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Transport Processes and Trace Constituents in the Stratosphere

Progress Report

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The research program has been subdivided into three tasks:

- (I) a numerical model
- (II) a stratospheric climatological data analysis
- (III) the effect of trace constituents on heating rates

During the first contract period less progress was made than was originally anticipated because of the difficulty encountered in freeing researchers from prior commitments and because of the departure from M. I. T. of several research personnel. At the present time a replacement for Dr. Boer-who was to be involved with Task (II) - has not yet been found; otherwise no other personnel difficulties are anticipated during the remainder of the current contract term. The principal investigator has been working full time on the project since June 1, 1972. As a result progress sufficient to merit discussion was confined to Task (I) and to the details of the development of the numerical model.

In preparation for the actual programming of the numerical dynamical model by F. Alyea (who has begun work on September 1), the basic framework of the model has been written in considerable detail. This write-up is enclosed as the appendix to this progress report. It includes the choice of vertical levels, and the associated mean state parameters, the non-dimensional finite-difference equations, details of the simplest photochemical model, and a new efficient procedure for solving the spherical quasi-geostrophic equations (Chapter 6 in the appendix).

An important change in computation is represented by our decision that, at each time step, the ozone and temperature field will be first transferred to physical space, where heating and chemical computation will be made, and then transferred back to spherical harmonic space. As mentioned elsewhere this will allow more accurate computation of ultraviolet absorption and of the temperature-dependent chemical reactions. This has opened up the possibility of more accurate infrared heating computation than originally envisaged. It is for this reason that the projected Chapter 5 on Heating is incomplete at the present time.

In August Dr. Cunnold attended the Arosa Symposium on Atmospheric Ozone. At this meeting observations of the global distribution of ozone in the natural stratosphere and troposphere were presented. Observations such as these form a basis against which our model results will be compared. Predictions of the perturbing effects on the stratosphere of a fleet of SSTs were also made with the use of very simple dynamical models and the latest chemical reaction schemes relating to the atmospheric ozone distribution were discussed.

During September Drs. Cunnold and Alyea attended a CIAP workshop in Gaithersburg, Maryland. The interrelationships of the various phases of the CIAP program were discussed and the type of results to be expected during the short time space of the program were outlined.

## Appendix

### Dynamical-photochemical Model of the Ozone Layer

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1. Basic dynamical equations and coordinate system.

The horizontal coordinate system will be longitude (positive eastward) and latitude, denoted by  $\lambda$  and  $\phi$ . This dependence will be represented in spherical surface harmonics, except that certain terms, such as part of the heating and photochemistry will be evaluated point-wise at selected values of  $\lambda$  and  $\phi$ . In the vertical direction pressure ( $p$ ) will be used as a coordinate with finite-differences being employed. These pressure levels will be distributed at equal intervals of  $\log P$  in order to give roughly equal intervals in height. We define

$$P = p \div (100 \text{ cbar})$$

$$Z = -\ln P, \quad P = e^{-Z} \quad (1.1)$$

From the hydrostatic relation  $dp = -\rho g dz$  and  $\rho = p/RT$ , we have

$$dZ = -\frac{dp}{p} = \frac{g}{RT} dz \quad (1.2)$$

The vertical levels will be separated by a uniform value of  $\Delta Z$ . To the extent that the temperature  $T$  is approximately uniform a change of one in  $Z$  corresponds to a height change of the order of 7 km. The bottom of the atmosphere will for simplicity be taken at  $Z = 0$ , i.e. at  $p = 100 \text{ cb}$  instead of at the conventional sea-level pressure of 101.325 cb. The top of the "atmosphere" will be artificially set at  $Z = Z_{TOP}$  corresponding to a geometric height of about 70 km.

The dynamical system not only assumes hydrostatic balance, but also a "quasi-geostrophic balance" in the horizontal equations of motion. Because we must consider global processes over the entire sphere, this balance must allow for complete variability of the coriolis parameter  $f$ :

$$f = 2\Omega \sin \phi \quad (1.3)$$

$$\Omega = 7.292 \times 10^{-5} \text{ rad sec}^{-1}$$

The quasi-geostrophic balance in question is obtained as follows (Lorenz, Tellus, 1960, P. 364). First, we divide the horizontal velocity  $\vec{v}$  into a non-divergent part  $\hat{k} \times \nabla \psi$  given by a stream function  $\psi$  and a divergent part  $-\nabla \chi$ , given by a velocity potential  $\chi$  :

$$\vec{v} = \hat{k} \times \nabla \psi - \nabla \chi \quad (1.4)$$

If the eastward and northward components of  $\vec{v}$  are represented by  $u$  and  $v$  and  $a$  is the radius of the earth, this is equivalent to

$$u = a \cos \phi \frac{d\lambda}{dt} = -\frac{1}{a} \frac{\partial \psi}{\partial \phi} - \frac{1}{a \cos \phi} \frac{\partial \chi}{\partial \lambda} \quad (1.5)$$

$$v = a \frac{d\phi}{dt} = \frac{1}{a \cos \phi} \frac{\partial \psi}{\partial \lambda} - \frac{1}{a} \frac{\partial \chi}{\partial \phi}$$

The vertical component of relative vorticity,  $\zeta$ , and the horizontal divergence of  $\vec{v}$  are related to  $\psi$  and  $\chi$  by

$$\zeta = \hat{k} \cdot \text{curl } \vec{v} = \nabla^2 \psi ; \quad \text{div } \vec{v} = -\nabla^2 \chi \quad (1.6)$$

where  $\nabla^2$  is the horizontal Laplacian operator on the sphere.

The condition of quasi-geostrophic balance is

$$\nabla \cdot f \nabla \psi = g \nabla^2 z \quad (1.7)$$

where  $g$  is gravity and  $z$  is the height of a constant pressure surface.

[Unless noted otherwise, all partial derivatives with respect to  $\lambda$ ,  $\phi$ , and  $t$  (time) are carried out at constant pressure (or  $Z$ )]. The hydrostatic relation,



3.  
(1.8a)

$$g \frac{\partial z}{\partial p} = - \frac{1}{\rho} = - \frac{RT}{p}$$

$$g \frac{\partial z}{\partial Z} = RT \quad \text{or} \quad (1.8b)$$

enables (1.7) to be rewritten as

$$\nabla \cdot f \nabla \frac{\partial \psi}{\partial Z} = \nabla^2 RT \quad (1.9)$$

Associated with this relation (which is a simplified form of the equation obtained by taking the horizontal divergence of the equations of motion) is the "vorticity equation":

$$\nabla^2 \frac{\partial \psi}{\partial t} = - \hat{k} \times \nabla \psi \cdot \nabla (f + \nabla^2 \psi) + \nabla \cdot f \nabla \chi + \nabla \cdot (\vec{F}_r \times \hat{k}) \quad (1.10)$$

where  $\vec{F}_r$  is the horizontal frictional force per unit mass.

The continuity equation (conservation of mass) is

$$\frac{\partial}{\partial p} \left( \frac{dp}{dt} \right) = \frac{\partial}{\partial p} \left( \frac{dp}{dt} \right) = - \nabla \cdot \vec{v} = \nabla^2 \chi \quad (1.11)$$

The upper boundary condition at  $Z = Z_{TOP}$  will be that  $dp/dt$  vanishes there. Let us define

$$X = - \int_{P_{TOP}}^P \chi dp, \quad \chi = - \frac{\partial X}{\partial p} \quad (1.12)$$

Equation (1.10) can then be rewritten as

$$\nabla^2 \frac{\partial \psi}{\partial t} = - \hat{k} \times \nabla \psi \cdot \nabla (f + \nabla^2 \psi) - \nabla \cdot f \nabla \left( \frac{\partial X}{\partial p} \right) + \nabla \cdot (\vec{F}_r \times \hat{k}) \quad (1.13)$$

If we use  $Z = - \ln P$  as the vertical coordinate, the appropriate vertical advection velocity is

$$W = \frac{dZ}{dt} = - \frac{1}{P} \frac{dP}{dt} \quad (1.14)$$

The continuity equation (1.11) in terms of  $W$  is:

$$\nabla \cdot P \vec{v} + \partial(PW)/\partial Z = 0 \quad (1.15)$$

From (1.11), (1.12) and (1.14) we get  $\partial(PW - \nabla^2 X)/\partial P = 0$ , or

$$PW = \nabla^2 X \quad (1.16)$$

Boundary conditions on  $W$  are that  $W$  vanishes at  $Z_{TOP}$  and that it is given by orographic upslope motion at the bottom:

$$Z = Z_{TOP} : W = 0 \quad (1.17)$$

$Z = 0$ :  $W \approx \frac{P g w}{H_0} = \frac{1}{H_0} \vec{v}_\psi \cdot \nabla h$   
 where  $h$  is the orography and  $\vec{v}_\psi$  is  $\frac{1}{k} \nabla \psi$  at the first interior level for  $\psi$ . Here

$$H_0 = \frac{RT_0}{g} = 7 \text{ km} \quad (1.18)$$

is a constant.

Friction will be represented by a vertical Austausch,  $\vec{E}_z = \frac{1}{\rho} \partial \vec{E} / \partial z$   
 $= -g \partial \vec{E} / \partial p$ . Thus  $\nabla \cdot \vec{E}_z \times \hat{k} = \frac{\partial}{\partial P} \left[ \nabla \cdot \left( -\frac{g}{P_0} \vec{E} \times \hat{k} \right) \right]$

In the interior regions of the model (but not at the ground) we set

$$\vec{E} = \rho K_m \partial(\hat{k} \times \nabla \psi) / \partial z, \text{ giving}$$

$$\nabla \cdot \left( -\frac{g}{P_0} \vec{E} \times \hat{k} \right) = \nabla \cdot \left[ \frac{g^2 \rho^2 K_m}{P_0^2} \frac{\partial \nabla^2 \psi}{\partial P} \right]$$

Replacing  $\rho$  by  $p/RT$  and replacing  $g/RT$  by  $1/H_0$  we get

$$\nabla \cdot \left[ -\frac{g}{P_0} \vec{E} \times \hat{k} \right] = -\frac{K_m}{H_0^2} p \frac{\partial \nabla^2 \psi}{\partial z}$$

At the ground, we can set  $\vec{E}$  equal to  $0.003 \bar{p} |\vec{v}| \vec{v}$ , with  $|\vec{v}|$

a suitable mean anemometer speed ( $5 \text{ m/sec}^{-1}$ ) and the anemometer vector wind

$\vec{v}$  equal to a rotated ( $\alpha = 22.5$  degrees) fraction (0.5) of  $\hat{k} \times \nabla \psi$

at the lowest interior level at which  $\psi$  is defined

$$\vec{E}_{grnd} = [0.003 \bar{p} |\vec{v}| (0.5)] [\cos \alpha \hat{k} \times \nabla \psi - \sin \alpha \nabla \psi]_{\text{interior}} \quad (1.19)$$

$$\nabla \cdot \left[ -\frac{g}{P_0} \vec{E} \times \hat{k} \right]_{grnd} = - \left\{ \frac{1}{H_0} [0.003 \bar{p} |\vec{v}| (0.5) \cos \alpha] \right\} \nabla^2 \psi_{int}$$

For  $H_0 = 7 \text{ km}$ ,  $|\vec{v}| = 5 \text{ m sec}^{-1}$  and  $\cos \alpha = 0.925$ , the coefficient here has the value  $10^{-6} \text{ sec}^{-1}$ .

The conventional quasi-geostrophic Taylor-Ekman theory (Charney and Eliassen, Tellus, 1949, Vol. 1, No. 2, P. 38) gives a corresponding term ("Ekman pumping") of

$$-\left\{ \frac{1}{H_0} \sqrt{\frac{K_m \bar{f}}{2}} \sin \alpha \right\} \nabla^2 \psi \quad (1.20)$$

For  $K_m = 5 \times 10^4 \text{ cm}^2 \text{ sec}^{-1}$  and  $\bar{f} = 10^{-4} \text{ sec}^{-1}$ , the coefficient in this derivation is  $1.6 \times 10^{-6} \text{ sec}^{-1}$ . To summarize the friction term we can write

$$\begin{aligned} \nabla \cdot \vec{F} \times \hat{k} &= \frac{\partial}{\partial p} (PF) \\ Z > 0: \quad F &= -\frac{K_m}{H_0^2} P \frac{\partial \nabla^2 \psi}{\partial Z} \\ Z = 0: \quad F &= -k_D \nabla^2 \psi_{int} \end{aligned} \quad (1.21)$$

where  $k_D$  refers to the "surface drag-coefficient" in (1.19) or (1.20).

At  $Z = Z_{TOP}$ ,  $F$  will vanish (no stress).

