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Relativistic Charged-Particle Ballistics in Constant, Uniform Electrostatic and Magnetic Fields

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by

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RELATIVISTIC CHARGED-PARTICLE BALLISTICS IN CONSTANT, UNIFORM ELECTROSTATIC AND MAGNETIC FIELDS

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ABSTRACT

The trajectory of a charged particle with arbitrary initial position and velocity is completely determined in the presence of constant uniform fields of the following types: (1) pure electrostatic, (2) pure magnetic, (3) electrostatic and magnetic fields superimposed, in parallel, perpendicular, and arbitrary orientation. The treatment, which is relativistic throughout, was motivated by recent Monte Carlo studies of electron transport, involved in laser design, and is supplemented by computational methods. Radiation losses are not considered.

I. DEFINITIONS, NOTATION, AND UNITS

A particle of rest mass $m > 0$ and velocity $V = (v_x, v_y, v_z)$ has <u>mass</u>

$$M = m\gamma;\ \gamma \equiv (1 - \beta^2)^{-1/2},\ \beta \equiv v/c,\ v \equiv |V| \quad .$$

Its <u>energy</u>, <u>rest energy</u>, and <u>kinetic energy</u> are respectively

$$E = Mc^2 \quad e = mc^2 \quad k = E - e = e(\gamma - 1) \quad .$$

The <u>momentum</u> $P = MV$ satisfies the equation

$$P^2 = c^2(M^2 - m^2)$$

from which it follows that $P \cdot \dot{P} = c^2 M\dot{M} = M\dot{E} = M\dot{k}$; hence,

$$\dot{k} = \dot{P} \cdot V \quad .$$

The <u>force</u> acting on a particle is

$$F = \dot{P}$$

which implies the relation

$$\dot{k} = F \cdot V \quad .$$

Thus, if $F \cdot V \equiv 0$, k, γ, β are constant on the trajectory. Moreover, for a force such that

$$F \cdot V = [-\text{grad}\ \phi(R)] \cdot V \equiv -\dot{\phi}$$

one has $\dot{k} = -\dot{\phi}$, and $k - k_o \equiv \phi_o - \phi$.

We adopt the convenient notation

$$\overline{V} = V/c = (v_x/c,\ v_y/c,\ v_z/c) \equiv (\beta_x, \beta_y, \beta_z);\ |\overline{V}| = \beta$$

and the parameters

$$X = \gamma\beta_x, \quad Y = \gamma\beta_y, \quad Z = \gamma\beta_z \quad ,$$

which satisfy the identity

$$X^2 + Y^2 + Z^2 = \gamma^2\beta^2 = \gamma^2 - 1 \quad .$$

The independent variable $\tau = ct$ is also used, in terms of which

$$\beta_x = dx/d\tau, \quad \beta_y = dy/d\tau, \quad \beta_z = dz/d\tau \quad .$$

Finally, the variable

$$\lambda = \int_o^\tau d\tau/\gamma; \quad \gamma \geq 1$$

adopted in place of τ, greatly simplifies much of the analysis. No use is made of the Lorentz transformation.

In the derivations, the cgs-esu system of units is used exclusively. The basic relations appear in the schematic equation

$$q\mathfrak{E} = F = q\bar{V} \times \mathfrak{H}$$

where q is the charge (±) in esu, and

$\mathfrak{E}$ electrostatic field (dyne/esu = volt/cm)

$\mathfrak{H}$ magnetic field (gauss)

$q\mathfrak{E}$ electrostatic force (dyne)

$q\bar{V} \times \mathfrak{H}$ Lorentz force (dyne) .

We also introduce the constants (both in cm^{-1})

$$\varepsilon = q\mathcal{E}/e;\ \mathcal{E} = |\mathfrak{E}| \qquad \mu = qH/e;\ H = |\mathfrak{H}|,\ (e\ \text{erg}) \quad .$$

In computation, these may be evaluated numerically, in terms of $\mathcal{E}'$ in MV (million Volt)/cm, and e' in MeV, by observing that

$$\varepsilon = q\mathcal{E}/e = q\mathcal{E}'\ 10^6\ (10^8/c)/e'\ 10^6(q\ 10^8/c) = \mathcal{E}'/e'$$

$$\mu = qH/e = qH/e'\ 10^6(q\ 10^8/c) = 3 \times 10^{-4}\ H/e' \quad (H\ \text{gauss}) \ .$$

Also, we note that an equation of form

$$k - k_o = q\mathcal{E}(x - x_o)$$

implies $k - k_o = e(q\mathcal{E}/e)(x - x_o) = e(\mathcal{E}'/e')(x - x_o)$ and hence

$$k' - k'_o = \mathcal{E}'(x - x_o)$$

with k', k'_o in MeV.

