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On Minimal Surfaces and Their Representations

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Abstract. We consider the problem of representation of minimal surfaces in the euclidean space and provide a proof of Bernstein's theorem. This paper serves as a concise and self-contained reference to the theory of minimal surfaces.

Contents

Preface

The study of minimal surfaces was initially motivated by the Plateau Problem of finding a surface that minimizes the area given a fixed boundary. The analysis of this variational problem reveals that surfaces that minimize area, hence called *minimal surfaces*, have constant zero curvature. In this paper, we provide an analysis of minimal surfaces focusing on the problem of how to represent surfaces and we provide a proof of Bernstein's theorem which says that the plane is the only minimal surface in $\mathbb{R}^3$ that is the graph of a function. Many of the results we draw upon can be found in [**5**] and [**2**].

In the development of Differential Geometry, we always thought of surfaces as objects sitting in the Euclidean space. In our work, we begin with an abstract treatment of the theory of differentiable manifolds and establish all the classical geometric objects independently of how these objects sit in the Euclidean space. Nevertheless, these concepts are intimately connected as we will show from Whitney's that states that every n-dimensional manifold may be embedded in $\mathbb{R}^{2n+1}$. This will provide us with motivation to introduce the theory of minimal surfaces as spaces embedded into the Euclidean space. The remainder of the paper is organized to describe the solutions of the variational problem and how to represent minimal surfaces. The project culminates in Chapter 7, where we provide a proof of Bernstein's theorem.

Part 1

Differentiable Manifolds

CHAPTER 1

Preliminaries

In this chapter, we will provide the basic definitions and constructions relevant to the study of differentiable manifolds. We begin by studying differentiable manifolds in order to prove Whitney's theorem, which states that every n-dimensional manifold may be embedded in $\mathbb{R}^{2n+1}$.

1. Differentiable Manifolds

Let us begin with some preliminaries that will motivate our definition of a differentiable manifold.

Definition 1.1. Let M be a non-empty set.

A pair $(\Omega, \mathbf{x})$ consisting of an open set $\Omega \subset \mathbb{R}^n$ and an injective map $\mathbf{x} : \Omega \to M$ is a *local parametrization* of M, *at* p if $p \in \mathbf{x}(\Omega)$. A pair (U, φ) consisting of a subset $U \subset M$ and a map $\varphi : U \to \mathbb{R}^n$ such that $\big(\varphi(U), \varphi^{-1}\big)$ is a local parametrization of M is a *chart* on M, *at* p if $p \in U$; here the map φ is a *coordinate chart*.

Two charts (U, φ) and (V, ψ) on M have a C^r*-overlap* if the *coordinate change*

$$\varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$$

is a C^r-diffeomorphism. Here, r can be a nonnegative integer, ∞, or ω (meaning real-analytic).

An *atlas* on M is a collection of charts $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}$ on M such that $\bigcup U_\alpha = M$. An atlas on M is C^r*-compatible* if any two charts in the atlas have a C^r overlap.

A C^r*-differential structure* on M is a maximal C^r-compatible atlas on M.

Our primary objective is to study sets with a differential structure defined on them. Though differential structures are not always easy to explicitly find, this is not a problem since, given a set M, we may define an equivalence relation $\sim$ on the collection of C^r-compatible atlases on M by $\mathscr{A}_1 \sim \mathscr{A}_2$ if and only if $\mathscr{A}_1 \cup \mathscr{A}_2$ is a C^r compatible atlas. Each equivalence class under $\sim$ contains a unique maximal C^r-compatible atlas on M, so it suffices to give a single C^r-compatible atlas on M to define a differential structure on M. For technical purposes, this extension will be done without further comment. That is, we shall assume that an atlas $\mathscr{A}$ on a set M is a differential structure wherever it is convenient.

Notice that the way we have defined differential structures on a set M makes no mention of a topology on M. Since we eventually want to study general objects that "look like" Euclidean spaces, it will be useful to define a topology on M. Fortunately, a C^0 atlas $\mathscr{A}$ on a set M induces a natural topology on M. Here, it suffices to define a subset $A \subset M$ to be *open* if and only if $\varphi(A \cap U)$ is open in $\mathbb{R}^n$ for every chart $(U\varphi) \in \mathscr{A}$. Notice that this topology is defined in such a way that every coordinate chart of $\mathscr{A}$ is a homeomorphism onto its image. We summarize this discussion with the following proposition.

Proposition 1.1. *Let M be a topological space. Then the topology of M is induced by a C^0 compatible atlas if and only if each point of M has a neighborhood homeomorphic to $\mathbb{R}^n$.*

It is precisely the property described in Proposition 1.1 that will motivate our definition of a manifold:

Definition 1.2. A C^r-differential structure $\mathscr{A}$ on a set M is a C^r*-manifold structure* on M if the topology on M induced by $\mathscr{A}$ is Hausdorff and second-countable.

A C^r*-differential manifold of dimension* n is a pair $(M, \mathscr{A})$ consisting of a set M and a C^r-manifold structure $\mathscr{A}$ on M.

Before giving some examples of manifolds, we give a word on notation. Although a manifold is an ordered pair $(M, \mathscr{A})$, we will often suppress the manifold structure in our notation and simply refer to the manifold as M. This abuse of notation is harmless enough and will often prove convenient. Also, until further notice, fix $r \geq 1$ and assume that all manifolds mentioned are C^r manifolds.

Example 1.1. Observe that $\mathbb{R}^n$ together with the manifold structure given by the identity map is a manifold.

Example 1.2 (The Product Manifold). If $(M_1, \mathscr{A}_1)$ and $(M_2, \mathscr{A}_2)$ are manifolds of dimensions m and n respectively, then there exists a natural manifold structure $\mathscr{A}$ on $M_1 \times M_2$ making $(M_1 \times M_2, \mathscr{A})$ an $m+n$-dimensional manifold. Namely this structure is given by the collection

$$\mathscr{A} = \{(U_1 \times U_2, \varphi_1 \times \varphi_2) : (U_i, \varphi_i) \in \mathscr{A}_i\}$$

where $\varphi_1 \times \varphi_2 : U_1 \times U_2 \to \mathbb{R}^{m+n}$ is the induced map.

Example 1.3. If $(M, \mathscr{A})$ is a manifold and $W \subseteq M$ is open, then there exists a natural manifold structure $\mathscr{A}_W$ on W. Namely, this structure is given by the collection

$$\mathscr{A}_U = \{(U, \varphi) \in \mathscr{A} : U \subset W\}.$$

Example 1.4 (Submanifolds). We will often find ourselves interested in manifolds that are contained in other manifolds. A subset A of a manifold $(M, \mathscr{A})$ is a C^r*-submanifold* of $(M, \mathscr{A})$ if there exists an integer $k \geq 0$ such that each point of A belongs to the domain of a chart $(U, \varphi) \in \mathscr{A}$ such that

$$U \cap A = \varphi^{-1}\big(\mathbb{R}^k\big)$$

where $\mathbb{R}^k \subset \mathbb{R}^n$ is the set of vectors whose last $n-k$ are 0. Such a chart (U, φ) is a *submanifold chart* for (M, A). If A is a submanifold of M, then the collection of coordinate charts

$$\varphi|_{\,U\cap A} : U \cap A \to \mathbb{R}^k$$

form a C^r atlas of A, where (U, φ) runs over all submanifold charts. Thus A is a C^r manifold of dimension k. The *codimension* of A is $n-k$.

We now turn our attention to the study of maps between manifolds. One of our main objectives is to extend the ideas of differential calculus in $\mathbb{R}^n$ to differentiable manifolds. The first step in this direction is defining what it means for a map between two manifolds to be differentiable.

Definition 1.3. Let M^m and N^n be manifolds and let $f : M \to N$.

A pair of charts (U, φ) for M and (V, ψ) for N is *adapted to* f if $f(U) \subset V$. Here, the map

$$\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$$

is defined; it is the *expression* of f in the given charts, *at* p if $p \in U$.

The map f is *differentiable at a point* $p \in M$ if it has an expression at p which is differentiable. Similarly, f is *differentiable of class* C^r if it has a C^r expression at every point of M.

We observe that it is not necessary for every expression to be C^r in order for a map to be C^r. To this extent, we have the following result.

PROPOSITION 1.2. *Given manifolds M^m and N^n, let $f : M \to N$ be of class C^r. If (U, φ) and (V, ψ) are a pair of charts on M and N, respectively, adapted to f, then the expression of f in (U, φ) and (V, ψ) is of class C^r.*

PROOF. Let $p \in \varphi(U)$ and say $q = \varphi^{-1}(p)$. Since f is C^r, there exist charts (U_0, φ_0) on M and (V_0, ψ_0) on N such that the map $\psi \circ f \circ \varphi^{-1}$ is a C^r-expression of f at q. It follows that $p \in U \cap U_0 = W$ so that $W \neq \varnothing$. By replacing U_0 with W, we may assume that $U_0 \subset U$. Similarly, we may assume that $V_0 \subset V$. In this way, our new restricted charts (U_0, φ_0) and (V_0, ψ_0) still adapt to f and the expression of f in these charts is still a C^r-expression of f at q.

Now, observe that

$$\psi \circ f \circ \varphi^{-1} = \left(\psi \circ \psi_0^{-1}\right) \circ \left(\psi_0 \circ f \circ \varphi_0^{-1}\right) \circ \left(\varphi_0 \circ \varphi^{-1}\right) \tag{1.1}$$

on $\varphi(U_0)$. The first and third maps on the right of (1.1) are C^r since they are coordinate changes. Hence $\psi \circ f \circ \varphi^{-1}$ is C^r in some neighborhood of every point of $\varphi(U)$. Thus the expression of f in (U, φ) and (V, ψ) is C^r as required. $\square$

Let $f : M \to N$ and $g : N \to P$ be C^r maps between manifolds. Then it is clear from the definition of differentiability that the composition map $g \circ f$ is also C^r. In addition, it can be easily verified that the identity map and all constant maps are C^r. Finally, if (U, φ) is a chart on a manifold M, then observe that φ is a C^r-diffeomorphism onto its image.

From our observation above, we may define an equivalence relation $\approx$ on the family of manifolds by defining $M \approx N$ whenever there exists a C^r-homeomorphism $f : M \to N$ whose inverse is also C^r, which is the basic equivalence relation in differential topology.

2. The Tangent Space

Our next task will be to extend the idea of a tangent vector to a curve on a differentiable manifold. For surfaces in $\mathbb{R}^n$, we think of a tangent vector to a point p as the "velocity" in $\mathbb{R}^n$ of a curve on the surface passing through p. While one may develop the theory of tangent spaces via the study of curves on a manifold, we will develop the tangent space from the point of view of derivations.

Definition 1.4. Given a manifold M with $p \in M$, let $\mathcal{C}_pM$ be the collection of all real-valued functions defined in a neighborhood of p that are C^r-differentiable at p and let $\sim$ be the equivalence relation on $\mathcal{C}_pM$ defined by $f \sim g$ whenever there exists a neighborhood U of p such that $f = g$ on U. A *germ* is an equivalence class of $\sim$ and the collection of germs of $\mathcal{C}_pM$ is denoted by $\mathcal{D}_pM$.

A *derivation* at p is a linear map $X : \mathcal{D}_pM \to \mathbb{R}$ such that

$$X(fg) = g(p)Xf + f(p)Xg$$

for every $f, g \in \mathcal{D}_pM$. A *tangent vector* at p is a derivation at p. The *tangent space of M at p* is the real vector space of all tangent vectors at p and is denoted by T_pM.

Although this definition does not rely on our usual geometric intuition, we will see shortly that the tangent space has the expected geometrical properties. Namely, the vector space structure that coincides with the *tangent plane* structure of real surfaces.

Before showing this, we give an example of a tangent space and an important derivation.

EXAMPLE 1.5. Let U be a neighborhood of a point p in a manifold M. Then the collection of germs of C^r functions in U at p is the same as $\mathcal{D}_p M$. Hence $T_p M = T_p U$.

EXAMPLE 1.6. Given a chart (U, φ) on a manifold M at a point p, let $D_i|_p^{\varphi} : \mathcal{D}_p M \to \mathbb{R}$ be the map defined by

$$D_i|_p^{\varphi} f = D_i|_{\varphi(p)} f \circ \varphi^{-1}.$$

Then $D_i|_p^{\varphi} \in T_p M$. Indeed, if $f, g \in \mathcal{D}_p M$, then the chain rule implies

$$\begin{aligned} D_i|_p^{\varphi} fg &= D_i|_{\varphi(p)} fg \circ \varphi^{-1} \\ &= D_i|_{\varphi(p)} \left(f \circ \varphi^{-1}\right)\left(g \circ \varphi^{-1}\right) \\ &= f(p)\ D_i|_{\varphi(p)} g \circ \varphi^{-1} + g(p)\ D_i|_{\varphi(p)} f \circ \varphi^{-1} \\ &= f(p)\ D_i|_p^{\varphi} g + g(p)\ D_i|_p^{\varphi} f \end{aligned}$$

and the linearity of $D_i|_p^{\varphi}$ can easily be verified.

We will often encounter the operator $D|_p^{\varphi}$. It is the *partial derivative at* p *with respect to* φ.

In the case that $M = \mathbb{R}^n$, we observe that $T_p \mathbb{R}^n$ is isomorphic to $\mathbb{R}^n$, which agrees with our intuition. In this direction, we prove the following result.

Proposition 1.3. *Given $p \in \mathbb{R}^n$, let $F : \mathbb{R}^n \to T_p\mathbb{R}^n$ be the map defined by*

$$F(v_1, \ldots, v_n) = \sum v_i \, D_i|_p.$$

Then F is a linear isomorphism.

Proof. The fact that F is linear is clear, so it suffices to show that F is bijective.

First, we see that that F is injective. Suppose that $Fv = 0$ and let $I = (I_1, \ldots, I_n)$ be the identity on $\mathbb{R}^n$. Then

$$v_j = \sum_i v_i \delta_{ij} = \sum_i v_i \, D_i|_p I_j = Fv(I_j) = 0$$

for every j so that $v = 0$. Hence F is injective.

Next, we show that F is surjective. Let $X \in T_pM$ and let $f : U \to \mathbb{R}$ be a representative of a germ in $\mathcal{D}_p\mathbb{R}^n$. Since U is a neighborhood of p, it contains an open ball B with center p. If we restrict f to B, then f is still a representative of the same germ, so we may assume this is done.

