \supset [a, 1]$ ,  $\bar{T}_\omega^{m+2}(J) \supset [0, a]$ ,  $\bar{T}_\omega^{m+3}(J) \supset [2 - \omega, 1]$  and  $[2 - \omega, 1]$  has  $a$  in its interior because  $2 - \omega < a = \omega/(\omega + 1)$  for  $\omega > \sqrt{2}$ . But we are now in the situation contemplated in the previous paragraph. A second possibility is that  $J = [c, a]$ ,  $c < a$ . In this case, either  $\bar{T}_\omega(J) = [a, b]$ ,  $a < b$  (the previous case), or plainly  $\bar{T}_\omega(J)$  contains  $a$  in its interior. Again, we are in one of the previous two cases. Finally, it could happen that  $a \notin J$ . If in addition  $a \notin \bar{T}_\omega(J)$  then  $|\bar{T}_\omega^2(J)| \geq \frac{\omega^2}{2}|J| > |J|$ . Therefore, repeated application of  $\bar{T}_\omega$  eventually yields  $m$  such that  $a \in \bar{T}_\omega^m(J)$  and again we get the desired conclusion by invoking the earlier cases.  $\square$

Now assume that  $1 < \omega \leq \sqrt{2}$ . It will be convenient to examine the map  $\overline{T}_\omega^2$ . The graph of one such map is sketched in the figure below.

It can be checked by straightforward computations that the interval  $Q_1 = [0, \overline{T}_\omega^2(0)] = [0, \omega^2 - \omega]$  remains invariant under the action of  $\overline{T}_\omega^2$ , i.e. that if  $x \in [0, \overline{T}_\omega^2(0)]$  then  $\overline{T}_\omega^2(x) \in [0, \overline{T}_\omega^2(0)]$ . Furthermore, if we define the new function  $F_\omega : I \rightarrow I$  by  $F_\omega = g^{-1} \circ \overline{T}_\omega^2 \circ g$ , where  $g : I \rightarrow Q_1$  is the homeomorphism  $g(x) = (1-x)\overline{T}_\omega^2(0)$  (so  $F_\omega$  and  $(\overline{T}_\omega^2)|_{Q_1}$  are conjugate) then, by another elementary computation, we see that  $F_\omega$  (a piecewise linear function) has the graph shown in Figure 4 below. It is apparent from this

graph that  $F_\omega = \overline{T}_\omega^2$ . Therefore, we have proved the following:

**Theorem 1.**  $(\overline{T}_\omega^2)|_{Q_1}$  is conjugate to  $\overline{T}_{\omega^2}$ .

Using Theorem 1 along with some of our previous results we can draw some conclusions. For example, if  $\omega < \sqrt{2}$  and  $\omega^2 > \sqrt{2}$  then  $\overline{T}_\omega^2|_{Q_1}$ , being conjugate to  $\overline{T}_{\omega^2}$ , is chaotic. Furthermore, for such  $\omega$ , if we set  $Q_2 = \overline{T}_\omega(Q_1)$  then  $Q_1$  and  $Q_2$  are disjoint intervals,  $Q_2 = \overline{T}_\omega(Q_1)$  (so  $Q_1 \cup Q_2$  is  $\overline{T}_\omega$ -invariant),  $\overline{T}_\omega$  is chaotic on  $Q_1 \cup Q_2$ , and every point in  $I \setminus Q_1 \cup Q_2$  is eventually mapped by  $\overline{T}_\omega$  into  $Q_1 \cup Q_2$ . In other words,  $(\overline{T}_\omega)|_{Q_1 \cup Q_2}$  is a chaotic attractor.

More generally, the following result can be obtained from judicious, repeated application of Theorem 1.