Computational methods for parts II, III, IV are given in Appendix A.

II. MOTION IN CONSTANT ELECTROSTATIC FIELD

Suppose a particle of rest mass m > 0 ($e = mc^2$) and charge q starts from $R = R_o$ at time t = 0 with velocity $V_o = (v_x^o, v_y^o, v_z^o)$, and is subject thereafter to an electrostatic field $\mathfrak{E} = (\mathcal{E}\ 0,0)$, $\mathcal{E} > 0$ constant. Its trajectory is then determined by the law

$$\dot{P} = F = (q\mathcal{E},0,0) = -\ \text{grad}\ \phi; \quad \phi = -\ q\mathcal{E}x \ . \qquad (1)$$

Since $\dot{k} = F \cdot V = [-\ \text{grad}\ \phi] \cdot V = -\dot{\phi}$, we have at all times

$$k - k_o = q\mathcal{E}(x - x_o) \qquad (2)$$

$$\gamma - \gamma_o = \varepsilon(x - x_o);\ \varepsilon \equiv q\mathcal{E}/e \quad . \qquad (3)$$

In the notation of I, we may write (1) in the form

$$dX/d\tau = \varepsilon \qquad dY/d\tau = 0 \qquad dZ/d\tau = 0 \qquad (4)$$

where

$$X^2 + Y^2 + Z^2 = \gamma^2 = 1 \quad . \qquad (5)$$

Integration of (4) yields

$$X = \varepsilon\tau + X_o \qquad Y = Y_o \qquad Z = Z_o \qquad (6)$$

and hence from (5), we obtain

$$\gamma = \{(\varepsilon\tau + X_o)^2 + W_o^2\}^{1/2} \qquad (7)$$

$$W_o^2 = Y_o^2 + Z_o^2 + 1 = \gamma_o^2 - X_o^2 = \gamma_o^2\ \{1 - (\beta_x^o)^2\} \ .$$

Thus γ, x, and k are known as functions of τ, from (7), (3), (2), as is also the velocity, since (6) implies

$$\beta_x = (\varepsilon\tau + X_o)/\gamma, \quad \beta_y = Y_o/\gamma, \quad \beta_z = Z_o/\gamma \quad . \qquad (8)$$

Since $\beta_y = dy/d\tau$, etc., we find from (8) and (3) the trajectory

$$x - x_o = \varepsilon^{-1}(\gamma - \gamma_o), \quad y - y_o = Y_o\lambda, \quad z - z_o = Z_o\lambda \qquad (9)$$

$$\gamma = \{(\varepsilon\tau + X_o)^2 + W_o^2\}^{1/2}$$

$$\lambda \equiv \int_o^\tau d\tau/\gamma = \varepsilon^{-1}\ln\{[(\varepsilon\tau + X_o) + \gamma]/(X_o + \gamma_o)\} \ .$$

This is a curve in the plane of $\mathfrak{E}$ and V_o, with y, z monotone as indicated by (8). If $\beta_x^o \geqq 0$, x is increasing without bound, However, for $\beta_x^o < 0$, γ and x first decrease to their minimal values at $\tau \equiv \tau^* = -\ X_o/\varepsilon$, the turning point of the trajectory, at which

$$\beta_x^* = 0 \qquad \gamma^* = \gamma_o\{1 - (\beta_x^o)^2\}^{1/2} \geqq 1 \qquad (10)$$

$$x^* - x_o = \varepsilon^{-1}(\gamma^* - \gamma_o) < 0,\ y^* - y_o = Y_o\lambda^*,\ z^* - z_o = Z_o\lambda^*$$

$$\lambda^* = (2\varepsilon)^{-1}\ \ln\ \{(1 - \beta_x^o)/(1 + \beta_x^o)\} \quad .$$

Thereafter, γ and x increase toward + ∞.

