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A Monte Carlo Sampler

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of the University of California
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ABSTRACT

Methods are given for sampling the standard probability densities arising in physical, chemical, and statistical problems, by means of machine generated "random numbers." The probability theory underlying each device is briefly indicated. The collection is intended as a reference work for Monte Carlo practice. No attempt is made to quote original sources, and no claim to priority is intended in any case.

FOREWORD

In all cases, the density to be sampled is followed by a rule (R) for choice of the variable, in terms of random numbers $r_0, r_1, \ldots$, uniform on (0,1). A justification (J) for the rule is usually given, frequently supported by various formulas (F). The indices D, C, R provide "key words" which may help in locating a desired density, but details in this direction are omitted.

FORMULAS

F1. $$\frac{d}{du}\int_{g(u)}^{h(u)} f(u,\nu)\,d\nu =$$

$$h'(u)f(u,h(u)) - g'(u)f(u,g(u)) + \int_{g(u)}^{h(u)} \frac{\partial}{\partial u} f(u,\nu)\,d\nu$$

F2. $$\frac{d}{du}\int_{\{\nu_1+\nu_2<u,\,\nu_i>0\}} p_1(\nu_1)d\nu_1 p_2(\nu_2)d\nu_2 = \frac{d}{du}\int_0^u p_1(\nu_1)d\nu_1 \int_0^{u-\nu_1} p_2(\nu_2)d\nu_2$$

$$\equiv \frac{d}{du}\int_0^u f(u,\nu_1)d\nu_1 = f(u,u) + \int_0^u \frac{\partial}{\partial u} f(u,\nu_1)d\nu_1 = 0 + \int_0^u p_1(\nu_1)d\nu_1 p_2(u-\nu_1)$$

F3. $$\int_0^\nu \nu^{n-1} e^{-b\nu}d\nu = (n-1)!b^{-n}\left\{1 - e^{-b\nu}\sum_0^{n-1}(b\nu)^i/i!\right\};$$

$0 < \nu < \infty$, $b > 0$, $n = 1,2,3,\ldots$

$$D_b \equiv \int_1^\infty \nu^{n-1}e^{-b\nu}\,d\nu = (n-1)!\,b^{-n}\,e^{-b}\,S_b,\quad S_b \equiv \sum_0^{n-1} b^i/i!$$

F4. $$\Gamma(n) \equiv \int_0^\infty u^{n-1}e^{-u}\,du = 2\int_0^\infty \nu^{2n-1}e^{-\nu^2}\,d\nu\ ;\ n \text{ real} > 0.$$

$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$, $\Gamma(n+1) = n\Gamma(n)$, $2^{n-1}\,\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n+1}{2}\right) = \Gamma(\tfrac{1}{2})\Gamma(n)$.

For $n = 0,1,2,\ldots$, $\Gamma(n+1) = n!$, $0! \equiv 1$.

F5. $$B(m,n) \equiv \int_0^1 \nu^{m-1}(1-\nu)^{n-1}\,d\nu = 2\int_0^{\pi/2}\sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta$$

$= \Gamma(m)\Gamma(n)/\Gamma(m+n)$; m,n real > 0.

F6. $$\int_{\{\Sigma_1^n \mu_i<1,\ \mu_i>0\}} \Pi_1^n\ \mu_i^{s_i-1}\,d\mu_i = \Pi_1^n\,\Gamma(s_i)/\Gamma\left(1 + \sum_1^n s_i\right);\ s_i \text{ real} > 0.$$

F7. $$V(u) = \int_{\{\Sigma_1^n \nu_i < u,\ \nu_i>0\}} \Pi_1^n \, d\nu_i = u^n/n!,$$

$$dV/du = u^{n-1}/(n-1)! = A(u)$$

F8. $$V(u) = \int_{\{(\Sigma_1^n \nu_i^2)^{1/2} < u,\ \nu_i>0\}} \Pi_1^N \, d\nu_i = \pi^{N/2}\, u^N/2^{N-1}\, N\Gamma(N/2),$$

$$dV/du = \pi^{N/2}\, u^{N-1}/2^{N-1}\, \Gamma(N/2) = A(u)$$

Note. $V(u) = 2\pi^{N/2}\, u^N/N\Gamma(N/2)$ is the volume of the full sphere of radius u, $A(u) = 2\pi^{N/2}\, u^{N-1}/\Gamma(N/2)$ its area. Area of unit sphere is $A(1) = 2\pi^{N/2}/\Gamma(N/2) = \int_\Omega d\Omega$, where Ω is the direction vector in N-space.

F9. $$\zeta(n) \equiv \sum_1^\infty j^{-n} \ ; \ n \text{ real} > 1.$$

$$\zeta_a(n) \equiv \sum_1^\infty (-1)^{j+1}\, j^{-n} = (1 - 1/2^{n-1})\ \zeta(n)$$

$$\zeta_u(n) \equiv \sum_1^\infty (2j-1)^{-n} = (1 - 1/2^n)\ \zeta(n)$$

$$\zeta(2n) = (-1)^{n-1}(2\pi)^{2n}\, B_{2n}/2(2n)!\ ,\ n = 1,2,3\ldots$$

$B_2 = 1/6$ $\quad$ $B_4 = -1/30$ $\quad$ $B_6 = 1/42$ $\quad$ (Bernoulli numbers)

$\zeta(2) = \pi^2/6$ $\quad$ $\zeta(4) = \pi^4/90$ $\quad$ $\zeta(6) = \pi^6/945$ $\quad$ $\zeta(3) = 1.2021\ldots$

F10. $\sum_{1}^{n} F_i = 0$, where

$F_i \equiv 1/(a_1 - a_i) \cdots (a_{i-1} - a_i) \cdot (a_{i+1} - a_i) \cdots (a_n - a_i)$, $n > 2$.

Proof. For $f(Z) \equiv 1/(a_1 - Z) \cdots (a_n - Z)$, one has

$(1/2\pi i) \int_C f(Z) dZ = \sum_{1}^{n} \mathrm{Res}\ (a_i)$, where C is any circle of radius $R > \max |a_i|$,

and $-\mathrm{Res}\ (a_i) = F_i$, as defined. But $\left| \int_C f(Z)\ dZ \right| \leq 2\pi R \max_C |f(Z)| \to 0$.

as $R \to \infty$ $(n > 2)$

F11. $$\int_0^\infty \nu^{n-1}\, d\nu/(\Lambda^{-1} e^{\nu} + 1) = \sum_{1}^{\infty} (-1)^{j+1} (\Lambda^j/j^n) \int_0^\infty j^n \nu^{n-1} e^{-j\nu} d\nu$$

$= \zeta_a(\Lambda,n)\Gamma(n)$, where $\zeta_a(\Lambda,n) \equiv \sum_{1}^{\infty} (-1)^{j+1} \Lambda^j/j^n$, $0 < \Lambda \leq 1$, $n > 1$.

F12. $$\int_0^\infty \nu^{n-1}\, d\nu\, \Lambda e^{-\nu}/(1 - \Lambda^2 e^{-2\nu}) =$$

$$\sum_{1}^{\infty} (\Lambda^{2j-1}/(2j-1)^n) \int_0^\infty (2j-1)^n \nu^{n-1} e^{-(2j-1)\nu}\, d\nu = \zeta_u(\Lambda,n)\Gamma(n),$$

where $\zeta_u(\Lambda,n) \equiv \sum_{1}^{\infty} \Lambda^{2j-1}/(2j-1)^n$, $0 < \Lambda \leq 1$, $n > 1$.

F13. $K_N(u) \equiv \int_0^\infty \cosh N\theta \, e^{-u \cosh \theta} \, d\theta$; $(0,\infty)$, $N > 0$

(a) $K_N(u) = 1/2 \int_{-\infty}^{\infty} e^{-N\theta} e^{-u \cosh\theta} \, d\theta$ (trivial)

(b) $K_N(u) = 2^{-(N+1)} u^N \int_0^\infty x^{-(N+1)} e^{-(x+\frac{u^2}{4x})} \, dx$ $(e^\theta = 2x/u)$

(c) $K_N(u) = \dfrac{\Gamma(1/2)\, u^N}{2^N\, \Gamma(N + 1/2)} \int_1^\infty (\nu^2 - 1)^{N-1/2} e^{-u\nu} \, d\nu$ (cf. [12])

$= \dfrac{\Gamma(1/2)\, u^N}{2^N\, \Gamma(N + 1/2)} \int_0^1 x^{-(2N+1)} (1 - x^2)^{N-\frac{1}{2}} e^{-u/x} \, dx$ $(\nu = x^{-1})$

(d) $K_N(2v^{\frac{1}{2}}) = 2^{-1} v^{N/2} \int_0^\infty x^{-(N+1)} e^{-(x+\frac{v}{x})} \, dx$ (from (b))

F14. $\int_0^\infty x^{n-1} e^{-b\sqrt{x^2 + 1}} \, dx = \int_1^\infty e^{-b\nu} \nu(\nu^2-1)^{\frac{n}{2}-1} d\nu$

$\equiv \int_1^\infty u \, dv = \frac{b}{n} \int_1^\infty (\nu^2 - 1)^{\frac{n}{2}} e^{-b\nu} d\nu$

$= (\Gamma(n/2)/\Gamma(\frac{1}{2}))\, (2/b)^{\frac{n-1}{2}} K_{\frac{n+1}{2}}(b)$; $n, b > 0$ (cf. F13(c))

F15. $\int_0^\infty x^{n-1}\, dx \,/\left(\Lambda^{-1}\, e^{a\sqrt{x^2+1}} + 1\right)$

$$= \sum_1^\infty (-1)^{j+1}\, \Lambda^j \int_0^\infty x^{n-1}\, e^{-ja\sqrt{x^2+1}}\, dx$$

$$= 2^{\frac{n-1}{2}} \left(\Gamma(n/2)/\Gamma(\tfrac{1}{2})\right) \sum_1^\infty (-1)^{j+1}\, \Lambda^j\, K_{\frac{n+1}{2}}(ja) \,/\, (ja)^{\frac{n-1}{2}} \;;\; n,\, a > 0. \quad (\text{cf. F14})$$

F16. $\int_1^\infty \nu^{n-1}\, \Lambda e^{-a\nu}/(1-\Lambda^2\, e^{-2a\nu}) = \sum_1^\infty \Lambda^{2j-1} \int_1^\infty \nu^{n-1}\, e^{-(2j-1)a\nu}\, d\nu$

$= \sum_1^\infty \Lambda^{2j-1}\, D_{(2j-1)a}$; D_b defined as in F3.

F17. $\int_0^\infty u^{n-1}\, K_N(u)\, du =$

$$\left(\Gamma(\tfrac{1}{2})/2^N\, \Gamma(N + \tfrac{1}{2})\right) \int_1^\infty (\nu^2 - 1)^{N-\frac{1}{2}}\, d\nu \int_0^\infty u^{n+N-1}\, e^{-u\nu}\, du$$

$$= \left(\Gamma(\tfrac{1}{2})\, \Gamma(n + N)/2^N\, \Gamma(N + \tfrac{1}{2})\right) \int_1^\infty \nu^{-(n+N)}\, (\nu^2 - 1)^{N-\frac{1}{2}}\, d\nu$$

$$= \left(\Gamma(\tfrac{1}{2})\ \Gamma(n+N)\ /\ 2^{N+1}\ \Gamma(N+\tfrac{1}{2})\right)\int_0^1 \xi^{\frac{n-N}{2}-1}\ (1-\xi)^{(N+\frac{1}{2})-1}\ d\xi$$

$$= \left(\Gamma(\tfrac{1}{2})\ \Gamma(n+N)\ /\ 2^{N+1}\ \Gamma(N+\tfrac{1}{2})\right) B\left(\frac{n-N}{2}\ ,\ N+\tfrac{1}{2}\right)$$

(F5)

$$= \Gamma(\tfrac{1}{2})\ \Gamma(n+N)\ \Gamma\left(\frac{n-N}{2}\right)\ /\ 2^{N+1}\ \Gamma\left(\frac{n+N+1}{2}\right)$$

(F4)

$$= 2^{n-2}\ \Gamma\left(\frac{n-N}{2}\right)\ \Gamma\left(\frac{n+N}{2}\right)\ ;\quad 0 < N < n.$$

F18. $E_N(u) \equiv \int_1^\infty \nu^{-N}\ e^{-u\nu}\ d\nu = \int_0^1 x^{N-2}\ e^{-u/x}\ dx;$

$0 < u < \infty\ ,\ N > 0$. Cf. [1].