**Corollary 1.** *Assume that  $1 < \omega^k < \sqrt{2}$ ,  $1 \leq k \leq n-1$ , and  $\omega^n > \sqrt{2}$ . Then there exist disjoint subintervals  $Q_1, Q_2 \dots Q_{2^n}$  of  $I$  permuted by  $\overline{T}_\omega$  such that  $Q = Q_1 \cup Q_2 \cup \dots \cup Q_{2^n}$  is a chaotic attractor of  $\overline{T}_\omega$ . That is,  $\overline{T}_\omega|_Q$  is chaotic, and every point in  $I \setminus Q$  is eventually mapped by  $\overline{T}_\omega$  to  $Q$ .*

Similarly, if  $\omega = \sqrt{2}$  then  $Q_1 = [0, a]$  where  $a = \omega/(\omega + 1)$  is the fixed point of  $\overline{T}_\omega$ ,  $Q_2 = \overline{T}_\omega(Q_1) = [a, 1]$  (so  $Q_1 \cup Q_2 = I$ ), and  $\overline{T}_\omega$  permutes  $Q_1$  and  $Q_2$ . Applying Theorem 1 once again we obtain:

**Corollary 2.**  $\overline{T}_{\sqrt{2}}$  is chaotic on  $I = [0, 1]$ .

**3. PERIODIC POINTS.** In the previous section we learnt that when  $1 < \omega \leq 2$  there is a set  $Q_\omega \subseteq I_\omega$  on which  $T_\omega$  is chaotic. We know then, from **C2** of the definition of chaos, that there is a dense (hence infinite) subset of  $Q_\omega$  consisting of  $T_\omega$ -periodic points. However, in contrast to the case  $\omega > 2$ , we don't have as yet any specific information about them. Myriad interesting questions can be asked regarding the periodic points of  $T_\omega$ , and we devote this section to answering some of them.

Naturally, when considering such questions, the well-known theorem of Šarkovskii which establishes that the presence of certain periodic orbits forces the existence of certain other periodic orbits will be very useful. The precise statement of Šarkovskii's theorem is the following:

**Theorem 2** ([?]). *Let  $(\mathbb{N}, \prec)$  be the following transitive ordering of the positive integers:  $3 \prec 5 \prec 7 \prec \dots \prec 2 \cdot 3 \prec 2 \cdot 5 \prec 3 \cdot 7 \prec \dots \prec 2^2 \cdot 3 \prec 2^2 \cdot 5 \prec 2^2 \cdot 7 \prec \dots \prec 2^n \cdot 3 \prec 2^n \cdot 5 \prec 2^n \cdot 7 \prec \dots \prec 2^n \prec 2^{n-1} \prec \dots \prec 2^2 \prec 2^1 \prec 2^0$ . Assume that  $J$  is an interval and that  $f : J \rightarrow J$  is continuous. If  $f$  has a periodic point of period  $k$  then it has a periodic point of period  $n$  for all  $k \prec n$ .*

We will find it convenient to make the following definition:  $\overline{\omega}_k = \inf\{\omega > 1 | T_\omega \text{ has a point of period } k\}$ . A few of the  $\overline{\omega}_k$ 's can be found by explicit algebraic computation. For example,  $\overline{\omega}_2 = 1$ ,  $\overline{\omega}_3 = (1 + \sqrt{5})/2$ ,  $\overline{\omega}_4 = 1$ ,  $\overline{\omega}_5 \approx 1.5129$ , and  $\overline{\omega}_6 = \sqrt{(1 + \sqrt{5})/2}$ . The relation observed between  $\overline{\omega}_3$  and  $\overline{\omega}_6$  is an instance of a general property which is consequence of our work above. In fact, several simple properties of  $\overline{\omega}_k$  can be found easily using the geometry of  $T_\omega$ , Šarkovskii's theorem, and our previous results. For example, the reader can check that if  $\omega > \overline{\omega}_k$  then  $T_\omega$  has a periodic point of period  $k$ , and that  $k \prec n$  implies  $\overline{\omega}_n \leq \overline{\omega}_k$ , i.e. that the map  $\overline{\omega} : (\mathbb{N}, \prec) \rightarrow (\mathbb{R}, \leq)$  is order-reversing. Also, as we saw in the proof of Theorem 1, when  $1 < \omega \leq \sqrt{2}$  there is a set  $Q_1$  which intersects all the periodic orbits of  $T_\omega$ , for which  $Q_2 = T_\omega(Q_1)$  and  $Q_1$  intersect in a set containing no more than one point, and for which  $(T_\omega^2)|_{Q_1}$  is conjugate to  $T_\omega^2$  (on an appropriate set.) These facts yield the following:

**Proposition 1.** (i)  $\overline{\omega}_{2n+1} \geq \sqrt{2}$ ; (ii)  $\overline{\omega}_k = \overline{\omega}_{2k}^2$