The next physical statement is the thermodynamic law  $d$  (entropy) /  $dt$  = rate of heating  $\div$  temperature. For our perfect gas system this would be

$$C_p \frac{d}{dt} [\ln(T p^{-\chi})] = \frac{q}{T} ; \quad \chi = \frac{R}{C_p} = \frac{2}{7} \quad (1.22)$$

where  $q$  is the rate of heating per unit mass and  $T$  the temperature. In terms of  $T$ , this becomes

$$\frac{\partial T}{\partial t} = -(\hat{k} \times \nabla \psi - \nabla \chi) \cdot \nabla T - w \frac{\partial T}{\partial Z} - \chi w T + \frac{q}{C_p} \quad (1.23)$$

We will however use a simplified form of this, obtained by ignoring  $\nabla \chi \cdot \nabla T$  and by replacing  $T$  in  $w \partial T / \partial Z$  and  $\chi w T$  by  $\bar{T}$ , where  $\bar{T}$  is the horizontal average:

$$T = \bar{T}(p, t) + T'(\lambda, \phi, p, t)$$

$$\bar{T} = \frac{1}{4\pi a^2} \int_{-\pi/2}^{\pi/2} \cos \phi d\phi \int_0^{2\pi} T d\lambda ; \quad \bar{T}' = 0 \quad (1.24)$$

[This definition of  $\bar{(\quad)}$  and  $(\quad)'$  will be applicable to any variable.]

This greatly simplifies the computations, and is reasonably accurate because

$\nabla \psi \gg \nabla \chi$  and  $\partial T'/\partial z + \kappa T'$  is generally small compared to  $\partial \bar{T}/\partial z + \kappa \bar{T}$ . The result is

$$\frac{\partial \bar{T}}{\partial t} = -\frac{1}{2} \kappa \nabla \psi \cdot \nabla \bar{T} - W \left( \frac{d\bar{T}}{dz} + \kappa \bar{T} \right) + g/c_p \quad (1.25)$$

However this simplification has the result that we can no longer interpret

(1.25) as forecasting  $\bar{T}$ , the horizontally averaged  $T$ ; this is

because the horizontal average of (1.25) gives simply

$$\frac{\partial \bar{T}}{\partial t} = \bar{g}/c_p$$

whereas the horizontal average of the exact equation (1.23) gives

$$\frac{\partial \bar{T}}{\partial t} = \frac{\bar{g}}{c_p} - \kappa \overline{W'T'} - \frac{1}{p} \frac{\partial}{\partial z} (p \overline{W'T'}), \quad (1.26)$$

showing the effect of vertical transports of entropy by the motion. We

expect little change in  $\bar{T}$  from the observed annual average  $\bar{T}(z)$ ,

however, either with season or with changes in the ozone chemistry.

[The effect of the latter will be investigated separately, as discussed later.]

In passing, we note that

$$\begin{aligned} \frac{\partial T}{\partial z} + \kappa T &= \frac{RT}{g} \left( \frac{\partial T}{\partial z} + \frac{g}{c_p} \right) \\ &= T \frac{\partial}{\partial z} \left[ \ln(T p^{-\kappa}) \right] \\ &= \frac{N^2}{R} \left( \frac{RT}{g} \right)^2 \end{aligned} \quad (1.27)$$

where  $N$  is the buoyancy frequency.

Finally, we describe the basic form of the equation for the (number density) mixing ratio of a trace substance such as  $O_3$ . Define

$$X_i = n_i \div n_m \quad (1.28)$$

where  $n_i$  is the number density of the  $i$ -th trace substance,  $n_m$  is the total number density, assumed to be equivalent to the "normal" constituents  $N_2$ ,  $O_2$  and  $CO_2$  since  $n_i$  is very small.

$$n_m \approx p/kT$$

$$k = \text{Boltzman constant} = 1.380 \times 10^{-26} \text{ kilojoules deg}^{-1} \quad (1.29)$$

The equation for  $dn_i/dt$  (the rate of change following the motion) is

$$\begin{aligned} \frac{dn_i}{dt} &= \frac{\partial n_i}{\partial t} + (\vec{k} \times \nabla \psi - \nabla \chi) \cdot \nabla n_i + W \frac{\partial n_i}{\partial z} \\ &= \frac{1}{n_m} \left( \frac{dn_i}{dt} \right)_c + \frac{1}{P} \frac{\partial}{\partial z} \left( P K_d \frac{\partial n_i}{\partial z} \right) \end{aligned}$$

where  $(dn_i/dt)_c$  is the net rate of local photo-chemical generation of the substance (number per unit volume per unit time) and  $K_d$  is the vertical eddy-diffusion coefficient [with dimensions  $(\text{length})^2 \div \text{time}$ ].  $K_d$  will vary only with  $P$ .

The vertical diffusion term can be rewritten by using the hydrostatic equation as

$$\frac{\partial}{\partial P} \left[ K_d \left( \frac{gP}{RT} \right)^2 \frac{\partial n_i}{\partial P} \right] \approx \frac{\partial}{\partial P} \left[ - \frac{K_d}{H_0^2} P \frac{\partial n_i}{\partial z} \right] \quad (1.30)$$

where we have again absorbed the variation of density with  $T$  into  $H_0$  on the recognition that  $K_d$  itself is not a precisely known quantity.  $K_d$  (and the momentum Austausch  $K_m$ ) will be prescribed functions of  $P$ . The equation for  $n_i$  is now

$$\frac{\partial n_i}{\partial t} = -(\vec{k} \times \nabla \psi - \nabla \chi) \cdot \nabla n_i - W \frac{\partial n_i}{\partial z} + \frac{1}{n_m} \left( \frac{dn_i}{dt} \right)_c + \frac{\partial}{\partial P} \left[ - \frac{K_d}{H_0^2} P \frac{\partial n_i}{\partial z} \right] \quad (1.31)$$

$$\text{or } \frac{\partial \kappa_i}{\partial t} = -\frac{1}{P} \left[ \nabla \cdot (P \vec{v} \kappa_i) + \frac{\partial (P W \kappa_i)}{\partial z} \right] + \frac{1}{n_m} \left( \frac{d n_i}{dt} \right)_c + \frac{\partial}{\partial P} \left[ -\frac{\kappa_d}{H_0^2} P \frac{\partial \kappa_i}{\partial z} \right] \quad (1.32)$$

[having made use of (1.4) and (1.15) to obtain the last form].

The rate of change of  $\overline{\kappa_i}$  (the horizontal average) is obtained from the horizontal average of (1.32):

$$\frac{\partial \overline{\kappa_i}}{\partial t} = \frac{\partial}{\partial P} \left[ P \overline{W' \kappa_i'} \right] + \overline{\left[ \frac{1}{n_m} \left( \frac{d n_i}{dt} \right)_c \right]} + \frac{\partial}{\partial P} \left[ -\frac{\kappa_d}{H_0^2} P \frac{\partial \overline{\kappa_i}}{\partial z} \right] \quad (1.33)$$

The rate of change of  $\kappa_i'$  will however be obtained from a simplified form of (1.31), much as was done in the thermodynamic equation (1.25):

$$\frac{\partial \kappa_i'}{\partial t} = -\vec{h} \times \nabla \psi \cdot \nabla \kappa_i' - W \frac{\partial \overline{\kappa_i}}{\partial z} + \left[ \frac{1}{n_m} \frac{d n_i}{dt} \right]' + \frac{\partial}{\partial P} \left[ -\frac{\kappa_d}{H_0^2} P \frac{\partial \kappa_i'}{\partial z} \right] \quad (1.34)$$

In contrast to  $\overline{T}$ , where we are for the most part content to take  $\overline{T}$  as given, we must predict  $\overline{\kappa_i}$  as well as  $\kappa_i'$ . Equation (1.33) will therefore be used as well as (1.34).

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Presumably (1.33) need not be applied every time step in the numerical integration,  $\overline{\kappa_i}$  being a slowly changing function of time. However, the term  $\overline{W' \kappa_i'}$  must be put equal to zero at  $P=1$  to ensure no net creation of  $\kappa_i$  by the large-scale motion.

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The form of  $(d \kappa_i / dt)_c$  is discussed later. However, a special treatment must be used for the lower boundary condition on the vertical eddy flux of ozone. Galbally (Quart. J. Roy. Meteor. Soc., 1971, P. 18) shows

that in the very lowest layer the vertical flux (over land) of ozone is proportional to the ground concentration

$$K_d \frac{\partial \kappa}{\partial z} \approx \frac{K_d}{H_0} \frac{\partial \kappa}{\partial Z} = d \kappa_{\text{ground}} \quad (1.35)$$

(the surface destruction of ozone being proportional to  $O_3$ ). The coefficient  $d$  has a value of about  $1 \text{ cm sec}^{-1}$ . We will apply this formulation to the lowest layer in our model ( $0 \leq Z \leq \Delta Z$ ). Values of  $\kappa$  are defined at the top of the layer ( $Z = \Delta Z, j = J-1$ ) and at the ground ( $Z=0, j = J$ ). Thus (dropping the  $i$ -subscript on  $\kappa$ ),

$$\left( \frac{K_d}{H_0} \frac{\partial \kappa}{\partial Z} \right)_{J-1/2} = \left( \frac{K_d}{H_0 \Delta Z} \right)_{J-1/2} (\kappa_{J-1} - \kappa_J) = d \kappa_J \quad (1.36)$$

whence

$$\kappa_J = \kappa_{J-1} \div \left[ 1 + \frac{d H_0 \Delta Z}{K_d} \right]_{J-1/2} \quad (1.37)$$

and

$$\left( \frac{K_d}{H_0 \Delta Z} \right)_{J-1/2} (\kappa_{J-1} - \kappa_J) = \frac{d \kappa_{J-1}}{1 + (d H_0 \Delta Z / K_d)} \quad (1.38)$$

Galbally cites values of the vertical number flux of ozone molecules over land in the range  $1$  to  $6 \times 10^{11} \text{ mol cm}^{-2} \text{ sec}^{-1}$ . Aldaz (J. Geo. Res., 1969, P. 6943) estimates a global average of  $1$  to  $1.7 \times 10^{11} \text{ mol cm}^{-2} \text{ sec}^{-1}$ . Pichinga representative value of  $2 \times 10^{11} \text{ mol cm}^{-2} \text{ sec}^{-1}$  and equating this to  $n_m K \partial \kappa / \partial z$ , we find, for  $n_m = 4.55 \times 10^{19} \text{ cm}^{-3}$  and  $K = 10^5 \text{ cm}^2 \text{ sec}^{-1}$ , that a vertical gradient of ozone number mixing ratio of

$$\frac{\partial \kappa}{\partial z} \sim 0.5 \times 10^{-13} \text{ cm}^{-1} = \frac{5 \times 10^{-8}}{10 \text{ km}} \quad (1.39)$$

is required. Galbally's data show a typical ground value for  $\kappa$  of

$5 \times 10^{11} \div 4.5 \times 10^{19} \sim 10^{-8}$  . The typical inferred downward flux of ozone observed near the ground is compatible then with a tropospheric  $K$  of  $10^5 \text{ cm}^2 \text{ sec}^{-1}$  and a tropopause (10km) value for  $\kappa$  of  $6 \times 10^{-8}$  or a 10-km value for  $M_3$  of  $(6 \times 10^{-8}) \times (8 \times 10^{18}) \sim 50 \times 10^{10} \text{ cm}^{-3}$ . This value is not greatly inconsistent with values of  $10^{12} \text{ cm}^{-3}$  which seem characteristic of the tropopause level in the meridional cross section prepared by D. Wu from the data of Mering and Borden (1967).

A special treatment of the ozone equation will be necessary at high levels. At these heights, Lindzen and Goody (J. Atmos. Sci., 1965, P. 341) show that the photo-dissociation of ozone is extremely rapid, with a time constant becoming less than 1 hour at heights above 45 km. (They presumably use typical values of incident solar radiation.) The conventional methods of "time-stepping" equations such as (1.34) require a computational time step no longer than the characteristic physical times associated with terms on the right side of (1.34). Since the advective time scale is of the order of an hour or so, we must consider replacing (1.33) and (1.34) at upper levels by the equilibrium condition.

$$\kappa_i = (\kappa_i)_{\text{equil}} \iff \frac{d\kappa_i}{dt} = 0 \tag{1.40}$$

For use in radiation computations, we need  $N_i$ , the number of molecules of  $\kappa_i$  in the vertical column of unit cross-section above a given pressure surface:

$$N_i = \int_z^\infty n_i dz = \int_z^\infty \kappa_i n_m dz = \int_z^\infty \kappa_i \frac{R}{k} p dz$$

$$= \frac{R p_0}{g k} \int_0^P \kappa_i dP$$

where  $R = 287 \text{ kJ ton}^{-1} \text{ deg}^{-1}$  is the gas constant for air.



This gives numerically

11.

$$\begin{aligned} N_i &= 2.12 \times 10^{29} \int_0^P \nu_i dP \text{ in (meter)}^{-2} \\ &= 2.12 \times 10^{25} \int_0^P \nu_i dP \text{ in (cm)}^{-2} \end{aligned} \quad (1.35)$$

In the case of molecular oxygen,  $\nu_i$  is taken as uniform and equal to 0.2096, giving

$$N_{O_2} = 0.4444 \times 10^{25} P \text{ cm}^{-2} \quad (1.36)$$

## 2. Choice of vertical levels.

We want the vertical domain to extend well above the actual ozone layer. We also want it high enough that there is some opportunity for the damping effects of ozone and radiation to absorb mechanical energy generated in the baroclinic processes of the lower atmosphere. On the other hand, we cannot for practical reasons get involved in the more complicated processes of the upper atmosphere and lower thermosphere. An upper limit of about 70 km seems reasonable.