F19. $\int_0^\infty u^{n-1}\ E_N(u)\ du = \int_1^\infty \nu^{-N}\ d\nu \int_0^\infty u^{n-1}\ e^{-u\nu}\ d\nu$

$$= \Gamma(n) \int_1^\infty \nu^{-(n+N)}\ d\nu = \Gamma(n)\ /\ (n+N-1)\ ;\ N > 0,\ n > 0\ ,\ n+N > 1.$$

D-INDEX

Discrete Densities

D1. $p(\nu);\ \nu = 0, 1, 2, \ldots$	Discrete, ν finite or countable, Bernoulli ($\nu = 0, 1$)
D2. $e^{-a}\ a^{\nu}\ /\nu!$	Poisson
D3. $P\{f(\nu) = u\}$	Density for value of a function
D4. $\Pi_1^n\ p_i\ (\nu_i)$	Vector density, independent variables
D5. $P\{f(\nu_1, \ldots, \nu_n) = u\}$	f-value density, independent variables

D6.	$C_u^n p^u (1 - p)^{n-u}$	Binomial, drawing with replacement
D7.	$C_{s-1}^{\nu-1} p^s (1 - p)^{\nu-s}$	Negative binomial
D8.	$(1 - q) q^{\nu-1}$	Geometric
D9.	$a^{\nu-1}/(1 + a)^{\nu}$	Pascal
D10.	$(1 + \alpha\beta)^{-1/\beta}(\alpha/1 + \alpha\beta)^{\mu} \times (1 + \beta)\cdots[1 + (\mu - 1)\beta]/\mu!$	Polya (β = 1, 1/2, 1/3, ...)
D11.	$(n!/n_1!\cdots n_f!)\ p_1^{n_1}\cdots p_f^{n_f}$	Multinomial, macrostate, particles in boxes.
D12.	$C_\nu^M C_{m-\nu}^N / C_m^{M+N}$	Hypergeometric, drawing without replacement
D13.	$C_k^n \frac{b\cdots[b+(k-1)s]\cdot c\cdots[c+(n-k-1)s]}{N\cdot(N+s)\cdots[N+(n-1)s]}$	Polya's urn
D14.	$1/N\ (N-1)\cdots(N - n + 1)$	Random permutation
D15.	$1/C_n^N$	Random combination
D16.	$\Pi_1^\infty p_i(\nu_i)$	Random sequences of integers

Discrete Densities

D1. $p(\nu)$; $\nu = 0, 1, 2, \ldots$

R. Set $\nu = \min\left\{ k;\ \sum_0^k p(\nu) > r_0 \right\}$

D2. $p(\nu) = e^{-a} a^\nu/\nu!$; $\nu = 0, 1, 2, \ldots, a > 0$.

R1. Set $\nu = \min\left\{ n\ ;\ \sum_0^n a^\nu/\nu! > r_0\ e^a \right\}$

R2. Set $\nu = -1 + \min\left\{ n\ ;\ \prod_1^n r_i < e^{-a} \right\}$

J2. By C2, C7, F7, F3, we have

$$P\left\{r_1\cdots r_n < e^{-a}\right\} = \int_{\left\{\Pi_1^n r_i < e^{-a}\right\}} \Pi_1^n \, dr_i = \int_{\left\{\sum_1^n \nu_i > a,\, \nu_i > 0\right\}} \Pi_1^n e^{-\nu_i} \, d\nu_i$$

$$= (1/(n-1)!) \int_a^\infty u^{n-1} e^{-u} \, du = e^{-a} \sum_0^{n-1} a^i/i! \quad . \quad \text{Cf. C22.}$$

D3. $q(u) = P\{f(\nu) = u\}$ $= \sum_{\{\nu;\, f(\nu) = u\}} p(\nu)$

R. Sample $p(\nu)$ for ν; set $u = f(\nu)$.

D4. $p(\nu) = p(\nu_1,\ldots,\nu_n) = \Pi_1^n p_i(\nu_i)$

R. Sample each $p_i(\nu_i)$ for ν_i; set $\nu = (\nu_1,\ldots,\nu_n)$

D5. $q(u) = P\{f(\nu_1,\ldots,\nu_n) = u\}$ $= \sum_{\{f(\nu) = u\}} p_1(\nu_1)\cdots p_n(\nu_n)$

R. Sample each $p_i(\nu_i)$ for ν_i; set $u = f(\nu_1,\ldots,\nu_n)$.

D6. $q(u) = C_u^n (1-p)^{n-u} p^u$; $u = 0, 1, \ldots, n$, $0<p<1$.

R. Set u = number of $r_1,\ldots,r_n$ such that $r_i < p$.

J. For $\nu_i \,\varepsilon\{0,1\}$, $p_i(0) \equiv 1-p$, $p_i(1) \equiv p$, $f(\nu) = \nu_1+\cdots+\nu_n$,

one has $P\{f(\nu_1,\cdots,\nu_n) = u\} = \sum_{\{\nu_1+\cdots+\nu_n = u\}} p_1(\nu_1)\cdots p_n(\nu_n)$

$= C_u^n (1-p)^{n-u} p^u$. (D5)

<u>D7. $p(\nu) = C_{s-1}^{\nu-1} p^s q^{\nu-s}$</u> ; $\nu = s, s+1, \ldots, 0<p<1, q = 1-p, s\geq 1$.

℞. Set ν = first n for which s of the random numbers $r_1,\ldots,r_n$ are $< p$.

J. $p(\nu) = \left| C_{s-1}^{\nu-1} p^{s-1} q^{(\nu-1)-(s-1)} \right|$. p is the probability of exactly s "successes" occurring for the first time at the ν-th trial.

Note. $1 = p^s(1-q)^{-s} = \sum_0^\infty \frac{(-s)(-s-1)\cdots(-s-\mu+1)}{\mu!} p^s (-q)^\mu$

$$= \sum_0^\infty \frac{(s+\mu-1)\cdots(s+1)s}{\mu!} p^s q^\mu = \sum_0^\infty C_\mu^{s+\mu-1} p^s q^\mu$$

$= \sum_s^\infty C_{\nu-s}^{\nu-1} p^s q^{\nu-s} = \sum_s^\infty C_{s-1}^{\nu-1} p^s q^{\nu-s}$. Hence the term "negative binomial."

<u>D8. $g(\nu) = pq^{\nu-1}$</u>; $\nu = 1,2,3,\ldots, 0<p<1, q = 1-p$.

℞ 1. Set $\nu = \min\{n; r_n<p, n = 1,2,\ldots\}$.

J 1. Case s = 1 of D7.

℞ 2. Set $\nu = k$, where $k-1 < \ell n\, r_1/\ell n\, q < k$, $k = 1,2,\ldots$

J 2. The rule follows from D1 (with $\nu = 1,2,\ldots$)

For, $\sum_1^k g(\nu) = 1-q^k$. (We have replaced r_0 by $r_1 = 1-r_0$).

<u>D9. $g(\nu) = a^{\nu-1}/(1+a)^\nu$</u>; $\nu = 1,2,3,\ldots, a > 0$.

R 1. Set $\nu = \min \{n; r_n \leq 1/(1+a),\ n = 1, 2, \ldots\}$

R 2. Set $\nu = k$, where $k-1 < \ell n\, r_1/\{\ell n\, a - \ell n(1+a)\} \leq k$.

J. Special case of D8, with $p = 1/(1+a)$.

<u>D10. $h(\mu) = (1+\alpha\beta)^{-1/\beta} \left(\frac{\alpha}{1+\alpha\beta}\right)^{\mu} \frac{1\cdot(1+\beta)\cdots(1+(\mu-1)\beta)}{\mu!}$</u>,

$h(0) = (1+\alpha\beta)^{-1/\beta}$; $\mu = 0, 1, 2, \ldots, \alpha > 0, \beta = 1, 1/2, 1/3, \ldots$

R. Define $s = 1/\beta$, $p = 1/(1+\alpha\beta)$. Set μ =

$-$ s+{first n for which s of the random numbers $r_1, \ldots, r_n$ are $< p$}.

J. One finds that $h(\mu) = C_{s-1}^{\nu-1}\, p^s\, q^{\nu-s}$, where $\nu = s+\mu$.

The rule follows from D7.

<u>D11. $p(n_1, \ldots, n_f) = (n!/n_1!\cdots n_f!)\ p_1^{n_1} \cdots p_f^{n_f}$</u> ;

$n_h \geq 0$, $n_1 + \ldots + n_f = n$, $p_1 + \ldots + p_f = 1$.

R. One follows the steps:

1. Put $0 \to n_1, \ldots, 0 \to n_f$; $1 \to t$.
2. Set $K = \min \left\{k; \sum_1^k p_i \geq r_t\right\}$. Put $1+n_K \to n_K$.
3. If $t < n$, put $t+1 \to t$, return to (2). Otherwise exit with $(n_1, \ldots, n_f)$.

J. If $\nu_1, \ldots, \nu_n$ are independent, each with density given by

$$\begin{array}{c} \nu_i = 1, \ldots, f \\ p_i(\nu_i) = p_1, \ldots, p_f \end{array}$$

then the vector $(\nu_1,\ldots,\nu_n)$ has density $p(\nu_1,\ldots,\nu_n) =$

$p_{\nu_1}\cdots p_{\nu_f}$, where $\sum_{\nu} p_{\nu_1} \cdots p_{\nu_f} = (p_1+\ldots+p_f)^n = 1$. Hence, the probability, under this density, of the class C of vectors with n_1 components 1, ..., n_f components f, is

$$\sum_{\nu\varepsilon C} p_{\nu_1}\cdots p_{\nu_f} = \sum_{\nu\varepsilon C} p_1^{n_1} \cdots p_f^{n_f} = (n!/n_1!\cdots n_f!)\, p_1^{n_1} \cdots p_f^{n_f},$$

and the rule follows.

Note. The density $q(u)$ for the <u>sum</u> $u = \nu_1+\cdots+\nu_f$ is not easily expressed (although easily sampled). Only for $f = 2$ do we have the connection with the binomial D6.

<u>D12. $p(\nu) = C_\nu^M C_{m-\nu}^N / C_m^{M+N}$</u>; $\max(0,m-N) \leq \nu \leq \min(m,M)$, $M,N \geq 1$, $1 \leq m \leq M+N$.

R. One follows the steps:

1. List the integers $1,2,\ldots,M+N$.
2. Put $0\to\nu$, $M+N\to b$, $1\to t$.
3. Set $K = \min\{k;\ k \geq br_t\}$, $(k = 1,2,\ldots,b)$.
4. Delete K^{th} integer $\equiv a_K$ of the <u>remaining</u> list. If $a_K \leq M$, put $\nu+1\to\nu$.
5. If $\nu = M$, or if $\nu < M$ and $t = m$, exit with ν. Otherwise, go to (6).
6. Put $b-1\to b$, $t+1\to t$ and return to (3).

J. One notes that $C_\nu^M C_{m-\nu}^N / C_m^{M+N} = P_\nu^M P_{m-\nu}^N \cdot C_\nu^m / P_m^{M+N}$

where $P_r^n \equiv n(n-1)\cdots(n-r+1)$; i.e., $p(\nu)$ is the chance of m drawings without replacement *yielding* exactly ν integers $\leq M$.

<u>D13. $p(k) = C_k^n \dfrac{b(b+s)\cdots(b+(k-1)s)\, c(c+s)\cdots(c+(n-k-1)s)}{N(N+s)\cdots(N+(n-1)s)}$</u>;

$k = 0,1,2,\cdots,n$, $b,c\varepsilon\{1,2,3,\ldots\}$, $N = b+c$, $s = 0,1,2,\ldots$

R. One follows the steps:

1. Put $0 \to k$, $1 \to t$, $b \to b'$, $N \to N'$
2. If $r_t \leq b'/N'$, put $k+1 \to k$, $b' + s \to b'$; go to (3).
   If $r_t > b'/N'$, go to (3).
3. Put $N' + s \to N'$
4. If $t < N$, put $t+1 \to t$, return to (2). If $t = n$, exit with k.

J. p(k) is the probability of drawing k black balls in n trials from an urn initially containing b black and c white balls ($b+c = N$), subject to the condition:

(c) On the t-th drawing, the ball drawn is replaced, and s more balls of the same color as that drawn are added to the urn. (For $s = 0$, p(k) is the binomial density).