*Proof.* (i) The properties of  $Q_1$  imply that  $T_\omega$  has no periodic points of odd period; (ii) by the conjugacy referred to before the statement of the

proposition, corresponding to a  $T_{\omega^2}$ -periodic point of period  $n \neq 1$  there is a point  $x$  in  $Q_1$  with  $T_{\omega^2}$ -period  $n$ . Since  $Q_1$  and its  $T_{\omega}$ -image  $Q_2$  intersect at at most one point,  $x$  must have  $T_{\omega}$ -period  $2n$ .  $\square$

We now describe a class of polynomials which will appear often in our discussion. We say that a polynomial *alternates* if it has the form  $p(x) = x^{a_1} - x^{a_2} + x^{a_3} - \dots \pm x^{a_i}$ , where  $a_1 > a_2 > a_3 > \dots > a_i \geq 1$ , including the zero polynomial. We will denote arbitrary alternating polynomials of degree less than or equal to  $k$  by  $p_k$ ; similarly, by  $q_k$  we will denote arbitrary alternating polynomials of degree exactly  $k$ . To indicate that  $p_k$  or  $q_k$  have an even number of nonzero terms we will write  $\hat{p}_k$  or  $\hat{q}_k$ ; and to indicate that they have an odd number of nonzero terms we will write  $\tilde{p}_k$  or  $\tilde{q}_k$ . The relevance of alternating polynomials arises from the fact that  $T_{\omega}^k(x)$  can be expressed in terms of them. The following result states this fact precisely.

**Lemma 2.**  $T_{\omega}^k(x) = \tilde{p}_k(\omega) - \omega^k x$  or  $T_{\omega}^k(x) = -\hat{p}_k(\omega) + \omega^k x$ . Furthermore, in the expressions above we can make sure that  $\deg(p_k) < k$  if  $x \leq 1/2$  and that  $\deg(p_k) = k$  if  $x \geq 1/2$ .

*Proof.* The proof is by induction. If  $k = 1$  then  $T_{\omega}(x)$  is  $\omega x$  or  $\omega - \omega x$  depending on whether  $x \leq 1/2$  or  $x \geq 1/2$ . In either case, the lemma is valid for  $k = 1$ . Now assume that the statement of the lemma is true for  $k = n - 1$ , so

$$T_{\omega}^{n-1}(x) = \tilde{p}_{n-2}(\omega) - \omega^{n-1}x \text{ or } T_{\omega}^{n-1}(x) = -\hat{p}_{n-2}(\omega) + \omega^{n-1}x$$

if  $x \leq 1/2$ , and

$$T_{\omega}^{n-1}(x) = \tilde{q}_{n-1}(\omega) - \omega^{n-1}x \text{ or } T_{\omega}^{n-1}(x) = -\hat{q}_{n-1}(\omega) + \omega^{n-1}x$$

if  $x \geq 1/2$ . To verify the lemma for  $k = n$  we consider different cases. For example, if  $x \leq 1/2$  and  $T_{\omega}^{n-1}(x) \leq 1/2$  then  $T_{\omega}^n(x)$  is  $\omega T_{\omega}^{n-1}(x)$ , that is,

$$T_{\omega}^n(x) = \omega(\tilde{p}_{n-2}(\omega) - \omega^{n-1}x) = \omega\tilde{p}_{n-2}(\omega) - \omega^n x = \tilde{p}_{n-1}(\omega) - \omega^n x$$

or

$$T_{\omega}^n(x) = \omega(-\hat{p}_{n-2}(\omega) + \omega^{n-1}x) = -\omega\hat{p}_{n-2}(\omega) + \omega^n x = -\hat{p}_{n-1}(\omega) + \omega^n x.$$

Or if  $x \geq 1/2$  and  $T_{\omega}^{n-1}(x) \geq 1/2$  then  $T_{\omega}^n(x)$  is  $\omega - \omega T_{\omega}^{n-1}(x)$ , that is,

$$T_{\omega}^n(x) = \omega - \omega(\tilde{q}_{n-1}(\omega) - \omega^{n-1}x) = \omega - \omega\tilde{q}_{n-1}(\omega) + \omega^n x = -\hat{q}_n(\omega) + \omega^n x$$

or

$$T_{\omega}^n(x) = \omega - \omega(-\hat{q}_{n-1}(\omega) + \omega^{n-1}x) = \omega + \omega\hat{q}_{n-1}(\omega) - \omega^n x = \tilde{q}_n(\omega) - \omega^n x.$$

