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Wilf Equivalence in Interval Embeddings

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Abstract

Consider the alphabet A and define A^* as the set of words over A. Define a vector of sequences of subsets of $\mathbb{N}$ as $\vec{u} = (u_1, u_2, \dots, u_k)$. Consider a word $w \in A^*$. Define their to be an embedding of $\vec{u}$ in w, $\vec{u} \leq w$ if there is some i such that, $w_i \in u_j, w_{i+1} \in u_{j+1}, \dots w_{i+k-1} \in u_{j+k-1}$. Define a word that avoids the vector $\vec{u}$ as a word where there is no such i, such that $w_i \in u_j, w_{i+1} \in u_{j+1}, \dots w_{i+k-1} \in u_{j+k-1}$.

We define the weight of a function as $\mathrm{wt}(w) = t^{|w|} x^{\Sigma(w)}$. We define the generating function for a certain pattern $\vec{u}$ as $F(u; x, t) = \sum_{u \leq w} \mathrm{wt}(w)$. We consider two patterns $\vec{u}$ and $\vec{v}$ to be Wilf Equivalent if $F(\vec{u}; x, t) = F(\vec{v}; x, t)$. We then prove some properties for Wilf Equivalence of patterns. We use these properties to then try to describe classes of Wilf Equivalent objects.

1 Introduction

Given an alphabet A, let: $A^* = \{w = w_1 w_2 \dots w_l : n \geq 0 \text{ and } w_j \in A\ \forall j\}$ and $A^i = \{w = w_1 w_2 \dots w_i : n \geq 0 \text{ and } w_j \in A\ \forall j\}$. That is A^* is the set of all words over A and A^i is the set of all words of length i over A. Let $\mathbb{N} = \{1, 2, 3, \dots\}$ and for any $k \in \mathbb{N}$ let $[k] = \{1, 2, \dots, k\}$. Define the length of a word $|w|$, as the number of characters in the word. Define the norm of a word w, $\sum(w) = \sum_{i=1}^{l} w_i$. For example, given the word $w = 14252$, $|w| = 5$ and $\sum(w) = 14$. A permutation of a non-empty set A is a bijection between A and itself. We will consider permutations of the set $[k]$. We will denote the set of permutations of $[k]$ as $\mathcal{S}_k$. The permutations of $[3]$ are the set $\{123, 132, 213, 231, 321, 312\}$.

Given a word $w \in [n]^*$ such that $|w| = \ell$, given an i such that $1 \leq i \leq \ell$, consider a subword to be $w' = w_i, w_{i+1}, \dots, w_{i+k-1}$. Let a subsequence of a word $w \in [n]^*$ and $|w| = \ell$, be $w' = w_{m_1}, w_{m_2}, \dots, w_{m_k}$ such that $1 \leq m_i \leq \ell$, and $m_i < m_{i+1}$. For example given the word $w = 14252$ an example of a subword and a subsequence is $w_1 w_2 w_3 = 142$, while $w_1 w_3 w_5 = 122$ is an example of a subsequence.

One of the first considered problems in the field of pattern avoidance was classical permutation avoidance. Given permutations $\tau \in \mathcal{S}_n$ and $\sigma \in \mathcal{S}_k$, we say σ embeds in τ if there is some subsequence of length k in τ such that the letters of this subsequence are in the same relative order as the permutation σ. That is if $\sigma_i < \sigma_{i+j}$ then $\tau_{m_i} < \tau_{m_{i+j}}$, and if $\sigma_i > \sigma_{i+j}$ then $\tau_{m_i} > \tau_{m_{i+j}}$. As an example, $\sigma = 312$ and $\tau = 42568317$, σ embeds in τ in three places: 42568317, $4256\underline{83}1\underline{7}$, and $4256\underline{8}3\underline{17}$. If there is no subsequence where σ embeds in τ we say σ avoids τ. For instance $\sigma = 54321$ avoids $\tau = 312$ since $\sigma_1 > \sigma_2 > \dots > \sigma_5$, and to embed

312 there must be an increase in the permutation. Let the number of words of length n that avoid a permutation σ be $A_n(312)$. It is known that $A_n(312) = C_n$ the Catalan numbers, but this problem is generally very difficult for a general σ.