<u>D14. $p(\Pi) = 1/N(N-1)\cdots(N-n+1)$</u>; $1 \leq n \leq N$, $\Pi = (C_1,\ldots,C_n)$ permutation of $1,\ldots,N$, "taken n at a time".

℞ 1. Follow the steps:

1. List $1,2,\ldots,N$.
2. Put $1 \to t$, $N \to D$
3. Set $K = \min\{k;\ k \geq Dr_t,\ k = 1,2,\ldots,D\}$
4. Set C_t = K-th integer of <u>remaining</u> list, and delete this integer from list.
5. If $t < n$, put $t+1 \to t$, $D-1 \to D$, and return to (3).

Otherwise, exit with random permutation $\Pi = (C_1,\ldots,C_n)$.

℞ 2. (For $n = N$ only!). Follow the steps:

1. Put $1 \to A_1,\ldots,N \to A_N$ (A_i storage position)
2. Put $1 \to t$, $N \to D$.
3. Set $K = \min\{k;\ k \geq Dr_t,\ k = 1,2,\ldots,D\}$
4. Interchange <u>contents</u> of A_D and A_K. (if $D \neq K$)
5. If $t < n$, put $t+1 \to t$, $D-1 \to D$ and return to (3)

Otherwise, exit with random permutation $\Pi = (C_1,\ldots,C_N)$,
C_i = <u>content</u> of A_i.

D15. $p(c) = 1/C_n^N$; $1 \leq n \leq N$, $c = \{C_1,\ldots,C_n\}$ unordered set of n integers from the set $\{1,\ldots,N\}$.

℞ 1. Follow D14, ℞ 1, and take the set of integers $\{C_1,\ldots,C_n\}$ in the permutation $\Pi = (C_1,\ldots,C_n)$ there obtained.

℞ 2. Follow the steps:

1. Put $0 \to m$ $\quad 1 \to t$
   $n \to C$ $\quad N \to D$
2. If $r_t \leq C/D$, put $m+1 \to m$; then set $C_m = t$; go to (3). If not go to (6).
3. If $m < n$, go to (4). If $m = n$, go to (9).
4. Put $C-1 \to C$, $D-1 \to D$; go to (5).
5. Put $t+1 \to t$; return to (2).
6. Put $D-1 \to D$; go to (7).
7. If $C < D$, go to (5). If $C = D$, go to (8).
8. Set $C_{m+1} = t+1,\ldots,C_n = N$; go to (9).
9. Exit with random set $c = (C_1,\ldots,C_n)$ (arranged in order of magnitude).

J. If m choices have been made through the t-th trial, $(t = 1,2,\ldots,N-1)$, one accepts t+1 as the m+1-st choice with probability $(n-m)/(N-t)$, where N-t is the number of remaining candidates $t+1,\ldots,N$, and n-m is the number of choices still to be made from these candidates.

D16. $\Pi_1^\infty p_i(\nu_i)$

℞. Methods are given in [7, 8], based on "Poisson sequences of trials", for producing random sequences of integers of stipulated asymptotic density.

C-INDEX

Continuous Densities

C1. $p(\nu)$	General density, continuous on open interval, finite or infinite.
C2. $q(\mu)d\mu = p(\nu)d\nu$	Change of variable.
C3. $\sum_1^J a_i(\nu)$	Sum of positive terms, interpolated densities.

C4.	$q(u) = \frac{d}{du} P\{f(\nu) < u\}$	Density for <u>value</u> of a function.
C5.	$\Pi_1^n p_i(\nu_i)\, d\nu_i$	Vector density, independent variables.
C6.	$q(u) = \frac{d}{du} P\{f(\nu_1,\ldots,\nu_n) < u\}$	f-value density, independent variables.
C7.	$q(u) = F(u)\, A(u)$	Case $\Pi_1^n p_i(\nu_i) = F(f(\nu_1,\ldots,\nu_n))$ of C6.
C8.	$s(u), p(u), q(u), q_1(u)$	Densities for $\nu_1+\nu_2$, $\nu_1\nu_2$, ν_2/ν_1
C9.	$p(\nu)$	A general device.
C10.	$1/(b-a)$	Uniform.
C11.	$c_o+c_1\nu$	Linear, disk radius
C12.	$c_o+c_1\nu+c_2\nu^2$	Quadratic, shell radius.
C13.	u^{m-1}	Power (m>0), sphere radius, completely degenerate gas momentum.
C14.	u^{-1}	Hyperbola
C15.	v^{-m-1}	Power (m>0)
C16.	$(y+1)^{-m-1}$	Pareto (m>0)
C17.	$e^{-a\nu}$	Exponential, Laplace I, decay time, collision distance (free path $\lambda=1/a$).
C18.	$e^{-a_1u} - e^{-a_2u}$	Difference of exponentials.
C19.	$\sum_1^n F_i\, e^{-a_iu}$	Exponential convolute.
C20.	$\sum_1^n B_j e^{-a_ju}$	Sum of exponentials (B_j>o).
C21.	$B_1e^{-a_1u} + B_2e^{-a_2u}$	Hyperexponential, residence times.
C22.	$u^{n-1}e^{-u}$, n = 1,2,3,...	Γ-type, Erlangian
C23.	$\nu^{n-1}/(e^{\nu}-1)$, n = 2,3,...	Planck type, Bose-Einstein, photons
C24.	$u^{2n-1}/(e^{u^2}-1)$	Version of C23.
C25.	$\nu^{2n-1}e^{-\nu^2}$, n = 1,2,3,...	Gauss type, Rayleigh, Maxwell
C26.	Re^{-R^2}	Gauss type, *n* = 1.
C27.	$e^{-\nu^2}$, (o,∞)	Error function
C28.	$e^{-\nu^2}$, $(-\infty,\infty)$	Normal
C29.	$u^{2n-1}e^{-u^2}$, n = 1/2, 3/2, ...	Gauss type, Maxwell speed.

C30.	$\nu^{2n-1}/(e^{\nu^2}-1)$, $n = 3/2, 5/2, \ldots$	Planck version.
C31.	$u^{n-1}/(e^{u}-1)$, $n = 3/2, 5/2, \ldots$	Planck type.
C32.	$\nu^{n-1}e^{-\nu}$, $n = 1/2, 3/2, \ldots$	Γ-type, Maxwell energy, fission spectrum.
C33.	$y^{A-1}\ln^{n-1}(1/y)$	Power-log power.
C34.	$t^{np-1}e^{-t^p}$	Γ-version
C35.	$u^{m-1}/(1+u)^{m+n}$ $\nu^{m-1}(1-\nu)^{n-1}$ $\sin^{2m-1}\theta\cos^{2n-1}\theta$ $w^{mp-1}(1-w^p)^{n-1}$	B-types, "arc sin", powers of sin, cos.
C36.	$\exp\left(-\sum_1^N \nu_i^2\right)$	N-normal, Maxwell velocity.
C37.	$p(\Omega), \Omega = (\omega_1, \ldots, \omega_N)$	Uniform (isotropic) direction, point on unit N-sphere.
C38.	$F\left(\sqrt{\sum_1^N \xi_i^2}\right)$	Radially symmetric density.
C39.	$u^{N/2-1}e^{-u/2b}$	χ^2 density.
C40.	$\chi^{N-1}e^{-\chi^2/2b}$	χ density, Rayleigh.
C41.	$\nu^{N/2-1}e^{-N\nu/2b}$	Mean square, χ^2/N.
C42.	$x^{N-1}e^{-Nx^2/2b}$	Root mean square, $\sqrt{\chi^2/N}$.
C43.	$1/(1+t^2/N)^{\frac{N+1}{2}}$	Student's t.
C44.	$1/(1+t^2)$	Cauchy
C45.	$F^{M/2-1}/(1+MF/N)^{\frac{M+N}{2}}$	Snedecor's F
C46.	$\nu^{n-1}e^{-\nu}/(1-\Lambda^2e^{-2\nu})$	Lemma for R8.
C47.	$\nu^{n-1}e^{-b\nu}$, $(1,\infty)$	Lemma for R9.
C48.	$\nu^{n-1}\Lambda e^{-a\nu}/1-\Lambda^2e^{-2a\nu}$	Lemma for R 10.

C49.	$u^{n-1}E_N(u)$	Schlömilch, neutron diffusion
C50.	$u^{n-1}K_N(u)$	Bessel
C51.	$e^{-a\|u-b\|}$	Bilateral exponential, Laplace II.
C52.	$x^{b-1}e^{-ax^b}$	Weibull
C53.	$u^{-1}\exp\{-(\ln^2 u)/2b\}$	Log normal.
C54.	$x^{-(n+1)}e^{-a^2/2x}$	One sided stable, recurrence times.
C55.	$s(u) = \begin{cases} u; & 0 < u < 1 \\ 2-u; & 1 < u < 2 \end{cases}$	Triangular.
C56.	$\cosh\theta$	Hyperbolic
C57.	$\sinh\theta$	Hyperbolic
C58.	$e^{Bx}/(1+Bx)^2$	Kahn, approximate normal.
C59.	$y^{a_1-1} - y^{a_2-1}$	Difference of powers.
C60.	$\sum F_i\, y^{a_i-1}$	Power convolute.
C61.	$P(\alpha'/\alpha)$	Klein-Nishina total cross section. See R 7C,D and R 14 for polarized case.

Continuous Densities

C1. $p(\nu);\ (a,b)$

R. Define $P(\nu) = \int_a^\nu p(\nu)d\nu$, $P_1(\nu) = \int_\nu^b p(\nu)d\nu$. Set $\nu = P^{-1}(r_0)$ or $\nu = P_1^{-1}(r_1)$.

C2. $q(\mu)d\mu = p(\nu)d\nu$; $\mu = f(\nu)$ monotone

R. If preferable, sample $p(\nu)$ for ν; set $\mu = f(\nu)$.

Note that $p(\nu) = q(f(\nu))\, d\mu/d\nu$. A similar rule applies to the n variable case, with $d\mu/d\nu$ replaced by the absolute Jacobian $|\det[\partial\mu_i/\partial\nu_j]|$ of the transformation $\mu = f(\nu)$.

C3. $p(\nu) = \sum_1^J a_j(\nu)$; (a,b), $a_j(\nu) \geq 0$, J finite or infinite.

R. Define $A_j = \int_a^b a_j(\nu)d\nu$. Set $K = \min\{k; \sum_1^k A_j \geq r_0\}$. Sample density $a_K(\nu)/A_K$ for ν.

Note. This provides a quick way of sampling an interpolated density $p(\nu) = \alpha_1 p_1(\nu) + \alpha_2 p_2(\nu)$, $\alpha_i > 0$, $\alpha_1 + \alpha_2 = 1$.

C4. $q(u) = \frac{d}{du} P\{f(\nu) < u\} = \frac{d}{du} \int_{f(\nu)\leq u} p(\nu)d\nu$; (a,b), $p(\nu)$ density for ν,

$P\{f(\nu) < a\} = 0$ (always assumed!)

R. Sample $p(\nu)$ for ν; set $u = f(\nu)$.

J. Since $\int_a^u q(u)du = P\{f(\nu) < u\}$, $q(u)$ is the density for the value u of the function $f(\nu)$.

C5. $p(\nu)d\nu = \Pi_1^n\, p_i(\nu_i)d\nu_i$

R. Sample each $p_i(\nu_i)$ for ν_i. Set $\nu = (\nu_1,\ldots,\nu_n)$

C6. $q(u) = \frac{d}{du} P\left|f(\nu_1,\ldots,\nu_n) < u\right| = \frac{d}{du} \int_{f(\nu_1,\ldots,\nu_n) < u} \Pi_1^n\, p_i(\nu_i)d\nu_i$; (a,b),

$p_i(\nu_i)$ density for ν_i, $i = 1, \ldots, n$.