Now, by Taylor's theorem (see [**7**]), there are C^r functions g_i defined in a neighborhood $V \subset B$ of p such that

$$g_i(p) = D_i|_p f$$

and

$$f = f(p) + \sum (I_i - p_i) \, g_i \tag{1.2}$$

for every $x \in V$. Applying X to (1.2) and observing that a derivation of a constant function is 0 gives us

$$Xf = \sum (XI_i) \, g_i(p) + \sum (p_i - p_i) \, Xg_i(p)$$

$$= \sum (XI_i)\, D_i|_p f.$$

Hence $X = Fv$ where $v = (XI_1, \ldots, XI_n)$ so that F is surjective. □

COROLLARY 1.1. *For $p \in \mathbb{R}^n$, the collection $\{D_1|_p, \ldots, D_n|_p\}$ is a basis for $T_p\mathbb{R}^n$.*

PROOF. Let F be the map defined in Proposition 1.3 and let $\{e_1, \ldots, e_n\}$ be the standard basis for $\mathbb{R}^n$. Then $Fe_i = D_i|_p$ for every i. Since F is an isomorphism, it maps the basis $\{e_1, \ldots, e_n\}$ to a basis $\{D_1|_p, \ldots, D_n|_p\}$ of $T_p\mathbb{R}^n$. □

One of the most useful properties of the tangent space is that it allows us to generalize the notion of derivatives of smooth functions between manifolds. This notion is known as the differential.

DEFINITION 1.5. Given manifolds M and N with $p \in M$, let $F : M \to N$ be C^r. Then the *differential of F at p* is the linear map $dF_p : T_pM \to T_{F(p)}N$ given by

$$dF_p(X)f = X(f \circ F)$$

for every $f \in \mathcal{D}_{F(p)}N$.

It is easy to verify that the differential map is well-defined and linear. For details see [**7**].

We now consider some of the familiar and important properties of differentiable functions on manifolds.

THEOREM 1.1 (The Chain Rule). *Given manifolds M, N, and P with $p \in M$, let $F : M \to N$ and $G : N \to P$ be C^r. Then*

$$d\,(G \circ F)_p = dG_{F(p)} \circ dF_p.$$

Proof. Let $X \in T_pM$ and let f be a germ at $(G \circ F)(p)$ in P. Then

$$\begin{aligned}\left(d\,(G \circ F)_p\, X\right) f &= X\,(f \circ G \circ F) = (dF_pX)\,(f \circ G) \\ &= \left(dG_{F(p)}(dF_pX)\right) f = \left(\left(dG_{F(p) \circ dF_p}X\right)\right) f\end{aligned}$$

as required. □

Proposition 1.4. *Given a manifold M with $p \in M$, let I be the identity map on M. Then dI_p is the identity map on T_pM.*

Proof. Let $X \in T_pM$ and let $f \in \mathcal{D}_pM$. Then

$$(dI_pX)\,f = X(f \circ I) = Xf$$

as required. □

Corollary 1.2. *Let M and N be manifolds, let $p \in M$, and let $F : M \to N$ be a C^r-diffeomorphism. Then $dF_p : T_pM \to T_{F(p)}N$ is a linear isomorphism.*

Proof. Since dF_p is linear, it suffices to show that dF_p is invertible. Since F is a C^r-diffeomorphism, it has a C^r inverse F^{-1}. Now, the chain rule and Proposition 1.4 imply that

$$dF^{-1}_{F(p)} \circ dF_p = d\left(F^{-1} \circ F\right)_{F(p)} = d\,(I_N)_p = I_{T_{F(P)}N}$$

and that

$$dF_p \circ dF^{-1}_{F(p)} = d\left(F \circ F^{-1}\right)_p = d\,(I_M)_p = I_{T_pM}.$$

Hence dF_p is invertible as required. □

Corollary 1.3. *Let U and V be open in R^m and R^n respectively and let $F : U \to V$ be a C^r-diffeomorphism. Then $m = n$.*

PROOF. By Corollary 1.2, $dF_p : T_pU \to T_{F(p)}V$ is a linear isomorphism. Since T_pU is isomorphic to $\mathbb{R}^m$ and $T_{F(p)}V$ is isomorphic to $\mathbb{R}^n$, it follows that $m = n$ as required. $\square$

Corollary 1.2 gives us a very important property of the tangent space. Given a manifold M^n with $p \in M$, let (U, φ) be a chart of M at p. Since $\varphi : U \to \mathbb{R}^n$ is a diffeomorphism onto its image, Corollary 1.2 implies that $d\varphi_p : T_pM \to T_{\varphi(p)}\mathbb{R}^n$ is a linear isomorphism.

PROPOSITION 1.5. *Let (U, φ) be a chart of a manifold M at a point $p \in M$. Then the collection $B = \left\{ D_1|_p^\varphi, \ldots, D_n|_p^\varphi \right\}$ is a basis for T_pM.*

PROOF. Let $f \in \mathcal{D}_{\varphi(p)}\mathbb{R}^n$ and observe that

$$d\varphi_p\, D_i|_p^\varphi f = D_i|_p^\varphi f \circ \varphi = D_i|_{\varphi(p)} f \circ \varphi \circ \varphi^{-1} = D_i|_p f.$$

Since $d\varphi_p$ is a linear isomorphism, B is a basis as required. $\square$

PROPOSITION 1.6 (Transition matrix for coordinate vectors). *Let M be a manifold, let (U, φ) and (V, ψ) be charts on M, and let $p \in U \cap V$. Then*

$$D_j|_p^\varphi = \sum_i D_j|_p^\varphi \psi_i\, D_i|_p^\psi.$$

PROOF. The collections $\left\{ D_1|_p^\varphi, \ldots, D_n|_p^\varphi \right\}$ and $\left\{ D_1|_p^\psi, \ldots, D_n|_p^\psi \right\}$ are bases for T_pM, so there exists a matrix (a_{ij}) of real numbers such that

$$D_j|_p^\varphi = \sum_k a_{kj}\, D_k|_p^\psi. \tag{1.3}$$

Now, applying ψ_i to both sides of (1.3), we obtain the relation

$$D_j|_p^\varphi \psi_i = \sum_k a_{kj}\, D_k|_p^\psi = \sum_k a_{kj}\delta_{ik} = a_{ij}$$

as required. □

We are now ready to show that the differential of a map between manifolds generalizes the notion of the Jacobian:

Proposition 1.7. *Given manifolds M^m and N^n with $p \in M$, let $F : M \to N$ be C^r, let (U, φ) be a chart on M at p, and let (V, ψ) be a chart on N at $F(p)$. Then, relative to the bases $B_M = \left\{ D_1|^{\varphi}_{p}, \dots, D_m|^{\varphi}_{p} \right\}$ for T_pM and $B_N = \left\{ D_1|^{\psi}_{F(p)}, \dots, D_n|^{\psi}_{F(p)} \right\}$ for $T_{F(p)}N$, the differential $dF_p : T_pM \to T_{F(p)}N$ is represented by the matrix (a_{ij}) where*

$$a_{ij} = D_j|^{\varphi}_{p} \psi_i \circ F.$$

Proof. Since B_M and B_N are bases for M and N respectively, there exists a matrix (a_{ij}) of real numbers such that

$$dF_p\big(D_j|^{\varphi}_{p} \big) = \sum_k a_{kj} \; D_k|^{\psi}_{F(p)}. \tag{1.4}$$

Now, apply φ_i to both sides of (1.4) to obtain the relation

$$\begin{aligned} a_{ij} &= \sum_k a_{kj} \; D_k|^{\psi}_{F(p)} \psi_i \\ &= dF_p\big(D_j|^{\varphi}_{p} \big) \psi_i \\ &= D_j|^{\varphi}_{p} \psi_i \circ F \end{aligned}$$

as required. □

Let $F : \mathbb{R}^m \to \mathbb{R}^n$ be a C^r map. Then Proposition 1.7 implies that the matrix representation (a_{ij}) of dF_p relative to the bases $\{ D_1|_{p}, \dots, D_m|_{p} \}$ for $T_p\mathbb{R}^m$ and

$\left\{ D_1|_{F(p)}, \ldots, D_n|_p \right\}$ for $T_{F(p)}\mathbb{R}^n$ is given by

$$a_{ij} = D_j|_p F_i,$$

which is precisely the Jacobian of F. Hence the differential is a generalization of the Jacobian.

We conclude this section by observing that the classical geometric interpretation of the tangent space via tangent vectors to a curve in a manifold is equivalent to our definition.

Definition 1.6. Given a manifold M with $p \in M$, for $\varepsilon > 0$, a *curve in M at p* is a differentiable function $\alpha : (-\varepsilon, \varepsilon) \to M$ such that $\alpha(0) = p$. The *velocity vector* $\alpha'(t)$ at time $t \in (-\varepsilon, \varepsilon)$ is the vector in $T_{\alpha(t)}M$ given by

$$\alpha'(t) = d\alpha_p(D|_t).$$

For curves in $\mathbb{R}^n$, let $\varepsilon > 0$ and consider a curve $\alpha : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$. For $I = (I_1, \ldots, I_n)$ the identity map on $\mathbb{R}^n$ and $t \in (-\varepsilon, \varepsilon)$, since $\alpha'(t) \in T_{\alpha(t)}\mathbb{R}^n$ and $\left\{ D_1|_{\alpha(t)}, \ldots, D_n|_{\alpha(t)} \right\}$ is a basis for $T_{\alpha(t)}\mathbb{R}^n$, there exist real numbers $a_1, \ldots, a_n$ such that

$$\alpha'(t) = \sum a_i \, D_i|_{\alpha(t)}. \tag{1.5}$$

Applying I_j to both sides of (1.5) we obtain the relations

$$\alpha'(t) I_j = d\alpha_p(D|_t) I_j = D|_t I_j \circ \alpha = D|_t \alpha_j \tag{1.6}$$

and

$$\alpha'(t) I_j = \sum_i a_i \, D_i|_{\alpha(t)} I_j = \sum_i a_i \delta_{ij} = a_j. \tag{1.7}$$

Hence (1.6) and (1.7) imply that

$$\alpha'(t) = \begin{bmatrix} D|_t\alpha_1 \\ \vdots \\ D|_t\alpha_n, \end{bmatrix}. \tag{1.8}$$

Observe that (1.8) is precisely the tangent vector as defined for curves in $\mathbb{R}^n$. This shows that every curve α at a point p in a manifold M gives rise to a tangent vector $\alpha'(0)$ in T_pM. Conversely, one can show that every tangent vector $X \in T_pM$ is the velocity vector of some curve at p. In fact, this is the content of the following proposition.

PROPOSITION 1.8. *Given a manifold M^n with $p \in M$, let $X \in T_pM$. Then there exists a curve α in M at p such that $\alpha'(0) = X$.*

PROOF. Let (U, φ) be a chart of M at p, let $I = (I_1, \ldots, I_n)$ be the identity map on $\mathbb{R}^n$, and put $q = \varphi^{-1}(p)$. Since $\left\{ D_1|^{\varphi}_p, \ldots, D_n|^{\varphi}{}_p \right\}$ is a basis for T_pM, there exists a vector $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ such that $X = \sum a_i \, D_i|^{\varphi}_p$. Now, there exists a $\varepsilon > 0$ such that the trace of the curve $c : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ given by

$$c(t) = at + q$$

is contained in $\varphi(U)$. Put $\alpha = \varphi^{-1} \circ c$. Then $\alpha(0) = p$ and

$$\begin{aligned} \alpha'(0)f &= d\alpha_p(D|_0)f = D|_0 f \circ \alpha \\ &= D|_0 f \circ \varphi^{-1} \circ \alpha = D|_p f \circ \varphi^{-1} \, D|_0 c \\ &= \sum a_i \, D_i|_p f \circ \varphi^{-1} = \sum a_i \, D_i|_{\varphi(p)} f \\ &= Xf \end{aligned}$$

for every $f \in \mathcal{D}_p M$ so that $\alpha'(0) = X$ as required. $\square$

Therefore the tangent space at a point can be viewed as the collection of velocity vectors of curves at the point.

3. The Tangent Bundle

DEFINITION 1.7. Let M be a manifold. The *tangent bundle* of M is the disjoint union TM of all the tangent spaces of M. Namely,

$$TM = \bigsqcup_{p \in M} \{p\} \times T_p M.$$

We claim that the tangent bundle of a manifold M is itself a manifold. Indeed, let us provide a topology for TM.

Given a manifold M with $p \in M$, let $X \in T_p M$ and let (U, φ) be a chart of M at p. Since the collection $\left\{ D_1|_p^\varphi, \ldots, D_n|_p^\varphi \right\}$ is a basis for $T_p M$, X may be written uniquely as

$$X = \sum a_i(X)\ D_i|_p^\varphi$$

where $a(X) = (a_1(X), \ldots, a_n(X)) \in \mathbb{R}^n$. Because $d\varphi_p X = \sum a_i(X)\ D|_{\varphi(p)} \in T_{\varphi(p)}\mathbb{R}^n$, we have that $d\varphi_p X$ may be identified with the vector $a(X)$. Now, let

$$TU = \bigcup_{p \in U} T_p M$$

and let $\widetilde{\varphi} : TU \to \varphi(U) \times \mathbb{R}^n$ be the map given by

$$\widetilde{\varphi}(p, X) = (d\varphi_p X, a(X))\,.$$

Then $\widetilde{\varphi}$ is a bijection and $\widetilde{\varphi}^{-1} : \varphi(U) \times \mathbb{R}^n \to TU$ is the map given by

$$\widetilde{\varphi}^{-1}(\varphi(p), a(X)) = \left(p, \sum a_i(X)\ D_i|_p^\varphi\right).$$

Using the map $\widetilde{\varphi}$, we can define a topology on TU by letting a subset $A \subset TU$ be open if and only if $\widetilde{\varphi}(A)$ is open in $\varphi(U) \times \mathbb{R}^n$.

Let $\mathcal{B}$ be the collection of all open subsets of TU_α, where U_α runs over all coordinate open sets in M. One may check that, in this way, the topology $\mathcal{B}$ on TM is Hausdorff and second-countable (see [**7**] for details). We will now provide a manifold structure for TM.