The problem of classical permutation pattern avoidance was then generalized to words avoiding words in this relative order. Given a pattern u is a word which contains all letters, and only letters from the alphabet $[k]$. Examples of such patterns using the alphabet $[k]$ are 1243 and 1442113. We say a word w pattern embeds u if there is some subsequence w' which is order isomorphic to the pattern u. That is if $|u| = |w'|$ and if $u_i < u_{i+j}$ then $w_{m_i} < w_{m_{i+j}}$, if $u_i > u_{i+j}$ then $w_{m_i} > w_{m_{i+j}}$, and if $u_i = u_{i+j}$ then $w_{m_i} = w_{mi+j}$. For example: the word $w = 17356348$ pattern embeds $u = 1243$, since $w' = w_1 w_3 w_5 w_7 = 1364$ is order isomorphic to u. Again, we say a word w avoids u if there is no subsequence such that u pattern embeds in w. The word $w = 12345$ avoids the pattern $u = 1321$ because there is no way to get a decreasing pair of letters for the 32 portion of the pattern $u = 1321$.

All of these definitions of avoidance consider themselves with classical avoidance. There is another type of avoidance which we will concern ourselves with more, which is consecutive avoidance. In this case, instead of using subsequences of words we will use subwords. So for instance permutation pattern avoidance is redefined to say that σ embeds in τ if there is some subwords of length k in τ such that the letters of this subword are in the same relative order as σ. So even though if $\sigma = 312$ and $\tau = 42568317$, σ embeds using classical pattern avoidance in τ in three places: 42568317, $4256\underline{83}1\underline{7}$, and $4256\underline{8}3\underline{17}$. We have that σ avoids τ in consecutive permutation avoidance, because none of these embeddings consist of subwords.

Previously Kitaev et al published a paper called Rationality, irrationality, and Wilf equivalence in generalized factor Order. In this paper, they describe a style of embedding called factor order, where given a word w, w' embeds in w if there are words u, v such that $w = uw'v$. For example if $w = 12321422$, $w' = 214$ is a factor of w since it appears in w, as seen here $123\underline{214}22$. Then the authors sought to generalize factor order, by defining a word u to factor embed in w denoted $u \leq w$, if $|u| = |w'|$, and $|u_i| \leq |w'_i|$ for $1 \leq i \leq |u|$. For example, if $u = 132$, and $w = 12321422$, then u factor embeds in w. It factor embeds in the place $1232\underline{142}2$, since $1 \leq 1$, $3 \leq 4$, and $2 \leq 2$. We will consider this definition of embedding until we introduce our generalization.

In Section 2, we will introduce generating functions and Wilf equivalence, and give the findings found in [Kitaev et. al., 2009]. In section 3, we will introduce our generalization of factor embedding, and give definitions. In section 4, we will introduce the theorems which give properties for which Wilf equivalence holds.

2 Generating Functions on Words

In an effort to describe properties of a certain pattern, authors have considered generating functions. Let t, x be commuting variables, and consider the a word w, we let the weight of w, $\mathrm{wt}(w)$ be defined as $\mathrm{wt}(w) = t^{|w|}x^{\sum(w)}$. For example, given the word $w = 14252$, $\mathrm{wt}(w) = t^5x^{14}$.

In [Kitaev et. al., 2009], new sets are defined. Using generalized factor embedding we denote the set $\mathcal{F}(u) = \{w | u \text{ embeds in } w\}$. Using generalized factor embedding, u is suffix embedded into w if the first time u embeds in w is in the last spot in w. We let $\mathcal{S}(u) = \{w | \text{the first embedding of } u \text{ is in the suffix of } w\}$. We let $\mathcal{A}(u) = \{w | w \text{ avoids } u\}$. Generating functions for these sets are also defined as below, we have

$$\begin{aligned} F(u;x,t) &= \sum_{w \in \mathcal{F}(u)} wt(w), \\ S(u;x,t) &= \sum_{w \in \mathcal{S}(u)} wt(w) \text{ and} \\ A(u;x,t) &= \sum_{w \in \mathcal{A}(u)} wt(w). \end{aligned}$$

Note that the generating function for all words is

$$\frac{1}{1 - \sum_{n \geq 1} tx^n} = \frac{1-x}{1-x-tx}.$$

From this, the authors in [Kitaev et. al., 2009] defined these relations between $F(u;x,t)$, $S(u;x,t)$, and $A(u;x,t)$.