III. MOTION IN CONSTANT MAGNETIC FIELD

A particle of charge q and velocity V in a magnetic field $\mathfrak{H}$ obeys the law

$$\dot{P} = F = q\bar{V} \times \mathfrak{H}\ ; \quad \bar{V} = V/c \qquad (11)$$

where F is the Lorentz force. Since $\dot{k} \equiv F \cdot V = 0$, all scalar parameters preserve their initial values on the resulting trajectory. Thus

$$k \equiv k_o, \qquad \gamma \equiv \gamma_o, \qquad \beta \equiv \beta_o \tag{12}$$

For a field $\mathfrak{H} = (-H,0,0)$, $H > 0$ constant, (11) may then be written as

$$d\beta_x/d\tau = 0, \quad d\beta_y/d\tau = -\omega\beta_z, \quad d\beta_z/d\tau = \omega\beta_y \tag{13}$$

$$\omega = \mu/\gamma_o \qquad \mu = qH/e \quad .$$

Thus we have at once

$$\beta_x = \beta_x^o \qquad x - x_o = \beta_x^o\tau \quad . \tag{14}$$

From (13), we also obtain

$$d^2\beta_y/d\tau^2 = -\omega^2\beta_y \qquad d^2\beta_z/d\tau^2 = -\omega^2\beta_z$$

and therefore

$$\beta_y = \beta_y^o \cos\omega\tau - \beta_z^o \sin\omega\tau, \tag{15}$$

$$\beta_z = \beta_z^o \cos\omega\tau + \beta_y^o \sin\omega\tau \quad .$$

The first constants are obviously necessary, and the second ones are obtained by substitution in (13), with $\tau = 0$.

From (14) and integration of (15) we obtain the trajectory

$$x - x_o = \beta_o a_x^o\tau \tag{16}$$

$$y - \eta_o = \mu^{-1}\gamma_o\beta_o(a_y^o \sin\omega\tau + a_z^o \cos\omega\tau) \ ,$$

$$\eta_o = y_o - \mu^{-1}\gamma_o\beta_o a_z^o$$

$$z - \zeta_o = \mu^{-1}\gamma_o\beta_o(a_z^o \sin\omega\tau - a_y^o \cos\omega\tau) \ ,$$

$$\zeta_o = z_o + \mu^{-1}\gamma_o\beta_o a_y^o$$

where $\omega = \mu/\gamma_o$, and $\psi_o = (a_x^o, a_y^o, a_z^o)$ is the initial direction.

Moreover, from (14), (15), the direction ψ at τ has components

$$a_x = a_x^o \tag{17}$$

$$a_y = a_y^o \cos\omega\tau - a_z^o \sin\omega\tau$$

$$a_z = a_z^o \cos\omega\tau + a_y^o \sin\omega\tau \quad .$$

If $\psi_o = (\pm 1,0,0)$, the trajectory is the line $x = x_o \pm \beta_o\tau$, $y = y_o$, $z = z_o$, parallel to $\mathfrak{H}$. For $\psi_o \neq (\pm 1,0,0)$, we define A, R_o, θ_o by

$$A = \{(a_y^o)^2 + (a_z^o)^2\}^{1/2}, \qquad R_o = \mu^{-1}\gamma_o\beta_o A$$

$$\sin\theta_o = a_y^o/A, \qquad \cos\theta_o = a_z^o/A$$

and write the trajectory (16) in the form

$$x - x_o = \beta_o a_x^o\tau \tag{18}$$

$$y - \eta_o = R_o \cos(\omega\tau - \theta_o)$$

$$z - z_o = R_o \sin(\omega\tau - \theta_o)$$

If $a_x^o = 0$, this is a circle in the plane $x = x_o$. Otherwise, it is a uniform circular spiral, the time of rotation being given by $\omega\tau = 2\pi$, namely

$$t = 2\pi\gamma_o/\mu c, \qquad \mu = qH/e \tag{19}$$

and the (cyclotron) frequency by

$$f = 1/t = \mu c/2\pi\gamma_o \quad . \tag{20}$$

IV. MOTION IN SIMPLY ORIENTED, SUPERIMPOSED FIELDS

A charged particle in superimposed fields $\mathfrak{E}$ and $\mathfrak{H}$ is governed by the law

$$\dot{P} = F = q\mathfrak{E} + q\overline{V} \times \mathfrak{H} \quad . \tag{21}$$

For the constant field $\mathfrak{E} = (\mathcal{E},0,0)$, $\mathcal{E} > 0$, considered here, we have $q\mathfrak{E} = -\text{grad}\,\phi$, $\phi = -q\mathcal{E}x$, and therefore $\dot{k} = F \cdot V = q\mathfrak{E} \cdot V = -\dot{\phi}$. Hence, the relations

$$k - k_o = q\mathcal{E}(x - x_o) \tag{22}$$

$$\gamma - \gamma_o = \varepsilon(x - x_o) \tag{23}$$

apply, just as in II. We obtain next the trajectories when a constant uniform field $\mathfrak{H}$ acts in directions parallel to, or perpendicular to $\mathfrak{E}$. Equations (21-23) are valid throughout this section and the next, where arbitrary orientations are studied.