We see that in the two cases considered the statements of the lemma are verified. There are two more cases to consider, and we leave them to the diligent reader. In each case the statements of the lemma are again validated for  $k = n$ , and the proof is complete.  $\square$

**Corollary 3.** *Assume that  $x$  is a point of period  $k$ . If  $x \leq 1/2$  then it can be represented in the form*

$$x = \frac{\hat{p}_{k-1}(\omega)}{\omega^k - 1} \text{ or } \frac{\tilde{p}_{k-1}(\omega)}{\omega^k + 1},$$

and if  $x \geq 1/2$  then it can be represented in the form

$$\frac{\hat{q}_k(\omega)}{\omega^k - 1} \text{ or } \frac{\tilde{q}_k(\omega)}{\omega^k + 1}.$$

*Proof.* If  $x$  has  $T_\omega$ -period  $k$  then  $T_\omega^k(x) = x$ . Now use the expressions for  $T_\omega^k(x)$  found in the preceding lemma and solve for  $x$ .  $\square$

This last corollary raises hope that the problem of clarifying the nature of the periodic orbits of  $T_\omega$  may be aided by the study of expressions of the form

$$\frac{\hat{p}_k(\omega)}{\omega^k - 1} \text{ and } \frac{\tilde{p}_k(\omega)}{\omega^k + 1}.$$

Such hope is further strengthened by the following observations:

(a) Given  $k$  and  $\omega$ , the (finite) set  $\left\{ \frac{\hat{p}_k(\omega)}{\omega^k - 1} \right\}$  is closed under the following operation  $R$ :

$$\frac{\hat{p}_k(\omega)}{\omega^k - 1} \mapsto \omega \frac{\hat{p}_k(\omega)}{\omega^k - 1} \text{ if } \deg(\hat{p}_k) < k, \text{ and}$$

$$\frac{\hat{p}_k(\omega)}{\omega^k - 1} \mapsto \omega - \omega \frac{\hat{p}_k(\omega)}{\omega^k - 1} \text{ if } \deg(\hat{p}_k) = k.$$

(b) Given  $k$  and  $\omega$ , the (finite) set  $\left\{ \frac{\tilde{p}_k(\omega)}{\omega^k + 1} \right\}$  is closed under the following operation  $R$ :

$$\frac{\tilde{p}_k(\omega)}{\omega^k + 1} \mapsto \omega \frac{\tilde{p}_k(\omega)}{\omega^k + 1} \text{ if } \deg(\tilde{p}_k) < k, \text{ and}$$

$$\frac{\tilde{p}_k(\omega)}{\omega^k + 1} \mapsto \omega - \omega \frac{\tilde{p}_k(\omega)}{\omega^k + 1} \text{ if } \deg(\tilde{p}_k) = k.$$

(Notice that the description of the action of  $T_\omega$  on  $x$ , whose form depends on whether  $x \leq 1/2$  or  $x \geq 1/2$ , is identical to the description of the given operation  $R$ , only now the dependence is on  $\deg(p_k)$ .)

The operation  $R$  just introduced partitions each of the sets  $\left\{ \frac{\hat{p}_k(\omega)}{\omega^k - 1} \right\}$  and  $\left\{ \frac{\tilde{p}_k(\omega)}{\omega^k + 1} \right\}$  into *cycles* (or orbits.) We will say that one of these cycles in  $\left\{ \frac{\hat{p}_k(\omega)}{\omega^k - 1} \right\}$  (resp. in  $\left\{ \frac{\tilde{p}_k(\omega)}{\omega^k + 1} \right\}$ ) satisfies the *1/2-condition* for  $\omega$  if

$$\deg(\hat{p}_k) < k \text{ if } \frac{\hat{p}_k(\omega)}{\omega^k - 1} \leq 1/2 \text{ and } \deg(\hat{p}_k) = k \text{ if } \frac{\hat{p}_k(\omega)}{\omega^k - 1} \geq 1/2$$

$$\text{(resp. } \deg(\tilde{p}_k) < k \text{ if } \frac{\tilde{p}_k(\omega)}{\omega^k + 1} \leq 1/2 \text{ and } \deg(\tilde{p}_k) = k \text{ if } \frac{\tilde{p}_k(\omega)}{\omega^k + 1} \geq 1/2)$$

for all the elements of the cycle. (The ambiguity concerning the value  $1/2$  is irrelevant.) We can now state the following important result.