1. $F(u;x,t) = \dfrac{1-x}{1-x-tx} - A(u;x,t)$
2. $F(u;x,t) = S(u;x,t)\dfrac{1-x}{1-x-tx}$

Two words u and v are defined as Wilf equivalent if $F(u;x,t) = F(v;x,t)$. Using the relationships between the three generating functions it is easy to see that it Wilf equivalence also follows if $A(u;x,t) = A(v;x,t)$, or $S(u;x,t) = S(v;x,t)$. The authors in [Kitaev et. al., 2009] proved these three properties about Wilf equivalence of words:

1. given a word u, $u \sim u^r$,
2. if $u \sim v$, then $1u \sim 1v$, and
3. if $u \sim v$, then $u^+ \sim v^+$, where u^+ is gotten by increasing every element in u by 1.

Now we will seek to generalize the notion of generalized factor embeddings.

3 Generalized Interval Embeddings

Define the sequence of subsets of $\mathbb{N}$ as $\vec{u} = (u_1, u_2, \dots u_k)$, and an element $w \in A^*$. Define set operations on two sequences $\vec{u}$ and $\vec{v}$ to be pairwise. Define there to be an embedding of $\vec{u}$ in w if there is some i such that, $w_i \in u_j, w_{i+1} \in u_{j+1}, \dots w_{i+k-1} \in u_{j+k-1}$. For example, if $w = 153244$, and a set of intervals $\vec{u} = \{[1,5],[2,4],[3,5]\}$, this set of intervals is embedded consecutively in the word in two places: $15\underline{324}4$, and $153\underline{244}$. This is because $3 \in [1,5]$, $2 \in [2,4]$, and $4 \in [3,5]$ for the first embedding, and similarly, $2 \in [1,5]$, $4 \in [2,4]$, and $4 \in [3,5]$. An example of a word that avoids this set of intervals is, 152142. Call this type of embedding generalized interval embedding. Given the set of intervals $\vec{u}$, in the case that $u_i = [a_i, b_i]$, define the distance of u_i, denoted $d(u_i) = b_i - a_i + 1$. Define $d(\vec{u}) = (d(u_1), d(u_2), \dots d(u_k))$. For example, in the case that $\vec{u} = ([1,5],[2,4],[3,5])$, $d(\vec{u}) = (5,3,3)$. Given two interval sets u and v for which $d(\vec{u}) = d(\vec{v})$, let $\Delta(\vec{u},\vec{v}) = (\delta_1, \delta_2, \dots, \delta_k)$ where given $u_i = [a_i, b_i]$, and $v_i = [a'_i, b'_i]$, $\delta_i = a'_i - a_i$. For example consider $\vec{u} = ([2,3],[3,5],[6,9])$ and $\vec{v} = ([2,3],[5,7],[4,7])$ Then $d(\vec{u}) = (1,2,3)$ and $d(\vec{v}) = (1,2,3)$. Since $d(\vec{u}) = d(\vec{v})$, then we may consider $\Delta(\vec{u},\vec{v}) = (0,2,-2)$. If we consider the possibility that $b_i = \infty$ for all i we now see that our embedding definition is a generalization of generalized factor order, which was used in the paper [Kitaev et. al., 2009]. Consider the example where we wish to embed $u = 132$. This correlates to avoiding the interval sets $u = \{[1,\infty],[3,\infty],[2,\infty]\}$.

4 Wilf Equivalence of Interval Embeddings

Our previous definitions of generating functions and wilf equivalence will hold in the case of generalized interval embeddings. Let $\mathcal{A}(\vec{u})$ represent the set of all words which avoid $\vec{u}$, and let $\mathcal{S}(\vec{u})$ represent all words which avoid $\vec{u}$ in all possible places to embed but the last one, that is the suffix. Let

$$A(\vec{u};x,t) = \sum_{w\in\mathcal{A}(\vec{u})} wt(w) \text{ and } S(\vec{u};x,t) = \sum_{w\in\mathcal{A}(\vec{u})} wt(w)$$

. In order to prove properties that show $\vec{u} \sim \vec{v}$ we want a bijection $\phi : \mathcal{A}(\vec{u}) \to \mathcal{A}(\vec{v})$, such that if $\phi(w) = w'$, $wt(w) = wt(w')$.

Theorem 1. *Given sets of intervals $\vec{u}$ and $\vec{v}$ with the following properties,*

1. $d(\vec{u}) = d(\vec{v})$,
2. $\sum \delta_i = 0$,
3. *there exists an m such that* $\bigcup_{r=1}^{m} u_r \bigcap \bigcup_{s=m+1}^{k} u_s = \emptyset$, *and*
4. *there exists an n such that* $\bigcup_{r=1}^{n} v_r \bigcap \bigcup_{s=n+1}^{k} v_s = \emptyset$,

then $\vec{u} \sim \vec{v}$.