A. Parallel Case

$\mathfrak{H} = (-H,0,0)$, $H > 0$ constant. (The case $(+H,0,0)$ is obtained by changing H to $-H$, and μ to $-\mu$ throughout.) In the present case, (21) may be written in the form

$$dX/d\tau = \varepsilon, \quad dY/d\tau = -\mu\gamma^{-1}Z, \quad dZ/d\tau = \mu\gamma^{-1}Y. \tag{24}$$

$$\varepsilon = q\mathcal{E}/e \qquad \mu = qH/e$$

From (24) it appears that

$$X = \varepsilon\tau + X_o \quad (25)$$

and moreover, $YdY/d\tau + ZdZ/d\tau = 0$. Hence, $Y^2 + Z^2 \equiv Y_o^2 + Z_o^2$ (constant), and therefore $\gamma^2 - 1 = X^2 + Y_o^2 + Z_o^2$, giving

$$\gamma = \{(\varepsilon\tau + X_o)^2 + W_o^2\}^{1/2}, \; W_o^2 = \gamma_o^2\{1 - (\beta_x^o)^2\} \; . \quad (26)$$

It follows from (22), (23), (25), (26), that k, x, β_x, and γ are unaffected by the field $\mathfrak{H}$, being the same functions of τ as in II.

To determine y and z, we change independent variable from τ to

$$\lambda = \lambda(\tau) = \int_o^\tau d\tau/\gamma(\tau); \quad \gamma \geq 1, \quad \lambda(0) = 0 \quad (27)$$

thus obtaining from (24) the system

$$dY/d\lambda = -\mu Z \qquad dZ/d\lambda = \mu Y \quad (28)$$

and therefore

$$d^2Y/d\lambda^2 = -\mu^2 Y \qquad d^2Z/d\lambda^2 = -\mu^2 Z \quad . \quad (29)$$

The solution of (28), (29) is

$$Y = Y_o \cos\mu\lambda - Z_o \sin\mu\lambda, \; Z = Z_o \cos\mu\lambda + Y_o \sin\mu\lambda \quad (30)$$

the second constants being obtained by substitution in (28), with $\lambda = 0$ ($\tau = 0$).

Since $\gamma(\tau)$ is known explicitly (26), so is the velocity from (25), (30), namely

$$\beta_x = (\varepsilon\tau + X_o)/\gamma, \quad \beta_y = Y/\gamma, \quad \beta_o = Z/\gamma \quad . \quad (31)$$

The functions $y(\tau)$, $z(\tau)$ are obtained by integration in (31). For example, $y - y_o = \int_o^\tau Y d\tau/\gamma = \int_0^\lambda Y d\lambda = \mu^{-1}(Y_o \sin\mu\lambda + Z_o \cos\mu\lambda) - \mu^{-1}Z_o$. In this way we arrive at the trajectory

$$x - x_o = \varepsilon^{-1}(\gamma - \gamma_o) \quad (32)$$

$$y - \eta_o = \mu^{-1}\gamma_o\beta_o(a_y^o \sin\mu\lambda + a_z^o \cos\mu\lambda);$$

$$\eta_o = y_o - \mu^{-1}\gamma_o\beta_o a_z^o$$

$$z - \zeta_o = \mu^{-1}\gamma_o\beta_o(a_z^o \sin\mu\lambda - a_y^o \cos\mu\lambda);$$

$$\zeta_o = z_o + \mu^{-1}\gamma_o\beta_o a_y^o$$

$$\varepsilon = q\mathcal{E}/e \qquad \mu = qH/e$$

$$\gamma = \{(\varepsilon\tau + X_o)^2 + W_o\}^{1/2}; \quad W_o^2 = \gamma_o^2\{1 - (\beta_x^o)^2\}$$

$$\lambda = \varepsilon^{-1} \ln\{[(\varepsilon\tau + X_o) + \gamma]/\gamma_o(1 + \beta_x^o)\} \quad .$$

Defining A, R_0, θ_o just as in (18), the present trajectory becomes

$$x - x_o = \varepsilon^{-1}(\gamma - \gamma_o) \quad (33)$$

$$y - \eta_o = R_o \cos(\mu\lambda - \theta_o)$$

$$z - \zeta_o = R_o \sin(\mu\lambda - \theta_o)$$

and the curve is seen to be a spiral on the same circular cylinder as in the absence of $\mathcal{E}$, but now of nonuniform pitch, the x-displacement being just what it was in the absence of $\mathfrak{H}$.