**Theorem 3.** *Given  $\omega$  and  $k$ , if  $x = \frac{\tilde{p}_k(\omega)}{\omega^k+1}$  or  $x = \frac{\hat{p}_k(\omega)}{\omega^k-1}$  and the  $R$ -cycle of the rational expression satisfies the 1/2-condition then  $T_\omega^k(x) = x$ .*

*Proof.* Suppose  $x = \frac{\tilde{p}_k(\omega)}{\omega^k+1}$ . We study the orbit of  $x$  under  $T_\omega$  by studying the orbit of the rational expression under the operation  $R$ ; this is possible since both actions are identical given that the orbit of the rational expression satisfies the 1/2-condition. To simplify notation, we consider only the coefficients of the polynomial  $\tilde{p}_k(\omega)$ , which we present as an element of  $S^k$ , where  $S = \{-1, 0, 1\}$ . Thus, if  $\vec{v} = (v_1, v_2, \dots, v_k)$  is the element of  $S^k$  containing the coefficients of  $\tilde{p}_k(\omega)$  then the action  $R : S^k \rightarrow S^k$  is given by

$$R(\vec{v})_i = v_{i-1} \text{ for } i > 1, \text{ and } R(\vec{v})_1 = 0 \text{ if } v_k = 0,$$

and

$$R(\vec{v})_i = -v_{i-1} \text{ for } i > 1, \text{ and } R(\vec{v})_1 = 1 \text{ if } v_k \neq 0.$$

That is,

$$R(\vec{v}) = (|v_k|, (-1)^{|v_k|}v_1, (-1)^{|v_k|}v_2, \dots, (1)^{|v_k|}v_{k-1})$$

and, continuing the same way

$$R^k(\vec{v}) = (|v_1|, (-1)^{|v_1|}|v_2|, (-1)^{|v_1|+|v_2|}|v_3|, \dots, (1)^{|v_1|+\dots+|v_{k-1}|}|v_k|).$$

Now, for any polynomial of the form  $\tilde{p}_k$ ,  $v_i = (-1)^X|v_i|$ , where  $X = \sum_{j=1}^{i-1} |v_j|$  because for every term with coefficient 1 there is an even number of terms of smaller degree in the polynomial, while for every term with coefficient  $-1$  there is an odd number of terms of smaller degree in the polynomial. Therefore, since  $R^k(\vec{v}) = \vec{v}$  we must have  $T_\omega^k(x) = x$ .

The proof for the case when  $x = \frac{\hat{p}_k(\omega)}{\omega^k-1}$  is very similar. We just use the appropriate form for  $R$ , namely

$$R(\vec{v})_i = v_{i-1} \text{ for } i > 1, \text{ and } R(\vec{v})_1 = 0 \text{ if } v_k = 0,$$

and

$$R(\vec{v})_i = v_{i-1} \text{ for } i > 1, \text{ and } R(\vec{v})_1 = -1 \text{ if } v_k \neq 0,$$

and the specific properties of the polynomials  $\hat{p}_k$ .  $\square$

Suppose now that  $k$  is an odd number  $\geq 5$ , and consider the polynomial  $s_1(\omega) = \omega^{k-1} - \omega^{k-2} + \omega^{k-3} - \dots - \omega$ . This polynomial is of the form  $\hat{p}_k(\omega)$ .