Proof. Given a pair of interval sets $\vec{u}$ and $\vec{v}$, which satisfy the stated conditions, consider the function $\phi : \mathcal{A}(\vec{u}) \to \mathcal{A}(\vec{v})$. Given a word $w \in \mathcal{A}(\vec{u})$, let $\phi(w)$ be defined by the following rules:

- if $w \in \mathcal{A}(u) \cap \mathcal{A}(v)$, let $\phi(w) = w$,
- if $w \in \mathcal{A}(u) \setminus \mathcal{A}(v)$, for any an occurrence of $\vec{v}$ which is embedded in the word w, where the first letter of the occurrence is at w_i, let $w'_{i+j-1} = \delta_j + w_{i+j-1}$,
- if you get a word $w \notin \mathcal{A}(u) \cup \mathcal{A}(v)$ repeat the process until the embedding is in $\mathcal{A}(v) \setminus \mathcal{A}(u)$. Once this algorithm terminates, we have $\phi(w)$.

Lemma 1. *No embeddings of $\vec{u}$ or $\vec{v}$ with the above properties in a word w will overlap with embeddings of themselves.*

Proof. Noting properties 3 and 4, we see that each interval set must have a place where the left is disjoint from the right, let this be u_m. Assume towards a contradiction there are two embeddings of $\vec{u}$ in w which overlap and are not in the same place. Consider that u_m correlates to w_{i+m-1} for the first embedding and some other letter $w_{i+m-1+j}$ for the second embedding. Consider any letter w_l between w_{i+m-1} and $w_{i+m-1+j}$. Letter $w_l \in \bigcup_{r=1}^{m} u_r$ since for the second embedding of $\vec{u}$ w_l is to the left of u_m. Also $w_l \in \bigcup_{s=m+1}^{k} u_s$ since for the first embedding of $\vec{u}$, w_l is to the right of u_m. But this means $w_l \in \bigcup_{r=1}^{m} u_r \bigcap \bigcup_{s=m+1}^{k} u_s = \emptyset$, therefore no embeddings will overlap. □

Lemma 2. *ϕ is a bijection.*

Proof. Note that if $\vec{u} = \vec{v}$ then it is trivially true that $\vec{u} \sim \vec{v}$, so assume now that $\vec{u} \neq \vec{v}$ with the above properties. Since $w \in \mathcal{A}(\vec{u})$, then $w \in \mathcal{F}(\vec{v}) \cup \mathcal{A}(\vec{u})$. That is, any embedding of $\vec{v}$ in w must not also be an embedding of v. From Lemma 1, we realize that if we consider words of where with the same length as the number of intervals in $\vec{u}$ let us call this length l, since we can consider these independently of each other. Given a word of length l, since $\vec{u} \neq \vec{w}$, then there exists $\delta_i < 0$. Since no element u_i or v_i in $\vec{u}$ or $\vec{v}$ has a lower bound $a_i < 1$, then after the algorithm runs for some time, some w_i will eventually leave the interval v_i. At this point the algorithm will terminate.

Now if we consider any $y \in \mathcal{A}(\vec{v})$, and wish to to send it to some element $w \in \mathcal{A}(\vec{u})$ since $\Delta(\vec{v}, \vec{u}) = -\Delta(\vec{u}, \vec{v})$ then we consider the function $\phi : \mathcal{A}(\vec{v}) \to \mathcal{A}(\vec{u})$, we realize it uses the same steps, except backwards. Therefore we realize $\phi(\phi(w)) = w$, so $\phi = \phi^{-1}$. Therefore ϕ is a bijection.

□

Since ϕ is a bijection, we have that $\vec{u} \sim \vec{v}$ □

Consider the following example. Given the interval sets $\vec{u} = ([1,1],[2,7],[7,12])$, and $\vec{v} = ([1,1],[4,9],[5,10])$. Certainly properties 1 and 2 hold. Note that $d(\vec{u}) = d(\vec{v}) = (1,6,6)$, $\delta(\vec{u}, \vec{v}) = (0,2,-2)$, and$0+2-2 = 0$, so properties 3 and 4 hold. For $\vec{u}$ and $\vec{v}$, $[1,1]$ is disjoint from everything else, so properties 5 and 6 hold. Therefore $\vec{u} \sim \vec{v}$.