R. Sample each $p_i(\nu_i)$ for ν_i. Set $u = f(\nu_1,\ldots,\nu_n)$.

C7. $q(u) = F(u)A(u)$; (a,b), $\Pi_1^n\, p_i(\nu_i) = F(f(\nu_1,\ldots,\nu_n))$, $A(u) = dV/du$,

$$V(u) = \int_{f(\nu)<u} \Pi_1^n\, d\nu_i .$$

R. Sample each $p_i(\nu_i)$ for ν_i. Set $u = f(\nu_1,\ldots,\nu_n)$.

J. If ν_i has density $p_i(\nu_i)$, and $\Pi_1^n\, p_i(\nu_i) = F(f(\nu_1,\ldots,\nu_n))$, where $F(u)$ is a function of one variable, and if moreover $A(u)$ is defined as above, then consideration of the one-parameter family of surfaces $f(\nu_1,\ldots,\nu_n) = u$ leads to the result

$$P\{f(\nu) < u\} = \int_{f(\nu)<u} \Pi_1^n\, p_i(\nu_i)\, \Pi_1^n\, d\nu_i = \int_a^u F(u)A(u)du.$$

The rule therefore follows from C6.

C8. $s(u) = \int_0^u p_1(\nu_1)\, p_2(u-\nu_1)d\nu_1$; $(0,\infty)$, p_1, p_2 densities on $(0,\infty)$

$p(u) = \int_0^\infty p_1(\nu_1)\nu_1^{-1}\, p_2(\nu_1^{-1}u)d\nu_1$; (o,∞), p_1, p_2 densities on (o,∞)

$q(u) = \int_0^\infty p_1(\nu_1)\nu_1 p_2(\nu_1 u)d\nu_1$; (o,∞), p_1, p_2 densities on (o,∞)

$q_1(u) = \int_0^\infty p_1(\nu_1)\, \nu_1 p_2(\nu_1 u)d\nu_1$; $(-\infty,\infty)$, p_1, p_2 densities on (o,∞) and $(-\infty,\infty)$, respectively.

R. In all cases, sample $p_1(\nu_1)$ for ν_1, $p_2(\nu_2)$ for ν_2. For s(u), set $u = \nu_1+\nu_2$; for p(u) set $u = \nu_1\nu_2$; for q(u) or $q_1(u)$, set $u = \nu_2/\nu_1$.

J. In each of the functions $f(\nu_1,\nu_2) = \nu_1+\nu_2,\ \nu_1\nu_2,\ \nu_2/\nu_1$, one verifies that $\frac{d}{du}\int_{f(\nu)\leq u} p_1(\nu_1)d\nu_1\ p_2(\nu_2)d\nu_2$ has the form of the corresponding density above. Verification for $\nu_1+\nu_2$ is given in F2.

C9. p(ν) ; (a,b)

R. Define $P(\nu)$, $P_1(\nu)$ as in C1, and

$$f(r) = r^{-1}\ P(a+(b-a)r)$$

$$g(r) = r^{-1}\ P_1(b-(b-a)r) \qquad 0 < r \leq 1.$$

(a) If f(r), in particular if p(ν), is increasing, set

$$u = \begin{cases} r_1 & \text{if } r_2 \leq f(r_1) \\ f^{-1}(r_2) & \text{if } r_2 > f(r_1) \end{cases} \qquad \nu = a+(b-a)u$$

(b) If g(r) is increasing, in particular if p(ν) is decreasing, set

$$u = \begin{cases} r_1 & \text{if } r_2 \leq g(r_1) \\ g^{-1}(r_2) & \text{if } r_2 > g(r_1) \end{cases} \qquad \nu = b-(b-a)u$$

J. For example, if f(r) is increasing, and we define the function

$$F(r,s) = \begin{cases} r & \text{if } s \leq f(r) \\ f^{-1}(s) & \text{if } s > f(r) \end{cases}$$

on the unit square,

then for $p_1(r) \equiv 1 \equiv p_2(s)$, we have $\int_0^v q(u)du \equiv$

$$\int_{F(r,s)\leq v} p_1(r)p_2(s)\, drds = vf(v) = P(a+(b-a)v)$$

$$= \int_a^{a+(b-a)v} p(\nu)d\nu$$, and the rule in (a) follows from C6, C2.

Note. The method is practical only if f, g are more easily invertible than P, P_1, which is indeed true for all linear densities, and for quadratic densities on certain intervals (Cf. C11,12). For details, see [2].

C10. $p(\nu) = 1/(b-a)$; (a,b)

R. Set $\nu = a+(b-a)r_0$. (C1)

C11. $p(\nu) = C^{-1}(c_0+c_1\nu)$; (a,b), $c_1 \neq 0$, $C = (b-a)\left\{c_0+\frac{1}{2} c_1(a+b)\right\}$.

R. (a) If $c_1>0$, set $\nu = a + \max\left\{(b-a)r_1,\ (b+a+2c_0c_1^{-1})\ r_2-2(a+c_0c_1^{-1})\right\}$.

(b) If $c_1<0$, set $\nu = b - \max\left\{(b-a)r_1,\ -(b+a+2c_0^{-1}c_1)r_2+2(b+c_0c_1^{-1})\right\}$. (C9)

Note. For $p(\nu) = 2\nu/(b^2-a^2)$, set $\nu = a+\max\left\{(b-a)r_1,\ (b+a)r_2-2a\right\}$. (C11) or $\nu = \left\{a^2+(b^2-a^2)r_0\right\}^{1/2}$ (C1) (radius, uniform circular disk).

C12. $p(\nu) = C^{-1}p_1(\nu)$, $p_1(\nu) = c_0+c_1\nu+c_2\nu^2$; (a,b), $c_2 \neq 0$,

$C = (b-a)\left\{c_0 + \frac{1}{2} c_1(b+a) + \frac{1}{3} c_2(b^2+ab+a^2)\right\}$, for certain intervals.

B. (a) If $c_2>0$, $p_1'(a)>0$: or if $c_2<0$, $a<-\frac{1}{2}c_1c_2^{-1}$, $b<-\frac{1}{2}\left(a+\frac{3}{2}c_1c_2^{-1}\right)$

$$\text{set } \nu = \begin{cases} a+(b-a)r_1 & \text{if } r_2 < f(r_1) \\ a+\lambda(r_2) & \text{if } r_2 > f(r_1) \end{cases}$$

where $f(r) \equiv (b-a)C^{-1}\left|p_1(a) + \frac{1}{2}p_1'(a)\ (b-a)r + \frac{1}{3}c_2(b-a)^2\ r^2\right|$,

$$\lambda(s) \equiv \frac{1}{2}\left|-\alpha + \text{sgn } c_2\sqrt{\alpha^2+12c_2^{-1}\left(\frac{C}{b-a}\ s-p_1(a)\right)}\right|$$

and $\alpha \equiv 3\left(a + \frac{1}{2}c_1c_2^{-1}\right)$

(b) If $c_2>0$, $p_1'(b) < 0$; or if $c_2<0$, $b > -\frac{1}{2}c_1c_2^{-1}$, $a > -\frac{1}{2}\left(b + \frac{3}{2}c_1c_2^{-1}\right)$,

$$\text{set } \nu = \begin{cases} b-(b-a)r_1 & \text{if } r_2 < g(r_1) \\ b-\mu(r_2) & \text{if } r_2 > g(r_1) \end{cases}$$

where $g(r) \equiv (b-a)C^{-1}\left|p_1(b) - \frac{1}{2}p_1'(b)(b-a)r + \frac{1}{3}c_2(b-a)^2r^2\right|$,

$$\mu(s) \equiv \frac{1}{2}\left|\beta + \text{sgn } c_2\sqrt{\beta^2+12c_2^{-1}\left(\frac{C}{b-a}\ s-p_1(b)\right)}\right|$$

and $\beta \equiv 3\left(b + \frac{1}{2}c_1c_2^{-1}\right)$. (C9)

Note. The method, when applicable, is indicated if C1 involves a difficult cubic. For $p(\nu) = 3\nu^2/(b^3-a^3)$ (radius, uniform spherical shell), C1 sets

$\nu = \left|a^3+(b^3-a^3)r_o\right|^{1/3}$, whereas C12 sets

$$\nu = \begin{cases} a+(b-a)r_1 & \text{if } r_2 < f(r_1) \\ a+\lambda(r_2) & \text{if } r_2 > f(r_1)\text{, where} \end{cases}$$

$$f(r) = \{3a^2+3a(b-a)r + (b-a)^2\ r^2\}/(b^2+ba+a^2) \equiv \bar{A}+r(\bar{B}+r\bar{C})$$

$$\lambda(s) = \frac{1}{2}\left|-3a + \sqrt{4(b^2+ba+a^2)s-3a^2}\right| \equiv \bar{D} + \sqrt{\bar{E}s-\bar{F}}\ ;\quad \bar{A},\ldots,\bar{F} \text{ stored.}$$

Machine times should be compared. For $a = 0$, see also C13, $m = 3$.

<u>C13. $q(u) = mb^{-m}u^{m-1}$</u>; (o,b), $m = k/\ell$, k, $\ell \in \{1,2,3,\ldots\}$

R. Set $u = b\ (\max\ \{r_1,\cdots,r_k\})^{\ell}$

J. For $\ell = 1$, $m = k$, let $p_i(\nu_i) \equiv 1$, $i = 1,\ldots,k$, and $f(\nu) =$ $b \max\ \{\nu_1,\ldots,\nu_k\}$. Then $\displaystyle\int_{f(\nu)\leq u} \Pi_1^k\ p_i(\nu_i)d\nu_i = b^{-k}u^k = \int_0^u kb^{-k}\ u^{k-1}\ du$,

and the rule follows from C6. For $\ell > 2$, and $u = b^{1-\ell}\ v^{\ell}$, note that $q(u)du = kb^{-k}v^{k-1}dv$, which reduces the second case to the first.

<u>C13A. $q(u) = C^{-1}u^{m-1}$</u>; (a,b), m real > 0, $C = (b^m-a^m)/m$, $a > 0$.

R. Set $u = \{a^m+(b^m-a^m)r_o\}^{1/m}$. (C1)

Note. For $m = 1/\ell$, $a = o$, this gives the rule of C13.

<u>C14. $q(u) = C^{-1}u^{-1}$</u>; $o < a < u < b < \infty$, $C = \ln(b/a)$

R. Set $u = ae^{Cr_o}$ (C1)

$\underline{C15.\ \ p(v) = m\beta^m\ v^{-m-1}}$; $o<\beta<v<\infty$, $m = 1/2, 1, 3/2, 2, \ldots$

R. For $m = 1, 2, 3, \ldots$, set $v = \beta/\max\{r_1,\ldots,r_m\}$.

For $m = 1/2, 3/2, \ldots$, set $v = \beta/(\max\{r_1,\ldots,r_{2m}\})^2$.

J. For $v = u^{-1}$, one finds $p(v)dv = m(1/\beta)^{-m}\ u^{m-1}(-du)$, $o<u<1/\beta$. and the rule follows from C13.

$\underline{C15A.\ p(v) = C^{-1}\ v^{-m-1}}$; $o<\beta<u<\alpha\leq\infty$, m real > o, $C = (\beta^{-m}-\alpha^{-m})/m$.

R. Set $v = 1/\{\alpha^{-m}+(\beta^{-m}-\alpha^{-m})r_o\}^{1/m}$. (C1)

$\underline{C16.\ \ q(y) = m(y+1)^{-(m+1)}}$; (o,∞), m real > o.

R. Set $y = v-1$, where v is obtained from C15 or C15A with $\beta = 1$.

J. For $y = v-1$, one has $q(y)dy = mv^{-m-1}dv$, $1<y<\infty$. (C1)

$\underline{C17.\ \ p(\nu) = ae^{-a\nu}}$; (o,∞), a>o.

R. Set $\nu = -a^{-1}\ \ln r_o$ (C1)

$$\underline{C18.\ \ q(u) = \frac{a_1a_2}{a_2-a_1}\left(e^{-a_1u}-e^{-a_2u}\right);\ (o,\infty),\ o<a_1<a_2 \text{ real.}}$$

R. Set $u = -(a_1^{-1}\ln r_1 + a_2^{-1}\ \ln r_2)$

J. For $p_1(\nu_1) = a_1e^{-a_1\nu_1}$, $p_2(\nu_2) = a_2e^{-a_2\nu_2}$ on (o,∞), one finds $\int_o^u p_1(\nu_1)\ p_2(u-\nu_1)d\nu_1 = q(u)$ above. The rule follows from C8(s), C17.

C19. $q(u) = \sum_1^n \frac{(a_1 \ldots a_n)}{(a_1-a_i)\cdots(a_{i-1}-a_i)\cdot(a_{i+1}-a_i)\cdots(a_n-a_i)} e^{-a_i u}$; (o,∞),

$a_i > o$, distinct, $n > 2$.