Finally, one may show that the collection $\{(TU_\alpha, \widetilde{\varphi}_\alpha)\}$ is a manifold structure on TM (see [**7**]) making TM a $2n$-dimensional manifold.

CHAPTER 2

The Topology of Manifolds

In this chapter, we will address some of the background results that will be used to establish Whitney's Theorem. We divide this chapter into two parts. In the first, we consider analytical results of measure theory on general manifolds. In particular, subsets of measure zero. In the second part, we consider general topological results that will play a fundamental role in the proof of Whitney's Theorem.

1. Measure Theory

We denote by μ the Lebesgue measure in $\mathbb{R}^n$ and for background and related results we refer to [**6**]. Our goal is to show that if M^m and N^n are manifolds with $m < n$ and $f : M \to N$ is a map of class C^1, then $N \setminus F(M)$ is dense in N. Let us now consider some preliminaries in this direction.

Lemma 2.1. *Given an open set $U \subset \mathbb{R}^n$ and a compact and convex subset $B \subset U$, let $f : U \to \mathbb{R}^m$ be C^1. Then f is Lipschitz continuous on B with Lipschitz constant $\kappa = \sup \{|Df(x)| : x \in B\}$.*

Proof. Let $a, b \in B$ and let $g : [0,1] \to \mathbb{R}^n$ be the map $g(t) = a + t\,(b - a)$. Since B is convex, $g([0,1]) \subset B$. It follows that

$$
\begin{aligned}
|f(b) - f(a)| &= \left| \int_0^1 D\,(f \circ g)\,(t)\,dt \right| \\
&= \left| \int_0^1 Df(g(t))Dg(t)\,dt \right|
\end{aligned}
$$

$$\leq \int_0^1 |Df(g(t))|\,|b-a|\;dt$$

$$\leq \kappa\,|b-a|$$

where $\kappa = \sup\{|Df(x)| : x \in B\}$. Hence f is Lipschitz continuous on B with Lipschitz constant κ as required. $\square$

LEMMA 2.2. *Given an open set $U \subset \mathbb{R}^n$ and a subset $A \subset U$ of measure zero, let $f : U \to \mathbb{R}^n$ be C^1. Then $f(A)$ has measure zero in $\mathbb{R}^n$.*

PROOF. Let $\varepsilon > 0$. For every $p \in E$ there exists an open ball $B_p \subset U$ such that $p \in B_p$. It follows that $\{B_p : p \in E\}$ is an open cover of E. Since E is second-countable, $\{B_p : p \in A\}$ contains a countable subcover $\{B_i\}$. Since

$$f(A) = \bigcup f\big(A \cap \overline{B}_i\big),$$

it suffices to show that $f\big(A \cap \overline{B}\big)$ has measure zero for every $B \in \{B_i\}$.

In this direction, let $B \in \{B_i\}$ and put $\kappa = \sup\big\{|Df(x)| : x \in \overline{B}\big\}$. By Lemma 2.1,

$$|f(x) - f(y)| \leq \kappa\,|x-y| \tag{2.1}$$

for every $x, y \in \overline{B}$.

Since $A \cap \overline{B}$ has measure zero, there exists a countable cover $\{C_i\}$ of $A \cap \overline{B}$ by open balls such that

$$\sum \mu(C_i) < \frac{\varepsilon}{\kappa^n}. \tag{2.2}$$

By (2.1), the set $f\big(\overline{B} \cap C_i\big)$ is contained in a ball $\widetilde{B}_i$ such that

$$\mu\Big(\widetilde{B}_i\Big) \leq \kappa^n \mu(C_i). \tag{2.3}$$

Now, since

$$f(A \cap \overline{B}) \subset \bigcup f(\overline{B} \cap C_i) \subset \bigcup \widetilde{B}_i,$$

equations (2.2) and (2.3) imply that

$$\mu\big(f(A \cap \overline{B})\big) \leq \sum \mu\Big(\widetilde{B}_i\Big) \leq \kappa^n \sum \mu(C_i) < \varepsilon.$$

Hence $f\big(A \cap \overline{B}\big)$ has measure zero as required. $\square$

LEMMA 2.3. *Given an open set $U \subset \mathbb{R}^m$ with $m < n$, let $f : U \to \mathbb{R}^n$ be C^1. Then $f(U)$ has measure zero in $\mathbb{R}^n$.*

PROOF. Let $\pi : \mathbb{R}^n \to \mathbb{R}^m$ be the projection on the first m coordinates, let $V = \pi^{-1}(U)$, and let $g = f \circ \pi : V \to \mathbb{R}^n$. Then

$$f(U) = g(V \cap \mathbb{R}^m). \tag{2.4}$$

Since $V \cap \mathbb{R}^m$ has measure zero in $\mathbb{R}^n$, (2.4) and Lemma 2.2 imply that $f(U)$ has measure zero as required. $\square$

Although our results are using the Lebesgue measure in $\mathbb{R}^n$, we can extend the measure to a manifold via charts and define sets of measure zero. For a complete study of measures on a manifold, see [**3**].

DEFINITION 2.1. Given a manifold M, let $A \subset M$. Then A has *measure zero* if there exists a collection of charts $\{(U_\alpha, \varphi_\alpha)\}$ on M such that $\{U_\alpha\}$ covers A and $\varphi_\alpha(U_\alpha \cap A)$ has measure zero in $\mathbb{R}^n$ for every α.

Observe that the definition above requires that A have measure zero only with respect to one collection of charts. However, we see from the result below that this implies that A has measure zero with respect to *every* chart. Hence Definition 2.1

is *well-defined*, in the sense that a subset A of a manifold M has measure zero if and only if $\varphi(A \cap U)$ has measure zero in $\mathbb{R}^n$ for every chart (U, φ) on M.

LEMMA 2.4. *Given a manifold M and a subset $A \subset M$ of measure zero, let (V, ψ) be a chart on M. Then $\psi(A \cap V)$ has measure zero in $\mathbb{R}^n$.*

PROOF. Since A has measure zero, there exists a collection $\{U_\alpha, \varphi_\alpha\}$ of charts on M such that $\{U_\alpha\}$ covers A and

$$\mu(\varphi_\alpha(A \cap U_\alpha)) = 0 \tag{2.5}$$

for every α. Since M is second-countable, $\{U_\alpha\}$ has a countable subcover $\{U_i\}$. Since

$$\psi(A \cap V) = \bigcup \psi(A \cap V \cap U_i),$$

it suffices to show that $\psi(A \cap V \cap U)$ has measure zero in $\mathbb{R}^n$ for every $(U, \varphi) \in \{(U_i, \varphi_i)\}$.

To do so, let $(U, \varphi) \in \{(U_i, \varphi_i)\}$ and observe that

$$\psi(A \cap V \cap U) = \left(\psi \circ \varphi^{-1}\right)(\varphi(A \cap V \cap U)). \tag{2.6}$$

Since $\varphi(A \cap V \cap U) \subset \varphi(A \cap U)$, (2.5) implies

$$\mu(\varphi(A \cap V \cap U)) = 0. \tag{2.7}$$

Hence (2.6), (2.7), and Lemma 2.2 imply that $\psi(A \cap V \cap U)$ has measure zero in $\mathbb{R}^n$ as required. $\square$

Therefore, a subset A of a manifold M has measure zero if and only if $\varphi(A \cap U)$ has measure zero in $\mathbb{R}^n$ for every chart (U, φ) on M.

Before proving our main result for this section, we observe the close connection between sets of measure zero and the topology of a manifold.

Lemma 2.5. *Given a manifold M, let $A \subset M$ have measure zero. Then $M \setminus A$ is dense in M.*

Proof. Seeking a contradiction, suppose that $M \setminus A$ is not dense in M. Then there exists a nonempty open set $U \subset A$. It follows that there exists a chart (U, φ) on M such that $\varnothing \neq \varphi(U) = \varphi(A \cap U)$. Since nonempty open sets of $\mathbb{R}^n$ have positive Lebesgue measure, $\varphi(A \cap U)$ has positive measure, a contradiction since A has measure zero. Hence $M \setminus A$ is dense in M. $\square$

We are now ready to prove the main result of this section, which generalizes the result in Lemma 2.3 to arbitrary manifolds. As mentioned earlier, this result will be useful in our proof of Whitney's theorem.

Proposition 2.1. *Given manifolds M^m and N^n with $m < n$, let $f : M \to N$ be C^1. Then $N \setminus f(M)$ is dense in N.*

Proof. By Lemma 2.5, it suffices to show that $f(M)$ has measure zero in N. Let $\{(U_i, \varphi_i)\}$ be a countable collection of charts on M such that $\{U_i\}$ covers M and let (V, ψ) be a chart on N. It suffices to show that $\psi(f(M) \cap V)$ has measure zero in $\mathbb{R}^n$.

Observe that

$$\psi(f(M) \cap V) = \bigcup \left(\psi \circ f \circ \varphi_i^{-1}\right)\left(\varphi_i\big(f^{-1}(V) \cap U_i\big)\right). \tag{2.8}$$

Finally, Lemma 2.3 and equation (2.8) imply that $\psi(f(M) \cap V)$ has measure zero in $\mathbb{R}^n$ as required. $\square$

2. Partitions of Unity

The second-countability condition on the topology of manifolds plays a fundamental role in the proofs of Lemma 2.5 and Proposition 2.1. This condition, which is also a fundamental topological property of $\mathbb{R}^n$, reveals the intimate connection between the structure of manifolds and the structure of $\mathbb{R}^n$. This connection will become even more apparent with Whitney's theorem.

The remainder of this section is devoted to investigating some of the topological properties of manifolds through a list of lemmas. Since the arguments required to prove the following lemmas are purely topological in nature, we will omit them and reference [**3**] and [**6**] for the technical details of the proofs. We include them for the sake of completeness.

LEMMA 2.6. *Let M be a manifold. Then M is a locally compact topological space.*

LEMMA 2.7. *Given a manifold M and an open subset $U \subset M$, let $K \subset U$ be compact. Then there exists an open subset $V \subset M$ with compact closure such that*

$$K \subset V \subset \overline{V} \subset U.$$

DEFINITION 2.2. Let X be a topological space. The *support* of a complex function f on X is the closure of the set

$$\{x \in X : f(x) \neq 0\}.$$

The collection of all continuous complex functions on X whose support is compact is denoted by $C_c(X)$. If V is open in X, then the notation

$$f \prec V$$

will mean that $f \in C_c(X)$, that $0 \le f \le 1$, and that the support of f lies in V.

LEMMA 2.8. *Given a manifold M and a collection $V_1, \dots, V_n$ of open subsets of M, let K be a compact subset of M such that*

$$K \subset V_1 \cup \cdots \cup V_n.$$

Then there exist smooth functions $h_i \prec V_i$ such that

$$h_1 + \cdots + h_n = 1 \tag{2.9}$$

on K.

Because of (2.9), the collection $\{h_1, \dots, h_n\}$ is called a *partition of unity on K*, subordinate to the cover $\{V_1, \dots, V_n\}$.

CHAPTER 3

Embeddings

We are now ready to introduce the main nomenclature in our goal to show that manifolds can be embedded in Euclidean space.

Definition 3.1. Given manifolds M and N with $p \in M$, let $f : M \to N$ be a map of class C^1.

We say that f is *immersive* at p if the linear map df_p is injective. We say that f is *submersive* at p if df_p is surjective.

Definition 3.2. A C^1 map $f : M \to N$ between manifolds M and N is an *immersion* if f is immersive at every point of M. Similiarly, f is a *submersion* if f is submersive at every point of M.

Definition 3.3. A C^1 map $f : M \to N$ between manifolds M and N is an *embedding* if it is an immersion which maps M homeomorphically onto its image.

Geometrically, we view embeddings as instances of one manifold being contained in another. Indeed, it is a result that a subset M of a manifold N is a C^r submanifold if and only if M is the image of a C^r-embedding. For the details, see [**1**]. The result that we are most interested in is Whitney's theorem which states that every n-dimensional manifold embeds into $\mathbb{R}^{2n+1}$. In this direction, we have the following lemma.

LEMMA 3.1. *Let M^n be a compact manifold. Then there exists a C^r embedding of M into $\mathbb{R}^N$ for some $N \in \mathbb{N}$.*

PROOF. For every $p \in M$ there exists a chart (U_p, φ_p) of M at p. By Lemma 2.7, for every $p \in M$, there exists an open set V_p such that

$$p \in V_p \subseteq \overline{V}_p \subseteq U_p.$$

It follows that $\{V_p : p \in M\}$ is an open cover of M and therefore has a finite subcover $\{V_1, \dots, V_m\}$. Re-indexing if necessary, we conclude that

$$M = V_1 \cup \cdots \cup V_m$$

and $\overline{V}_i \subset U_i$ for every $i \in \{1, \dots, m\}$. Lemma 2.8 ensures functions $\lambda_i : M \to [0, 1]$ such that $\operatorname{supp}(\lambda_i) \subset U_i$ and $\lambda_i = 1$ on V_i for every $i \in \{1, \dots, m\}$.

Now, let $\widehat{\varphi}_i : M \to \mathbb{R}^n$ be the map

$$\widehat{\varphi}_i(x) = \begin{cases} \lambda_i(x)\varphi_i(x) & \text{if } x \in U_i; \\ \\ 0 & \text{if } x \notin U_i \end{cases}$$

for every $i \in \{1, \dots, m\}$ and let $F : M \to \mathbb{R}^{m(n+1)}$ be the map

$$F = (\widehat{\varphi}_1, \dots, \widehat{\varphi}_m, \lambda_1, \dots, \lambda_m).$$

It suffices to show that F is injective and that dF_p is injective for every $p \in M$.

To show that F is injective, suppose that $F(x) = F(y)$. Since $\{V_1, \dots, V_m\}$ covers M, there exists an i such that $x \in V_i$. Then $\lambda_i(x) = \lambda_i(y) = 1$ so that $\widehat{\varphi}_i(x) = \varphi(x)$ and $\widehat{\varphi}_i(y) = \varphi(y)$. It follows that $\varphi(x) = \varphi(y)$ so that $x = y$. Hence F is injective.

Finally, to see that dF_p is injective for every $p \in M$, observe that if $p \in U_i$ then $\left(\widehat{\phi}_i, \lambda_i\right)$ is immersive at p. Hence F is an immersion as required. $\square$

We are now ready to prove Whitney's theorem.