We obtain equal intervals in  $Z = -\ln P$  ( $P = \text{pressure} \div 100 \text{ cb}$ ) by defining

$$\left. \begin{aligned} Z_j &= \Delta Z (J-j) \\ P_j &= e^{-\Delta Z (J-j)} \end{aligned} \right\} j=1, 2, \dots, J. \quad (2.1)$$

$j=1$  is at the "top" of our atmosphere, and  $j=J$  at the bottom, whence

$$\Delta Z = \frac{Z_1}{J-1} = \frac{Z_{TOP}}{J-1}$$

A convenient choice is obtained by choosing

$$\begin{aligned} e^{\Delta Z} &= r, \quad r = 3/2 \\ \Delta Z &= \ln r = 0.40547 \\ J &= 26 \end{aligned} \quad (2.2)$$

so that

$$\begin{aligned} Z_1 &= Z_{TOP} = (J-1) \ln r = 10.13675 \\ P_1 &= r^{-(J-1)} = 3.9605 \times 10^{-5} \end{aligned} \quad (2.3)$$

$P_1$  corresponds to a pressure of 39.605 dynes/cm<sup>2</sup>, typical of the height  $z \sim 71.5$  km. Successive pressure levels are separated by (roughly)  $72/25 \sim 2.9$  km. The relations

$$P_j = r^{-(J-j)}; \quad P_{j+1} = r P_j \quad (2.4)$$

are useful.

At these levels, the following basic variables will be represented

$$j = 1, 2, \dots, J: \quad T_j, W_j, (\mathcal{N}_i)_j,$$

together with the heating rate, the photo-chemical term, and the vertical turbulent fluxes of momentum. At the intermediate levels the streamfunction  $\psi_j$  will be represented

$$j = \frac{3}{2}, \frac{5}{2}, \dots, J - \frac{1}{2} : \quad \psi_j.$$

For convenience in notation, however,  $\psi$  will be labeled with an integer subscript according to the convention

$$\psi (P = P_{j+1/2}) \equiv \psi_j.$$

This results in the following scheme.

	$1$	$P_1, Z_1$	$W_1 (=0), (\mathcal{N}_i)_1, T_1$	$F_1$		
			$\psi_1, S_1$		$G_1$	
	$2$	$P_2, Z_2$	$W_2, (\mathcal{N}_i)_2, T_2$	$F_2$		
			$\psi_2, S_2$		$G_2$	
	$3$	$P_3, Z_3$	$W_3, (\mathcal{N}_i)_3, T_3$	$F_3$		
	$j-1$	$P_{j-1}, Z_{j-1}$	$W_{j-1}, (\mathcal{N}_i)_{j-1}, T_{j-1}$	$F_{j-1}$		
			$\psi_{j-1}, S_{j-1}$		$G_{j-1}$	
	$j$	$P_j, Z_j$	$W_j, (\mathcal{N}_i)_j, T_j$	$F_j$		
			$\psi_j, S_j$		$G_j$	
	$j+1$	$P_{j+1}, Z_{j+1}$	$W_{j+1}, (\mathcal{N}_i)_{j+1}, T_{j+1}$	$F_{j+1}$		
	$J-1$	$P_{J-1}, Z_{J-1}$	$W_{J-1}, (\mathcal{N}_i)_{J-1}, T_{J-1}$	$F_{J-1}$		
			$\psi_{J-1}, S_{J-1}$		$G_{J-1}$	
	$J$	$P_J, Z_J$	$W_J, (\mathcal{N}_i)_J, T_J$	$F_J$		

(F and G are defined on pages 17 and 18.)

The following table lists the values of the more basic variables for the 14. choice  $n = 3/2$ ,  $J = 26$ . Values of  $\bar{T}$  above 30 km were taken from the 1965 CIRAS annual mean, values at lower elevation coming from data based on statistics gathered by the Planetary Circulation Project at M.I.T. (To be precise, they were obtained from the latter as shown in a figure based on them in the thesis by A. Hollingsworth.) The static stability parameter  $S$  is defined later in equation (3.20).

$j$	$P_j$	$Z_j$	$z$ (km) (approx)	$\bar{T}_j$	$n_m$ (cm) <sup>-3</sup>	$S_j / \Delta Z$
1	.0000396	10.137	71.5	211	$136 \times 10^{13}$	$137 \times 10^{-4}$
2	.0000594	9.731	69.0	219	196	144
3	.0000891	9.326	66.3	226.5	285	154
4	.000134	8.920	63.5	234	415	161
5	.000200	8.515	60.6	241.5	600	166
6	.000301	8.109	57.6	249.5	$877 \times 10^{13}$	167
7	.000451	7.704	54.5	258.5	$126 \times 10^{14}$	174
8	.000677	7.298	51.4	267	184	217
9	.00101	6.893	48.2	267.5	274	277
10	.00152	6.488	45.1	261.5	421	302
11	.00228	6.082	42.0	254.5	$649 \times 10^{14}$	295
12	.00343	5.677	38.8	248.5	$100 \times 10^{15}$	285
13	.00514	5.271	35.9	242.5	154	277
14	.00771	4.866	33.1	237	236	272
15	.0116	4.460	30.2	231	364	269
16	.0173	4.055	27.5	225	557	261
17	.0260	3.649	24.8	219.5	$855 \times 10^{15}$	251
18	.0390	3.244	22.2	214.5	$132 \times 10^{16}$	237
19	.0585	2.838	19.6	211.5	201	217
20	.0878	2.433	17.0	210.5	302	194
21	.132	2.027	14.4	213	449	155
22	.198	1.622	11.8	222	646	125
23	.296	1.216	9.2	234	$913 \times 10^{16}$	116
24	.444	0.811	6.6	248	$130 \times 10^{17}$	104
25	.667	0.405	3.4	266	182	105
26	1.000	0.0	0.1	284	$455 \times 10^{17}$	$122 \times 10^{-4}$

3. Non-dimensional finite-difference equations.

In this section we write the basic equation in a non-dimensional form (primarily to simplify the dynamical computations) and simultaneously introduce the vertical finite-difference representation defined in section 2. We define

$$\begin{aligned} \mu &= \sin \phi \\ \nabla(\text{dim}) &= \frac{1}{a} \nabla(\text{non-dim}) \\ \nabla^2(\text{dim}) &= \frac{1}{a^2} \nabla^2(\text{non-dim}) \\ \psi(\text{dim}) &= 2\Omega a^2 \psi(\text{non-dim}) \\ X(\text{dim}) &= 2\Omega a^2 X(\text{non-dim}) \\ t(\text{dim}) &= \frac{1}{2\Omega} t(\text{non-dim}) \\ W(\text{dim}) &= 2\Omega W(\text{non-dim}) \end{aligned} \quad (3.1)$$

$$T(\text{dim}) = \bar{T}(z) + \left( \frac{4\Omega^2 a^2}{R \Delta z} \right) T(\text{non-dim})$$

In the last expression  $T(\text{dim})$  is the "total" temperature in absolute degrees,  $\bar{T}(z)$  is the "standard atmosphere" temperature (also in degrees) given in the table at the end of section 2, while the quantity  $(4\Omega^2 a^2 / R \Delta z)$   $T(\text{non-dim})$  is the variable  $T$  appearing in (1.25), having a zero horizontal average. [The total  $T(\text{dim})$  is of course used in all chemical computations.]

$$\begin{aligned} \Omega &= 7.292 \times 10^{-5} \text{ rad sec}^{-1} \\ a &= 6.371 \times 10^6 \text{ meters} \\ R &= 287 \text{ kJ ton}^{-1} \text{ deg}^{-1} \\ C_p &= (7/2) R \end{aligned} \quad (3.2)$$

One day,  $(2\pi/\Omega)$  secs, corresponds to

$$\Delta t(\text{non-dim}) = 2\Omega \left( \frac{2\pi}{\Omega} \right) = 4\pi \quad (3.3)$$

The non-dimensional  $\nabla^2$  operator is

$$\nabla^2(\cdot) = \frac{1}{\cos^2 \phi} \frac{\partial^2(\cdot)}{\partial \lambda^2} + \frac{1}{\cos \phi} \frac{\partial}{\partial \phi} \left[ \cos \phi \frac{\partial(\cdot)}{\partial \phi} \right] \quad (3.4)$$

The relation

$$PW = \nabla^2 X \quad (1.16)$$

between  $W$  and  $X$  can be used to eliminate  $X$  in favor of  $W$  [in equation (1.13)] by defining the inverse Laplacian operator

$$\begin{aligned} \mathcal{L} &\equiv \nabla^{-2} \\ X &= P \mathcal{L} W \end{aligned} \quad (3.5)$$

We also have:

$$\zeta = \nabla^2 \psi, \quad \psi = \mathcal{L} \zeta \quad (3.6)$$

A further convenient arrangement is useful for evaluating terms of the form  $\partial(PF)\partial P$ , which appears in the vertical diffusion terms for vorticity and trace substances and in the term

$$\frac{\partial X}{\partial P} = \frac{\partial}{\partial P} [P(\mathcal{L}W)]$$

in the vorticity equation (1.13). We have

$$\left[ \frac{\partial}{\partial P} (PF) \right]_j = \frac{P_{j+1/2} F_{j+1/2} - P_{j-1/2} F_{j-1/2}}{P_{j+1/2} - P_{j-1/2}} = \left( \frac{2}{2-1} \right) F_{j+1/2} - \left( \frac{1}{2-1} \right) F_{j-1/2} \quad (3.7)$$

where we have made use of (2.4).

The horizontal advection of a quantity  $F$  can be written as the

Jacobian

$$\begin{aligned} -\vec{v}_\psi \cdot \nabla F &= -\vec{k} \times \nabla \psi \cdot \nabla F = \frac{\partial F}{\partial \lambda} \frac{\partial \psi}{\partial \mu} - \frac{\partial \psi}{\partial \lambda} \frac{\partial F}{\partial \mu} \\ &\equiv J(F, \psi) \end{aligned} \quad (3.8)$$

The non-dimensional form of the vorticity equation (1.13), with regard to the subscript labelling defined in section 2, together with equation (1.21) and (3.5) - (3.8) is as follows.

For  $j = 1, 2, \dots, J-1$ :

$$\begin{aligned} \frac{\partial \zeta_j}{\partial t} &= J(\mu + \zeta_j, \psi_j) - \nabla \cdot \left\{ \mu \nabla \mathcal{L} \left[ \left( \frac{2}{2-1} \right) W_{j+1} - \left( \frac{1}{2-1} \right) W_j \right] \right. \\ &\quad \left. + \left( \frac{2}{2-1} \right) F - \left( \frac{1}{2-1} \right) F. \right\} \end{aligned} \quad (3.9)$$

$$\psi_j = \mathcal{L} \zeta_j \quad - \quad - \quad - \quad - \quad (3.10)$$

$$F_1 = 0 \quad - \quad - \quad - \quad - \quad (3.11)$$

$$F_J = -D \zeta_{J-1} \quad - \quad - \quad - \quad - \quad (3.12)$$

$$F_j = E_j (\zeta_j - \zeta_{j-1}) \quad (j = 2, 3, \dots, J-1) \quad (3.13)$$

$$E_j = (K_m)_j \div [H_0^2 2\Omega \Delta Z] \quad - \quad - \quad - \quad - \quad (3.14)$$

$$D = k_0 \div 2\Omega \quad - \quad - \quad - \quad - \quad (3.15)$$

$$W_1 = 0 \quad - \quad - \quad - \quad - \quad (3.16)$$

$$W_J = -\int \left( \frac{h}{H_0}, \psi_{J-1} \right) \quad - \quad - \quad - \quad (3.17)$$

The non-dimensional form of the "thermal wind equation" (1.9) becomes

for  $j = 2, 3, \dots, J-1$ :

$$\nabla \cdot \mu \nabla (\psi_j - \psi_{j-1}) = -\nabla^2 T_j \quad (3.18)$$

The non-dimensional form of the thermal equation (1.25) becomes

for  $j = 2, 3, \dots, J-1$ :

$$\frac{\partial T_j}{\partial t} = \frac{1}{2} \int (T_j, \psi_j + \psi_{j-1}) - S_j W_j + \left[ \frac{R \Delta Z}{c_p 8 \Omega^3 a^2} \right] q_j \quad (3.19)$$

where

$$S_j = \left( \frac{R \Delta Z}{4 \Omega^2 a^2} \right) \left[ \frac{d\bar{T}}{dz} + \frac{R}{c_p} \bar{T} \right]_j \quad (3.20)$$

is tabulated at the end of section 2 (but without the factor  $\Delta Z$ ).  $q_j$  the rate of heating per unit mass, is still in dimensional form in (3.19). It is considered later in section 5.