Also consider the  $R$ -cycle of  $\frac{s_1(\omega)}{\omega^k-1}$ , consisting, in addition to that rational expression itself, of  $\frac{s_2(\omega)}{\omega^k-1}, \frac{s_3(\omega)}{\omega^k-1}, \dots, \frac{s_i(\omega)}{\omega^k-1}, \dots, \frac{s_{k-1}(\omega)}{\omega^k-1}$ , and  $\frac{s_k(\omega)}{\omega^k-1}$ . Here

$$s_2(\omega) = \omega^k - \omega^{k-1} + \omega^{k-2} \dots - \omega^2,$$

$$s_3(\omega) = \omega^k - \omega^{k-1} + \dots + \omega^3 - \omega,$$

$\vdots$

$$s_i(\omega) = \omega^k - \omega^{k-1} + \dots + (-1)^{(i+1)}\omega^i + (-1)^i\omega^{i-2} + \dots - \omega,$$

$\vdots$

$$\begin{aligned} s_{k-1}(\omega) &= \omega^k - \omega^{k-1} + \omega^{k-3} - \dots - \omega, \\ s_k(\omega) &= \omega^k - \omega^{k-2} + \omega^{k-3} - \dots - \omega. \end{aligned}$$

**Lemma 3.** *If  $\frac{s_{k-1}(\omega)}{\omega^{k-1}} \geq 1/2$  for some  $\omega > 1$  then the  $R$ -cycle above satisfies the  $1/2$ -condition for that  $\omega$ .*

*Proof.* First consider those  $s_i$  having degree  $k$ . If  $2 \leq i \leq k-2$ , we would like that  $s_{k-1}(\omega) \leq s_i(\omega)$ . That this is so follows from the fact that  $s_i(\omega) - s_{k-1}(\omega) = (\omega-1)(\omega^{k-3} - \omega^{k-4} + \dots + (-1)^{(i+1)}\omega^{i+1}) > 0$  for all  $\omega > 1$ . Thus,  $\frac{s_{k-1}(\omega)}{\omega^{k-1}} \geq 1/2$  forces  $\frac{s_i(\omega)}{\omega^{k-1}} > 1/2$ . Likewise,  $s_k(\omega) - s_{k-1}(\omega) = \omega^{k-1} - \omega^{k-2} > 0$  for  $\omega > 1$  so  $\frac{s_k(\omega)}{\omega^{k-1}} > 1/2$  as well.

Finally, we need to show that  $\frac{s_{k-1}(\omega)}{\omega^{k-1}} \geq 1/2$  implies that  $\frac{s_1(\omega)}{\omega^{k-1}} \leq 1/2$ . By cross-multiplying and subtracting in the last inequality, we reformulate our desideratum as  $\omega^k - 2\omega^{k-1} + 2\omega^{k-2} - \dots + 2\omega - 1 \geq 0$ . But, again cross-multiplying and subtracting, the first inequality can be written as  $\omega^k - 2\omega^{k-1} + 2\omega^{k-3} - 2\omega^{k-4} + \dots - 2\omega + 1 > 0$  and this inequality is easily seen to imply the previous one when  $\omega > 1$ .  $\square$

It is now a consequence of Theorem 3 and Lemma 3 that if for some  $\omega > 1$ ,  $x = \frac{s_{k-1}(\omega)}{\omega^{k-1}} \geq 1/2$ , then  $x = T_\omega^k(x)$ . One such  $\omega$  is, of course, the largest real root of  $f_k(x) = x^k - 2x^{k-1} + 2x^{k-3} - \dots - 2x + 1$  which will henceforth be denoted by  $\bar{\eta}_k$ . Thus,  $x = \frac{s_1(\bar{\eta}_k)}{\bar{\eta}_k^{k-1}}$  satisfies  $x = T_{\bar{\eta}_k}^k(x)$ . We have not excluded the a priori possibility that  $x = T_{\bar{\eta}_k}^n$  for some  $1 \leq n < k$ , i.e. that  $x$  have (prime) period  $< k$ . If this indeed happened then for some  $2 \leq i \leq k$  we would have  $s_1(\bar{\eta}_k) = s_i(\bar{\eta}_k)$ . However, these two polynomials (and in fact any pair  $s_j, s_l, j \neq l$ ) can have the same value at only finitely many points. Therefore we can select  $\eta_k$  arbitrarily close to, but larger than  $\bar{\eta}_k$ , i.e. in  $[\bar{\eta}_k, \bar{\eta}_k + \delta)$  for arbitrarily small  $\delta$ , so that  $f_k(\eta_k) \geq 1/2$  ( $f_k$  is nondecreasing in a neighborhood of its largest real root), and so that  $s_i(\eta_k) \neq s_j(\eta_k), j \neq l$ . In this case  $x = \frac{s_1(\eta_k)}{\eta_k^{k-1}}$  is  $T_{\eta_k}$ -periodic of (prime) period  $k$ . From this the inequality  $\bar{\eta}_k \geq \bar{\omega}_k$  follows. This inequality is useful because we will be able to show that  $\lim_{n \rightarrow \infty} \bar{\eta}_{2n+1} = \sqrt{2}$ .