THEOREM 3.1 (Whitney). *Let M be a compact n-dimensional manifold. Then there exists a C^r embedding $f : M \to \mathbb{R}^{2n+1}$.*

PROOF. By Lemma 3.1, there exists an embedding $F : M \to \mathbb{R}^N$ for some $N \in \mathbb{N}$. If $N \leq 2n+1$, then we are done, so assume that $N > 2m+1$. We will show that M embeds in $\mathbb{R}^{N-1}$, then by an iteration of the argument we conclude that M will be embedded into $\mathbb{R}^{2n+1}$.

Define the map $G : M \times M \times \mathbb{R} \to \mathbb{R}^N$ by

$$G(x, y, t) = t\left(F(x) - F(y)\right),$$

define the map $H : TM \to \mathbb{R}^N$ by

$$(x, v) \mapsto dF_x(v),$$

and let G^* and H^* be the images of G and H respectively. Note that $M \times M \times \mathbb{R}$ has dimension $2n+1$. Since TM has dimension $2n$ and $N > 2n+1$, Proposition 2.1 implies that there exists an $a \in \mathbb{R}^N \setminus (F^* \cup G^*)$.

Now, let $\pi : \mathbb{R}^N \to \mathbb{R}^{N-1}$ be the linear projection parallel to a. It suffices to show that $\pi \circ F$ is injective and that $d\left(\pi \circ F\right)_p$ is injective for every $p \in M$.

Seeking a contradiction, suppose that $\pi \circ F$ is not injective. Then there exist $x, y \in M$ such that $x \neq y$ and $\pi(F(x)) = \pi(F(y))$. Since π is linear, it follows that

$$\pi(F(x) - F(y)) = 0. \tag{3.1}$$

We observe that $\ker \pi$ consists of all scalar multiples of a, hence (3.1) implies that there exists a $t \in \mathbb{R}$ with $t \neq 0$ such that $F(x) - F(y) = ta$. In other words,

$$\frac{1}{t}\left(F(x) - F(y)\right) = a, \tag{3.2}$$

a contradiction since $a \notin G^*$. Thus $\pi \circ F$ is injective.

Finally, to show that $d(\pi \circ F)_p$ is injective for every p, assume towards a contradiction that there exists a $p \in M$ such that $d\left(\pi \circ F\right)_p$ is not injective. Since $d\left(\pi \circ F\right)_p$ is linear, there exists a nonzero $v \in T_pM$ such that $d\left(\pi \circ F\right)_p(v) = 0$. It follows that

$$\pi \circ dF_p(v) = 0. \tag{3.3}$$

Since the kernel of π consists of all scalar multiples of a, (3.3) implies that there exists a scalar t such that

$$dF_p(v) = ta. \tag{3.4}$$

Since F is an immersion, dF_p is injective and so (3.4) implies that $t \neq 0$. Hence

$$dF_p\left(\frac{v}{t}\right) = a,$$

a contradiction since $a \notin H^*$. Hence $d\left(\pi \circ F\right)_p$ is injective for every $p \in M$ as required. $\square$

Part 2

Minimal Surfaces

CHAPTER 4

Differential Geometry

A *surface* is a two-dimensional manifold. Given our previous work on embeddings, from Whitney's theorem we can consider a surface to be embedded in some $\mathbb{R}^N$. Indeed, we shall consider surfaces as subsets of a Euclidean space. Since we are interested in surfaces that have a differential structure, we restrict ourselves to the study of a particular class of surfaces called *regular parameterized surfaces.* Nevertheless, we note that there exist surfaces, even C^0, that have no differential structure whatsoever (see [**1**]), but these do not have the desirable properties we are looking for.

1. Euclidean Surfaces

Let us now define the surfaces and structures that we are interested in.

Definition 4.1. Let $S \subset \mathbb{R}^n$ be nonempty. Then S is a C^r*-regular parameterized surface* if there exists a C^r-immersion $\mathbf{x} : \Omega \to \mathbb{R}^n$ of an open domain $\Omega \subset \mathbb{R}^2$ into $\mathbb{R}^n$ such that $\mathbf{x}(\Omega) = S$. In this case, S is *parameterized by* $\mathbf{x}$.

From the definition above, we restrict ourselves to the study of regular parameterized surfaces, however, for the sake of brevity we refer to them simply as *surfaces.*

DEFINITION 4.2. For $v, w \in \mathbb{R}^n$, denote the *inner product* of $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$ by the real number

$$v \cdot w = \sum v_i w_i$$

and the *exterior product* of v and w by

$$v \wedge w; \quad v \wedge w \in \mathbb{R}^N, \quad N = \binom{n}{2},$$

where the components of $v \wedge w$ are the determinants

$$\det \begin{bmatrix} v_i & v_j \\ w_i & w_j \end{bmatrix}, \quad 1 \leq i < j \leq n.$$

NOTATION 4.1. Given a surface parameterized by $(\Omega, \mathbf{x})$, define the map $G : \Omega \to \mathscr{M}_2(\mathbb{R})$, where $\mathscr{M}_2(\mathbb{R})$ denotes the family of all 2×2 matrices with real entries, by $G(p) = (d\mathbf{x}_p)^\top (d\mathbf{x}_p)$. Observe that

$$\det G = |D_1\mathbf{x} \wedge D_2\mathbf{x}|^2 = \sum_{1 \leq i < j \leq n} [\det(x_i, x_j)]^2 . \tag{4.1}$$

For the surfaces we are interested in, we see that conditions on G are geometrically significant. In fact, we have the following result, whose proof follows immediately from (4.1) and elementary properties of the rank of a matrix.

LEMMA 4.1. *Let $\mathbf{x} : \Omega \to \mathbb{R}^n$ be a C^r mapping of an open domain $\Omega \subset \mathbb{R}^2$ into $\mathbb{R}^n$. Then at every point of Ω the following are equivalent:*

(*a*) *the vectors $D_1\mathbf{x}$ and $D_2\mathbf{x}$ are independent;*

(*b*) *the Jacobian matrix $D\mathbf{x}$ has rank 2;*

(*c*) *the differential $d\mathbf{x}$ is injective;*

(*d*) *there exist $1 \leq i < j \leq n$ such that $\det d(x_i, x_j) \neq 0$;*

(*e*) $D_1\mathbf{x} \wedge D_2\mathbf{x} \neq 0$;

(*f*) $\det G > 0$.

For a surface parameterized by $\mathbf{x}$, the conditions of Lemma 4.1 hold everywhere. We are now interested in showing that every surface has a natural manifold structure. For the rest of this section, let us fix $r \geq 1$ and assume that our surface S is C^r. Also, unless otherwise defined, we reserve the symbols $\mathbf{x} = (x_1, \ldots, x_n)$ and Ω for the parametrization and domain of parametrization respectively of our surface whose parametrization may not be explicitly given.

Definition 4.3. Given a surface S, let $\phi : \widetilde{\Omega} \to \Omega$ be a C^r-diffeomorphism of an open domain $\widetilde{\Omega} \subset \mathbb{R}^2$ onto Ω. Then the surface $\widetilde{S}$ parameterized by $\widetilde{\mathbf{x}} = \mathbf{x} \circ \phi$ is *obtained from S by a change of parameter.* A property of S is *independent of parameters* if it holds at corresponding points of all surfaces $\widetilde{S}$ obtained from S by a change of parameter.

It is the object of differential geometry to study those properties which are independent of parameter. Since we are interested in these properties, we will often find that there are several choices of parameters that will often be convenient to use.

Definition 4.4. A surface S is defined *explicitly* if there exists a C^r map $\mathbf{f} : \Omega \to \mathbb{R}^{n-2}$ such that $\mathbf{x} = (I, bf)$ where I is the identity on Ω. In this case, the map $\mathbf{x}$ is said to be *explicit.*

Of course, if there exist $1 \leq i < j \leq n$ such that one of the maps (x_i, x_j) and (x_j, x_i) are the identity on Ω, a relabeling of the axes immediately gives us an explicit map $\widetilde{\mathbf{x}}$ defining a surface $\widetilde{S}$, which is simply our original surface up to a translation. For this reason, we assume that all explicit maps $\mathbf{x}$ are of the form $\mathbf{x} = (I, \mathbf{f})$ where I and $\mathbf{f}$ are as in Definition 4.4, and we assume that appropriate relabeling is done without further comment.

Observe that not every surface can be expressed in explicit form, since its original parameter is not required to be injective. We do, however, have the following important lemma:

Lemma 4.2. *Given a surface S, let $p \in \Omega$. Then there exists a neighborhood $\Delta \subset \Omega$ of p such that the the surface Σ obtained by restricting $\mathbf{x}$ to Δ may be expressed explicitly.*

Proof. By Lemma 4.1 and by relabeling the axes if necessary, $\det D\,(x_1, x_2) \neq 0$. By the inverse function theorem, there exists a neighborhood Δ of p such that the restriction of the map (x_1, x_2) to Δ is a C^r-diffeomorphism. Now, letting $\phi = (x_1, x_2)^{-1}$ we obtain an explicit parameterization $\mathbf{x} \circ \phi$ of Σ. $\square$

Lemma 4.2 has several important consequences. The first is that when studying the local behavior of a surface we may assume that the surface is given in explicit form. The second is that every immersion is locally injective. It is this second property that will give insight into the manifold structure of surfaces.

Definition 4.5. Let S be a surface and let $\mathscr{A}$ be the collection of all pairs $(\Delta, \mathbf{y})$ consisting of an open domain $\Delta \subset \mathbb{R}^2$ and an embedding $\mathbf{y} : \Delta \to S$. Then $\mathscr{A}$ is the *natural atlas* of S.

That the natural atlas of a surface is indeed an atlas is implied by Lemma 4.2. In fact, this will be the atlas that we are interested in as it will induce a C^r-manifold structure on S.

PROPOSITION 4.1. *Given a surface S with natural atlas $\mathscr{A}$, let $(\Omega, \mathbf{x}), (\Delta, \mathbf{y}) \in \mathscr{A}$ such that $\mathbf{x}(\Omega) \cap \mathbf{y}(\Delta) = W \neq \varnothing$. Then the change of parameter $h = \mathbf{x}^{-1} \circ \mathbf{y} : \mathbf{y}^{-1}(W) \to \mathbf{x}^{-1}(W)$ is a C^r-diffeomorphism.*

PROOF. Since h is obtained by a composition of homeomorphisms, we trivially have that h and h^{-1} are homeomorphisms. It now suffices to show that h and h^{-1} are C^r.

To do so, let $r \in \mathbf{y}^{-1}(W)$ and put $q = h(r)$. Since $\mathbf{x}$ is an embedding, we may assume, by renaming the axis if necessary, that $\det d(x_1, x_2)_q \neq 0$. Let $F : \Omega \times \mathbb{R}^{n-2} \to \mathbb{R}^n$ be the map

$$(u, v, t_3, \ldots, t_n) \mapsto \mathbf{x}(u, v) + \sum_{i=3}^{n} t_i e_i$$

where $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$. Then F is a C^r map, $F|_{\Omega \times \{0\}^{n-2}} = \mathbf{x}$, and

$$\det(dF_q) = \det \begin{bmatrix} D_1 x_1 & D_2 x_1 & 0 & \cdots & 0 \\ D_1 x_2 & D_2 x_2 & 0 & \cdots & 0 \\ D_1 x_3 & D_2 x_3 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ D_1 x_n & D_2 x_n & 0 & \cdots & 1 \end{bmatrix} = \det d(x_1, x_2)_q \neq 0.$$

By the inverse function theorem, there exists a neighborhood U of $\mathbf{x}(q)$ such that F^{-1} exists and is C^r on U.

Since $\mathbf{y}$ is continuous, there exists a neighborhood $V \subset \Delta$ of r such that $\mathbf{y}(V) \subset U$. Now, observe that, when restricted to V, the map $h|_V = F^{-1} \circ \mathbf{y}|_V$ is a composition of C^r maps and is therefore C^r. Hence h is C^r. Similarly, h^{-1} is C^r as required. □

COROLLARY 4.1. *The natural atlas of a surface induces a C^r-manifold structure on the surface.*

With Corollary 4.1, we may view every surface as a C^r-manifold and therefore apply all the notions of tangent spaces and differentiable curves as defined in Section 1. These ideas will be integral in our development of the local theory of curvature of surfaces.

2. Mean Curvature

We will develop the local theory of curvature on a surface S by investigating curves in S. Since we are interested in local properties of S, we will assume throughout that its parameter map is injective, which is allowed by Lemma 4.2. Recall from Section 1 that for a surface S with $p \in S$, the tangent space T_pS is isomorphic to the collection of all vectors $\alpha'(0)$ such that α is a curve in S at p.

PROPOSITION 4.2. *Let S be a surface with $p \in S$ and let $\alpha : (-\varepsilon, \varepsilon) \to S$ be a curve in S at p. Then there exists a unique curve $\gamma : (-\varepsilon, \varepsilon) \to \Omega$ such that $\alpha = \mathbf{x} \circ \gamma$.*

PROOF. Put $\gamma = \mathbf{x}^{-1} \circ \alpha$. Then $\alpha = \mathbf{x} \circ \left(\mathbf{x}^{-1} \circ \alpha\right) = \mathbf{x} \circ \gamma$. Furthermore, if $\alpha = \mathbf{x} \circ \gamma_0$ as well, then

$$\gamma = \mathbf{x}^{-1} \circ (\mathbf{x} \circ \gamma) = \mathbf{x} \circ \alpha = \mathbf{x}^{-1} \circ (\mathbf{x} \circ \gamma_0) = \gamma_0.$$

Hence $\alpha = \mathbf{x} \circ \gamma$ for a unique γ as required. $\square$

Proposition 4.2 establishes that in order to understand curves in a surface S it suffices to understand curves in its parameter domain Ω. Since every curve γ obviously defines a curve $\alpha = \mathbf{x} \circ \gamma$ in S, Proposition 4.2 gives a natural one-to-one correspondence between curves in S and curves in Ω.

PROPOSITION 4.3. *Given a surface S, let $q \in S$. Then T_qS is isomorphic to $\mathbb{R}^2$ and to $T_p\Omega$.*

PROOF. Since $T_p\Omega$ is isomorphic to $\mathbb{R}^2$ and since $\mathbf{x}$ is an embedding, it suffices to show that the differential $d\mathbf{x}_p : T_p\Omega \to T_qS$ is surjective.