The trace substance equation is

For  $j = j_0, j_0+1, \dots, J-1$ :

$$\frac{\partial \kappa_j}{\partial t} = \frac{1}{2} \int (\kappa_j \psi_j + \psi_{j-1}) - W_j \left( \frac{d\bar{\kappa}}{dz} \right)_j + \left( \frac{2}{2-1} \right) G_j - \left( \frac{1}{2-1} \right) G_{j-1} + \frac{1}{2\Omega} \left[ \frac{1}{n_m} \left( \frac{d\kappa}{dt} \right)_c \right]_j \quad (3.21)$$

$$G_j = D_j (\kappa_{j+1} - \kappa_j) \quad ; \text{ for } j = j_0, \dots, J-2.$$

$$D_j = (K_d)_{j+1/2} \div (2\Omega H_0^2 \Delta Z). \quad (3.22)$$

[The vertical diffusion coefficient  $K_d$  is defined at the  $Z_j$ -levels corresponding to  $j = \text{integer plus } 1/2$ , whereas the vertical exchange coefficient  $K_m$  for vorticity appearing in (3.14) is defined at integer values of  $j$ ]. At the bottom, the relation (1.38) gives

$$G_{J-1} = - \frac{\kappa_{J-1}}{\left[ \frac{2\Omega H_0}{d} + \frac{2\Omega H_0^2 \Delta Z}{(K_d)_{J-1/2}} \right]} \quad (3.23)$$

The integer  $j_0$  sets the level above which (3.21) may be replaced by a photochemical equilibrium statement, as discussed near the end of section one.

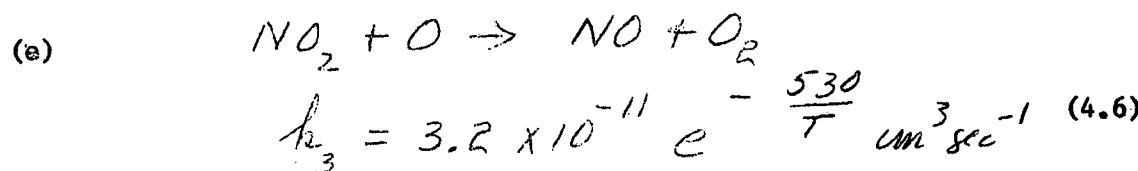
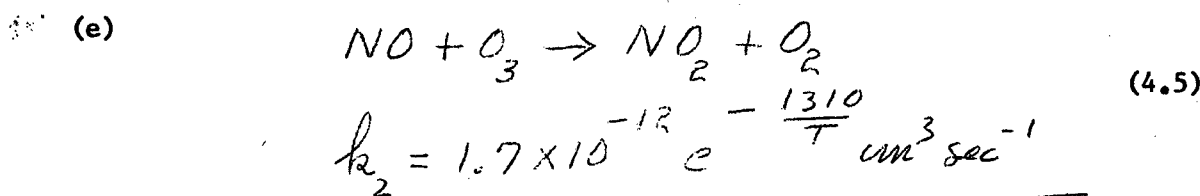
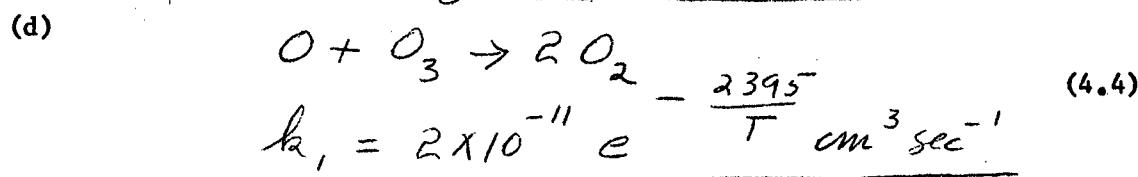
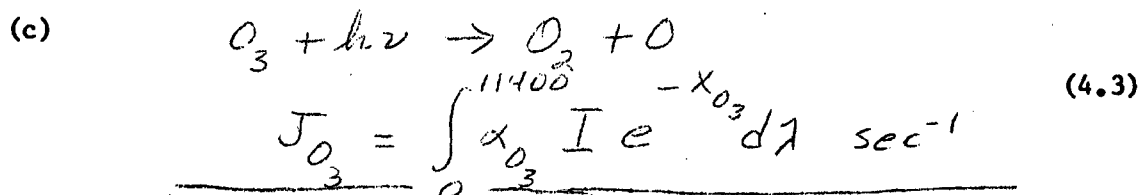
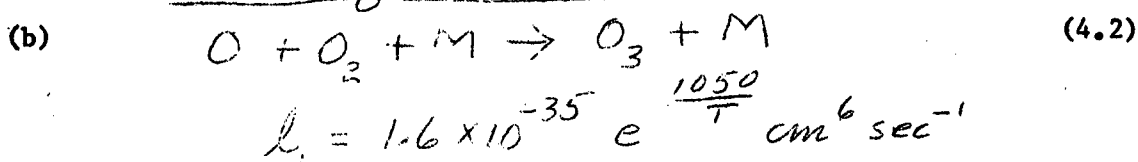
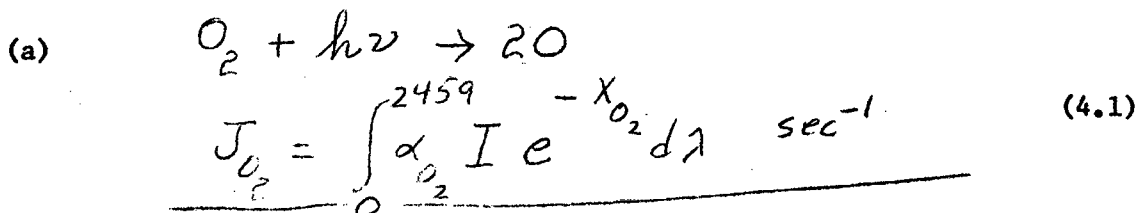


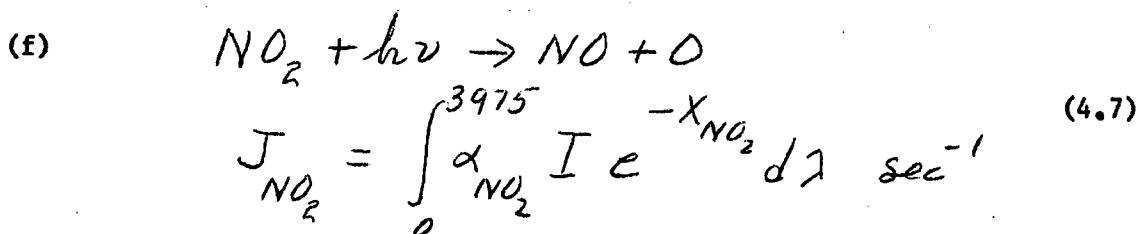
4. Photochemistry.

To begin with, the photochemistry will involve only oxygen and odd nitrogen compounds, with the latter being specified as given functions of pressure (or possibly pressure and latitude).  
 pressure  $\wedge$  (Equations to predict the nitrogen compounds may be added later.)

The reactions involved are the four main Chapman reactions and reactions of NO and NO<sub>2</sub> with oxygen. (The following write-up is based on analysis by R. Prinn.)

## Reference:





In the three radiation integrals the exponent  $X$  is

$$X_i = \frac{\alpha_i(\lambda) N_i}{\cos \psi} \quad (4.8)$$

where  $\alpha_i(\lambda)$  is the absorption cross-section,  $N_i$  is the number of  $i$ -molecules in the  $\text{cm}^2$  vertical column above the point [See equation (1.35)] and  $\psi$  is the solar zenith angle. (The diurnal variation of  $\psi$  will be averaged as described below.)  $I$  is the incident solar radiation, a function of the wavelength  $\lambda$ .

#### References:

- (a) Kockarts, 1970: Proc. 4th Joint ESRO-ESRIN Symposium (ed. G. Fiocco).  
D. Reidel, Dordrecht, 1971.

Ackerman, 1970: As for Kockarts.

- (b) Johnston, 1971: Science, 173, P. 517.

Schiff, 1969: Can. J. Chem., 47, P. 1903.

( $l_1$  is an average of their values)

- (c) Ackerman, 1970: As for Kockarts.

- (d) Schiff, 1969: As above.

- (e) Schofield, 1967: Planet. Space Sci., 15, P. 643.

- (f) Hall and Blacet, 1952: J. Chem. Phys. 20, P. 1745.

NOTE ADDED Sept. 25, 1972: The CIAP newsletter dated Sept. 8, 1972 reports two revised rate constants (Davis, Herron and Huie) as follows:

$$k_3 = (9.12 \pm 0.44) \times 10^{-12} \text{ cm}^3 \text{ sec}^{-1}$$

$$k_1 = (6.19 \pm 0.50) \times 10^{-35} e^{\frac{523 \pm 19}{T}} \text{ cm}^6 \text{ sec}^{-1}$$

with the result that  $k_3$  is about twice as large and  $k_1$  half as large as the values cited above on page 19. Use of these new values would modify the ratio  $K$  defined in (4.19) and (4.21) below.

Typical Variation of rate coefficients with height  
and from day to night based upon results in Crutzen (1971)

Coefficient	Height (km)				Comments CGS units
	50	40	30	20	
$J_{O_2}$	$10^{-9}$	$3 \times 10^{-10}$	$2 \times 10^{-11}$	$4 \times 10^{-14}$	day only
$J_{O_2}[O_2]$	$4 \times 10^6$	$5 \times 10^6$	$1.6 \times 10^6$	$1.4 \times 10^4$	day only(=0 at night)
$\rho_1[O_2][M]$	0.08	1.3	32	600	same day and night
$J_{O_3}$	$8 \times 10^{-3}$	$5.5 \times 10^{-3}$	$3 \times 10^{-3}$	$1.3 \times 10^{-3}$	day only
$J_{O_3}[O_3]$	$5 \times 10^8$	$3 \times 10^9$	$7 \times 10^9$	$5 \times 10^9$	day only
[O] (day)	$9 \times 10^9$	$7 \times 10^8$	$4 \times 10^7$	$1.3 \times 10^6$	
(night)	0	0	0	0	
[O <sub>3</sub> ] (day)	$7 \times 10^{10}$	$6 \times 10^{11}$	$2.6 \times 10^{12}$	$3.8 \times 10^{12}$	
(night)	$1.4 \times 10^{11}$	$6 \times 10^{11}$	$2.6 \times 10^{12}$	$3.8 \times 10^{12}$	
[NO] (day)	$7 \times 10^8$	$2 \times 10^9$	$3 \times 10^9$	$4 \times 10^9$	
(night)	0	0	0	0	
[NO <sub>2</sub> ] (day)	$2 \times 10^7$	$1 \times 10^9$	$8 \times 10^9$	$1 \times 10^{10}$	
(night)	$5 \times 10^8$	$2 \times 10^9$	$6 \times 10^9$	$1 \times 10^{10}$	
$J_{NO_2}$	$7 \times 10^{-3}$	$7 \times 10^{-3}$	$7 \times 10^{-3}$	$6 \times 10^{-3}$	day only
$k_3[O]$	$45 \times 10^{-3}$	$35 \times 10^{-4}$	$20 \times 10^{-5}$	$6.5 \times 10^{-6}$	( $k_3 = 5 \times 10^{-12}$ )
$k_3[O] + J_{NO_2}$	$5 \times 10^{-2}$	$1 \times 10^{-2}$	$7 \times 10^{-3}$	$6 \times 10^{-3}$	
$k_2[O_3]$	$7 \times 10^{-4}$	$6 \times 10^{-3}$	$2.6 \times 10^{-2}$	$3.8 \times 10^{-2}$	( $k_2 = 10^{-14}$ )

Coefficient	Height (km)				Comments CGS units
	50	40	30	20	
$\frac{k_3[O] + J_{NO_2}}{k_2[O_3]}$	70	2	0.3	.15	day only
$\bar{T}$	267	250	231	212	
$k_1$	$2.5 \times 10^{-15}$	$1.4 \times 10^{-15}$	$0.6 \times 10^{-15}$	$0.2 \times 10^{-15}$	
$k_3$	$0.44 \times 10^{-11}$	$0.38 \times 10^{-11}$	$0.32 \times 10^{-11}$	$0.26 \times 10^{-11}$	
$k_1[O_3] + k_3[NO_2]$	$19 \times 10^{-4}$	$65 \times 10^{-4}$	$240 \times 10^{-4}$	$270 \times 10^{-4}$	
$K$	$2.4 \times 10^{-2}$	$5 \times 10^{-3}$	$7 \times 10^{-4}$	$5 \times 10^{-4}$	
$J_{O_2}[O_2] + J_{O_3}[O_3]$	$0.8 \times 10^{-2}$	$1.7 \times 10^{-3}$	$2.3 \times 10^{-4}$	$2.8 \times 10^{-4}$	

The equation for  $d[O]/dt$  (we denote temporarily number densities  $n_i$  by a square bracket) is

$$\frac{d[O]}{dt} = 2[O_2]J_{O_2} + [O_3]J_{O_3} + [NO_2]J_{NO_2} - [O] \left\{ l_1[O_2][M] + k_1[O_3] + k_3[NO_2] \right\} \quad (4.9)$$

( $[M]$  is the neutral number density, equal to  $n_m$  in the table at the end of section 2.) The term  $l_1[O_2][M]$  increases with decreasing elevation from a minimum of  $10^{-3} \text{ sec}^{-1}$  at 70 km, and reaches a value as large as  $40 \text{ sec}^{-1}$  at 30 km. We can therefore assume equilibrium for  $[O]$ :

$$[O] = \frac{2[O_2]J_{O_2} + [O_3]J_{O_3} + [NO_2]J_{NO_2}}{L(1+K)} \quad (4.10)$$

where

$$L = l_1[O_2][M] \quad (4.11)$$

$$K = \frac{k_1[O_3] + k_3[NO_2]}{l_1[O_2][M]}$$

A similar thing happens in the equation for  $[NO]$ ,

$$\frac{d[NO]}{dt} = k_3[NO_2][O] + J_{NO_2}[NO_2] - [NO] \left\{ k_2[O_3] \right\} \quad (4.12)$$

where the product  $k_2[O_3]$  is again large enough ( $10^{-3} \text{ sec}^{-1}$  at 50 km,  $.04 \text{ sec}^{-1}$  at 20 km) to allow us to assume  $d[NO]/dt = 0$ . Thus

$$[NO] = \left\{ \frac{k_3[O] + J_{NO_2}}{k_2[O_3]} \right\} [NO_2] \quad (4.13)$$

In the nighttime  $[NO]$  vanishes according to this expression. In the daytime the factor multiplying  $[NO_2]$  in (4.13) has the typical values 70, 2, 0.3 and 0.15 at 50, 40, 30 and 20 km (see table). We may then argue that at the levels of major  $[O_3]$ , the major portion of  $[NO + NO_2]$  averaged over a day is in the form of  $[NO_2]$ . That is to say, an assumption that  $[NO + NO_2]$  is a given function of pressure can be replaced by an assumption that  $[NO_2]$  is an assigned function of pressure (possibly a different function, of course).