R. Set $u = -\sum_1^n a_i^{-1} \ln r_i$.

J. The relation $q(u) = \frac{d}{du} \int_{\sum_1^n \nu_i < u} \Pi_1^n a_i e^{-a_i \nu_i} d\nu_i$ may be proved by induction, using C18 as a basis. If F_i^n denotes the above coefficient of $e^{-a_i u}$, the induction step requires that (C8)

$\int_0^u \sum_1^n F_i^n e^{-a_i \nu} d\nu \; a_{n+1} e^{-a_{n+1}(u-\nu)} = \sum_1^{n+1} F_i^{n+1} e^{-a_i u}$. This result follows at once from an identity proved in F10.

C20. $p(u) = \sum_1^n B_j e^{-a_j u}$; (o,∞), $B_j > o$.

R. Set $K = \min \left\{ k; \sum_1^k (B_j/a_j) > r_o \right\}$ and $u = -a_K^{-1} \ln r_1$. (C3, C17).

C21. $p(u) = (2a^2/\tau)e^{-2au/\tau} + 2((1-a)^2/\tau)\, e^{-2(1-a)u/\tau}$; (o,∞), $\tau > o$, $o < a < 1/2$.

R. Define $B_1 = 2a^2/\tau$, $a_1 = 2a/\tau$, $B_2 = 2(1-a)^2/\tau$, $a_2 = 2(1-a)/\tau$, whence $B_1/a_1 = a$, $B_2/a_2 = (1-a)$. Set K (= 1 or 2) and u as in C20.

C22. $q(u) = u^{n-1} e^{-u}/\Gamma(n)$; (o,∞), $n = 1,2,3,\ldots$

R. Set $u = -\ln \Pi_1^n r_i$.

J. For $p_i(\nu_i) = e^{-\nu_i}$, $(0,\infty)$, and $f(\nu) = \sum_1^n \nu_i$, we have

$$\frac{d}{du}\int_{f(\nu)<u} \Pi_1^n\, p_i(\nu_i)\, \Pi_1^n\, d\nu_i = \frac{d}{du}\int_{f(\nu)<u} e^{-f(\nu)}\, \Pi_1^n\, d\nu_i = \frac{d}{du}\int_0^u e^{-u} A(u)du$$

$= e^{-u}A(u) = e^{-u}u^{n-1}/(n-1)! = q(u)$ where $A(u) = dV/du$

and $V = \int_{f(\nu)\leq u} \Pi_1^n\, d\nu_i = u^n/n!$ as in F7. The rule follows from C7, C17.

C23. $p(\nu) = \nu^{n-1}/(e^{\nu}-1)\,\zeta(n)\,\Gamma(n)$; $(0,\infty)$, $n = 2,3,4,\ldots$

R. Set $K = \min\left\{k;\ \sum_1^k j^{-n} > r_o\zeta(n)\right\}$, $\nu = -K^{-1}\,\ell n\, \Pi_1^n\, r_i$.

J. Since $p(\nu)d\nu = \sum_1^\infty (j^{-n}/\zeta(n))\ \left|(j\nu)^{n-1}e^{-j\nu}\, d(j\nu)/\Gamma(n)\right|$, the rule follows from C3, C22, C2. (Noted for $n = 4$ by C. Barnett, E. Canfield of LRL.)

C24. $p(u) = 2u^{2n-1}/(e^{u^2}-1)\,\zeta(n)\Gamma(n)$; $(0,\infty)$, $n = 2,3,4,\ldots$

R. Set ν as in C23, and $u = \sqrt{\nu}$. (C2)

C25. $p(\nu) = 2\nu^{2n-1}e^{-\nu^2}/\Gamma(n)$; $(0,\infty)$, $n = 1,2,3,\ldots$

R. Set $\nu = \{-\ell n\, \Pi_1^n\, r_i\}^{1/2}$ (C2, C22)

C26. $p(R) = 2Re^{-R^2}$; $(0,\infty)$

R. Set $R = \{-\ell n\, r\}^{1/2}$ (C25)

$\underline{\text{C27.}\ \ p(\nu_1) = 2e^{-\nu_1^2}/\sqrt{\pi}}$; $(0,\infty)$ (See also R12)

℞ 1. Set $\nu_1 = R\cos\theta$, $\nu_2 = R\sin\theta$, where $R = (-\ell n\ r_0)^{1/2}$, $\theta = \frac{\pi}{2} r_1$. (Two independent samples ν_1, ν_2 are obtained.)

J1. Under the indicated transformation,

$\frac{2}{\sqrt{\pi}} e^{-\nu_1^2} d\nu_1 \cdot \frac{2}{\sqrt{\pi}} e^{-\nu_2^2} d\nu_2 = 2Re^{-R^2} dR \cdot \frac{2}{\pi} d\theta$. The rule follows from C2, C26.

℞ 2. Obtain $S = r_1^2+r_2^2 < 1$ as in R1. Set

$\nu_1 = \{(-\ell n\ r_0)/S\}^{1/2}\ r_1$, $\nu_2 = \{(-\ell n\ r_0)/S\}^{1/2}\ r_2$. (Two samples)

℞ 3. Obtain $S = r_1^2+r_2^2 < 1$ as in R1. Set

$\nu_1 = \{(-\ell n\ S)/S\}^{1/2}\ r_1$, $\nu_2 = \{(-\ell n\ S)/S\}^{1/2}\ r_2$. (Two samples)

J3. Under the transformation $(\nu_1,\nu_2) \leftrightarrow (\rho,\theta)$:

$$\nu_1 = R\cos\theta,\quad \nu_2 = R\sin\theta,\quad R \equiv \{-2\ \ell n\ \rho\}^{1/2}\ ,$$

one finds $\frac{2}{\sqrt{\pi}} e^{-\nu_1^2} d\nu_1\ \frac{2}{\sqrt{\pi}} e^{-\nu_2^2} d\nu_2 = \frac{4}{\pi} \rho\ d\rho\ d\theta$. To sample the latter is to sample the unit disk in quadrant I, uniformly in area, and this is just what R1 does. The method avoids the additional r_0 of ℞ 2. (Box, Muller, Marsaglia.)

$\underline{\text{C28.}\ \ p_0(\nu_1) = e^{-\nu_1^2}/\sqrt{\pi}}$; $(-\infty,\infty)$ (See also R12)

℞ 1. Set $\nu_1 = R\cos\theta$, $\nu_2 = R\sin\theta$, where $R = (-\ell n\ r_0)^{1/2}$, $\theta = 2\pi r_1$.

℞ 2. Obtain $S < 1$, x, y as in ℛ3, ℞ 3. Set

$\nu_1 = \{(-\ln r_0)/S\}^{1/2}\, x, \quad \nu_2 = \{(-\ln r_0)/S\}^{1/2}\, y.$

R 3. Obtain $S < 1$, x, y as in R3, R 3. Set

$\nu_1 = \{(-\ln S)/S\}^{1/2}\, x, \quad \nu_2 = \{(-\ln S)/S\}^{1/2}\, y.$

J. Cf. C27.

Note. Two independent samples ν_1, ν_2 are obtained in each rule.

<u>C29. $p(u) = 2u^{2n-1}\, e^{-u^2}/\Gamma(n)$;</u> $(0,\infty)$, $n = 1/2, 3/2, 5/2, \ldots$

R. Define $h = n-1/2$ ($h = 0,1,2,\ldots$). Sample $2e^{-\tau^2}/\sqrt{\pi}$ for $(\tau)^2$ by any of the rules in C27. (Two samples. Save one. Avoid squaring square roots!). Set

$$u = \left|-\ln \Pi_1^h r_i + (\tau)^2\right|^{1/2}$$

J. For $p_i(\nu_i) = 2e^{-\nu_i^2}/\sqrt{\pi}$ on $(0,\infty)$, $i = 1,\ldots,2n$, and

$$f(\nu) = \left|\sum_1^{2n} \nu_i^2\right|^{1/2}, \text{ one has } \Pi_1^{2n}\, p_i(\nu_i) = 2^{2n}\, e^{-f^2(\nu)}/\pi^n,$$

$$A = \frac{d}{du} \int_{f(\nu)<u} \Pi_1^{2n}\, d\nu_i = \pi^n\, u^{2n-1}/2^{2n-1}\, \Gamma(n),$$ (F8), and hence the rule follows from C7, C27.

<u>C30. $p(\nu) = 2\nu^{2n-1}/(e^{\nu^2}-1)\zeta(n)\Gamma(n)$;</u> $(0,\infty)$, $n = 3/2, 5/2, \ldots$

R. Set $K = \min\left\{k;\ \sum_1^k j^{-n} > r_0\, \zeta(n)\right\}$, $h = n-1/2$.

Sample $2e^{-\tau^2}/\sqrt{\pi}$ for $(\tau)^2$ by C27 (two samples, save one. Avoid squaring square roots.) Set

$\nu = \{K^{-1}[-\ln \Pi_1^h r_i + (\tau)^2]\}^{1/2}$.   (C3, C29. Cf. C23)

C31. $p(u) = u^{n-1}/(e^u-1)\zeta(n)\Gamma(n)$; $(0,\infty)$, $n = 3/2, 5/2, \ldots$

R. Obtain $(\nu)^2$ as in C30. (Avoid squaring the square root!) Set $u = (\nu)^2$.

C32. $p(\nu) = \nu^{n-1}e^{-\nu}/\Gamma(n)$; $(0,\infty)$, $n = 1/2, 3/2, 5/2, \ldots$

R. Obtain $(u)^2$ as in C29. (Avoid squaring the square root.)
Set $\nu = (u)^2$.   (C2, C29).

C33. $g(y) = A^n y^{A-1} \ln^{n-1}(1/y)/\Gamma(n)$; $(0,1)$, $A > 0$, $n = 1/2, 1, 3/2, 2, \ldots$

R. Set $y = e^{-x/A}$, where x is the u of C22 or the ν of C32.

J. $g(y)dy = x^{n-1}e^{-x}(-dx)/\Gamma(n)$.   (C2)

C34. $g(t) = pt^{np-1}e^{-t^p}/\Gamma(n)$; $(0,\infty)$, p real > 0, $n = 1/2, 1, 3/2, 2, \ldots$

R. Sample $x^{n-1}e^{-x}/\Gamma(n)$ for x on $(0,\infty)$ by C22 or C32.
Set $t = x^{1/p}$.   (C2)   For $p = 2$, cf. C25, C29.

C35. $b(u) = u^{m-1}/(1+u)^{m+n} \cdot B(m,n)$; $(0,\infty)$

$p(\nu) = \nu^{m-1}(1-\nu)^{n-1}/B(m,n)$; $(0,1)$

$q(\theta) = 2 \sin^{2m-1}\theta \cos^{2n-1}\theta/B(m,n)$; $(0,\pi/2)$

$C(w) = pw^{mp-1}(1-w^p)^{n-1}/B(m,n)$; $(0,1)$, p real > 0

$m,n \in \{1/2, 1, 3/2, 2, \ldots\}$ in all.

R. Sample $p_1(\nu_1) = \nu_1^{n-1} e^{-\nu_1}/\Gamma(n)$, $p_2(\nu_2) = \nu_2^{m-1} e^{-\nu_2}/\Gamma(m)$

for ν_1, ν_2 on $(0,\infty)$ by C22 and/or C32. Set $u = \nu_2/\nu_1$, $\nu = u/(1+u)$,

$\theta = \text{arc sin} \sqrt{\nu}$, $w = \nu^{1/p}$. (The densities are equivalent under the last 3 substitutions.)

J. The rule follows from C8(q), since

$\int_0^\infty p_1(\nu_1)\ \nu_1 p_2(u\nu_1) d\nu_1 = b(u)$ above. Note. The same rule results from the

equivalence $\frac{2}{\Gamma(n)} \xi^{2n-1} e^{-\xi^2} d\xi \cdot \frac{2}{\Gamma(m)} \eta^{2m-1} e^{-\eta^2} d\eta =$

$\frac{2}{\Gamma(m+n)} \rho^{2(m+n)-1} e^{-\rho^2} d\rho \cdot \frac{2}{B(m,n)} \sin^{2m-1} \theta \cos^{2n-1} \theta\ d\theta.$

Note. For $n = 1$, $m = 1/2, 1, 3/2, 2, \ldots$, see C13.