To do so, let $\alpha'(0) \in T_qS$. By Proposition 4.2, there exists a unique curve γ in Ω such that $\alpha = \mathbf{x} \circ \gamma$. It follows that

$$d\mathbf{x}_p\gamma'(0) = \sum \gamma_i'(0)D_i\mathbf{x}(q) = \alpha'(0).$$

Hence $d\mathbf{x}_p$ is surjective as required. $\square$

From Proposition 4.3, we refer to T_pS as the *tangent plane*.

PROPOSITION 4.4. *Given a surface S with $p \in S$, let α be a curve in S at p and let γ be the unique curve in Ω such that $\alpha = \mathbf{x} \circ \gamma$. Then*

$$|\alpha'(0)|^2 = \sum_{i,j=1}^{2} g_{ij}\gamma_i'(0)\gamma_j'(0).$$

PROOF. Observe that

$$\begin{aligned}|\alpha'(0)|^2 &= \alpha'(0) \cdot \alpha'(0)\\ &= d(\mathbf{x} \circ \gamma)_0 d(\mathbf{x} \circ \gamma)_0\end{aligned}$$

$$= \sum_{i,j=1}^{2} \gamma_i'(0)\gamma_j'(0) \left[D_i|_{\gamma(0)}\mathbf{x} \cdot D_j|_{\gamma(0)}\mathbf{x} \right]$$

$$= \sum_{i,j=1}^{2} g_{ij}\gamma_i'(0)\gamma_j'(0)$$

as required. □

Proposition 4.4 shows that the square of a tangent vector may be expressed as a quadratic form in the corresponding tangent vector with the matrix G as defined in Notation 4.1. This quadratic form is referred to as the *first fundamental form* of the surface S.

LEMMA 4.3. *Given a symmetric $n \times n$ matrix A, let $\lambda_1 \geq \cdots \geq \lambda_n$ be the eigenvalues of A. Then*

$$\max_{\mathbf{x} \neq 0} \frac{\mathbf{x}^T A\mathbf{x}}{\mathbf{x}^T\mathbf{x}} = \lambda_1$$

and the maximum is attained at any eigenvector of A corresponding to λ_1.

PROOF. Since A is symmetric it may be diagonalized into $A = P^T \Lambda P$ where P is an orthogonal matrix. Put $\mathbf{y} = P\mathbf{x}$ for $\mathbf{x} \neq 0$ and compute

$$\begin{aligned} \frac{\mathbf{x}^T A\mathbf{x}}{\mathbf{x}^T\mathbf{x}} &= \frac{\mathbf{y}^T \Lambda \mathbf{y}}{\mathbf{y}^T\mathbf{y}} \\ &= \frac{\lambda_1 y_1^2 + \cdots + \lambda_n y_n^2}{Y_1^2 + \cdots + y_n^2} \\ &\leq \frac{\lambda_1 y_1^2 + \cdots + \lambda_1 y_n^2}{y_1^2 + \cdots + y_n^2} \\ &= \lambda_1. \end{aligned}$$

It follows that

$$\max_{\mathbf{x} \neq 0} \frac{\mathbf{x}^T A\mathbf{x}}{\mathbf{x}^T\mathbf{x}} \leq \lambda_1.$$

Furthermore, if $\mathbf{x}$ is an eigenvector corresponding to λ_1, then

$$\frac{\mathbf{x}^T A\mathbf{x}}{\mathbf{x}^T\mathbf{x}} = \lambda_1$$

as required. $\square$

LEMMA 4.4. *Let A and B be $n \times n$ matrices where A is symmetric and B is positive definite. Then*

$$\max_{\mathbf{x}\neq 0} \frac{\mathbf{x}^T A\mathbf{x}}{\mathbf{x}^T B\mathbf{x}} = \mu$$

where μ is the largest eigenvalue of AB^{-1}.

PROOF. Observe that

$$\max_{\mathbf{x}\neq 0} \frac{\mathbf{x}^T A\mathbf{x}}{\mathbf{x}^T B\mathbf{x}} = \max_{\mathbf{x}\neq 0} \frac{\mathbf{x}^T A\mathbf{x}}{\left(\mathbf{x}^T B^{1/2}\right)\left(B^{1/2}\mathbf{x}\right)} = \max_{\mathbf{y}\neq 0} \frac{\mathbf{y}^T B^{-1/2} A B^{-1/2}\mathbf{y}}{\mathbf{y}^T\mathbf{y}}.$$

By Lemma 4.3, the last expression equals the maximum eigenvalue of $B^{-1/2}AB^{-1/2}$. Since $B^{-1/2}AB^{-1/2}$ and AB^{-1} have the same eigenvalues, it follows that

$$\max_{\mathbf{x}\neq 0} \frac{\mathbf{x}^T A\mathbf{x}}{\mathbf{x}^T B\mathbf{x}} = \mu$$

as required. $\square$

DEFINITION 4.6. Given a surface S with $p \in S$, let α be a curve in S at p. Then the *length* of alpha is the real number

$$L(\alpha) = \int_{-\varepsilon}^{\varepsilon} |\alpha'(t)|\; dt.$$

We say α is *regular* if $\alpha' > 0$ everywhere. In this case, the map $s : (-\varepsilon, \varepsilon) \to (0, L(\alpha))$ given by

$$s(t) = \int_{-\varepsilon}^{t} |\alpha'(\tau)|\; d\tau$$

is a diffeomorphism. The map $\widehat{\alpha} = \alpha \circ s^{-1}$ is the *parameterization of* α *with respect to arclength*. Whenever α is C^2 and regular, the *curvature vector* is the vector $\widehat{\alpha}'' \circ s$.

We would like to describe the totality of curvature vectors with respect to all regular C^2 curves in S evaluated at the point $q = \mathbf{x}(p)$. By the Projection theorem (see [**6**]), every vector is determined by its projections in T_qS and $T_qS^\perp$. We will examine the projection of the curvature vectors into $T_q^\perp$.

Proposition 4.5. *Given a surface* S *with* $q = \mathbf{x}(p) \in S$, *let* α *be a* C^2 *regular curve in* S *at* q *and let* $N \in T_qS$. *Then*

$$\widehat{\alpha}''(s(0)) \cdot N = \frac{\sum b_{ij}(N)\gamma_i'(0)\gamma_j'(0)}{\sum g_{ij}\gamma_i'(0)\gamma_j'(0)} \tag{4.2}$$

where γ *is the unique curve such that* $\alpha = \mathbf{x} \circ \gamma$ *and*

$$b_{ij}(N) = D_{ij}\mathbf{x}(q) \cdot N.$$

Proof. Observe that

$$\begin{aligned} \widehat{\alpha}' \circ s &= D|_{s}\alpha \circ s^{-1} = \frac{1}{s'}\alpha' \\ &= \frac{1}{s'} D\mathbf{x} \circ \gamma = \frac{1}{s'} \sum \gamma_i' \, D_i|_{\gamma}\mathbf{x}. \end{aligned} \tag{4.3}$$

Since T_qS is spanned by $D_1\mathbf{x}(p)$ and $D_2\mathbf{x}(p)$, (4.3) implies

$$\begin{aligned} \widehat{\alpha}''(s(0)) \cdot N &= \frac{1}{s'(0)} \, D|_{s(0)}\widehat{\alpha}' \cdot N \\ &= \frac{1}{[s'(0)]^2} \sum \gamma_i'(0) \left[D|_0 \, D_i|_{\gamma}\mathbf{x} \right] \\ &= \frac{\sum b_{ij}(N)\gamma_i'(0)\gamma_j'(0)}{\sum g_{ij}\gamma_i'(0)\gamma_j'(0)} \end{aligned}$$

as required. $\square$

The numerator on the right side of (4.2) is a quadratic form in the tangent vector $\gamma'(0)$ whose matrix $b_{ij}(N)$ only depends on the point on the surface and the normal N. This is called the *second fundamental form* of S with respect to N. Since the right hand side of (4.2) depends on α only to the extent of the tangent vector to α at the point, we may define a function $k_q : T_qS \times T_qS^{\perp} \to \mathbb{R}$ by

$$k_q(v, N) = \widehat{\alpha}''(s(0))$$

where α is the curve in S at q associated with v. From (4.2), we may write

$$k_q(v, N) = \frac{v^T B(N) v}{v^T G v} \tag{4.4}$$

where $B(N)$ is second fundamental form of S with respect to N and G is the first fundamental form of S. Using the function k_q, we can define curvature of S at q as follows.

Definition 4.7. Given a surface S with $q \in S$, let $N \in T_qS^{\perp}$. Then the *principal curvatures* of S at q with respect to N are the quantities

$$k_1(N) = \max_v k_q(v, N), \quad k_2(N) = \min_v k_q(v, N).$$

The *mean curvature* of S at q with respect to N is the average value

$$H(N) = \frac{k_1(N) + k_2(N)}{2}.$$

By Lemma 4.4, $k_1(N)$ and $k_2(N)$ are given by the eigenvalues of the matrix $B(N)G^{-1}$. That is, $k_1(N)$ and $k_2(N)$ are the roots of the equation

$$\det\left(B(N) - \lambda G\right) = 0,$$

which we may rewrite as

$$\det G\lambda^2 - (g_{22}b_{11}(N) + g_{11}b_{22}(N) - 2g_{12}b_{12}(N))\,\lambda + \det B(N) = 0. \tag{4.5}$$

This gives us an expression for the mean curvature of S at q with respect to N:

$$H(N) = \frac{g_{22}b_{11}(N) + g_{11}b_{22}(N) - 2g_{12}b_{12}(N)}{2\det G}. \tag{4.6}$$

Thus H is linear in $T_qS^\perp$ and, by the Riesz Lemma (see [**6**]), there exists a unique vector $\mathcal{H}$ such that $H(N) = \mathcal{H} \cdot N$ for every $N \in T_qS^\perp$. We call $\mathcal{H}$ the *mean curvature vector* of S at q.

CHAPTER 5

Minimal Surfaces

In this chapter, we will develop the notion of what it means for a surface to be minimal. We will do so by investigating the problem which historically led to the theory of minimal surfaces.

1. The Variational Problem

Given a surface S, let Γ be a closed curve in Ω which bounds a subdomain Δ and let Σ be the surface parameterized by the restriction of $\mathbf{x}$ to Δ. Suppose that the area of Σ is less than or equal to the area of every surface $\widetilde{\Sigma}$ parameterized by $(\Delta, \widetilde{\mathbf{x}})$ with $\mathbf{x} = \widetilde{\mathbf{x}}$ on Γ^*. Then what are the properties of Σ? In particular, can we say anything about the mean curvature vector of Σ?

It turns out that Σ will be a minimal surface and its mean curvature vector will vanish everywhere. Let us give a formal explanation and properly define this interesting problem.

Definition 5.1. Let S be a surface, let Δ be a subdomain of Ω with $\overline{\Delta} \subset \Omega$, and let Σ be the surface parameterized by the restriction of $\mathbf{x}$ to Δ. Then the *area* of Σ is the real number

$$\mathcal{A}(\Sigma) = \iint_{\Delta} \sqrt{\det G}.$$

If $f : \overline{\Delta} \to \mathbb{R}$ is C^1, then the *integral of f with respect to surface area* is the real number

$$\iint_{\Sigma} f\, d\mathcal{A} = \iint_{\Delta} f\sqrt{\det G}.$$

Now, let $N : \Sigma \to \mathbb{R}^n$ be C^1 such that $N(p) \in T_{\mathbf{x}(p)}S$ for every $p \in \Omega$ and let h be a C^2 real-valued function on Ω. For every real number λ, let S_λ be the surface parameterized by the map

$$\mathbf{x}_\lambda = \mathbf{x} + \lambda h N.$$

Here, we say that $\mathbf{x}_\lambda$ is a *normal variation* of $\mathbf{x}$.

THEOREM 5.1. *There exists an $\varepsilon > 0$ such that the map $A : (-\varepsilon, \varepsilon) \to \mathbb{R}$ given by*

$$A(\lambda) = \mathcal{A}(\Sigma_\lambda)$$

is well defined and

$$A'(0) = -2\iint_{\Sigma} H(N)h\, d\mathcal{A}.$$

PROOF. Let g_{ij}^λ be the entries of the first fundamental form of S_λ. Then

$$g_{ij}^\lambda = g_{ij} - 2\lambda b_{ij}(N) + \lambda^2 c_{ij}$$

where c_{ij} is continuous in Ω. It follows that

$$\det G_\lambda = a_0 + a_1\lambda + a_2\lambda^2$$

where

$$\begin{aligned} a_0 &= \det G \\ a_1 &= -2h\left(g_{11}b_{22}(N) + g_{22}b_{11}(N) - 2g_{12}b_{12}(N)\right) \end{aligned}$$

and a_2 is continuous in Ω and λ.

Since $\det G > 0$ everywhere, $\det G$ has a positive minimum on $\overline{\Delta}$ so that there exists an $\varepsilon > 0$ such that $\det G_\lambda > 0$ on $\overline{\Delta}$ whenever $|\lambda| < \varepsilon$. That is, for $|\lambda| < \varepsilon$, the conditions of Lemma 4.1 hold for S_λ so that A is well defined on $(-\varepsilon, \varepsilon)$.

The Taylor series expansion of the determinant function ensures an $M > 0$ such that

$$\left| \det G_\lambda - \left(\sqrt{a_0} + \frac{a_1}{2\sqrt{a_0}} \right) \lambda \right| < M\lambda^2. \tag{5.1}$$

Integrating (5.1) over Δ ensures an $M_1 > 0$ such that

$$\left| \frac{A(\lambda) - A(0)}{\lambda} - \iint_\Delta \frac{a_1}{2\sqrt{a_0}} \right| < M_1 \lambda. \tag{5.2}$$

Now, by (4.6),

$$\frac{a_1}{2\sqrt{a_0}} = H(N) \tag{5.3}$$

and combining (5.2) with (5.3) and letting $\lambda \to 0$ gives

$$A'(0) = -2 \iint_\Delta H(N)\, d\mathcal{A} \tag{5.4}$$

as required. □

Corollary 5.1. *If S minimizes area, then its mean curvature vanishes everywhere.*

Proof. Seeking a contradiction, suppose that the mean curvature does not vanish everywhere. Then there exists a point $p \in \Delta$ with $q = \mathbf{x}(p)$ and a normal $N_0 \in T_q S^\perp$ such that $H(N_0) \neq 0$. Assume that $H(N_0) > 0$. By Lemma 2.2 in [**5**], there exists a neighborhood V_1 of p and a C^1 map $N : V_1 \to \mathbb{R}^n$ such that $N(a) \in T_{\mathbf{x}(a)}S$ for every a and $N(p) = N_0$. It follows that $H(N) > 0$ on a neighborhood V_2 of p where $V_2 \subset V_1$.