Calculations by Johnston (1971; Science, 173, P. 517) and Crutzen (1971; J. Geophys. Res., 76, P. 7311) suggest that the total odd nitrogen mixing ratio  $([NO] + [NO_2]) \div [M]$  increases considerably with height. These results do depend however on assumptions about the vertical eddy diffusion coefficient and the set of assumed reactions and reaction rates, (see Crutzen, *ibid* plus 1972, AMBIO, in press). The calculations by Crutzen (1971), which include  $HNO_3$  suggest that  $[NO_2] \div [M]$  is approximately constant above about 25 km while decreasing by a factor of about 10 from 25 km down to 15 km.

For the moment then, we assume that  $[NO_2]$  or  $[NO_2] \div [M]$  is a prescribed function of pressure, with  $[NO]$  determined from (4.13).

The equation for  $[O_3]$  is

$$\frac{d[O_3]}{dt} = k_1 [O_2][M][O] - J_{O_3} [O_3] - k_1 [O][O_3] - k_2 [NO][O_3] \quad (4.14)$$

Substitution for  $[O]$  and  $[NO]$  from (4.10) and (4.13) gives

$$\begin{aligned} \frac{d[O_3]}{dt} &= \frac{1-K}{1+K} 2 [O_2] J_{O_2} - \frac{2K}{1+K} ([O_3] J_{O_3} + [NO_2] J_{NO_2}) \\ &= \frac{1}{1+K} \left\{ 2 [O_2] J_{O_2} - 2K ([O_2] J_{O_2} + [O_3] J_{O_3} + [NO_2] J_{NO_2}) \right\} \quad (4.15) \end{aligned}$$

The previous table shows that  $K$  and the ratio  $( [O_2] J_{O_2} ) / ( [O_3] J_{O_3} )$  are both equally small, of order  $10^{-2}$  or less. A case can then be made for simplifying the above expression to

$$\frac{d[O_3]}{dt} \approx 2[O_2]J_{O_2} - 2K([O_3]J_{O_3} + [NO_2]J_{NO_2}) \tag{4.16}$$

where we have retained only the leading terms. In terms of the number mixing ratios  $\chi_i$ ,

$$\begin{aligned} \chi_{O_2} &= [O_2] / n_m = 0.2096 \\ \chi_{O_3} &= [O_3] / n_m \\ \chi_{NO_2} &= [NO_2] / n_m \end{aligned} \tag{4.17}$$

we have

$$\left[ \frac{1}{n_m} \frac{dn_{O_3}}{dt} \right] = 2(0.2096)J_{O_2} - 2K[\chi_{O_3}J_{O_3} + \chi_{NO_2}J_{NO_2}] \tag{4.18}$$

$$K = \frac{k_1 \chi_{O_3} + k_3 \chi_{NO_2}}{(0.2096) l_1 n_m} = K(p, T) \tag{4.19}$$

At constant pressure the logarithmic derivative of  $n_m$  with  $T$  is  $T^{-1}$ , whereas the logarithmic variation of  $(k_1/l_1)$  and  $(k_3/l_1)$  with  $T$  is  $3445/T^2 \sim 10/T$  and  $1580/T^2 \sim 5/T$  respectively. We therefore treat  $n_m$  as a known function of pressure ( $= p/kT$ ; see table at end of section 2), but will compute the exact dependence of the rate constants on  $T$ .

A proper way to compute  $K$  is suggested as follows, using the particular numerical values given in (4.2), (4.4) and (4.6). We define a scaled neutral number density  $\hat{n}$ :



$$\hat{M} \equiv 10^{-17} M_m \quad (0.0136 \leq \hat{M} \leq 455 \text{ cm}^{-3}) \quad (4.20)$$

$\hat{M}$  is stored for each  $j$ -level. Then

$$K = \frac{2 \times 10^{-11} e^{-\frac{2395}{T}} N_{O_3} + 3.2 \times 10^{-11} e^{-\frac{530}{T}} N_{NO_2}}{(0.2096) (1.6 \times 10^{-35}) e^{\frac{1050}{T}} (10^{17} \hat{M})}$$

$$= \frac{1}{\hat{M}} \left\{ \left[ 5.96374 \times 10^{-14} e^{(14 - \frac{3445}{T})} \right] e N_{O_3} + \left[ 9.54198 \times 10^{-6} e^{(6 - \frac{1580}{T})} \right] e N_{NO_2} \right\} \quad (4.21)$$

$$= \frac{1}{\hat{M}} \left\{ 49.603 e^{(14 - \frac{3445}{T})} N_{O_3} + 2.3654 \times 10^{-5} e^{(6 - \frac{1580}{T})} N_{NO_2} \right\}$$

The exponents have been chosen so that the exponent in both cases has an average value — considering variations in  $T$  from 200 to 300 — which is close to zero. They range from  $-3.5$  to  $+3$ .  $T$  itself is computed from the last equation in (3.1).

Temporarily denoting the exponent by  $\kappa$

$$\kappa = 14 - \frac{3445}{T} \quad \text{or} \quad 6 - \frac{1580}{T}$$

we can get  $e^\kappa$  efficiently by interpretation in a stored table of  $f = e^\kappa$ .

$$f_i = e^{\kappa_i}$$

$$\kappa_i = -4, -4 + \delta, -4 + 2\delta, \dots, 0, \dots, 3 - \delta, 3$$

from the formula

$$\kappa = \kappa_i + \xi, \quad 0 \leq \xi \leq \delta$$

$$f(\kappa)_{\text{inter}} = f_i [1 + \xi (A + B\xi)] \quad (4.22)$$

where  $A$  and  $B$  are constants depending only on  $\delta$  :

$$A = \frac{1}{\delta^2} [(6-2\delta)e^\delta - (6+4\delta)] \sim 1 - \frac{1}{12}\delta^2 + \dots$$

$$B = \frac{3}{\delta^3} [2 + \delta - (2-\delta)e^\delta] \sim \frac{1}{2} + \frac{\delta}{4} + \dots \quad (4.23)$$

$f(x)_{int}$  agrees with  $f_i$  and  $f_{i+1}$  when  $\xi=0$  and  $\xi=\delta$  and has the same mean value as  $f(x)$ :

$$\int_0^\delta f_{int} d\xi = f_{i+1} - f_i = \int_{x_i}^{x_{i+1}} e^x dx$$

The percentage error is

$$E = \frac{f_{int} - f}{f} = [1 + \xi(A+B\xi)]e^{-\xi} - 1 \sim (A-1)\xi + (B-A + \frac{1}{2})\xi^2 + \dots$$

$$\leq \frac{(A-1)^2}{4(B-A + \frac{1}{2})} \sim \frac{\delta^3}{144}$$

For  $\delta = 1/2$ , the percentage error is at most 0.00115:

$\xi$ :	.05	.1	.15	.2	.25	.3	.35	.4	.45
$10^5 E$ :	-89	-115	-98	-55	-3	45	79	88	64

In the photochemical radiation integrals  $J$  (and in the solar heating integrals), we have the problem of averaging over a day, during which the zenith angle varies.

$$I(x) = \int_{\lambda_1}^{\lambda_2} f(\lambda) e^{-x \alpha(\lambda)} d\lambda \quad (4.24)$$

where

$$x = N \div \cos \psi$$

$N$  = vertical column number density ( $\text{cm}^{-2}$ ) [see (1.35)]

$\psi$  = zenith angle

$\phi$  = latitude

$\delta$  = solar declination

$h$  = hour angle

$\alpha$  = cross-section

$f$  = function of  $\lambda$

Using the relation

$$\cos \psi = \sin \phi \sin \delta + \cos \phi \cos \delta \cos h \quad (4.25)$$

we find sunrise and sunset at  $h = -H$  and  $h = +H$ , with

$$\cos H = -\tan \phi \tan \delta \quad (4.26)$$

Considering only the Northern Hemisphere ( $\tan \phi \geq 0$ ) we have

$$\tan \phi \tan \delta \leq -1 : H \equiv 0 \quad (\text{Polar winter night})$$

$$-1 < \tan \phi \tan \delta < 0 \quad 0 < H < \frac{\pi}{2}$$

$$\tan \phi \tan \delta = 0 \quad H = \frac{\pi}{2} \quad (\text{Equinox})$$

$$0 < \tan \phi \tan \delta < 1 \quad \frac{\pi}{2} < H < \pi$$

$$1 \leq \tan \phi \tan \delta \quad H \equiv \pi \quad (\text{Polar summer day})$$

Using the relation  $\Omega = dh/dt$ , the 24-hour average of  $I$  is

$$\begin{aligned} I_{24} &= \frac{\Omega}{2\pi} \int_{-H}^H I dt = \frac{1}{2\pi} \int_{-H}^H I dh \\ &= \frac{1}{\pi} \int_0^H I dh \end{aligned}$$

Dividing the interval  $0 < h < H$  into ten portions gives

$$I_{24} = \frac{H}{10\pi} \sum_{i=1}^{10} I \left( X = \frac{N}{\cos \psi_i} \right)$$

$$(\cos \psi)_i = \sin \phi \sin \delta + \cos \phi \cos \delta \cos \left[ \frac{H}{10} \left( i - \frac{1}{2} \right) \right] \quad (4.27)$$

This results finally in a function of latitude, declination and  $N$ .

$$I_{24} = I(\phi, \delta, N) \quad (4.28)$$

By symmetry, Southern Hemispheric values are given by

$$I_{24}(-\phi, \delta, N) = I_{24}(\phi, -\delta, N) \quad (4.29)$$

An important practical problem is to represent the function (4.28) at each of a series of selected latitudes as a moderately simple function of  $\delta$  (i.e., time of year) and  $N$ . Our current approach to this problem is to construct and store a table for  $I(X)$  as a function of  $X$  and then to use expression (4.27) to value  $I$ . Now this technique will only work provided that within each absorption band a single species of atom is responsible for the absorption of each incident solar photon.

Let us consider each of the photodissociation rates  $J_{O_2}$ ,  $J_{O_3}$ , and  $J_{NO_2}$ . There exists a small amount of overlap between the absorption bands of  $O_2$  and  $O_3$  and between  $NO_2$  and  $O_3$ . Now the absorption by  $NO_2$  is sufficiently weak that it does not significantly reduce the number of photons available for absorption at any altitude of the atmosphere. Thus for  $NO_2$ ,  $I(X)$  may be written as a function of the column concentration of  $O_3$  only. The absorption bands of  $O_2$  and  $O_3$  overlap in the region 2000-2450 Å. For ozone strong absorption also occurs in the region 2450-3200 Å and because there is more

solar energy in this portion of the spectrum than in the overlap region, absorption by  $O_2$  only reduces  $J_{O_3}$  at any altitude by a few % at most. Therefore  $I(X)$  for  $O_3$  will be calculated as a function of the columnar concentration of  $O_3$  only. Finally considering the photodissociation of  $O_2$ , it is found that at low altitudes at which absorption in the range 2000-2450 Å is significant, photodissociation of  $O_3$  is a more effective source of O atoms by 3 orders of magnitude. Therefore for  $O_2$ ,  $I(X)$  will be calculated as a function of the columnar concentration of  $O_2$  atoms only with the integration over wavelength being truncated with an upper limit of 2000 Å.