The algorithm works as follows, consider the element $w = (1,2,9,9,1,2,11)$. We see that $w \in \mathcal{A}(\vec{v})$, but there are two embeddings of $\vec{u}$: $(\underline{1,2,9},9,1,2,11)$, and $(1,2,9,9,\underline{1,2,11})$. We wish to have our function ϕ take this to an element in $\mathcal{A}(\vec{u})$. After step one, $w' = (1+0, 2+2, 9-2, 9, 1+0, 2+2, 11-2) = (1,4,7,9,1,4,9)$. We notice that in w' there are still places where $\vec{u}$ embeds. $w' = (\underline{1,4,7},9,1,4,9)$, and $w' = (1,4,7,9,\underline{1,4,9})$ so we do the algorithm again. This time in $w'' = (1+0, 4+2, 7-2, 9, 1+0, 4+2, 9-2) = (1,6,5,9,1,6,7)$ there is only one place where $\vec{u}$ embeds: (1,6,5,9,$\underline{1,6,7}$). So we operate again on this embedding to get: $w''' = (1,6,5,9,1+0,6+2,7-2) = (1,6,5,9,1,8,5) = \phi(w) = x$.

Theorem 2. *We may change property 1 from Theorem 1 from $d(\vec{u}) = d(\vec{v})$ to the following $d(\vec{u}) = \sigma(d(\vec{v}))$, where σ is some permutation of the elements of the vector, and and $\delta_i = a_{v\sigma_i} - a_{ui}$.*

Proof. A similar proof to Theorem 1 is considered with the following steps. Before you operate with the δ_i permute the embedding of v, y as $\sigma(y) = y'$, so we then have some subword y' of w such that $\sigma(v) \leq y'$. Now consider the same algorithm as above, then when you get to the end, every subsequence which was permuted, is reversed with σ^{-1}. Since σ is invertible, then it still holds with this algorithm that $\phi = \phi^{-1}$. □

Lemma 3. *Generalizing the Wilf equivalence properties in [Kitaev et. al., 2009], we have the following Wilf Equivalences:*

1. *$\vec{u} \sim (\vec{u})^r$ where $(\vec{u})^r$ is the reverse of $\vec{u}$,*
2. *if $\vec{u} \sim \vec{v}$ then $(U, u_1, u_2, \ldots, u_k) \sim (U, v_1, v_2, \ldots, v_k)$, and $(u_1, u_2, \ldots, u_k, U) \sim (v_1, v_2, \ldots, v_k, U)$,*
3. *if $\vec{u} \sim \vec{v}$ then if we add 1 to all a_i and b_i then the resulting interval vectors are wilf equivalent.*

The proofs for these follow very similarly to the proofs they given in the paper [Kitaev et. al., 2009] for generalized factor order.

Theorem 3 (Corollaries)**.** *By mixing our previous theorems we have the following corollaries: Let $(\vec{u}\vec{v}) = (u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_j)$, and generalize this operation for any number of interval sets as $(\vec{u_1}\vec{u_2} \ldots \vec{u_k})$. If $\cup(u_i) \bigcap \cup(v_i) = \emptyset$, $\cup(u'_i) \bigcap \cup(v'_i) = \emptyset$, $\vec{u} \sim \vec{u'}$ and $\vec{v} \sim \vec{v'}$*

1. *$(\vec{u}\vec{v}) \sim (\vec{u'}\vec{v'}) \sim (\vec{v}\vec{u}) \sim (\vec{v'}\vec{u'})$,*
2. *$(U\vec{u}\vec{v}) \sim (U\vec{u'}\vec{v'}) \sim (U\vec{v}\vec{u}) \sim (U\vec{v'}\vec{u'})$ and $(\vec{u}\vec{v}U) \sim (\vec{u'}\vec{v'}U) \sim (\vec{v}\vec{u}U) \sim (\vec{v'}\vec{u'}U)$*

Theorem 4 (Rearrangement Theorem)**.** *Let $\vec{u}$ and $\vec{v}$ be intervals such that there is at least 1 $u_i < \infty$, and $\vec{u} \sim \vec{v}$. Then $d(\vec{u}) = d(\vec{v})$ up to reordering.*