Now, pick h so that $h(p) > 0$, $h \geq 0$ everywhere, and $h = 0$ on V_2^c. Then the integral on the right of (5.4) is strictly positive. However, if V_2 is small enough such that $V_2 \subset \Delta$, then $\mathbf{x}_\lambda = \mathbf{x}$ on Γ^* so that Σ_λ is a surface with the same boundary as Σ. Since Σ minimizes area by hypothesis, $A(\lambda) \geq A(0)$ for every λ. Hence $A'(0)$ so that the integral to the right of (5.4) is 0, a contradiction. Hence the mean curvature vanishes everywhere. □

It is Corollary 5.1 that motivates our definition of a minimal surface.

Definition 5.2. A surface S is *minimal* if its mean curvature vector vanishes at every point.

Notice that if a surface S minimizes area in the sense of the situation described at the beginning of this chapter, then Corollary 5.1 implies that the surface is minimal. However, if a surface is minimal, then Theorem 5.1 only guarantees that 0 is a critical point of A, and not necessarily a minimum. This is an interesting fact and there exist minimal surfaces that do not minimize area. See [**4**] for the construction.

Note that by (4.6) that the mean curvature vector $\mathcal{H}$ of a surface S vanishes at a point p if and only if $H(N) = 0$ for every $N \in T_{\mathbf{x}(p)}S$. Therefore, minimal surfaces are characterized in terms of their first and second fundamental forms by the equation

$$g_{22}b_{11}(N) + g_{11}b_{22}(N) - 2g_{12}b_{12}(N) = 0. \tag{5.5}$$

That is, a surface S is minimal if and only if (5.5) holds at all points for every normal N.

2. The Minimal Surface Equation

For the rest of this section, we will investigate the properties of surfaces of class C^2 given in explicit form. This is not a restriction from a local prospective, since every surface may be locally represented explicitly by Lemma 4.2. For our explicit map $\mathbf{x} = (I, \mathbf{f})$, we can make the calculations

$$D_1\mathbf{x} = (1, 0, D_1 f_3, \ldots, D_1 f_n), \quad D_2\mathbf{x} = (0, 1, D_2 f_3, \ldots, D_2 f_n) \tag{5.6}$$

and

$$\begin{aligned} g_{11} &= 1 + \sum (D_1 f_i)^2 & g_{12} &= \sum D_1 f_i \cdots D_2 f_i \\ g_{21} &= g_{12} & g_{22} &= 1 + \sum (D_2 f_i)^2, \end{aligned} \tag{5.7}$$

where the g_{ij}'s are the entries of the first fundamental form.

If we further suppose that S is minimal, then a computation in [**5**] shows that (5.5) takes the form

$$\left(1 + |D_2\mathbf{f}|^2\right) D_{11}\mathbf{f} - 2\,(D_1\mathbf{f} \cdot D_2\mathbf{f})\, D_{12}\mathbf{f} + \left(1 + |D_1\mathbf{f}|^2\right) D_{22}\mathbf{f} = 0. \tag{5.8}$$

Equation (5.8) is the *minimal surface equation* for explicit minimal surfaces. By Lemma 4.2, every minimal surface provides local solutions to (5.8). This equation allows us to give some interesting examples of minimal surfaces.

EXAMPLE 5.1 (The Plane). We note that any affine linear function satisfies (5.8), implying that the plane is indeed minimal.

EXAMPLE 5.2 (The Catenoid). The *Catenoid* (Figure 1) may be represented explicitly by

$$f(x_1, x_2) = \cosh^{-1} \sqrt{x_1^2 + x_2^2}. \tag{5.9}$$

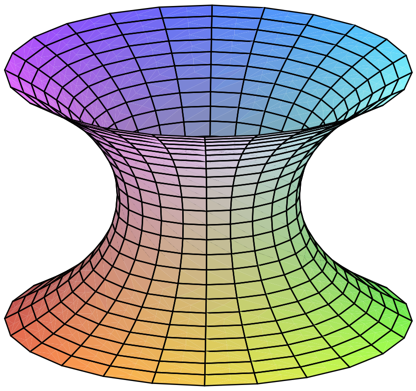

FIGURE 1. The Catenoid

EXAMPLE 5.3 (The Helicoid). The *Helicoid* (Figure 2) may be represented explicitly by

$$f(x_1, x_2) = \tan^{-1} \frac{x_2}{x_1}. \tag{5.10}$$

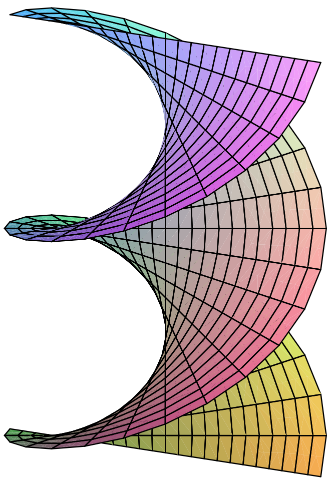

FIGURE 2. The Helicoid

Example 5.4 (Scherk's Surface). *Scherk's Surface* (Figure 3) may be represented explicitly by

$$f(x_1, x_2) = \log \frac{\cos x_2}{\cos x_1}. \tag{5.11}$$

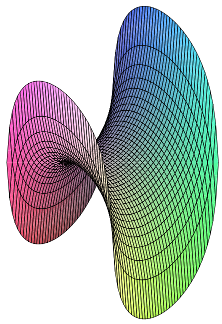

Figure 3. Scherk's Surface

Each of the equations (5.10), (5.9), and (5.11) satisfy the minimal surface equation (5.8). Though we have not specified the domain of definition for these surfaces, we observe that none of them are defined in the whole plane. This is not a coincidence. Bernstein's theorem, which we shall prove later, states that for $n = 3$, the only solution to (5.8) defined in all $\mathbb{R}^2$ is the plane.

Before concluding this section, we will derive a another form of the minimal surface equation for explicitly defined surfaces that will be of use to us later. We begin with a surface S parameterized by and explicit map $\mathbf{x} = (I, \mathbf{f})$ and adopt the notation

$$p = D_1\mathbf{f}, \quad q = D_2\mathbf{f}, \quad r = D_{11}\mathbf{f}, \quad s = D_{12}\mathbf{f}, \quad t = D_{22}\mathbf{f}, \quad W = \sqrt{\det G}. \tag{5.12}$$

Then the minimal surface equation (5.8) takes the form

$$\left(1 + |q|^2\right) r - 2\,(p \cdot q)\, s + \left(1 + |p|^2\right) t = 0. \tag{5.13}$$

Furthermore, we may rewrite (5.6) as

$$g_{11} = 1 + |p|^2, \quad g_{12} = g_{21} = p \cdot q, \quad g_{22} = 1 + |q|^2 \tag{5.14}$$

so that

$$W^2 = 1 + |p|^2 + |q|^2 + |p|^2 |q|^2 - (p \cdot q)^2 . \tag{5.15}$$

Now, (5.13) implies

$$\begin{aligned} & D_1 \left(\frac{1 + |q|^2}{W} \right) - D_2 \left(\frac{p \cdot q}{W} \right) \\ & = \frac{1}{W^3} \left[(p \cdot q)\, q - \left(1 + |q|^2 \right) p \right] \cdot \left[\left(1 + |q|^2 \right) r - 2\,(p \cdot q)\, s + \left(1 + |p|^2 \right) t \right] \\ & = 0 \end{aligned} \tag{5.16}$$

and similarly

$$D_1 \left(\frac{p \cdot q}{W} \right) - D_2 \left(\frac{1 + |p|^2}{W} \right) = 0. \tag{5.17}$$

Equations (5.16) and (5.17) will be of great importance to us, for together they imply that the two equations

$$\begin{aligned} D_1 \left(\frac{1 + |q|^2}{W} \right) &= D_2 \left(\frac{p \cdot q}{W} \right) \\ D_1 \left(\frac{p \cdot q}{W} \right) &= D_2 \left(\frac{1 + |p|^2}{W} \right) \end{aligned} \tag{5.18}$$

are satisfied by every explicit solution to the minimal surface equation (5.13).

It may seem that the identities in (5.16) and (5.17) have been introduced arbitrarily, however they arise in a quite natural setting. We may make a variation on S by putting

$$\widetilde{\mathbf{f}} = \mathbf{f} + \lambda \mathbf{h} \tag{5.19}$$

where $\lambda \in \mathbb{R}$ and $\mathbf{h} : \Omega \to \mathbb{R}^{n-2}$ is a C^1 map. Adopting the notation (5.12) to our new surface, we obtain

$$\widetilde{p} = p + \lambda D_1 \mathbf{h}, \quad \widetilde{q} = q + \lambda D_2 \mathbf{h}, \quad \widetilde{W} = \sqrt{\det \widetilde{G}} \tag{5.20}$$

so that

$$\widetilde{W}^2 = W^2 + 2\lambda X + \lambda^2 Y \tag{5.21}$$

where

$$X = \left[\left(1 + |q|^2\right) p - (p \cdot q)\, q\right] \cdot D_1 \mathbf{h} + \left[\left(1 + |p|^2\right) q - (p \cdot q)\, p\right] \cdot D_2 \mathbf{h} \tag{5.22}$$

and Y is continuous in Ω. By Taylor's theorem, (5.21) gives

$$\begin{aligned} \widetilde{W} &= \widetilde{W}(0) + \widetilde{W}'(0)\lambda + O\big(\lambda^2\big) \\ &= W + \frac{X}{W}\lambda + O\big(\lambda^2\big) \end{aligned} \tag{5.23}$$

where $O\big(\lambda^2\big)$ are terms in λ^2 and higher for λ small enough.

Now, as in the beginning of this section, let Γ be a closed curve in Ω bounding a subdomain $\Delta \subset \Omega$, let Σ be the restriction of $\mathbf{x}$ to Δ, and assume that Σ minimizes area among all surfaces with the same boundary. Then, for every $\mathbf{h}$ such that $\mathbf{h} = 0$ on Γ^*, (5.23) implies

$$\iint_\Delta \widetilde{W} \geq \iint_\Delta W \tag{5.24}$$

so that

$$\iint_\Delta \frac{X}{W} = 0. \tag{5.25}$$

Combining (5.12), (5.22), and (5.25), integrating by parts, and using the fact that $h = 0$ on Γ^*, we obtain the relation

$$\iint_\Delta \left[D_1 \left[\frac{1+|q|^2}{W} p - \frac{p \cdot q}{W} q \right] + D_2 \left[\frac{1+|p|^2}{W} q - \frac{p \cdot q}{W} p \right] \right] h = 0. \tag{5.26}$$

By using the same argument as in the proof of Corollary 5.1 on the integrand of the integral in (5.26), we find that the equation

$$D_1 \left[\frac{1+|q|^2}{W} p - \frac{p \cdot q}{W} q \right] + D_2 \left[\frac{1+|p|^2}{W} q - \frac{p \cdot q}{W} p \right] = 0 \tag{5.27}$$

holds everywhere. By the linearity of the D_i's in (5.27) and applying the product rule, we see that (5.27) implies that

$$\begin{aligned} W^{-1} \left[\left(1+|q|^2\right) r - 2\,(p \cdot q)\, s + \left(1+|p|^2\right) t \right] & \\ + \left[D_1 \left(\frac{1+|q|^2}{W} \right) - D_2 \left(\frac{p \cdot q}{W} \right) \right] p & \\ + \left[D_2 \left(\frac{1+|p|^2}{W} \right) - D_1 \left(\frac{p \cdot q}{W} \right) \right] q &= 0 \end{aligned} \tag{5.28}$$

The first term of (5.28) is the minimal surface equation, which vanishes by (5.13). Furthermore, the coefficients of p and q in (5.28) vanish by (5.18). Hence our original identities (5.16) and (5.17) arise quite naturally. As we will see in the next section, they will provide a connection of minimal surface theory to the study of holomorphic functions.

CHAPTER 6

Complex Analysis

In this chapter, we will show that there exists a strong connection between the study of minimal surfaces and the study of holomorphic functions. We begin by showing how this connection is made and then provide a brief review of some of the basic properties of holomorphic functions.

1. Isothermal Parameters

Since we are studying properties of a surface that are independent of parameters, it is often convenient to choose a parameter of our surface which makes computations easier. In particular, we are interested in choosing a parameter in such a way that geometric properties of our surface is reflected in our original domain. For instance, one condition that is useful is that our parameter $\mathbf{x}$ preserves angles between curves on the surface and angles between corresponding curves in Ω. This will motivate our definition of isothermal parameters.

Definition 6.1. Given a surface S, its parameterization $\mathbf{x}$ is an *isothermal parameterization* of S if the first fundamental form $G = (g_{ij})$ of S satisfies

$$G = \begin{bmatrix} \lambda^2 & 0 \\ 0 & \lambda^2, \end{bmatrix} \tag{6.1}$$

or, equivalently,

$$g_{ij} = \lambda^2 \delta_{ij} \tag{6.2}$$

where $\lambda : \Omega \to \mathbb{R}$.

Parameterizing a surface S in isothermal parameters considerably simplifies some of our previous computations. For example, (6.1) and (6.2) imply that

$$\det G = g_{11}^2 = \lambda^4. \tag{6.3}$$

Furthermore, our formula for mean curvature (4.6) becomes

$$H(N) = \frac{b_{11}(N) + b_{22}(N)}{2}, \tag{6.4}$$

which allows us to write the minimal surface equation (5.5) as

$$b_{11}(N) + b_{22}(N) = 0. \tag{6.5}$$

The following lemma also allows us to make a natural connection between surfaces given in isothermal parameters and holomorphic functions.

Lemma 6.1. *Let S be a surface with an isothermal parameterization $\mathbf{x}$. Then*

$$\Delta \mathbf{x} = 2\lambda^2 \mathcal{H} \tag{6.6}$$

where $\mathcal{H}$ is the mean curvature vector of S.