## 5. Heating

The heating of the atmosphere between the ground and 70 km consists of latent heat release, the absorption of short wave solar energy, and the absorption and remission of long wave radiation. The absorption of short wave radiation by ozone (including absorption in the Chappuis band) and the small contribution resulting from absorption by molecular oxygen may be treated in a straightforward manner.

$$Q_{O_3} = [O_3] \int_{2000 \text{ \AA}}^{7500 \text{ \AA}} \alpha_{O_3} I \varepsilon e^{-X_{O_3}} d\lambda \quad \text{joules/cm}^2/\text{sec}$$

where  $\varepsilon$  is the energy of a photon. A similar expression may be written down for heating by  $O_2$  - which makes a significant contribution to total heating only at altitudes above 50 km. It is convenient as it was in the discussion of photodissociation rates to construct a table of  $Q_{O_3}$  as a function of the columnar concentration of  $O_3$  and similarly to calculate  $Q_{O_2}$  as a function of the  $O_2$  column concentration. In these cases also the daily average values of these heating rates are needed rather than the instantaneous values; thus, by analogy with (4.27), the average value  $Q_{24}$  may be derived from

$$Q_{24} = \frac{H}{10\pi} \sum_{i=1}^{10} Q \left( X = \frac{\alpha N}{\cos \psi_i} \right)$$

with  $\cos \psi_i = \sin \phi \sin \delta + \cos \phi \cos \delta \cos \left[ \frac{H}{10} \left( i - \frac{1}{2} \right) \right]$  (5.1)

A simple algorithm for representing infrared heating and cooling is more difficult to derive. Our current approach which will be used in initial runs of the computer model is to separate the atmosphere into two regions - above and below 30 km - and to represent these terms slightly differently in the two regions. Above 30 km we plan to use a Newtonian cooling approximation with different coefficients at each altitude of the model. Kuhn and London have kindly supplied us with their detailed results of infrared cooling by  $\text{CO}_2$ ,  $\text{H}_2\text{O}$ , and  $\text{O}_3$  at each latitude and each altitude above 30 km. We have plotted total cooling versus temperature at each altitude and have made a linear fit to the resulting data. It has been found that straight line representations on the average account for approximately 50 % of the variance of the data points. Although absorption in the  $9.6\mu$  band of  $\text{O}_3$  is not the major cooling source above 30 km, it makes a sufficiently large contribution that it appears desirable to relate this cooling to the concentration of ozone molecules. Possible ways of doing this are under investigation at the present time.

Below 30 km we plan to represent infrared heating and latent heat release as Trenberth (M. I. T. thesis, 1972) did by a single linear term of the form

$$h_j (T_j^* - T)$$

where  $j$  refers to the altitude level  $j$ . The equilibrium temperature distribution,  $T^*$ , is obtained from calculations by Manabe and Strickler (J. A. S., 21, 361, 1964) in which using

climatological distributions of minor constituents a value of  $T^*$  corresponding to zero net heating was derived. Newell et. al. (RMS London, 1970) have made daily heating rate calculations including boundary layer heating and latent heat release. These computations may be used to derive values of  $h_j$ .



6. Determination of  $W$  in the dynamic equations.

31.

Defining

$$\begin{aligned} \epsilon &= \frac{r}{r-1} > 1 \quad (r = e^{\Delta z}) \\ \mathcal{M} &= \nabla \cdot \mu \nabla \mathcal{L} \\ \mathcal{L} &= \nabla^{-2} \\ \xi_j &= \nabla^2 \psi_j \end{aligned} \tag{6.1}$$

the basic equations can be written as

$$j=1, \dots, J-1: \quad \frac{\partial \xi_j}{\partial t} = A_j - \mathcal{M} [\epsilon W_{j+1} - (\epsilon-1) W_j] \tag{6.2}$$

$$j=2, \dots, J-1: \quad \mathcal{M} (\xi_j - \xi_{j-1}) = -\nabla^2 T_j \tag{6.3}$$

$$j=2, \dots, J-1: \quad \frac{\partial T_j}{\partial t} = -S_j W_j + B_j \tag{6.4}$$

where  $A_j$  and  $B_j$  symbolize the following quantities:

$$A_j = \int (\mu + \xi_j, \psi_j) + \epsilon F_{j+1} - (\epsilon-1) F_j \tag{6.5}$$

$$B_j = \frac{1}{2} \int (T_j, \psi_j + \psi_{j-1}) + \left[ \frac{R \Delta z}{c_p \rho \Omega^3 a^2} \right] \delta_j \tag{6.6}$$

As boundary conditions we have

$$\begin{aligned} W_1 &\equiv 0 \\ W_J &= -\int \left( \frac{h}{H_0}, \psi_{J-1} \right) \end{aligned} \tag{6.7}$$

Note that  $\mathcal{L}$  is not defined when it acts on a constant function. Considering

$A_j$ ,  $B_j$  and  $W_J$  as known (at a given time step), the problem is to solve for  $\partial \xi_j / \partial t$  and  $W_j$ . We propose to eliminate  $\frac{\partial S}{\partial t} = \nabla^2 \frac{\partial \psi}{\partial t}$  in

favor of  $W$ . Having found  $W$ ,  $\partial S/\partial t$  is determinable from (6.2) and can be used to extrapolate  $S$  in time.

In general we will represent  $S$ ,  $T$ ,  $W$  as a series in spherical harmonics  $\checkmark$

$$S_j(\lambda, \mu, t) = \sum_m \sum_n \tilde{S}_{jnm}(t) \bar{P}_n^m(\mu) e^{im\lambda}$$

For concreteness, let us focus attention on only one longitudinal wave number  $m$ , and consider only those parts of  $S$ ,  $T$ ,  $W$  etc which are proportional to  $\cos m\lambda$ . Thus

$$\begin{aligned} \nabla^2 \psi_j &= S_j = \cos m\lambda \sum_{n=m}^N \tilde{S}_{jnm} \bar{P}_n^m(\mu) \\ T_j &= \cos m\lambda \sum_{n=m}^N \tilde{T}_{jnm} \bar{P}_n^m(\mu) \\ W_j &= \cos m\lambda \sum_{n=m}^N \tilde{W}_{jnm} \bar{P}_n^m(\mu) \\ A_j &= \quad \quad \quad \tilde{A}_{jnm} \quad \quad \quad \\ B_j &= \quad \quad \quad \tilde{B}_{jnm} \quad \quad \quad \end{aligned} \quad (6.8)$$

$N = m + L$

The above sum limits hold for  $m \geq 1$ . For  $m = 0$  the sum will be from  $n = 1, 2, \dots, L+1$ . Thus there will be  $L+1$  latitudinal modes for each zonal wave number  $m = 0, 1, \dots$ . The discussion below is given primarily for  $m \geq 1$ , with occasional comments for  $m = 0$  where appropriate.

From Abramowitz and Stegun (pages 332-341) we have  $\checkmark$

$$\bar{P}_n^m(\mu) = \alpha_{nm} P_n^m(\mu) \quad (6.9)$$

$$\alpha_{nm} = (-1)^m \left[ \frac{(2m+1)(n-m)!}{2(n+m)!} \right]^{1/2} \quad (6.10)$$

$\checkmark$  Trenberth uses  $P_n^m$  and  $T_n^m$  in place of  $\sqrt{2} \bar{P}_n^m$  and  $P_n^m$  used here.

$$\int_{-1}^1 P_n^m(\mu) P_l^m(\mu) d\mu = \begin{cases} 1 & \text{if } l=n \\ 0 & \text{if } l \neq n \end{cases} \quad (6.11)$$

$$(n+1-m) P_{n+1}^m = (2n+1)\mu P_n^m - (n+m) P_{n-1}^m \quad (6.12)$$

$$(\mu^2-1) \frac{dP_n^m}{d\mu} = n\mu P_n^m - (n+m) P_{n-1}^m \quad (6.13)$$

The  $\nabla^2$  operator introduces a coefficient  $-m(m+1)$  inside each sum.

The  $\mathcal{M}$  operator is more complicated, however:

$$\mathcal{M}F = \mu F + (1-\mu^2) \frac{\partial}{\partial \mu} (\mathcal{L}F) \quad (6.14)$$

Applying this to the  $W_j$  sum in (6.8) we get, by using (6.9) - (6.14)

$$\begin{aligned} \mathcal{M}W_j &= \cos m\lambda \sum_{j=mm}^N \tilde{W}_j \left[ \mu \bar{P}_n^m + \frac{(\mu^2-1)}{n(n+1)} \frac{d\bar{P}_n^m}{d\mu} \right] \\ &= \cos m\lambda \sum_{n=m}^N \tilde{W}_j \left[ \beta_{nm} \bar{P}_{n-1}^m + \gamma_{nm} \bar{P}_{n+1}^m \right] \end{aligned} \quad (6.15)$$

where

$$\beta_{nm} = \left( \frac{n-1}{n} \right) \sqrt{\frac{n^2-m^2}{4n^2-1}} \quad (6.16)$$

$$\gamma_{nm} = \frac{n+2}{n} \beta_{n+1m} \quad (6.17)$$

$$\beta_{mm} = 0, \quad \gamma_{m-1m} = 0 \quad (6.18)$$

Note that  $\beta_{m,m}$  and  $\gamma_{m-1,m} = 0$ , so that  $\bar{P}_{m-1}^m$  does not appear in (6.15). We also drop the  $\bar{P}_{N+1}^m$  term. Thus we have (suppressing the  $j$ -subscript)

$$\mathcal{M}W = \cos m\lambda \sum_{n=m}^N \left[ \gamma_{m-1,m} \tilde{W}_{m-1,m} + \beta_{m+1,m} \tilde{W}_{m+1,m} \right] \bar{P}_n^m \quad (6.19)$$

in which

$$\gamma_{m-1,m} = 0, \quad \tilde{W}_{N+1,m} = 0$$

Combining this with (6.2) and (6.8), and using the orthonormal properties of

$\bar{P}_n^m$  we have

$$\begin{aligned} \frac{\partial \tilde{S}}{\partial t}_{j,m,m} &= \tilde{A}_{j,m,m} - \epsilon \left[ \gamma_{m-1,m} \tilde{W}_{j+1,m-1} + \beta_{m+1,m} \tilde{W}_{j+1,m+1} \right] \\ &\quad + (\epsilon - 1) \left[ \gamma_{m-1,m} \tilde{W}_{j,m-1} + \beta_{m+1,m} \tilde{W}_{j,m+1} \right] \end{aligned} \quad (6.20)$$

Consider now the  $\mathcal{M}$  operator on  $\tilde{S}$  in (6.3). By similarity with  $\mathcal{M}W$

we get

$$\mathcal{M}\tilde{S} = \cos m\lambda \sum_{n=m}^N \left[ \gamma_{m-1,m} \tilde{S}_{m-1,m} + \beta_{m+1,m} \tilde{S}_{m+1,m} \right] \bar{P}_n^m \quad (6.21)$$

in which again

$$\gamma_{m-1,m} = 0, \quad \tilde{S}_{N+1,m} = 0$$

Combining (6.3) and (6.4) to eliminate  $T_j$  gives

$$S_j \nabla^2 W_j = \nabla^2 B_j + \mathcal{M} \left( \frac{\partial S_j}{\partial t} - \frac{\partial S_{j-1}}{\partial t} \right) \quad (6.22)$$

which in terms of the  $\bar{P}_n^m$  series gives

$$-m(m+1) S_j \tilde{W}_{j,m,m} = -m(m+1) \tilde{B}_{j,m,m} \quad (6.23)$$

$$+ \gamma_{m-1,m} \left[ \frac{\partial \tilde{S}}{\partial t}_{j,m-1,m} - \frac{\partial \tilde{S}}{\partial t}_{j-1,m-1,m} \right] + \beta_{m+1,m} \left[ \frac{\partial \tilde{S}}{\partial t}_{j,m+1,m} - \frac{\partial \tilde{S}}{\partial t}_{j-1,m+1,m} \right]$$

Equations (6.20) and (6.23) together with the boundary conditions

$$\begin{aligned} \tilde{W}_{1mm} &\equiv 0 \\ \tilde{W}_{Jmm} &= \text{known} \end{aligned} \tag{6.24}$$

form a system of  $(J-1)(L+1) + (J-2)(L+1) \approx 294$  simultaneous equations (if  $J=26$  and  $L=5$ ) for  $\partial \tilde{S}/\partial t$  and  $\tilde{W}$ .