Proof. It is easy to show that if $\vec{u} \sim \vec{v}$ then $\vec{u}$ and $\vec{v}$ are of the same length, and $\sum a_u i = \sum a_v i$, so there exist permutations $\sigma = \sigma_1\sigma_2 \ldots \sigma_k$ and $\tau = \tau_1\tau_2 \ldots \tau_k$ so that $u_\sigma = u_{\sigma_1} \geq u_{\sigma_2} \geq \cdots \geq u_{\sigma_k}$ and $v_\tau = v_{\tau_1} \geq v_{\tau_2} \geq \cdots \geq v_\tau$ Suppose that u and v are not rearrangements. Then there exists some $1 \leq i \leq k$ such that $u_{\sigma_j} = v_{\tau_j}$ for each $1 \leq j \leq i-1$ and, without loss of generality, $v_{\tau_i} > u_{\sigma_i}$. Define A_{u_σ} to be the set of $w \in \mathcal{S}(u_\sigma)$ such that

1. $wt(w) = x^{k+n-v_{\tau_i}+1}t^k$, and
2. $w_z = u_{\sigma_z}$ for $1 \leq z \leq i-1$,

that is, u_σ and w have the same length, and the first $i-1$ letters of u_σ and w are the same. Now, define B_u to be the set of $w \in S_n(u_\sigma)$ such that

1. $wt(w) = x^{k+n-v_{\tau_i}+1}t^k$, and
2. $w_z > u_{\sigma_z}$ for at least one $z \leq i-1$.

Then the coefficient of $x^{k+n-v_{\tau_i}+1}t^k$ in $\mathcal{S}_n(u_\sigma, x, t)$ is $|A_{u_\sigma}|+|B_{u_\sigma}|$, and similarly, the coefficient of $x^{k+n-v_{\tau_i}+1}t^k$ in $\mathcal{S}(v_\tau, x, t)$ is $|A_{v_\tau}| + |B_{v_\tau}|$. By construction, each word in A_{u_σ} corresponds to an element $(a_1, \ldots, a_{k-i+1})$ of the set T_{u_σ} such that

1. for each $1 \leq r \leq k-i+1$, $0 \leq a_r \leq n - v_{\tau_i} + 1$, and
2. $a_1 + \cdots + a_{k-i+1} = n - v_{\tau_i} + 1$.

Each word in A_{v_τ} corresponds to an element $(a_1, \ldots, a_{k-i+1})$ of the set T_{v_τ} such that

1. $0 \leq a_1 < k-i+1$,
2. for each $2 \leq r \leq k-i+1$, $0 \leq a_r \leq n - v_{\tau_i} + 1$, and
3. $a_1 + \cdots + a_{k-i+1} = n - v_{\tau_i} + 1$.

We can see from these descriptions that $T_{v_\tau} \subset T_{u_\sigma}$, which gives that

$$|A_{v_\tau}| = |T_{v_\tau}| < |T_{u_\sigma}| = |A_{u_\sigma}|.$$

Now, every word of B_{u_σ} corresponds to an element $(b_1, \ldots, b_k)$ of the set L_{u_σ} such that

1. for each $1 \leq r \leq i-1$, $0 \leq b_r \leq n - u_{\sigma_r}$,
2. for each $i \leq r \leq k$, $0 \leq b_r < n - v_{\tau_i} + 1$, and
3. $b_1 + \cdots + b_k = n - v_{\tau_i} + 1$.

Each word in B_{v_τ} corresponds to an element $(b_1, \ldots, b_k)$ of the set L_{v_τ} such that

1. for each $1 \leq r \leq i-1$, $0 \leq b_r \leq n - v_{\tau_r}$,

2. for each $i \leq r \leq k$, $0 \leq b_r < n - v_{\tau_i} + 1$, and

3. $b_1 + \cdots + b_k = n - v_{\tau_i} + 1$.

We can see from these descriptions that, since $u_{\sigma_r} = v_{\tau_r}$ for $1 \leq r \leq i-1$, $L_{v_\tau} = L_{u_\sigma}$, which gives that

$$|B_{v_\tau}| = |L_{v_\tau}| = |L_{u_\sigma}| = |B_{u_\sigma}|.$$

Since we are dealing with embedding words of the same length of the original words, u and v, $|A_u| = |A_{u_\sigma}|$, $|B_u| = |B_{u_\sigma}|$, $|A_v| = |A_{v_\tau}|$, and $|B_v| = |B_{v_\tau}|$. Thus, $\mathcal{S}_n(u,x,t) \neq \mathcal{S}_n(v,x,t)$ as their coefficients of $x^{k+n-v_{\tau_i}+1}t^k$ do not agree, that is, $u \not\sim_n v$. $\square$

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