From (30), (31), the direction ψ at τ is found to be

$$a_x = \gamma_o^{-1}\beta_o^{-1}(\varepsilon\tau + X_o)/B \quad (34)$$

$$a_y = \{a_y^o \cos\mu\lambda - a_z^o \sin\mu\lambda\}/B$$

$$a_z = \{a_z^o \cos\mu\lambda + a_y^o \sin\mu\lambda\}/B$$

$$B = \{[\gamma_o^{-1}\beta_o^{-1}(\varepsilon\tau + X_o)]^2 + (a_y^o)^2 + (a_z^o)^2\}^{1/2}$$

B. Perpendicular case

$\mathfrak{H} = (0,H,0)$, $H > 0$ constant. The analogue of (24) is now

$$dX/d\tau = \varepsilon - \mu\gamma^{-1}Z, \quad dY/d\tau = 0, \quad dZ/d\tau = \mu\gamma^{-1}X \; . \quad (35)$$

From the Y relation we see that

$$Y = Y_o, \quad y - y_o = Y_o\lambda, \quad \lambda \equiv \int_o^\tau d\tau/\gamma \quad (36)$$

Changing variables from τ to λ, we obtain from (35)

$$dX/d\lambda = \varepsilon\gamma - \mu Z \qquad dZ/d\lambda = \mu X \quad (37)$$

and therefore

$$d^2X/d\lambda^2 + (\mu^2 - \varepsilon^2)X = 0 \quad d^2Z/d\lambda^2 + \mu^2 Z = \mu\varepsilon\gamma(\lambda). \quad (38)$$

Note here that we have used (23) to evaluate

$$d\gamma/d\lambda = (d\gamma/d\tau)(d\tau/d\lambda) = (\varepsilon dx/d\tau)(\gamma) = \varepsilon X \quad . \quad (39)$$

It can be shown, by the method of "variation of parameters," that the general solution of an equation of form

$$d^2Z/d\lambda^2 + \mu^2 Z = \mu f(\lambda), \quad \mu > 0 \text{ constant} \tag{40}$$

such as that in (38), is

$$Z = Z_o \cos\mu\lambda + B_2 \sin\mu\lambda + J \tag{41}$$

$$J = \int_o^\lambda f(\lambda') \sin \mu(\lambda - \lambda')d\lambda'$$

where $Z_o = Z(0)$, and B_2 is an undetermined constant.

We must consider separately now the three cases $\mu \gtreqless \varepsilon$, i.e., $H \gtreqless \mathcal{E}$, which have quite different solutions. Our method, in all three, consists in the steps: (a) Solution of (38a) for $X(\lambda)$, determining its constants from $X(0) = X_o$, and substitution in (37a), with $\lambda = 0$; (b) Determination of $x(\lambda)$ by integration of $X(\lambda)$; (c) Finding $\gamma(\lambda)$ from (23), and, in passing,

$$\tau = \int_o^\lambda \gamma(\lambda)d\lambda \equiv \Gamma(\lambda) \quad ;$$

(d) Solving (38b) for $Z(\lambda)$ by (41), determining B_2 by substitution in (37b); (e) Integration of $Z(\lambda)$ for $z(\lambda)$. The essential dependence of λ on τ is inherently of an implicit kind, as indicated in (c). (cf. Appendix B).

1. ($\mu = \varepsilon$). Here, (37), (38) become

$$dX/d\lambda = \mu(\gamma - Z) \qquad dZ/d\lambda = \mu X \tag{42}$$

$$d^2X/d\lambda^2 = 0 \qquad d^2Z/d\lambda^2 + \mu^2 Z = \mu^2\gamma \quad . \tag{43}$$

Following the above steps, we find

$$X = X_o + B_1\lambda; \quad B_1 = \mu(\gamma_o - Z_o) \tag{44}$$

$$x - x_o = \int_o^\tau X d\tau/\gamma = \int_o^\lambda X d\lambda = X_o\lambda + B_1\lambda^2/2 \tag{45}$$

$$\gamma = \gamma_o + \mu X_o\lambda + \mu B_1 \lambda^2/2 \tag{46}$$

$$\tau = \gamma_o\lambda + \mu X_o \lambda^2/2 + \mu B_1 \lambda^3/6 \equiv \Gamma(\lambda) \quad . \tag{47}$$

To obtain Z from (43), (41), we require

$$J = \int_o^\lambda \mu\gamma(\lambda') \sin\mu(\lambda - \lambda')d\lambda \quad .$$

Under the substitution $w = \mu(\lambda - \lambda')$, this becomes

$$J = \int_o^{\mu\lambda} \{\gamma(\lambda) - X(\lambda)w + B_1 w^2/2\mu\} \sin w dw$$

$$= \{Z_o + \mu X_o\lambda + \mu B_1\lambda^2/2\} - Z_o \cos\mu\lambda - X_o \sin\mu\lambda \quad .$$

Hence, from (41),

$$Z = \{Z_o + \mu X_o\lambda + \mu B_1\lambda^2/2\} + (B_2 - X_o)\sin\mu\lambda$$

and substitution in (42) shows that $B_2 = X_o$.