**Lemma 4.** (i)  $f_k(\sqrt{2}) = 2\sqrt{2} - 3$  for all odd  $k$ ; and (2)  $\lim_{k \rightarrow \infty} f_k(x) = \infty$  for  $x > \sqrt{2}$ .

*Proof.* (i)  $f_k(x) = x^k - 2x^{k-1} + 1 + (1 - \frac{1}{x})(x^{k-3} + x^{k-5} + \dots + x^2)$ . Thus  $f_k(x) = x^k - 2x^{k-1} + 1 + \frac{2x^{k-2}}{x+1} - \frac{2x}{x+1} = \frac{x^{k-2}}{x+1}(x^3 - x^2 - 2x + 2) + \frac{1-x}{1+x}$ . Now we compute  $f_k(\sqrt{2}) = 0 + \frac{1-\sqrt{2}}{\sqrt{2}+1} = 2\sqrt{2} - 3$ . (ii) When  $x > \sqrt{2}$  a simple calculation shows that  $x^3 - x^2 - 2x + 2 > 0$ . Thus,  $f_k(x) = \alpha \frac{x^{k-2}}{x+1} - \beta$  for some  $\alpha > 0$ , implying that  $\lim_{n \rightarrow \infty} f_{2n+1}(x) = \infty$ .  $\square$

**Theorem 4.**  $\lim_{n \rightarrow \infty} \bar{\eta}_{2n+1} = \sqrt{2}$ .

*Proof.* By the preceding lemma, given any  $\epsilon > 0$ , once  $n$  is large enough we will have  $f_{2n+1}(\sqrt{2} + \epsilon) > 1$ . Since  $f_{2n+1}(\sqrt{2}) = 2\sqrt{2} - 3 < 0$  for such  $n$ , we must have  $\sqrt{2} < \bar{\eta}_{2n+1} < (\sqrt{2} + \epsilon)$ . Now let  $\epsilon$  decrease to 0.  $\square$

**Corollary 4.**  $\lim_{n \rightarrow \infty} \bar{\omega}_{2n+1} = \sqrt{2}$ .

*Proof.* From Proposition 1(i) and the comment preceding Lemma 4 we have  $\sqrt{2} \leq \bar{\omega}_{2n+1} \leq \bar{\eta}_{2n+1}$ . Now use Theorem 4.  $\square$

Thus,  $T_{\sqrt{2}}$  has, besides the fixed point, periodic points of all even periods and none of odd period. Also,

**Corollary 5.** (i)  $\lim_{n \rightarrow \infty} \bar{\omega}_{(2n+1)2^m} = 2^{2^{-m-1}}$ ; (ii)  $\lim_{m \rightarrow \infty} \bar{\omega}_{(2n+1)2^m} = 1$ .

**Corollary 6.** For all  $m$ ,  $\bar{\omega}_{2^m} = 1$ .

*Proof.* This follows from part (ii) of the Corollary 5 and from the previously mentioned fact that  $\bar{\omega} : (\mathbb{N}, <) \rightarrow (\mathbb{R}, \leq)$  is order-reversing.  $\square$

#### REFERENCES

- [D] R. L. Devaney, *An Introduction to Chaotic Dynamical Systems*, Second Edition, Addison Wesley, Menlo Park, CA, **1989**.
- [VB] M. Vellekoop and R. Berglund, *On Intervals, Transitivity = Chaos*, Amer. Math. Monthly, **101**, No. 4 (1994), 353-355.
- [S] A. N. Šarkovskii, *Coexistence of cycles of a continuous mapping of a line into itself*, Ukr. Mat. Z., **16**, 61-71, 1964.

DEPARTMENT OF MATHEMATICS, TRINITY UNIVERSITY, SAN ANTONIO, TX 78212  
*E-mail address:* `jhasfura@trinity.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON, SEATTLE, WA 98195  
*E-mail address:* `lynch@math.washington.edu`