Proof. The definition of the first fundamental form and equation (6.1) allow us to write

$$\langle D_1\mathbf{x}, D_1\mathbf{x}\rangle = \langle D_2\mathbf{x}, D_2\mathbf{x}\rangle, \quad \langle D_1\mathbf{x}, D_2\mathbf{x}\rangle = 0. \tag{6.7}$$

Applying D_1 to the first equation in (6.7) and D_2 to the second gives

$$\langle D_{11}\mathbf{x}, D_1\mathbf{x}\rangle = \langle D_{12}\mathbf{x}, D_2\mathbf{x}\rangle = -\langle D_{22}\mathbf{x}, D_1\mathbf{x}\rangle \tag{6.8}$$

so that

$$\langle \Delta \mathbf{x}, D_1 \mathbf{x} \rangle = (D_{11}\mathbf{x} + D_{22}\mathbf{x}) \cdot D_1 \mathbf{x} = 0. \tag{6.9}$$

Similarly, applying D_2 to the first equation in (6.7) and D_1 to the second gives

$$\langle \Delta \mathbf{x}, D_2 \mathbf{x} \rangle = 0. \tag{6.10}$$

Now, since $D_1\mathbf{x}$ and $D_2\mathbf{x}$ span the tangent plane of S at each point, (6.9) and (6.10) imply that $\Delta\mathbf{x}$ is orthogonal to the tangent plane of S at every point. So, if N is an element of the orthogonal complement of the tangent plane to S at a point, then (6.4) implies

$$\langle \Delta \mathbf{x}, N \rangle = \langle D_{11}\mathbf{x}, N \rangle + \langle D_{22}\mathbf{x}, N \rangle = b_{11}(N) + b_{22}(N) = 2\lambda^2 H(N). \tag{6.11}$$

It follows that $\Delta\mathbf{x}/\left(2\lambda^2\right)$ is a normal vector which satisfies the defining equation for the mean curvature vector $\mathcal{H}$. Hence $\Delta\mathbf{x} = 2\lambda^2\mathcal{H}$ as required. □

Note that Lemma 6.1 gives a natural connection between isothermal parameters and harmonic functions. Namely, from (6.6), $\Delta\mathbf{x} = 0$ everywhere if and only if the mean curvature vector of S vanishes everywhere. Hence S is minimal if and only if the coordinate functions of $\mathbf{x}$ are harmonic. We summarize this discussion with the following lemma.

Lemma 6.2. *Let S be a surface with a C^2 isothermal parameterization $\mathbf{x}$. Then S is minimal if and only if the coordinate functions x_k of $\mathbf{x}$ are harmonic.*

From Lemma 6.2, we see that minimal surfaces arise in a quite different context than simply minimizing area. Indeed, Lemma 6.2 will allow us to make a connection between minimal surfaces and holomorphic functions.

Notice that Lemmas 6.1 and 6.2 assume that our surface is already parameterized in isothermal parameters. To apply these lemmas usefully, we must show that we can indeed represent our surface in this way. For the case of of minimal surfaces, this always holds.

Lemma 6.3. *Let S be a minimal surface with $a \in \Omega$. Then there exists a neighborhood $\Delta \subset \Omega$ of a such that the surfae Σ obtained by restricting $\mathbf{x}$ to Δ has an isothermal reparameterization.*

Proof. By Lemma 4.2, there exists an open ball $B \subset \Omega$ with $a \in B$ such that the surface $\Sigma_1 \subset S$ obtained by restricting $\mathbf{x}$ to B has a reparameterization in explicit form. Assume this is done where our explicit parameterization of Σ_1 is $\mathbf{x} = (I, \mathbf{f})$. Using the notation in (5.12), (5.18) implies that the equations

$$\begin{aligned} D_1\left(\frac{1+|q|^2}{W}\right) &= D_2\left(\frac{p\cdot q}{W}\right) \\ D_1\left(\frac{p\cdot q}{W}\right) &= D_2\left(\frac{1+|p|^2}{W}\right) \end{aligned} \tag{6.12}$$

hold throughout B.

Now, define a vector field $V : B \to \mathbb{R}^3$ by

$$V = \left(\frac{1+|p|^2}{W}, \frac{p\cdot q}{W}, 0\right).$$

Then (6.12) implies

$$|\nabla \times V| = D_2\left(\frac{1+|p|^2}{W}\right) - D_1\left(\frac{p\cdot q}{W}\right) = 0$$

so that V is conservative. It follows that there exists a map $F_1 : B \to \mathbb{R}$ such that

$$D_1F_1 = \frac{1+|p|^2}{W}, \quad D_2F_1 = \frac{p\cdot q}{W} \tag{6.13}$$

everywhere. Similarly, there exists a map $F_2 : B \to \mathbb{R}$ such that

$$D_1F_2 = \frac{p \cdot q}{W}, \quad D_2F_2 = \frac{1 + |q|^2}{W}. \tag{6.14}$$

Let $\xi : B \to \mathbb{R}^2$ be the map

$$\xi(x, y) = \big(x + F_1(x, y), y + F_2(x, y)\big) \tag{6.15}$$

and observe that

$$\begin{aligned}
J = \det D\xi &= \det \begin{bmatrix} D_1\xi_1 & D_2\xi_1 \\ D_1\xi_2 & D_2\xi_2 \end{bmatrix} \\
&= \det \begin{bmatrix} 1 + D_1F_1 & D_2F_1 \\ D_1F_2 & 1 + D_2F_2 \end{bmatrix} = \det \begin{bmatrix} 1 + \frac{1+|p|^2}{W} & \frac{p \cdot q}{W} \\ \frac{p \cdot q}{W} & 1 + \frac{1+|q|^2}{W} \end{bmatrix} \\
&= 2 + \frac{2 + |p|^2 + |q|^2}{W} \\
&> 0.
\end{aligned}$$

By the inverse function theorem, there exists a neighborhood $\Delta \subset B$ of a such that ξ is a diffeomorphism when restricted to Δ. By the chain rule,

$$\begin{aligned}
D\xi^{-1} &= [D\xi]^{-1} \\
&= \frac{1}{J} \begin{bmatrix} D_2\xi_2 & -D_2\xi_1 \\ -D_1\xi_2 & D_1\xi_1 \end{bmatrix}
\end{aligned}$$

so that

$$\begin{aligned}
D_1\xi_1^{-1} &= \frac{W + 1 + |q|^2}{JW}, & D_2\xi_1^{-1} &= -\frac{p \cdot q}{JW} \\
D_1\xi_2^{-1} &= -\frac{p\dot{q}}{JW}, & D_2\xi_2^{-1} &= \frac{W + 1 + |p|^2}{JW}.
\end{aligned}$$

Furthermore, by the inverse function theorem,

$$D\left(x_k \circ \xi^{-1}\right) = Dx_k D\xi^{-1}$$

$$= \begin{bmatrix} D_1 x_k D_1 \xi_1^{-1} + D_2 x_k D_1 \xi_2^{-1} \\ \\ D_1 x_k D_2 \xi_1^{-1} + D_2 x_k D_2 \xi_2^{-1} \end{bmatrix}$$

so that

$$\begin{aligned} D_1 \left(x_k \circ \xi^{-1} \right) &= \frac{W + 1 + |q|^2}{JW} p_k - \frac{p \cdot q}{JW} q_k \\ D_2 \left(x_k \circ \xi^{-1} \right) &= \frac{W + 1 + |p|^2}{JW} q_k - \frac{p \cdot q}{JW} p_k. \end{aligned}$$

We may compute

$$\left| D_1 \left(\mathbf{x} \circ \xi^{-1} \right) \right|^2 = \left| D_2 \left(\mathbf{x} \circ \xi^{-1} \right) \right|^2 = \frac{W}{J} \tag{6.16}$$

and

$$D_1 \left(\mathbf{x} \circ \xi^{-1} \right) \cdot D_2 \left(\mathbf{x} \circ \xi^{-1} \right) = 0. \tag{6.17}$$

If $G = (g_{ij})$ is the first fundamental form of the surface Σ with respect to the parameterization $\mathbf{x} \circ \xi^{-1}$, then (6.16) and (6.17) imply that $g_{11} = g_{22}$ and $g_{12} = 0$. Hence $\mathbf{x} \circ \xi^{-1}$ is an isothermal parameterization of Σ as required. □

2. Holomorphic Functions

In this section, we will review some of the basic notions of complex analysis and further investigate the connection between isothermal parameters and holomorphic functions. Recall that a complex function $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ on an open set Ω is *holomorphic* if it is complex-differentiable at each point of Ω (see [**6**]). A function $f : \mathbb{C} \to \mathbb{C}$ is *entire* if it is holomorphic in the whole complex plane. Since we may identify $\mathbb{C}$ with $\mathbb{R}^2$, we will naturally be interested in a connection between holomorphic functions and functions on defined in the real plane. In this direction, we have the following results, whose proof can be found in [**6**].

LEMMA 6.4. *Let $f = (u, v) : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ and suppose that the first partial derivatives of u and v exist and that the Cauchy-Riemann Equations*

$$D_1 u = D_2 v, \quad D_2 u = -D_1 v \tag{6.18}$$

hold on Ω. Then $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ is holomorphic.

LEMMA 6.5. *Given $g : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$, let $\phi : \Omega \subset \mathbb{C} \to \mathbb{C}$ be defined by*

$$\phi = D_1 g - iD_2 g.$$

Then ϕ is holomorphic if and only if g is harmonic.

As seen in Lemma 6.2, we have a natural connection between harmonic functions and minimal surfaces. Specifically, given a surface S parameterized isothermally by $\mathbf{x}$, then S is minimal if and only if the coordinate functions x_k of $\mathbf{x}$ are harmonic. Thus Lemma 6.5 gives a connection to holomorphic functions.

LEMMA 6.6. *Given a surface S let $\phi_k : \Omega \subset \mathbb{C} \to \mathbb{C}$ be defined by*

$$\phi_k = D_1 x_k - iD_2 x_k \quad (k = 1, \ldots, n)\,. \tag{6.19}$$

Then

(*a*) *ϕ_k is holmomorphic if and only if x_k is harmonic;*

(*b*) *$\mathbf{x}$ is isothermal if and only if*

$$\sum \phi_k^2 = 0 \tag{6.20}$$

on Ω;

(c) *if* $\mathbf{x}$ *is isothermal, then*

$$\sum |\phi_k|^2 \neq 0 \tag{6.21}$$

on Ω.

PROOF. The proof follows from Lemma 6.5 and the identities

$$\begin{aligned}
\sum \phi_k^2 &= \sum (D_1 x_k)^2 - \sum (D_2 x_k)^2 - 2i \sum (D_1 x_k D_2 x_k) \\
&= |D_1 \mathbf{x}|^2 - |D_2 \mathbf{x}|^2 - 2i \langle D_1 \mathbf{x}, D_2 \mathbf{x} \rangle \\
&= g_{11} - g_{22} - 2i g_{12}
\end{aligned}$$

and

$$\sum |\phi_k|^2 = \sum (D_1 x_k)^2 + \sum (D_2 x_k)^2 = g_{11} + g_{22}. \qquad \square$$

To strengthen the connection between minimal surfaces and holomorphic functions even further, we have the following result.

LEMMA 6.7. *Let S be a minimal surface with isothermal parameterization* $\mathbf{x}$. *Then the functions ϕ_k defined in* (6.19) *are holomorphic and satisfy* (6.20) *and* (6.21). *Conversely, if $\phi_1, \ldots, \phi_n$ are holomorphic functions in a simply connected domain $\Omega \subset \mathbb{C}$ satisfying* (6.20) *and* (6.21), *then there exists a map $\mathbf{x} : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^n$ that parameterizes a minimal surface such that* (6.19) *holds.*

PROOF. The first statement follows immediately from Lemmas 6.2 and 6.6. For the converse, let

$$x_k = \Re \int \phi_k$$

for $k = 1, \ldots, n$. Then each x_k is harmonic and we may apply Lemmas 6.2, 6.5, and 6.6 to obtain the result. $\square$

As mentioned at the beginning of this chapter, the representation of a surface in isothermal parameters preserves the geometry of angles between curves in the parameter plane as well as on the surface itself. We now make this notion more precise.

Definition 6.2. Let $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ where $\Omega \subset \mathbb{C}$ is open. Then f is *conformal* if f is holomorphic and $f'(z) \neq 0$ for every $z \in \Omega$.

Definition 6.3. Let $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ where $\Omega \subset \mathbb{C}$ is open. Then f is *anti-conformal* if $\overline{f}$ is conformal.

The following result will be useful to us later and the proof is straight forward. See [**6**] for details.

Lemma 6.8. *Let $f : \mathbb{C} \to \mathbb{C}$ be invertible. Then f is conformal if and only if f is entire. Additionally, f is conformal if and only if f^{-1} is conformal.*

We see that the geometry of a surface given in isothermal parameters is intimately connected to conformal maps. In fact, the following result makes this connection more precise.

Lemma 6.9. *Given a surface S with isothermal parameterization $\mathbf{x}$, let $\phi : \widetilde{\Omega} \to \Omega$ be a C^r-diffeomorphism. Then the parameterization $\mathbf{x} \circ \phi$ is also isothermal if and only if $\phi : \widetilde{\Omega} \subset \mathbb{C} \to \Omega \subset \mathbb{C}$ is conformal or anti-conformal.*

Proof. Suppose that $\mathbf{x} \circ \phi$ is isothermal, let G and $\widetilde{G}$ be the first fundamental forms of S with respect to $\mathbf{x}$ and $\mathbf{x} \circ \phi$ respectively, and let U and V be the Jacobians of $\mathbf{x}$ and ϕ respectively. Since $\mathbf{x}$ and $\mathbf{x} \circ \phi$ are isothermal, (6.2) implies that

$$G = \lambda^2 I_2, \quad \widetilde{G} = \widetilde{\lambda}^2 I_2.$$

Furthermore, by the chain rule,

$$I_2 = \frac{1}{\widetilde{\lambda}^2}\widetilde{G} = \frac{1}{\widetilde{\lambda}^2}(UV)^\top UV = \frac{1}{\widetilde{\lambda}^2}V^\top G V = \frac{\lambda^2}{\widetilde{\lambda}^2}V^\top V. \tag{6.22}$$

Let

$$a = D_1\phi_1, \quad b = D_1\phi_2, \quad c = D_2\phi_1, \quad d = D_2\phi_2$$

and observe that

$$V^\top V = \begin{bmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{bmatrix}. \tag{6.23}$$

Combining (6.22) and (6.23) gives

$$a^2+b^2 = c^2+d^2, \quad ac+bd = 0. \tag{6.24}$$

Since ϕ is a diffeomorphism, $\det V \neq 0$ so that one of a and b is nonzero. Assuming that $a \neq 0$, we may write $c = -bd/a$ so that

$$a^4 + \left(b^2 - d^2\right)a^2 - b^2d^2 = 0. \tag{6.25}$$

Viewing (6.25) as a quadratic polynomial in a^2, we may apply the quadratic formula to obtain $a^2 = d^2$.