Collecting these results, we find

$$X = X_o + \mu(\gamma_o - Z_o)\lambda \qquad = \gamma\beta_x \tag{48}$$

$$Y = Y_o \qquad = \gamma\beta_y$$

$$Z = Z_o + \mu X_o\lambda + \mu^2(\gamma_o - Z_o)\lambda^2/2 = \gamma\beta_z$$

from which β_x, β_y, β_z may be found, via (46), (47).

The trajectory is then given by

$$x - x_o = X_o\lambda + \mu(\gamma_o - Z_o)\lambda^2/2 \tag{49}$$

$$y - y_o = Y_o\lambda$$

$$z - z_o = Z_o\lambda + \mu X_o \lambda^2/2 + \mu^2(\gamma_o - Z_o)\lambda^3/6$$

$$= \tau - (\gamma_o - Z_o)\lambda \quad .$$

2. ($\mu > \varepsilon$). We have to solve (37), (38), namely

$$dX/d\lambda = \varepsilon\gamma - \mu Z \qquad dZ/d\lambda = \mu X \tag{50}$$

$$d^2X/d\lambda^2 + \delta^2 X = 0 \qquad d^2Z/d\lambda^2 + \mu^2 Z = \mu\varepsilon\gamma(\lambda) \tag{51}$$

$$\delta \equiv (\mu^2 - \varepsilon^2)^{1/2} > 0 \quad .$$

Following our strategy shows that

$$X = X_o \cos\delta\lambda + B_1 \sin\delta\lambda; \; B_1 = \delta^{-1}(\varepsilon\gamma_o - \mu Z_o) \tag{52}$$

$$x - x_o = \delta^{-1}B_1 + \delta^{-1}(X_o \sin\delta\lambda - B_1 \cos\delta\lambda) \tag{53}$$

$$\gamma = \delta^{-2}\mu(\mu\gamma_o - \varepsilon Z_o) + \varepsilon\delta^{-1}(X_o \sin\delta\lambda - B_1 \cos\delta\lambda) \tag{54}$$

$$\tau = \varepsilon\delta^{-2}X_o + \delta^{-2}\mu(\mu\gamma_o - \varepsilon Z_o)\lambda - \varepsilon\delta^{-2}(X_o\cos\delta\lambda + B_1\sin\delta\lambda)$$

$$= \Gamma(\lambda) \; . \tag{55}$$

For (41), we now require $J = \varepsilon\int_o^\lambda \gamma(\lambda')\sin \mu(\lambda - \lambda')d\lambda'$ for the γ of (54), and this turns out to be

$$J = \delta^{-2}\varepsilon(\mu\gamma_o - \varepsilon Z_o) - Z_o \cos\mu\lambda - X_o \sin u\lambda -$$

$$\delta^{-1}\mu(B_1 \cos\delta\lambda - X_o \sin\delta\lambda) \quad .$$

Substitution in (41) yields

$$Z = \delta^{-2}\,\varepsilon(\mu Y_o - \varepsilon Z_o) + (B_2 - X_o)\sin\mu\lambda - \delta^{-1}\mu\,(B_1\cos\delta\lambda - X_o\sin\delta\lambda)$$

and from (50) we find $B_2 = X_o$.

Hence we have obtained

$$X = X_o\cos\delta\lambda + B_1\sin\delta\lambda;\quad B_1 = \delta^{-1}(\varepsilon Y_o - \mu Z_o) \quad (56)$$

$$Y = Y_o$$

$$Z = \delta^{-2}\,\varepsilon(\mu Y_o - \varepsilon Z_o) - \delta^{-1}\mu(B_1\cos\delta\lambda - X_o\sin\delta\lambda)$$

so the trajectory is given by

$$x - \xi_o = \delta^{-1}(X_o\sin\delta\lambda - B_1\cos\delta\lambda);\ \xi_o = x_o + \delta^{-1}B_1 \quad (57)$$

$$y - y_o = Y_o\lambda$$

$$z - \zeta_o = \delta^{-2}\varepsilon(\mu Y_o - \varepsilon Z_o)\lambda - \delta^{-2}\mu(B_1\sin\delta\lambda + X_o\cos\delta\lambda);$$