For $n = 1/2$, observe that $q(\theta) = 2\Gamma(m + \frac{1}{2}) \sin^{2m-1}\theta/\sqrt{\pi}\,\Gamma(m)$,

$m = 1/2, 1, 3/2, 2, \ldots$. For $n = 1/2 = m$, set $\theta = \frac{\pi}{2} r_o$, $\nu = \sin^2\theta$, etc.

C36. $p(\nu_1,\ldots,\nu_N) = \pi^{-N/2} \exp - \sum_1^N \nu_i^2$; $-\infty < \nu_i < \infty$, $N = 1,2,\ldots$

R. For $N = 2h$, obtain $(\nu_1,\nu_2),\ldots,(\nu_{2h-1},\nu_{2h})$ from C28.

For $N = 2h+1$, obtain also (ν_{2h+1},ν_{2h+2}) from C28 (save ν_{2h+2}). (C5)

Note. The density of the value u of the function

$$f(\nu) = \left(\sum_1^N \nu_i^2\right)^{1/2} \text{ is } 2u^{N-1}\, e^{-u^2}/\Gamma(N/2) \qquad \text{(Cf. C40).}$$

C37. $p(\Omega) = \Gamma(N/2)/2\pi^{N/2}$

R. Set $\nu_1,\ldots,\nu_N$ as in C36, and $\Omega = (\omega_1,\ldots,\omega_N)$,

where $\omega_i = \nu_i/\rho$, $\rho = \left(\sum_1^N \nu_i^2\right)^{1/2}$.

Note 1. In C28 (the source of the ν_i), observe that $\nu_1^2+\nu_2^2 = -\ell n\ r_0$ or $-\ell n\ S$; this saves time in computing ρ.

Note 2. The rule determines a uniformly distributed direction Ω in N-space; equivalently, a point on the unit sphere $|\Omega| = 1$ (F8). See also R3, 4, 5.

C38. $p(\xi_1,\ldots,\xi_N) = F(u);\ u = \left(\sum_1^N \xi_i^2\right)^{1/2}$, $-\infty < \xi_i < \infty$.

R. Sample $q(u) = 2\pi^{N/2}\, u^{N-1}\, F(u)/\Gamma(N/2)$ for u on $(0,\infty)$. Sample unit sphere $|\Omega| = 1$ for Ω by C37. Set $\xi_i = u\omega_i$.

J. $$\int_{\left(\sum_1^N \xi_i^2\right)^{1/2} < u} p(\xi_1,\ldots,\xi_N)\ \Pi_1^N\, d\xi_i = \int_0^u F(u)\, A(u)du \qquad \text{(F8, C7)}$$

Note. For F(u) normal, use C36.

C39. $q(u) = u^{\frac{N}{2}-1}\ e^{-u/2b}/(2b)^{N/2}\ \Gamma(N/2)$; $(0,\infty)$, $b>0$, $N = 1,2,3,\ldots$

R. Sample $w^{N/2-1}\ e^{-w}/\Gamma(N/2)$ for w on $(0,\infty)$ by C22 or C32.

Set $u = 2bw$. (C2).

Note. $q(u)$ is the density of the value of $u = \chi^2 = \sum_1^N \nu_i^2$, defined in statistics as $\frac{d}{du} \int_{\sum_1^N \nu_i^2 < u} \Pi_1^N \frac{1}{\sqrt{2\pi b}} e^{-\nu_i^2/2b} d\nu_i$, $-\infty < \nu_i < \infty$, which may be evaluated using F8, C7.

C40. $p(\chi) = 2\chi^{N-1} e^{-\chi^2/2b}/(2b)^{N/2} \Gamma(N/2)$; $(0,\infty)$, $b > 0$, $N = 1,2,3,\ldots$

B. Sample $w^{N/2-1}e^{-w}/\Gamma(N/2)$ by C22 or C32. Set $\chi = \sqrt{2bw}$. (C2)

Note. $p(\chi)$ is the density for $\chi = \sqrt{\sum_1^N \nu_i^2}$.

C41. $q(\nu) = N^{N/2} \nu^{N/2-1} e^{-N\nu/2b}/(2b)^{N/2} \Gamma(N/2)$; $(0,\infty)$, $b > 0$, $N = 1,2,3,\ldots$

B. Sample $w^{N/2-1} e^{-w}/\Gamma(N/2)$ by C22 or C32. Set $\nu = 2bw/N$ (C2)

Note. $q(\nu)$ is the density of the mean square $\sum_1^N \nu_i^2/N = \chi^2/N$.

C42. $p(x) = 2N^{N/2} x^{N-1} e^{-Nx^2/2b}/(2b)^{N/2} \Gamma(N/2)$; $(0,\infty)$, $b > 0$, $N = 1,2,3,\ldots$

B. Sample $w^{N/2-1}e^{-w}/\Gamma(N/2)$ by C22 or C32. Set $x = \sqrt{2bw/N}$ (C2)

Note. $p(x)$ is the density of the "root mean square" $\sqrt{\sum_1^N \nu_i^2/N}$

$= \sqrt{\chi^2/N} = \chi/\sqrt{N}$.

C43. $q_1(t) = \Gamma(\frac{N+1}{2})/\sqrt{N\pi}\ \Gamma(N/2) \left(1 + \frac{t^2}{N}\right)^{\frac{N+1}{2}}$; $(-\infty,\infty)$, $N = 1,2,3,\ldots$

R. Sample $w^{N/2-1}\, e^{-w}/\Gamma(N/2)$ for w on $(0,\infty)$ by C22 or C32;

set $x = (w/N)^{1/2}$. Sample $e^{-y^2}/\sqrt{\pi}$ for y on $(-\infty,\infty)$ by C28. Set $t = y/x$.

J. $q_1(t)$ is the density for the function $y\Big/\left\{N^{-1}\sum_1^N \nu_i^2\right\}^{1/2}$ where the ν_i, y all have density $e^{-\xi^2/2b}/\sqrt{2\pi b}$ on $(-\infty,\infty)$. For, using the density $p_0(y) = e^{-y^2/2b}/\sqrt{2\pi b}$ for y and the $p(x)$ of C42 for $x = \sqrt{N^{-1}\sum_1^N \nu_i^2}$, one finds that $\int_0^\infty p(x)\, x p_0(tx)\, dx = q_1(t)$ above. The result being independent of b, the rule follows from C8, C28, C42 (with $2b = 1$).

C44. $q(t) = 1/\pi\,(1+t^2)$; $(-\infty,\infty)$

R. Set $t = \tan \frac{\pi}{2}(2r_0-1)$, or $t = y/x$ as in R2. (C1)

Note. Compare with C43 ($N = 1$).

C45. $q(F) = (M/N)^{M/2}\, F^{\frac{M-2}{2}}/(1 + \frac{M}{N}F)^{\frac{M+N}{2}}\, B(M/2, N/2)$; $(0,\infty)$, $M, N \in \{1,2,3,\ldots\}$

R. Define $m = M/2$, $n = N/2$. Sample $\nu_1^{n-1}e^{-\nu_1}/\Gamma(n)$, $\nu_2^{m-1}e^{-\nu_2}/\Gamma(m)$ for ν_1, ν_2 on $(0,\infty)$ by C22 and/or C32. Set $F = N\nu_2/M\nu_1$.

J. For $F = Nu/M$, one finds $q(F)dF = b(u)du$ as in C35.

Note. $q(F)$ is the density for the function $M^{-1}\sum_1^M \mu_i^2/N^{-1}\sum_1^N \nu_j^2$, where all μ_i, ν_j have density $e^{-\xi^2/2b}/\sqrt{2\pi b}$ on $(-\infty,\infty)$. Using the densities $q_M(\mu)$, $q_N(\nu)$ of C41 for numerator and denominator, it is easy to verify that $\int_0^\infty q_N(\nu)\nu\, q_M(F\nu)d\nu = q(F)$ above, independent of b. Cf. C8.

C46. $q(\nu) = C^{-1}\nu^{n-1}\Lambda e^{-\nu}/(1-\Lambda^2 e^{-2\nu})$; $(0,\infty)$, $0<\Lambda<1$, $n = 3/2, 2, 5/2, 3, \ldots$

$C = \zeta_u(\Lambda,n)\Gamma(n)$, $\zeta_u(\Lambda,n) \equiv \sum_1^\infty \Lambda^{2j-1}/(2j-1)^n$ (F12)

R. Set $K = \min\{k; \sum_1^k \Lambda^{2j-1}/(2j-1)^n > r_o\zeta_u(\Lambda,n)\}$. Sample $x^{n-1}e^{-x}/\Gamma(n)$ for x on $(0,\infty)$ by C22 or C32. Set $\nu = x/(2K-1)$.

J. One can write $q(\nu) = \sum_1^\infty \{\Lambda^{2j-1}/(2j-1)^n\ \zeta_u(\Lambda,n)\} \cdot$ $(2j-1)^n\nu^{n-1}e^{-(2j-1)\nu}\, d\nu/\Gamma(n)$ and apply C3, C2.

C47. $q(\nu) = D_b^{-1}\nu^{n-1}e^{-b\nu}$; $(1,\infty)$, $b>0$, $n = 1,2,3,\ldots$

$D_b = (n-1)!\ b^{-n}e^{-b}S_b$, $S_b = \sum_0^{n-1} b^i/i!$ (F3)

R. Set $L = \min\{\ell; \sum_0^\ell b^i/i! > r_oS_b\}$ $(0\le L\le n-1)$, and $\nu = 1-b^{-1}\ \ell n\ \Pi_1^{n-L}\ r_i$

J. Under the transformation $\nu = 1+b^{-1}u$, one finds $q(\nu)d\nu = p(u)du$ $\equiv D_b^{-1}b^{-n}e^{-b}(u+b)^{n-1}\ e^{-u}du = \sum_0^{n-1} \left|S_b^{-1}b^i/i!\right|\ u^{n-i-1}\ e^{-u}du/(n-i-1)!$, $0<u<\infty$, and the rule follows from C3, C22.

Note. The partial sums of S_b should be stored.

C48. $\bar{q}(\nu) = D^{-1}\nu^{n-1}\Lambda e^{-a\nu}/(1-\Lambda^2e^{-2a\nu})$; $(1,\infty)$, $0<\Lambda<1$, $n = 2,3,\ldots$,

$D = \sum_1^\infty \Lambda^{2j-1}\ D_{(2j-1)a}$, $D_b \equiv (n-1)!\ b^{-n}e^{-b}\ S_b$, $S_b = \sum_0^{n-1} b^i/i!$ (F16)

℞. Compute and store partial sums S_k of

$$S = \sum_1^\infty \Lambda^{2j-1} (2j-1)^{-n} e^{-2ja} S_{(2j-1)a},$$ where S_b is defined above.

Set $K = \min \{k;\ S_k > Sr_o\}$. Use C47, with $b \equiv (2K-1)a$, to obtain ν on $(1,\infty)$.

<u>C49. $q(u) = C^{-1}u^{n-1} E_N(u)$</u>; $(0,\infty)$, $N \geq 0$, $n+N>1$, $n = 1/2, 1, 3/2, 2, \ldots,$

$E_N(u) = \int_1^\infty \nu^{-N} e^{-u\nu}\, d\nu$, $C = \Gamma(n)/(n+N-1)$. (F18, 19).

℞. Sample $p_1(x) = (n+N-1)x^{n+N-2}$ for x on (0,1) by C13, and $p_2(y) = y^{n-1}e^{-y}/\Gamma(n)$ for y on $(0,\infty)$ by C22 or C32. Set $u = xy$.

J. $$\frac{d}{du} \int_{xy<u} p_1(x)dx\, p_2(y)dy = \frac{d}{du} \int_0^1 p_1(x)dx \int_0^{u/x} p_2(y)dy$$

$$= \int_0^1 p_1(x)dx\, x^{-1}\, p_2(u/x) = C^{-1}u^{n-1} \int_0^1 x^{N-2}\, e^{-u/x}\, dx = q(u) \qquad \text{(F18)}.$$

The rule follows from C6. (Cf. C8p)

Note. For $N = 0$, $n = 3/2, 2, 5/2, \ldots,$ sample $q(u) = u^{n-2}e^{-u}/\Gamma(n-1)$ for u on $(0,\infty)$ by C22 or C32.