If $a = d$, then (6.24) implies that $b = -c$ so that

$$D_1\phi_1 = D_2\phi_2, \quad D_2\phi_2 = -D_2\phi_1$$

and Lemma 6.4 implies that $\phi : \widetilde{\Omega} \subset \mathbb{C} \to \Omega \subset \mathbb{C}$ is holomorphic and hence conformal by Lemma 6.8. If $a = -d$, then it can readily be checked that $\phi : \widetilde{\Omega} \subset \mathbb{C} \to \Omega \subset \mathbb{C}$ is anti-conformal. Similarly, $\phi : \widetilde{\Omega} \subset \mathbb{C} \to \Omega \subset \mathbb{C}$ is conformal or anti-conformal if $b \neq 0$.

Conversely, suppose that ϕ is conformal or anti-conformal. Then Lemma 6.8 implies that either

$$D_1\phi_1 = D_2\phi_2, \quad D_2\phi_1 = -D_1\phi_2$$

or

$$D_1\phi_1 = -D_2\phi_2, \quad D_2\phi_1 = D_1\phi_2.$$

In either case, we conclude that

$$V^\top V = \frac{\tilde{\lambda}^2}{\lambda^2} I_2 \tag{6.26}$$

where $\tilde{\lambda}$ is a real function on $\widehat{\Omega}$, and combining (6.26) and (6.22) gives that $\mathbf{x} \circ \phi$ is isothermal. □

Finally, we state one of the most surprising and useful results of complex analysis, Picard's theorem, which we present without proof, but the details can be found in [**6**].

THEOREM 6.1 (Picard's Theorem). *Every nonconstant entire function $f : \mathbb{C} \to \mathbb{C}$ omits at most one point.*

We note that this is a very useful result. Indeed, one may use Picard's theorem to give a simple proof of the fundamental theorem of algebra (see [**6**]).

CHAPTER 7

Bernstein's Theorem

In this section, we will prove our main result, Bernstein's theorem. We begin with a few elementary lemmas.

LEMMA 7.1. *Given a C^2 map $E : B_r(0) \subset \mathbb{R}^2 \to \mathbb{R}$ with positive definite Hessian (h_{ij}), let $\phi : B_r(0) \subset \mathbb{R}^2 \to \mathbb{R}^2$ be the map*

$$\phi(x,y) = \big(D_1E(x,y), D_2E(x,y)\big)$$

and let a and b be distinct points of $B_r(0)$. Then

$$\big(\phi(b) - \phi(a)\big) \cdot (a - b) > 0. \tag{7.1}$$

PROOF. Let $g : [0,1] \to \mathbb{R}$ be the map $g\big(t\big) = E(tb + (1-t)\,a)$. Then g is well-defined since $B_r(0)$ is convex and

$$g'(t) = \sum_{i=1}^{2} D_iE\big(tb + (1-t)\,a\big)\,(b_i - a_i)\,.$$

It follows that

$$g''(t) = \sum_{i,j=1}^{2} \Big[D_{ij}E\big(tb - (1-t)\,a\big)\Big]\,(b_i - a_i)\,(b_j - a_j) > 0$$

since (h_{ij}) is positive-definite. Hence $g'(1) > g'(0)$ so that

$$\begin{aligned}\big(\phi(b) - \phi(a)\big) \cdot (b - a) &= \phi(b) \cdot (b - a) - \phi(a) \cdot (b - a)\\ &= \sum_{i=1}^{2} D_iE(b)\,(b_i - a_i) - \sum_{i=1}^{2} D_iE(a)\,(b_i - a_i)\end{aligned}$$

$$= g'(1) - g'(0)$$

$$> 0$$

as required. $\square$

Lemma 7.2. *Given the hypotheses in Lemma 7.1, let $\xi : B_r(0) \subset \mathbb{R}^2 \to \mathbb{R}^2$ be the map*

$$\xi(x, y) = \big(x + \phi(x, y), y + \phi(x, y)\big). \tag{7.2}$$

Then

$$|\xi(b) - \xi(a)| > |b - a| . \tag{7.3}$$

Proof. Observe that

$$\begin{aligned} \big(\xi(b) - \xi(a)\big) \cdot (b - a) &= \big(b - a + \phi(b) - \phi(a)\big) \cdot (b - a) \\ &= |b - a|^2 + \big(\phi(b) - \phi(a)\big) \cdot (b - a) . \end{aligned} \tag{7.4}$$

It follows from (7.1) and (7.4) that

$$\big(\xi(b) - \xi(a)\big) \cdot (b - a) > |b - a|^2 . \tag{7.5}$$

Now, applying the Cauchy-Schwarz inequality to (7.5) gives (7.3). $\square$

Lemma 7.3. *Given the hypotheses of Lemma 7.2, ξ is a diffeomorphism onto a domain $\Delta \subset \mathbb{R}^2$ such that $B_r\big(\xi(0)\big) \subset \Delta$.*

PROOF. Since E is C^2, ξ is C^1 and we may observe that

$$\begin{aligned}\det D\xi &= \det \begin{bmatrix} 1+h_{11} & h_{12} \\ h_{12} & 1+h_{22} \end{bmatrix} \\ &= 1 + h_{11} + h_{22} + h_{11}h_{22} - h_{12}^2 \\ &= 1 + \operatorname{tr} h_{ij} + \det h_{ij}.\end{aligned} \tag{7.6}$$

Since (h_{ij}) is positive-definite, it has positive eigenvalues so that

$$\operatorname{tr} h_{ij} + \det h_{ij} > 0. \tag{7.7}$$

Thus (7.6) and (7.7) imply that

$$\det D\xi > 1 \tag{7.8}$$

everywhere. By the inverse function theorem, ξ is a local diffeomorphism. Furthermore, ξ is injective by (7.3), so it is a global diffeomorphism onto a domain $\Delta \subset \mathbb{R}^2$.

It remains to show that $B_r\big(\xi(0)\big) \subset \Delta$. This is trivial if $\Delta = \mathbb{R}^2$, so assume that $\Delta \neq \mathbb{R}^2$. Then there exists a point $x \in \partial\Delta$ that minimizes the distance to $\xi(0)$. It follows that there exists a sequence $\{x_n\}$ in Δ such that $x_n \to x$. Letting $w_n = \xi^{-1}(x_n)$, we see that $\{w_n\}$ does not converge in $B_r(0)$. Therefore $|w_n| \to r$ and (7.3) implies that

$$|x_n - \xi(0)| > |w_n|. \tag{7.9}$$

Letting $n \to \infty$ in (7.9) gives $|x - \xi(0)| \geq r$. Hence $B_r\big(\xi(0)\big) \subset \Delta$ as required. $\square$

LEMMA 7.4. *Given an explicit parameterization* $\mathbf{x} : B_r(0) \subset \mathbb{R}^2 \to \mathbb{R}^n$ *of a minimal surface* S*, let* ξ *be the map defined in* (6.15)*. Then* ξ *is a diffeomorphism onto a domain* Δ *such that* $B_r\big(\xi(0)\big) \subset \Delta$*.*

PROOF. Let $F_1, F_2 : B_r(0) \subset \mathbb{R}^2$ be as in (6.13) and (6.14). By (6.13) and (6.14), there exists a C^2 map $E : B_r(0) \subset \mathbb{R}^2 \to$ such that

$$D_1 E = F_1, \quad D_2 E = F_2.$$

It follows that E is C^2 and if (h_{ij}) is the Hessian of E, then

$$h_{11} = \frac{1 + |p|^2}{W} > 0, \quad \det h_{ij} = \frac{1}{W^2}\left[1 + |p|^2 + |q|^2 + |p|^2\,|q|^2 - (p \cdot q)^2\right] = 1$$

by (5.15), (6.13), and (6.14). Thus (h_{ij}) is positive-definite and we may apply Lemma 7.3 to ξ to obtain the result. □

LEMMA 7.5. *Let* $S \subset \mathbb{R}^3$ *be a surface defined explicitly by* $\mathbf{x} = (I, f)$*. Then* S *lies on a plane if and only if there exists a nonsingular transformation* $A : \tilde{\Omega} \to \Omega$ *such that* $\mathbf{x} \circ A$ *is isothermal.*

PROOF. Suppose that S lies on a plane. Then there exist constants A, B, and C such that $f(x, y) = Ax + By + C$. Now, let L be the map

$$L(x, y) = (\lambda Ax + By, \lambda Bx - Ay)$$

where

$$\lambda = \frac{1}{1 + A^2 + B^2}$$

and let ϕ_1, ϕ_2, and ϕ_3 be as defined in (6.19) with respect to $\mathbf{x} \circ L$. Then

$$\begin{aligned} \phi_1 &= \lambda A - iB \\ \phi_2 &= \lambda B + iA \\ \phi_3 &= \lambda \left(A^2 + B^2\right) \end{aligned}$$

so that

$$\begin{aligned} \phi_1^2 + \phi_2^2 + \phi_3^2 &= \lambda^2 \left(A^2 + B^2\right) - \left(A^2 + B^2\right) + \lambda^2 \left(A^2 + B^2\right)^2 \\ &= \left(A^2 + B^2\right) \left[\lambda^2 - 1 + \lambda^2 \left(A^2 + B^2\right)\right] \\ &= \left(A^2 + B^2\right) \left[\lambda^2 \left(1 + A^2 + B^2\right) - 1\right] \\ &= \left(A^2 + B^2\right) (1 - 1) \\ &= 0. \end{aligned}$$

Hence (6.20) implies that $\mathbf{x} \circ L$ is isothermal.

Conversely, suppose that there exists a nonsingular linear transformation $L : \widetilde{\Omega} \to \Omega$ such that $\mathbf{x} \circ L$ is isothermal and let ϕ_1, ϕ_2, and ϕ_3 be as defined in (6.19) with respect to $\mathbf{x} \circ L$. Then ϕ_1 and ϕ_2 are constant since $x_1 \circ L$ and $x_2 \circ L$ are linear. It follows from (6.20) that ϕ_3 is also constant. Thus $f \circ L$ has constant gradient so that f also has constant gradient. Hence f is linear so that S lies on a plane. $\square$

We are now ready to prove Theorem 7.1, which is stated as follows.

THEOREM 7.1 (Osserman). *Let $\mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^n$ be an explicit parameterization of a minimal surface S defined in all $\mathbb{R}^2$. Then there exists a nonsingular linear transformation $A : \mathbb{R}^2 \to \mathbb{R}^2$ such that the parameterization $\mathbf{x} \circ A$ is isothermal.*

Proof. Let ξ be the map defined in the proof of Lemma 6.3, which is now defined in all $\mathbb{R}^2$. Then Lemma 7.4 implies that ξ is a diffeomorphism onto $\mathbb{R}^2$. From the proof of Lemma 6.3, $\mathbf{x} \circ \xi$ is isothermal.

By Lemma 6.7, the functions

$$\phi_k = D_1\left(x_k \circ \xi\right) - iD_2\left(x_k \circ \xi\right) \quad (k = 1, \ldots, n)$$

are holomorphic. Observing that

$$\begin{aligned}
\Im\left\{\overline{\phi_1}\phi_2\right\} &= D_2(x_1 \circ \xi)D_1(x_2 \circ \xi) - D_1(x_1 \circ \xi)D_2(x_2 \circ \xi) \\
&= -\left(D_1\xi_1 D_2\xi_2 - D_1\xi_2 D_2\xi_1\right) \\
&= -\det D\xi \\
&< 0,
\end{aligned}$$

we deduce that $\phi_1 \neq 0$ and $\phi_2 \neq 0$ everywhere. Furthermore, we may observe that

$$\Im\left\{\frac{\phi_2}{\phi_1}\right\} = \frac{1}{\left|\phi_1\right|^2}\Im\left\{\overline{\phi_1}\phi_2\right\} < 0.$$

Thus ϕ_2/ϕ_1 is an entire function with strictly imaginary part. By Picard's theorem, there exists $a \in \mathbb{R}$ and $b > 0$ such that

$$\phi_2 = (a - ib)\,\phi_1. \tag{7.10}$$

Taking the real and imaginary parts of (7.10) gives

$$D_1\left(x_2 \circ \xi\right) = aD_1\left(x_1 \circ \xi\right) + bD_2\left(x_1 \circ \xi\right) \tag{7.11}$$

$$-D_2\left(x_2 \circ \xi\right) = aD_2\left(x_1 \circ \xi\right) + bD_1\left(x_1 \circ \xi\right). \tag{7.12}$$

Now, let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map

$$A(x, y) = (x, ax + by)$$
$$A^{-1}(x, y) = (u(x, y), v(x, y)) = \left(x, \frac{y - ax}{b}\right).$$

Then (7.11) gives

$$D_1 (u \circ \xi) = D_2 (v \circ \xi), \quad D_2 (u \circ \xi) = -D_1 (v \circ \xi).$$

That is, $A^{-1} \circ \xi$ satisfy the Cauchy Riemann equations. Hence $A^{-1} \circ \xi : \mathbb{C} \to \mathbb{C}$ is holomorphic. It follows that $\left(A^{-1} \circ \xi\right)^{-1}$ is conformal. Since

$$\mathbf{x} \circ A = (\mathbf{x} \circ \xi) \circ \left(A^{-1} \circ \xi\right)^{-1},$$

Lemma 6.9 implies that $\mathbf{x} \circ A$ is isothermal as required. $\square$

For $n = 3$, Lemma 7.5 and Theorem 7.1 imply Bernstein's theorem.

Corollary 7.1 (Bernstein's Theorem). *In the case $n = 3$, the only solution to the minimal surface equation f defined in all of $\mathbb{R}^2$ is the trivial solution, f a linear function.*

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