$$\zeta_o = z_o + \delta^{-2}\,\mu X_o \quad .$$

Defining A, θ_o, a, b by

$$A = (X_o^2 + B_1^2)^{1/2},\ \sin\theta_o = B_1/A,\ \cos\theta_o = X_o/A \quad (58)$$

$$a = \delta^{-2}\,\mu A > b = \delta^{-1}A$$

we may write (57) in the form

$$x - \xi_o = b\sin(\delta\lambda - \theta_o) \quad (59)$$

$$y - y_o = Y_o\lambda$$

$$z - \zeta_o = \delta^{-2}\,\varepsilon(\mu Y_o - \varepsilon Z_o)\lambda - a\cos(\delta\lambda - \theta_o) \quad .$$

This may be visualized as an elliptical spiral with axis in the direction of $\mathfrak{H}$ (i.e., Y), undergoing a "drift" in the Z direction, indicated by the first term of $z - \zeta_o$.

3. ($\mu < \varepsilon$). In this case, we solve the equations (37), (38) in the form

$$dX/d\lambda = \varepsilon Y - \mu Z \qquad dZ/d\lambda = \mu X \quad (60)$$

$$d^2X/d\lambda^2 - \delta^2 X = 0 \qquad d^2Z/d\lambda^2 + \mu^2 Z = \mu\varepsilon Y(\lambda) \quad (61)$$

$$\delta \equiv (\varepsilon^2 - \mu^2)^{1/2} > 0 \quad .$$

Following the standard method, we obtain now

$$X = A_1e^{\delta\lambda} + B_1e^{-\delta\lambda} \quad (62)$$

$$A_1 = (X_o + D_1)/2,\ B_1 = (X_o - D_1)/2,\ D_1 = \delta^{-1}(\varepsilon Y_o - \mu Z_o)$$

$$x - x_o = -\delta^{-1}D_1 + \delta^{-1}(A_1e^{\delta\lambda} - B_1e^{-\delta\lambda}) \quad (63)$$

$$Y = \delta^{-2}\,\mu(\varepsilon Z_o - \mu Y_o) + \delta^{-1}\varepsilon(A_1e^{\delta\lambda} - B_1e^{-\delta\lambda}) \quad (64)$$

$$\tau = \delta^{-2}[-\varepsilon X_o + \mu(\varepsilon Z_o - \mu Y_o)\lambda + \varepsilon(A_1e^{\delta\lambda} + B_1e^{\delta\lambda})]$$

$$= \Gamma(\lambda) \quad . \quad (65)$$

In (41), we need the J integral for $f(\lambda') = \varepsilon Y(\lambda')$ as given by (64). Making the substitution $w = \mu(\lambda - \lambda')$ we find

$$J = \delta^{-2}\varepsilon(\varepsilon Z_o - \mu Y_o) + \delta^{-1}\mu(A_1e^{\delta\lambda} - B_1e^{-\delta\lambda})$$

$$- Z_o\cos\mu\lambda - X_o\sin\mu\lambda$$

and hence, from (41),

$$Z = \delta^{-2}\varepsilon(\varepsilon Z_o - \mu Y_o) + \delta^{-1}\mu(A_1e^{\delta\lambda} - B_1e^{-\delta\lambda})$$

$$+ (B_2 - X_o)\sin\mu\lambda$$

where again $B_2 = X_o$ from (60b).

We now know that

$$X = A_1e^{\delta\lambda} + B_1e^{-\delta\lambda} \quad (66)$$

$$Y = Y_o$$

$$Z = \delta^{-2}\varepsilon(\varepsilon Z_o - \mu Y_o) + \delta^{-1}\mu(A_1e^{\delta\lambda} - B_1e^{-\delta\lambda})$$

and we infer the trajectory

$$x - \xi_o = \delta^{-1}(A_1e^{\delta\lambda} - B_1e^{-\delta\lambda});\quad \xi_o = x_o - \delta^{-1}D_1 \quad (67)$$

$$y - y_o = Y_o\lambda$$

$$z - \zeta_o = \delta^{-2}\varepsilon(\varepsilon Z_o - \mu Y_o)\lambda + \delta^{-2}\mu(A_1e^{\delta\lambda} + B_1e^{-\delta\lambda});$$

$$\zeta_o = z_o - \delta^{-2}\mu X_o \quad .$$

V. MOTION IN ARBITRARILY ORIENTED FIELDS

Having considered in §4 the parallel and perpendicular cases, it is clear that all other orientations are included if we study the motion (21)