We want a less demanding solution than the straightforward matrix inversion of this system. The philosophy is suggested by the usual Cartesian "beta-plane" approximations in which the  $\mu = \sin \phi$  appearing explicitly in  $\mathcal{M} = \nabla \cdot \mu \nabla \mathcal{L}$  is replaced by a constant  $\mu_0$  so that  $\mathcal{M}$  is simply  $\mu_0$ . If in that system the horizontal variation is expressed by sines and cosines, with  $\nabla^2 = -k^2$ , the  $\partial \tilde{S}/\partial t$  and  $W$  equations can be combined into an operator on  $W_j$  alone

$$\nabla^2 W_j + \frac{\mu_0^2}{S_j} \left[ \epsilon W_{j+1} - (2\epsilon - 1) W_j + (\epsilon - 1) W_{j-1} \right] = F_j$$

and, by considering each component

$$\begin{aligned} W_j &\propto \tilde{W}_{j\alpha\beta} e^{i(\alpha x + \beta y)} \\ \alpha^2 + \beta^2 &= k^2 \end{aligned}$$

one gets

$$\begin{aligned} -\epsilon \tilde{W}_{j+1\alpha\beta} + \left[ (2\epsilon - 1) + \frac{k^2 S_j}{\mu_0^2} \right] \tilde{W}_{j\alpha\beta} - (\epsilon - 1) \tilde{W}_{j-1\alpha\beta} &= F_{j\alpha\beta} S_j / \mu_0^2 \end{aligned} \tag{6.25}$$

The "elliptic" character of this one-dimensional system allows rapid inversion for  $\tilde{W}_{j\alpha\beta}$  by the method given in Richtmyer and Morton (Difference Methods for Initial Value Problems, P. 199). About  $3J$  operations are needed instead of the  $J^2$  operations needed with a matrix inversion of (6.25). At this point it is convenient to change the  $n$  subscript to

$$\begin{aligned} l = n - m & \quad l = 0, 1, \dots, L \quad \text{for } m > 0 \tag{6.26} \\ l = 1, 2, \dots, L+1 & \quad \text{for } m = 0 \end{aligned}$$

and suppress the  $m$  subscript. We have

$$\beta_l \equiv \beta_{l+m m} = \left[ \frac{(m+l)^2 - m^2}{4(m+l)^2 - 1} \right]^{1/2} \left( \frac{l+m-1}{l+m} \right) \quad (6.27)$$

$$\gamma_l \equiv \gamma_{l+m m} = \left( \frac{m+l+2}{m+l} \right) \beta_{l+1}$$

$$\beta_0 = 0, \gamma_{-1} = 0 \text{ for } m > 0; \quad \beta_1 = 0, \gamma_0 = 0 \text{ for } m = 0$$

We also define

$$\nu_l = m(m+1) = (l+m)(l+m+1) \quad (6.28)$$

Equation (6.20) yields ( $m > 0$ )

$$\begin{aligned} l=0: \left( \frac{\partial \tilde{S}}{\partial t} \right)_{j_0} &= \tilde{A}_{j_0} - \beta_1 \left[ \epsilon \tilde{W}_{j+1,1} - (\epsilon-1) \tilde{W}_{j,1} \right] \\ l=1, \dots, L-1: \left( \frac{\partial \tilde{S}}{\partial t} \right)_{j,l} &= \tilde{A}_{j,l} - \gamma_{l-1} \left[ \epsilon \tilde{W}_{j+1,l-1} - (\epsilon-1) \tilde{W}_{j,l-1} \right] \\ &\quad - \beta_{l+1} \left[ \epsilon \tilde{W}_{j+1,l+1} - (\epsilon-1) \tilde{W}_{j,l+1} \right] \\ l=L: \left( \frac{\partial \tilde{S}}{\partial t} \right)_{j,L} &= \tilde{A}_{j,L} - \gamma_{L-1} \left[ \epsilon \tilde{W}_{j+1,L-1} - (\epsilon-1) \tilde{W}_{j,L-1} \right] \end{aligned} \quad (6.29)$$

while (6.23) gives

$$\begin{aligned} l=0: -S_j \nu_0 \tilde{W}_{j_0} &= -\nu_0 \tilde{B}_{j_0} + \beta_1 \left[ \frac{\partial \tilde{S}}{\partial t} \Big|_{j_1} - \frac{\partial \tilde{S}}{\partial t} \Big|_{j_0} \right] \\ l=1, \dots, L-1: -S_j \nu_l \tilde{W}_{j,l} &= -\nu_l \tilde{B}_{j,l} + \gamma_{l-1} \left[ \frac{\partial \tilde{S}}{\partial t} \Big|_{j,l-1} - \frac{\partial \tilde{S}}{\partial t} \Big|_{j-1,l-1} \right] \\ &\quad + \beta_{l+1} \left[ \frac{\partial \tilde{S}}{\partial t} \Big|_{j,l+1} - \frac{\partial \tilde{S}}{\partial t} \Big|_{j-1,l+1} \right] \\ l=L: -S_j \nu_L \tilde{W}_{j,L} &= -\nu_L \tilde{B}_{j,L} + \gamma_{L-1} \left[ \frac{\partial \tilde{S}}{\partial t} \Big|_{j,L-1} - \frac{\partial \tilde{S}}{\partial t} \Big|_{j-1,L-1} \right] \end{aligned} \quad (6.30)$$

Differencing (6.29) in  $j$  introduces the important operator

$$\mathcal{D}(\cdot)_j = \epsilon(\cdot)_{j+1} - (2\epsilon-1)(\cdot)_j + (\epsilon-1)(\cdot)_{j-1} \quad (6.31)$$

and results in the following equations:

$$\begin{aligned}
 l=0: \quad \frac{\partial \tilde{S}}{\partial t_{j_0}} - \frac{\partial \tilde{S}}{\partial t_{j_{-1}0}} &= \tilde{A}_{j_0} - \tilde{A}_{j_{-1}0} - \mathcal{D}(\beta_1 \tilde{W}_{j_1}) \\
 l=1, \dots, L-1: \quad \frac{\partial \tilde{S}}{\partial t_{j_l}} - \frac{\partial \tilde{S}}{\partial t_{j_{l-1}}} &= \tilde{A}_{j_l} - \tilde{A}_{j_{l-1}} - \mathcal{D}(\gamma_{l-1} \tilde{W}_{j_{l-1}} + \beta_{l+1} \tilde{W}_{j_{l+1}}) \\
 l=L: \quad \frac{\partial \tilde{S}}{\partial t_{j_L}} - \frac{\partial \tilde{S}}{\partial t_{j_{L-1}}} &= \tilde{A}_{j_L} - \tilde{A}_{j_{L-1}} - \mathcal{D}(\gamma_{L-1} \tilde{W}_{j_{L-1}})
 \end{aligned} \tag{6.32}$$

Using these to eliminate  $\partial \tilde{S} / \partial t$  in (6.30) gives us the desired system in  $\tilde{W}$  alone:

$$l=0: \quad \beta_1 \mathcal{D}(\gamma_0 \tilde{W}_{j_0} + \beta_2 \tilde{W}_{j_2}) - S_j \nu_0 \tilde{W}_{j_0} = -\nu_0 \tilde{B}_{j_0} + \beta_1 (\tilde{A}_{j_1} - \tilde{A}_{j_{-1}1}) \tag{6.33a}$$

$$\begin{aligned}
 l=1: \quad \mathcal{D}[(\gamma_0 \beta_1 + \gamma_1 \beta_2) \tilde{W}_{j_1} + \beta_2 \beta_3 \tilde{W}_{j_3}] - S_j \nu_1 \tilde{W}_{j_1} &= \gamma_0 (\tilde{A}_{j_0} - \tilde{A}_{j_{-1}0}) \\
 &+ \beta_2 (\tilde{A}_{j_2} - \tilde{A}_{j_{-1}2}) - \nu_1 \tilde{B}_{j_1}
 \end{aligned} \tag{6.33b}$$

$$\begin{aligned}
 l=2, \dots, L-2: \quad \mathcal{D}[\gamma_{l-2} \gamma_{l-1} \tilde{W}_{j_{l-2}} + (\gamma_{l-1} \beta_l + \gamma_l \beta_{l+1}) \tilde{W}_{j_l} + (\beta_{l+1} \beta_{l+2}) \tilde{W}_{j_{l+2}}] - S_j \nu_l \tilde{W}_{j_l} &= \\
 = -\nu_l \tilde{B}_{j_l} + \gamma_{l-1} (\tilde{A}_{j_{l-1}} - \tilde{A}_{j_{l-1}l-1}) + \beta_{l+1} (\tilde{A}_{j_{l+1}} - \tilde{A}_{j_{l+1}l+1})
 \end{aligned} \tag{6.33c}$$

$$\begin{aligned}
 l=L-1: \quad \mathcal{D}[\gamma_{L-3} \gamma_{L-2} \tilde{W}_{j_{L-3}} + (\gamma_{L-2} \beta_{L-1} + \gamma_{L-1} \beta_L) \tilde{W}_{j_{L-1}}] - S_j \nu_{L-1} \tilde{W}_{j_{L-1}} &= \\
 = -\nu_{L-1} \tilde{B}_{j_{L-1}} + \gamma_{L-2} (\tilde{A}_{j_{L-2}} - \tilde{A}_{j_{L-2}L-2}) + \beta_L (\tilde{A}_{j_L} - \tilde{A}_{j_{L-1}L})
 \end{aligned} \tag{6.33d}$$

$$\begin{aligned}
 l=L: \quad \mathcal{D}[\gamma_{L-2} \gamma_{L-1} \tilde{W}_{j_{L-2}} + \gamma_{L-1} \beta_L \tilde{W}_{j_L}] - S_j \nu_L \tilde{W}_{j_L} &= \\
 = -\nu_L \tilde{B}_{j_L} + \gamma_{L-1} (\tilde{A}_{j_{L-1}} - \tilde{A}_{j_{L-1}L-1})
 \end{aligned} \tag{6.33e}$$

(Note the equation for  $l=L$  is not obtained from the equation for  $l=2, \dots, L-2$  by simply putting  $l=L$  with  $\tilde{A}_{j,L+1}$  and  $\tilde{W}_{j,L+2} = 0$ .) We have now a system of only  $(J-2)(L+1) = 144$  equations for  $\tilde{W}_{j,l}$ .

We can go farther, however.

Let us concentrate on the case of odd  $L$ . First note that equations (6.33) divide into two independent sets for  $\tilde{W}_{j,l}$ , whether  $L$  is even or odd. For odd  $L$  —

$$\text{Set I: } l = 0, 2, \dots, L-1 \quad (m > 0) \quad (6.34)$$

$$\text{Set II: } l = 1, 3, \dots, L \quad (m > 0)$$

Let us define

$$G_l = \frac{\gamma_l \gamma_{l+1}}{\nu_{l+2}} = \frac{\beta_{l+1} \beta_{l+2}}{\nu_l} = \left[ \frac{[(m+l+1)^2 - m^2][m+l+2]^2 - m^2}{[4(m+l+1)^2 - 1][4(m+l+2)^2 - 1]} \right]^{1/2} \frac{1}{\nu_{l+1}} \quad (6.35)$$

$$H_l = \gamma_{l-1} \beta_l = \frac{\nu_l}{\nu_{l-1}} \beta_l^2 = \frac{[(l+m)^2 - m^2][l+m]^2 - 1}{[4(l+m)^2 - 1](l+m)^2} \quad (6.36)$$

$$\nu_l = (l+m)(l+m+1) \quad (6.37)$$

$$\mu_l = \frac{\gamma_l}{\nu_{l+1}} = \frac{\beta_{l+1}}{\nu_l} \quad (6.38)$$

By using the relation

$$\gamma_l = \frac{\nu_{l+1}}{\nu_l} \beta_{l+1}$$

from (6.27), equations (6.33), after division by the  $\nu_l$  appearing in each of them, can be rewritten as



$l=0:$ 

$$\mathcal{D} \left[ \frac{H_1}{2_0} \tilde{w}_{j_0} + G_0 \tilde{w}_{j_0} \right] - S_j \tilde{w}_{j_0} = -\tilde{B}_{j_0} + \mu_0 (\tilde{A}_{j_1} - \tilde{A}_{j-1,1}) \equiv \tilde{F}_{j_0} \quad (6.39a)$$

 $l=1:$ 

$$\mathcal{D} \left[ \left( \frac{H_1 + H_2}{2_1} \right) \tilde{w}_{j_1} + G_1 \tilde{w}_{j_1} \right] - S_j \tilde{w}_{j_1} = -\tilde{B}_{j_1} + \mu_0 (\tilde{A}_{j_0} - \tilde{A}_{j-1,0}) + \mu_1 (\tilde{A}_{j_2} - \tilde{A}_{j-1,2}) \equiv \tilde{F}_{j_1} \quad (6.39b)$$

 $l=2, \dots, L-2:$ 

$$\mathcal{D} \left[ G_{l-2} \tilde{w}_{j,l-2} + \left( \frac{H_l + H_{l+1}}{2_l} \right) \tilde{w}_{j,l} + G_l \tilde{w}_{j,l+2} \right] - S_j \tilde{w}_{j,l} = -\tilde{B}_{j,l} + \mu_{l-1} (\tilde{A}_{j,l-1} - \tilde{A}_{j-1,l-1}) + \mu_l (\tilde{A}_{j,l+1} - \tilde{A}_{j-1,l+1}) \equiv \tilde{F}_{j,l} \quad (6.39c)$$

$$l=L-1: \mathcal{D} \left[ G_{L-3} \tilde{w}_{j,L-3} + \left( \frac{H_{L-1} + H_L}{2_{L-1}} \right) \tilde{w}_{j,L-1} \right] - S_j \tilde{w}_{j,L} = -\tilde{B}_{j,L-1} + \mu_{L-2} (\tilde{A}_{j,L-2} - \tilde{A}_{j-1,L-2}) + \mu_{L-1} (\tilde{A}_{j,L} - \tilde{A}_{j-1,L}) \equiv \tilde{F}_{j,L-1} \quad (6.39d)$$

$$l=L: \mathcal{D} \left[ G_{L-2} \tilde{w}_{j,L-2} + \frac{H_L}{2_L} \tilde{w}_{j,L} \right] - S_j \tilde{w}_{j,L} = -\tilde{B}_{j,L} + \mu_{L-1} (\tilde{A}_{j,L-1} - \tilde{A}_{j-1,L-1}) \equiv \tilde{F}_{j,L} \quad (6.39e)$$

If we now define an index  $k$  common to both sets I and II, according to the following scheme (for  $m > 0$ ),

$k:$	1	2	3	--	$\frac{L+1}{2}$
Set I, $l=2k-2:$	0	2	4	--	$L-1$
Set II, $l=2k-1:$	1	3	5	--	$L$

(6.40)

we can define an  $(L+1)/2$  square matrix  $A_{2k}$  with non-zero elements only along the main and adjacent diagonals. These are, for the two sets I and II, as follows:

Set I ( $l=2k-2$ )Set II ( $l=2k-1$ )

$$\begin{array}{ccc}
 A_{1,1} & \frac{H_1}{\nu_0} & \frac{1}{\nu_1} (H_1 + H_2) \\
 A_{1,2} & G_0 & G_1 \\
 A_{k,k-1} & G_{l-2} & G_{l-2} \\
 A_{k,k} & \frac{1}{\nu_l} (H_l + H_{l+1}) & \frac{1}{\nu_l} (H_l + H_{l+1}) \\
 A_{k,k+1} & G_l & G_l \\
 A_{\frac{L+1}{2}, \frac{L-1}{2}} & G_{L-3} & G_{L-2} \\
 A_{\frac{L+1}{2}, \frac{L+1}{2}} & \frac{1}{\nu_{L-1}} (H_{L-1} + H_L) & \frac{1}{\nu_L} H_L
 \end{array} \tag{6.41}$$

We also designate  $\tilde{F}_{j,k}$  as the proper one of the  $\tilde{F}_{j,l}$  in (6.39) according to the scheme (6.40). Equations (6.39) can then be expressed as a single matrix equation, of the same symbolic form for Set I and II:

$$(\mathcal{D} A_{kr} - S_j S_{kr}) \tilde{W}_{jr} = \tilde{F}_{j,k} \tag{6.42}$$

where  $S_{kr}$  is the unit matrix.  $A_{kr}$  is symmetric and has a set of  $(L+1)/2$  real positive eigenvalues  $\lambda_p$  and orthonormal eigenvectors

$$Q_r^p \left[ r=1, 2, \dots, \frac{L+1}{2}; p=1, 2, \dots, \frac{L+1}{2} \right].$$

$$A_{kn} Q_n^p = \lambda_p Q_k^p \quad (6.43a)$$

$$\sum_{n=1}^{(L+1)/2} Q_n^p Q_n^q = \delta_{pq} \quad (6.43b)$$

We express  $\tilde{W}_{jn}$  as a sum of  $Q_n^p$

$$\tilde{W}_{jn} = \sum_{p=1}^{(L+1)/2} V_j^p Q_n^p \quad (6.44)$$

insert this into (6.42), use (6.43a), multiply by  $Q_k^g$  and sum over  $k$  and use (6.43b) to get

$$\lambda_g \mathcal{D} V_j^g - S_j V_j^g = \sum_{k=1}^{L+1} Q_k^g \tilde{F}_{jk} \equiv -\lambda_g E_j^g \quad (6.45)$$

For ease in applying the Richtmyer-Morton method we rewrite this by dividing by  $-\lambda_g E_j^g$  to obtain [Refer to the definition of  $\mathcal{D}$  in (6.31)]:

$$-V_{j+1}^g + \left[ \frac{2\epsilon-1}{\epsilon} + \frac{S_j}{\epsilon \lambda_g} \right] V_j^g - \left( \frac{\epsilon-1}{\epsilon} \right) V_{j-1}^g = D_j^g \quad (6.46)$$

With  $\lambda$  positive, this satisfies the requirements of the R.-M. method.

To use it we need for each  $g$  a pre-computed table of  $E_j^g$ :

$$\begin{aligned} j=1: & \quad E_1^g = 0 \\ j=2, \dots, J-1: & \quad E_j^g = \epsilon \left[ (2\epsilon-1) + \frac{S_j}{\lambda_g} - (\epsilon-1) E_{j-1}^g \right] \quad (6.47) \end{aligned}$$

Given these, we first compute  $F_j^g$ :

$$\begin{aligned} j=1: & \quad F_1^g = 0 \\ j=2, \dots, J-1: & \quad F_j^g = E_j^g \left[ D_j^g + \left( \frac{\epsilon-1}{\epsilon} \right) F_{j-1}^g \right] \quad (6.48) \end{aligned}$$

Then, in the reverse order  $j=J-1, J-2, \dots, 2$  we get  $V_j^g$  by

$$V_J = \sum_k \tilde{W}_{Jk} Q_k^g \quad (6.49)$$

$$V_j^{\theta} = E_j^{\theta} V_{j+1}^{\theta} + F_j^{\theta} \quad j = J-1, \dots, 2 \quad (6.50)$$

The required operations (multiplication plus addition) are  $3(J-2)$  in number.

To compare the matrix inversion with the eigenfunction method we have (defining 1 "operation" as 1 multiplication plus 1 addition), for  $J=26$  and  $L=5$ :

Matrix method: 2 sets of equations, each set  $(J-2) \left(\frac{L+1}{2}\right)$  in number;

$$2(J-2)^2 \left(\frac{L+1}{2}\right)^2 = 10,368 \text{ operations} \quad (6.51)$$

Eigenfunction method:

For each function --

"Right hand side":  $(J-1) \left(\frac{L+1}{2}\right)$  (includes  $V_J^{\theta}$ )

Inversion by R.&M. method:  $3(J-2)$

For  $(L+1)$  functions --  $(L+1) \left[ (J-2) \left(3 + \frac{L+1}{2}\right) + \frac{L+1}{2} \right]$

(coefficients of)

Reanalysis into  $(L+1)$  Legendre functions for  $j=2, \dots, J-1$ :

$$2 \left(\frac{L+1}{2}\right)^2 (J-2)$$

Total:  $(L+1) \left[ (J-2)(L+4) + \frac{L+1}{2} \right] = 1,314 \quad (6.52)$

A factor of 7.9 ( $\approx (J-2)/3$ ) is achieved in speed.

The eigenfunction method will also require storage of the  $\lambda'_2$  and their associated eigenvectors. For each  $m$  value there are  $(L+1)$   $\lambda'_2$  and  $(L+1)$  eigenvectors, each of the latter having  $(L+1)/2$  components.

Total storage required for each value of  $m$  is

$$\lambda'_2: (L+1), \text{ Eigenvectors: } \frac{1}{2} (L+1)^2, E_j: (L+1)(J-1)$$

The above analysis is for  $m > 0$ , where  $n = m + l$ ,  $l = 0, 1, \dots, L$ .  
 For  $m = 0$  we have instead the indices  $n = l = 1, 2, \dots, L + 1$ . The  
 definitions and relations (6.34) - (6.50) are similar for  $m = 0$  except  
 that  $l$  is increased by one

$$\text{Set I: } l = 2k - 1 \quad (m = 0)$$

$$\text{Set II: } l = 2k$$

The two sets for each  $m$  correspond to distributions of  $W_j$  which are either  
 even or odd functions of  $\mu$  (sine of latitude).

It is useful at this point to consider the energy integral for a wave number  $m$ . We expect the time-rate-of-change of the kinetic energy in a fixed zonal wave number  $m$  to be proportional to the  $m$ -sum of

$$\left( \sum_{j,m} \frac{\partial \tilde{S}_{j,m}}{\partial t} \right) (P_{j+1} - P_j) \div \nu_m, \text{ since}$$

$$\frac{\partial}{\partial t} \left( \frac{1}{2} [\nabla \psi]^2 \right) = \nabla \cdot \psi \nabla \frac{\partial \psi}{\partial t} - \psi \frac{\partial \nabla^2 \psi}{\partial t}.$$

By introducing

$$\tilde{\psi}_{j,m} = - \frac{1}{\nu_m} \tilde{S}_{j,m}$$

and using the relation

$$e W_{j+1} - (e-1) W_j = (P_{j+1} W_{j+1} - P_j W_j) \div (P_{j+1} - P_j)$$

we can write the vorticity equation (6.20) as

$$\begin{aligned} \frac{(P_{j+1} - P_j)}{\nu_m} \sum_{j,m} \frac{\partial \tilde{S}_{j,m}}{\partial t} &= \frac{P_{j+1} - P_j}{\nu_m} \tilde{S}_{j,m} \tilde{A}_{j,m} + \gamma_{m-1} \tilde{\psi}_{j,m} (P_{j+1} \tilde{W}_{j+1} - P_j \tilde{W}_{j,m-1}) \\ &+ \beta_{m+1} \tilde{\psi}_{j,m} (P_{j+1} \tilde{W}_{j+1,m+1} - P_j \tilde{W}_{j,m+1}). \end{aligned} \quad (6.53)$$

[We suppress the zonal wave number index  $m$ ].

By defining

$$\begin{aligned} \tilde{S}_{j,m} &= \tilde{\psi}_{j,m} - \tilde{\psi}_{j-1,m} \\ q_m &= \sum_{j=1}^{J-1} \left( \frac{P_{j+1} - P_j}{\nu_m} \right) \tilde{S}_{j,m} \frac{\partial \tilde{S}_{j,m}}{\partial t} \\ a_m &= \sum_{j=1}^{J-1} \frac{P_{j+1} - P_j}{\nu_m} \tilde{S}_{j,m} \tilde{A}_{j,m}; \quad g_m = \gamma_{m-1} \tilde{\psi}_{j-1,m} \tilde{W}_{j-1,m-1} + \beta_{m+1} \tilde{\psi}_{j-1,m} \tilde{W}_{j,m+1} \end{aligned}$$

and summing over  $j$ , we obtain

$$q_n = a_n + \gamma_n - \gamma_{n-1} \sum_{j=2}^{J-1} p_j \tilde{w}_{j, n-1} \tilde{s}_{j, n} - \beta_{n+1} \sum_{j=2}^{J-1} p_j \tilde{w}_{j, n+1} \tilde{s}_{j, n} \quad (6.54)$$

The "thermal wind equation" (6.3) and equation (6.21) together give

$$\frac{v_{n-1} \gamma_{n-1}}{v_n} \tilde{s}_{j, n-1} + \frac{v_{n+1} \beta_{n+1}}{v_n} \tilde{s}_{j, n+1} = -\tilde{T}_{j, n}$$

or, using the relation  $v_n \gamma_n = v_{n+1} \beta_{n+1}$  from (6.18),

$$\beta_n \tilde{s}_{j, n-1} + \gamma_n \tilde{s}_{j, n+1} = -\tilde{T}_{j, n} \quad (6.55)$$

We also have (6.4) in the component form [see also (6.23)]

$$\tilde{w}_{j, n} = -\frac{1}{s_j} \frac{\partial \tilde{T}_{j, n}}{\partial t} + \frac{\tilde{B}_{j, n}}{s_j}$$

Defining also

$$B_n = -\gamma_{n-1} \sum_{j=2}^{J-1} \frac{p_j}{s_j} \tilde{B}_{j, n-1} \tilde{s}_{j, n} - \beta_{n+1} \sum_{j=2}^{J-1} \frac{p_j}{s_j} \tilde{B}_{j, n+1} \tilde{s}_{j, n}$$

allows equation (6.54) to be written as

$$q_n = a_n + B_n + \gamma_n + \sum_{j=2}^{J-1} \frac{p_j}{s_j} \tilde{\Gamma}_{j, n} \quad (6.56)$$

where

$$\tilde{\Gamma}_{j, n} = \gamma_{n-1} \tilde{s}_{j, n} \frac{\partial \tilde{T}_{j, n-1}}{\partial t} + \beta_{n+1} \tilde{s}_{j, n} \frac{\partial \tilde{T}_{j, n+1}}{\partial t} \quad (6.57)$$

We now sum (6.57) over  $n$  (at fixed  $m$ ) from  $n = m$  to  $N$ :

$$\begin{aligned} \sum_{n=m}^N P_{jm} &= \sum_{n=m}^{N-1} \gamma_n \delta_{j,n+1} \frac{\partial \tilde{T}_{jm}}{\partial t} + \sum_{n=m+1}^N \beta_n \delta_{j,n-1} \frac{\partial \tilde{T}_{jm}}{\partial t} \\ &= \gamma_m \delta_{j,m+1} \frac{\partial \tilde{T}_{jm}}{\partial t} + \beta_N \delta_{j,N-1} \frac{\partial \tilde{T}_{jm}}{\partial t} \\ &\quad + \sum_{n=m+1}^{N-1} (\gamma_n \delta_{j,n+1} + \beta_n \delta_{j,n-1}) \frac{\partial \tilde{T}_{jm}}{\partial t} \end{aligned}$$

where we have recognized that  $\tilde{T}_{jm}$  is zero for  $n = m-1$  and  $n = N+1$ .

Using (6.55), with  $\delta_{jm} = 0$  for  $n = m-1$  and  $n = N+1$ , now

gives

$$\sum_{n=m}^N P_{jm} = - \sum_{n=m}^N \tilde{T}_{jm} \frac{\partial \tilde{T}_{jm}}{\partial t}$$

Using this result in the  $n$ -sum of (6.56) finally gives

$$\begin{aligned} \frac{1}{2} \frac{\partial}{\partial t} \sum_{n=m}^N \left\{ \sum_{j=1}^{J-1} \frac{(P_{j+1} - P_j)}{2s_n} (\tilde{s}_{jm})^2 + \sum_{j=2}^{J-1} \frac{P_j}{s_j} (\tilde{T}_{jm})^2 \right\} & \quad (6.58) \\ &= \sum_{n=m}^N (A_n + B_n + J_n). \end{aligned}$$

The left-hand side defines the time rate of change of the energy-- kinetic plus available -- associated with a single zonal wave number  $m$ , and the right side (symbolically) represents the net addition of energy into this wave number from heating, friction, orography, and non-linear transfer from other wave numbers.