<u>C50. $q(u) = C^{-1}u^{n-1}K_N(u)$</u>; $(0,\infty)$, $N = 0, 1/2, 1, 3/2, 2, \ldots,$

$n-N = 1,2,3,\ldots,$ $K_N(u) = \int_0^\infty \cosh N\theta\, e^{-u \cosh\theta}\, d\theta,$

$C = 2^{n-2}\, \Gamma\left(\frac{n-N}{2}\right) \Gamma\left(\frac{n+N}{2}\right)$. (F17)

℞ 1. Define $H = (n-N)/2$, $J = (n+N)/2$. Sample $p_1(x) = x^{H-1}e^{-x}/\Gamma(H)$

and $p_2(x) = x^{J-1}e^{-x}/\Gamma(J)$ for x, y on $(0,\infty)$ by C22 and/or C32.

Set $u = 2(xy)^{1/2}$.

J1. Transforming by $u = 2v^{1/2}$, we have $q(u)du = C^{-1}u^{n-1}\ K_N(u)\ du$

$= C^{-1}2^{n-1}\ v^{\frac{n}{2}-1}\ K_N(2v^{1/2}) = \int_0^\infty p_1(x)x^{-1}\ p_2(x^{-1}v)dx$ (cf. F13(d)), and the rule follows from C2, C8p.

> Note. The Legendre relation $2^{n-1}\ \Gamma(\frac{n}{2})\ \Gamma(\frac{n+1}{2}) = \Gamma(\frac{1}{2})\ \Gamma(n)$ may be regarded as a <u>consequence</u> of the above reduction!

R 2. Define $H = (n-N)/2$, $K = N + \frac{1}{2}$. Sample $p(\xi) = \xi^{H-1}(1-\xi)^{K-1}/B(H,K)$ for ξ on $(0,1)$ by C35, and $p_2(y) = y^{n+N-1}\ e^{-y}/\Gamma(n+N)$ for y on $(0,\infty)$ by C22. Set $u = y\xi^{1/2}$.

J2. Define $p_1(x) = \frac{2}{B(H,K)}\ x^{n-N-1}\ (1-x^2)^{N-1/2}$. Then by F13, we may write

$q(u) = \int_0^1 p_1(x)x^{-1}\ p_2(x^{-1}u)dx = \frac{d}{du} \int_{xy < u} p_1(x)dx\ p_2(y)dy$. Hence, we can sample $p_1(x)$ for x on (0,1) and $p_2(y)$ on $(0,\infty)$, and set $u = xy$, by C8p. But for $x = \xi^{1/2}$, we have $p_1(x)dx = p(\xi)d\xi$, as defined above; so we may sample $p(\xi)$ for ξ on (0,1) and set $x = \xi^{1/2}$. The rule follows. (Noted by Kalos [11] for $n = N+2$.)

<u>C51. $q(u) = (a/2)\ \exp{-a|u-b|}$</u>; $(-\infty,\infty)$, $a > 0$, $-\infty < b < \infty$.

R. Set $u = b \pm a^{-1}\ \ell n\ r_0$, the signs being equally likely. (C2, C17)

C52. $f(x) = abx^{b-1} e^{-ax^b}$; $(0,\infty)$, $a,b > 0$.

R. Set $x = \exp\{b^{-1} \ln(-a^{-1} \ln r_0)\}$. (C2, C17)

C53. $q(u) = u^{-1}\exp\{-(\ln^2 u)/2b\}/(2\pi b)^{1/2}$; $(0,\infty)$, $b > 0$.

R. Sample $e^{-\nu^2}/\sqrt{\pi}$ for ν on $(-\infty,\infty)$ by C28. Set $u = e^{\nu\sqrt{2b}}$.

J. For $u = e^{\nu\sqrt{2b}}$, one has $q(u)du = e^{-\nu^2} d\nu/\sqrt{\pi}$. (C2)

Note. Since $q(u) = \frac{d}{du} \int_{e^x < u} e^{-x^2/2b}\, dx/(2\pi b)^{1/2}$, $q(u) = \frac{d}{du} P\{e^x < u\}$

under the density $e^{-x^2/2b}/(2\pi b)^{1/2}$.

C54. $f(x) = a^{2n}x^{-(n+1)} e^{-a^2/2x}/2^n\Gamma(n)$; $(0,\infty)$, $a > 0$, $n = 1/2, 1, 3/2, 2, \ldots$

R. Sample $y^{n-1}e^{-y}/\Gamma(n)$ for y on $(0,\infty)$ by C22 or C32. Set $x = a^2/2y$. (C2)

C55. $s(u) = \begin{cases} u; & 0 < u < 1 \\ 2-u; & 1 < u < 2 \end{cases}$

R. Set $u = r_1+r_2$.

J. For $p_1(\nu_1) \equiv 1 \equiv p_2(\nu_2)$, ν_i on $(0,1)$, it is geometrically obvious that

$$P\{\nu_1+\nu_2 < u\} = \begin{cases} u^2/2; & 0 < u < 1 \\ 1-(2-u)^2/2; & 1 < u < 2 \end{cases}$$

Hence, $\frac{d}{du} P\{\nu_1+\nu_2 < u\} = s(u)$, and the rule follows from C6.

C56. $p(\theta) = S^{-1} \cosh \theta$; $(0,t)$, $S = \sinh t$.

R1. Define $A_1 = (e^t-1)/2S$, $A_2 = (1-e^{-t})/2S$. Set $K = \min\{k; \sum_1^k A_j > r_0\}$ (K = 1 or 2). For $K = 1$, set $\theta = \ln\{1+r_1(e^t-1)\}$. For $K = 2$, set $\theta = -\ln\{1-r(1-e^{-t})\}$ (C3, C1).

R2. Set $\theta = \ln\{Sr_0 + \sqrt{(Sr_0)^2+1}\}$ (C1)

<u>C57. $p(\theta) = C^{-1}\sinh\theta$</u>; $(0,t)$, $C = \cosh t - 1$

R. Set $\theta = \ln\{(Cr_0+1) + \sqrt{(Cr_0+1)^2-1}\}$

J. The rule follows from C1. Note that the (+) sign is required, since $x \equiv e^\theta > 1$ and $a = (x+x^{-1})/2 < (x+x)/2 = x$.

<u>C58. $f(x) = Be^{Bx}/(1+e^{Bx})^2$</u>; $(-\infty,\infty)$, $B > 0$.

R. Set $x = B^{-1}\ln(r_0^{-1}-1)$ (C1)

<u>C59. $p(y) = \dfrac{a_1a_2}{a_2-a_1}\left(y^{a_1-1} - y^{a_2-1}\right)$</u>; $(0,1)$, $0 < a_1 < a_2$

R. Set $y = \exp\{a_1^{-1}\ln r_1 + a_2^{-1}\ln r_2\}$

J. For $y = e^{-u}$, one finds $p(y)dy = q(u)du$ as in C18.

<u>C60. $p(y) = \sum_1^n \dfrac{(a_1\ldots a_n)y^{a_i-1}}{(a_1-a_i)\ldots(a_{i-1}-a_i)\cdot(a_{i+1}-a_i)\ldots(a_n-a_i)}$</u>;

$(0,1)$, a_i distinct, $n > 2$.

R. Set $y = \exp \sum_1^n a_i^{-1} \ln r_i$

J. For $y = e^{-u}$, one has $p(y)dy = q(u)du$ as in C19.

C61. $P(\alpha'/\alpha)$

R. A direct method is given in [5,6] for sampling the Klein-Nishina differential cross section, based on C1, and an accurate fit for the inverse of the associated probability distribution.

R-INDEX
Rejection Techniques

R1.	$\cos\theta$, $\sin\theta$; $p(\theta) = 2/\pi$, $(0,\pi/2)$	Uniform direction, quadrant I.
R2.	$\cos\theta$, $\sin\theta$, $\tan\theta$; $p(\theta) = 1/\pi$, $(-\pi/2, \pi/2)$	Direction in quadrants I, IV.
R3.	$\cos\theta$, $\sin\theta$; $p(\theta) = 1/2\pi$, $(0,2\pi)$	Uniform direction in plane, point on unit circle.
R4.	$\Omega = (\omega_1,\omega_2,\omega_3)$	Uniform direction in 3-space, point on unit sphere.
R5.	$\Omega = (\omega_1,\ldots,\omega_N)$	Uniform direction in N-space, point on unit sphere $\|\Omega\| = 1$.
R6.	$p_1(x)\ \{P_2(h(x))-P_2(g(x))\}$	Density x distribution
R7.	$p_1(x)\ h(x)$	Density x bounded function, various versions R7A-D.
R8.	$\nu^{n-1}/(\Lambda^{-1}e^{\nu}+1)$; $n = 3/2,1,5/2,2,\ldots$	(NR) non-degenerate electron gas energy.
R9.	$x^{n-1}\ e^{-b\sqrt{x^2+1}}$; $n = 2,3,\ldots$	(R) extreme non-degenerate electron gas momentum, Maxwell-Juttner.
R10.	$x^{n-1}/(\Lambda^{-1}e^{a\sqrt{x^2+1}} + 1)$; $n = 2,3,\ldots$	(R) non-degenerate electron gas momentum.

R11. $e^{-aE}\sinh\sqrt{bE}$ — Fission energy spectrum, Cranberg density.

R12. $e^{-x^2/2}$ — Normal.

R13. $P(\alpha'/\alpha, \theta)$ — Polarized Klein-Nishina.

Rejection Techniques

Note. In all cases, the process is iterated until the stated condition is satisfied.

R1. $\cos\theta$, $\sin\theta$ for $p(\theta) = 2/\pi$; $(0,\pi/2)$

R. If $S \equiv r_1^2 + r_2^2 < 1$, set $\cos\theta = r_1/S^{1/2}$, $\sin\theta = r_2/S^{1/2}$.

R2. $\cos\theta$, $\sin\theta$, $\tan\theta$ for $p(\theta) = 1/\pi$; $(-\pi/2,\pi/2)$

R. If $S \equiv x^2+y^2 < 1$, where $x = r_1$, $y = 2r_2-1$, set

$\cos\theta = x/S^{1/2}$, $\sin\theta = y/S^{1/2}$, and $\tan\theta = y/x$ $(x\neq 0)$.

R3. $\cos\phi$, $\sin\phi$ for $p(\phi) = 1/2\pi$; $(0,2\pi)$

R1. If $S \equiv x^2+y^2 < 1$, where $x = 2r_1-1$, $y = r_2$, set

$\cos\phi = (x^2-y^2)/S$, $\sin\phi = 2xy/S$ (von Neumann)

R2. Use R1 to obtain $\cos\phi$, $\sin\phi$; change sign of each independently with probability 1/2.

R3. If $S \equiv x^2+y^2 < 1$, where $x = 2r_1-1$, $y = 2r_2-1$,

set $\cos\phi = x/S^{1/2}$, $\sin\phi = y/S^{1/2}$.

R4. Uniform direction $\Omega = (\omega_1, \omega_2, \omega_3)$ in 3-space

R1. Obtain $\cos\phi$, $\sin\phi$ from R3. Set $\cos\theta = 2r_3 - 1$,

$\sin\theta = +(1-\cos^2\theta)^{1/2}$, $\Omega = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$.

J1. For spherical coordinates, $p(\theta,\phi)d\theta d\phi =$

$\frac{1}{2}\sin\theta d\theta \cdot \frac{1}{2\pi} d\phi$; $p_1(\theta)d\theta = \frac{1}{2}\sin\theta d\theta = -\frac{1}{2}d(\cos\theta) = -\frac{1}{2}d\mu$, $-1 < \mu < 1$.

R2. Obtain S, $\cos\phi$, $\sin\phi$ from R3. Set $\cos\theta$

$= 2S-1$, $\sin\theta = +(1-\cos^2\theta)^{1/2}$, and Ω as in R1.

J2. The density actually sampled in R3 is $p(\rho,\theta)d\rho d\theta = \rho d\rho d\theta/\pi$, with marginal density $p_1(\rho)d\rho = 2\rho d\rho$. Under the latter, the density of $S = \rho^2$ is $q(S) = 1$ on (0,1). Hence, S may be used in place of r_3 in R1.

Note. If R3 of R3 is used, one avoids unnecessary square roots by setting

$\omega_1 = x\,\{S^{-1}(1-\cos^2\theta)\}^{1/2}$, $\omega_2 = y\,\{S^{-1}(1-\cos^2\theta)\}^{1/2}$.

R5. Uniform direction $\Omega = (\omega_1, \ldots, \omega_N)$ in N-space.

R. If $S \equiv \sum_1^N \nu_i^2 < 1$, where $\nu_i = 2r_i - 1$, set $\omega_i = \nu_i/S^{1/2}$.

Note. Efficiency $= \pi^{N/2}/2^{N-1}N\Gamma(N/2) \to 0$, less than 1/2 for $N > 3$. (F8). See C37 for alternative.

R6. $p(x) = A^{-1}\,p_1(x)\,\{P_2(h(x)) - P_2(g(x))\}$; (a,b), $p_1(x)$ density on (a,b),

$p_2(x)$ density on (c,d), $P_2(y) = \int_c^y p_2(y)dy$, $c < g(x) < h(x) < d$.

R. Sample $p_1(x)$ for x on (a,b), and $p_2(y)$ for y on (c,d).

Accept x if $g(x) < y < h(x)$.

J. Since $\int_{g(x)<y<h(x)} p_1(x)dx\, p_2(y)dy = \int_a^b p_1(x)dx \int_{g(x)}^{h(x)} p_2(y)dy$

$= \int_a^b p_1(x) \{P_2(h(x))-P_2(g(x))\}\, dx = A$, A is the probability of acceptance (efficiency), and hence p(x)dx is the probability of an accepted x lying on (x,x+dx).

<u>R7. $p(x) = A^{-1}p_1(x)h(x)$</u> ; (a,b), $p_1(x)$ density on (a,b), $0 < h(x) < 1$.

R. Sample $p_1(x)$ for x on (a,b). Accept x if $r_1 < h(x)$.

J. Special case of R6, with c = 0, d = 1, $p_2(y) = 1$, $g(x) = 0$.
Efficiency = A.

> Note. The method is useful in Klein-Nishina and Thomson (incoherent and coherent) scattering modified by form factors. Due to the nature of the latter, efficiency considerations make it advisable to take the Klein-Nishina density for $p_1(x)$ and the form factor for h(x) in incoherent scattering, whereas in coherent scattering, $p_1(x)$ is based on the form factor and h(x) on the Thomson cross section. For details, see a forthcoming LA report on the MCP code, which incorporates the method referred to.

<u>R7A. $p(x) \equiv A^{-1}\cdot(b-a)^{-1}\cdot p(x)/M$</u> ; (a,b), $M = \max p(x)$, $A = 1/(b-a)M$.

R. Accept $x \equiv a+r_0(b-a)$ if $r_1 < p(x)/M$.

J. Special case of R7, with $p_1(x) = 1/(b-a)$ uniform, and $h(x) = p(x)/M \le 1$.

R7B. $p(x) = k^{-1}f(x)$; (a,b), $k > 0$, $k = \int_a^b f(x)dx$

R. Accept $x \equiv a+r_0(b-a)$ if $r_1 < f(x)/D$, where $D = \max f(x)$.

J. Special case of R7A, with $h(x) = p(x)/M = k^{-1}f(x)/k^{-1}D = f(x)/D$.

Note. If $C \equiv \min f(x)$, then efficiency =

$$A = 1/(b-a)M = k/(b-a)D = \int_a^b f(x)dx/(b-a)D > \int_a^b \{f(x)-C\}dx/(b-a)(D-C).$$

R7C. $p(\phi) = k^{-1}f(\phi)$, $f(\phi) = K-B\cos^2\phi$; $(0,2\pi)$, $K > B > 0$.

R1. Accept $\phi \equiv 2\pi r_0$ if $r_1 < f(\phi)/K$.

J1. Special case of R7B. Efficiency $\geq 1/2$. (Note, R7B)

R2. Obtain $\cos\phi$ from R3 and accept if next $r < f(\phi)/K$.

Note. This density occurs in polarized Compton scattering, where ϕ itself is not required.

R7D. $p(\phi) = k^{-1}f(\phi)$, $f(\phi) = K-S^2 (Q\cos 2\phi + U\sin 2\phi)$

$= K-H\cos 2(\phi-\phi_0)$; $(0,2\pi)$, $H = S^2(Q^2+U^2) < K$,

$C \equiv \min f(\phi) = K-H$, $D \equiv \max f(\phi) = K+H$.

R1. Accept $\phi = 2\pi r_0$ if $r_1 < f(\phi)/D$.

J1. Special case of R7B. Efficiency $> 1/2$ (Note, R7B)

R2. Obtain $\cos\phi$, $\sin\phi$ by R3; compute $\cos 2\phi = \cos^2\phi - \sin^2\phi$,

$\sin 2\phi = 2\sin\phi\cos\phi$. Accept if next $r < f(\phi)/D$.

Note. The density occurs in polarized Compton scattering, where ϕ itself is not required.

R8. $p(\nu) = C_1^{-1}\, \nu^{n-1}/(\Lambda^{-1}e^{\nu}+1)$; $(0,\infty)$, $0 < \Lambda < 1$, $n = 3/2, 2, 5/2, 3, \ldots,$

$C_1 = \zeta_a(\Lambda,n)\, \Gamma(n)$, $\zeta_a(\Lambda,n) \equiv \sum_1^{\infty} (-1)^{j+1}\Lambda^j/j^n$ (F11).

R. Sample $q(\nu)$ for ν on $(0,\infty)$ by C46; accept ν if $r_0 > \Lambda$, or if $\Lambda > r_0 > \Lambda e^{-\nu}$.

J. One can write $p(\nu) = c^{-1}h(\nu)q(\nu)$, where $c^{-1} = \zeta_u(\Lambda,n)/\zeta_a(\Lambda,n)$,

$\zeta_u(\Lambda,n) \equiv \sum_1^{\infty} \Lambda^{2j-1}/(2j-1)^n$, $0 < 1-\Lambda < h(\nu) \equiv 1-\Lambda e^{-\nu} < 1$, and

$q(\nu) = \nu^{n-1}\, \Lambda e^{-\nu}/(1-\Lambda^2 e^{-2\nu})\; \zeta_u(\Lambda,n)\Gamma(n)$ is the density of C46. The rule follows from R7.

R9. $\bar{p}(x) = E_b^{-1}\, x^{n-1}\, e^{-b\sqrt{x^2+1}}$; $(0,\infty)$, $b > 0$, $n = 2,3,4,\ldots,$

$E_b = 2^{\frac{n-1}{2}}\, (\Gamma(n/2)/\Gamma(1/2))\, K_{\frac{n+1}{2}}(b)/b^{\frac{n-1}{2}}$ (F14).

R. Sample $q(\nu) = D_b^{-1}\, \nu^{n-1}\, e^{-b\nu}$ for ν on $(1,\infty)$ by C47; accept ν with probability $h(\nu) = (1-1/\nu^2)^{\frac{n}{2}-1}$. For accepted ν, set $x = (\nu^2-1)^{1/2}$.

J. Under the transformation $x^2+1 = \nu^2$, one has

$\bar{p}(x)dx = p(\nu)d\nu \equiv (E_b^{-1}D_b)(1-1/\nu^2)^{\frac{n}{2}-1}\, (D_b^{-1}\nu^{n-1}e^{-b\nu}d\nu)$, and the rule follows from R7.

Note. For $n = 2$, $h(\nu) \equiv 1$, acceptance is certain, and $D_b = E_b$. For $n = 3$, $\overline{p}(x)$ is the Maxwell-Juttner relativistic momentum density [3].

R10. $\overline{p}(x) = A^{-1}x^{n-1}/(\Lambda^{-1}e^{a\sqrt{x^2+1}} + 1)$; $(0,\infty)$, $a > 0$, $0 < \Lambda < 1$, $n = 2,3,4,\ldots$

$$A = 2^{\frac{n-1}{2}}(\Gamma(n/2)/\Gamma(1/2))\sum_1^{\infty}(-1)^{j+1}\Lambda^j K_{\frac{n+1}{2}}(ja)/(ja)^{\frac{n-1}{2}} \qquad \text{(F15).}$$

R. Sample $\overline{q}(\nu)$ for ν on $(1,\infty)$ by C48. Accept ν if

$r_0 < (1-1/\nu^2)^{\frac{n}{2}-1}(1-\Lambda e^{-a\nu})$. For accepted ν, set $x = (\nu^2-1)^{1/2}$.

J. Under the transformation $x^2+1 = \nu^2$, one has $\overline{p}(x)dx = p(\nu)d\nu$

$= DA^{-1}\cdot(1-1/\nu^2)^{\frac{n}{2}-1}(1-\Lambda e^{-a\nu})\,\overline{q}(\nu)$ for the $\overline{q}(\nu)$ in C48, and the rule follows from R7.

R11. $\overline{p}(E) = C^{-1}e^{-aE}\sinh\sqrt{bE}$; $(0,\infty)$, $a,b > 0$.

R. Define $K = 1+(b/8a)$, $L = a^{-1}\{K+\sqrt{K^2-1}\}$, $M = aL-1 > 0$.

Set $x = -\ln r_1$, $y = -\ln r_2$. Accept x if $\{y-M(x+1)\}^2 < bLx$. For accepted x, set $E = Lx$.

J. The rule follows from R6 and C17. Let $E = Lx$ for arbitrary $L > 0$. Then $\overline{p}(E)dE = p(x)dx \equiv \frac{1}{2}C^{-1}Le^{aL-1}\cdot e^{-x}\cdot e^{-(aL-1)(x+1)}\{e^{\sqrt{bLx}}-e^{-\sqrt{bLx}}\}dx$
$= \frac{1}{2}C^{-1}Le^{M}\cdot e^{-x}\cdot\{e^{-g(x)}-e^{-h(x)}\} = A^{-1}p_1(x)\{P_2(h(x))-P_2(g(x))\}dx$,
where $M = aL-1$, $g(x) = M(x+1)-\sqrt{bLx}$, $h(x) = M(x+1)+\sqrt{bLx}$,
$p_1(x) = e^{-x}$, $p_2(y) = e^{-y}$, $P_2(y) = \int_0^y p_2(y)dy$. The above choice of L insures that $g(x) \geq 0$ on $(0,\infty)$. For a check, let $a = 3$, $b = 6$. (After Kalos [10])

R12. $p(x) = \sqrt{2/\pi}\, e^{-x^2/2}$; $(0,\infty)$

R. Set $x = -\ln r_1$, $y = -\ln r_2$. Accept x if $(x-1)^2 \leq 2y$.

J. Special case of R6, with $a = 0 = c$, $b = \infty = d$, $p_1(x) = e^{-x}$, $p_2(y) = e^{-y}$, $P_2(y) = 1-e^{-y}$, $g(x) = (x-1)^2/2$, $h(x) \equiv \infty$. Efficiency $A = (\pi/2e)^{1/2} = .76$.

Note. For $\bar{p}(x) = (2\pi)^{-1/2}\, e^{-x^2/2}$ on $(-\infty,\infty)$ choose $(\pm)$ sign for accepted x above with probability 1/2.

R13. $p(\frac{\alpha'}{\alpha}, \Theta)$; Polarized Compton scattering.

R. A method of sampling the Klein-Nishina cross section for polarized photons is given in [4]. This involves C61 and R7C,D. The fit used for C61 (cf. [5, 6]) is an improvement on that given in [4].

Note on Statistics

In judging the reliability of a sampling procedure, the following test may be useful. For a discrete density, pre-compute $p_i = p(i)$ for any desired set of argument values $\nu = i$. For a continuous density, compute $\int_{a_i}^{b_i} p(\nu)d\nu = p_i$ for a suitable set of intervals (a_i,b_i). Let the density $p(\nu)$ be sampled N times, according to the rule given, and tally the number f_i of times the sample $\nu = i$, or $\nu\varepsilon(a_i,b_i)$. Fixing attention on any one index i, this yields a Bernoulli sequence of N trials, with f_i the number of "successes", and p_i the probability of success, $q_i = 1-p_i$ the probability of "failure", in any one sampling.

In this situation, the law of large numbers states that

$P_N \equiv P\{|f_i/N-p_i|<\rho p_i\} \geq 1-(q_i/\rho^2 p_i N) \to 1$ while the central limit theorem asserts that $P_N = \phi(\rho\sqrt{p_i N/q_i}) + Z_N$, where $\phi(z) = \frac{2}{\sqrt{2\pi}} \int_0^z e^{-z^2/2}dz$, and $Z_N \to 0$.

One may note that, insofar as a method is correct, its statistical reliability depends only on the density itself.

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