$$\dot{P} = F = q\mathcal{E} + q\overline{V} \times \mathfrak{H}$$

where $\mathcal{E} = (\mathcal{E},0,0)$, $\mathcal{E} > 0$; $\mathfrak{H} = (HC, HS, 0)$, $H > 0$,

$$C = \cos\theta \neq 0, \quad S = \sin\theta \neq 0 \quad .$$

The relations $k - k_o = q\mathcal{E}(x - x_o)$ and $\gamma - \gamma_o = \varepsilon(x - x_o)$ of (22), (23) are still valid, and (21) now reads

$$dX/d\tau = \varepsilon - \gamma^{-1}\mu SZ, \quad dY/d\tau = \gamma^{-1}\mu CZ,$$
$$dZ/d\tau = \gamma^{-1}\mu(SX - CY) \quad . \qquad (68)$$

Denoting by primes differentiation with respect to

$$\lambda = \int_o^\tau d\tau/\gamma$$

we obtain

$$X' = \varepsilon\gamma - \mu SZ, \quad Y' = \mu CZ, \quad Z' = \mu(SX - CY) \qquad (69)$$

$$X'' = A_1X + B_1Y, \quad A_1 = \varepsilon^2 - \mu^2S^2, \quad B_1 = \mu^2SC \neq 0, (70)$$

$$C_1 = -\mu^2C^2 < 0, \quad Y'' = B_1X + C_1Y$$

$$\Delta \equiv A_1C_1 - B_1^2 = -\mu^2\varepsilon^2C^2 < 0 \qquad (70)$$

$$Z'' + \mu^2Z = \mu\varepsilon S\gamma(\lambda) \quad . \qquad (71)$$

The solution of (70) is found to be of the form

$$X = U + V \qquad Y = cU + dV \qquad (72)$$

$$U = U_1 \cos K\lambda + U_2 \sin K\lambda, \quad V = V_1e^{L\lambda} + V_2e^{-L\lambda}; \quad K,L > 0$$

$$K^2 = \frac{1}{2}\,[R + (\mu^2 - \varepsilon^2)] > 0$$

$$L^2 = \frac{1}{2}\,[R - (\mu^2 - \varepsilon^2)] > 0$$

$$R = [(\mu^2 - \varepsilon^2)^2 + 4\,\mu^2\varepsilon^2C^2]^{1/2}$$

$$c = -(A_1 + K^2)/B_1 \neq 0, \quad d = -(A_1 - L^2)/B_1 \neq 0$$

where

$$A_1 + K^2 > 0 > A_1 - L^2 \quad .$$

The constants U_i, V_i are determined by the initial conditions:

$$U_1 + V_1 + V_2 = X_o \qquad (73)$$
$$cU_1 + dV_1 + dV_2 = Y_o$$
$$KU_2 + LV_1 - LV_2 = X_o' \equiv \varepsilon\gamma_o - \mu SZ_o$$
$$cKU_2 + dLV_1 - dLV_2 = Y_o' \equiv \mu CZ_o$$

the determinant here being

$$\Delta' = 2KL(K^2 + L^2)^2/B_1^2 \neq 0 \quad .$$

Explicitly, we find

$$U_1 = -(Y_o - dX_o)/(d - c), \quad U_2 = -(Y_o' - dX_o')/K(d - c) \qquad (74)$$

$$V_1 = [L(Y_o - cX_o) + (Y_o' - cX_o')]/2L(d - c)$$
$$V_2 = [L(Y_o - cX_o) - (Y_o' - cX_o')]/2L(d - c) \quad .$$

From Y in (72) we obtain

$$y - \eta_o = cK^{-1}(U_1 \sin K\lambda - U_2 \cos K\lambda) + dL^{-1}(V_1e^{L\lambda} - V_2e^{-L\lambda}) \qquad (75)$$

$$\eta_o = y_o + Y_1, \quad Y_1 = cK^{-1}U_2 - dL^{-1}(V_1 - V_2) = (\varepsilon Z_o - \mu S\gamma_o)/\mu\varepsilon C \quad .$$

Similarly, we find

$$x - \xi_o = K^{-1}(U_1 \sin K\lambda - U_2 \cos K\lambda) + L^{-1}(V_1e^{L\lambda} - V_2e^{-L\lambda}) \qquad (76)$$

$$\xi_o = x_o + X_1, \quad X_1 = K^{-1}U_2 - L^{-1}(V_1 - V_2) = -\gamma_o/\varepsilon$$

From (76) and (23), it follows that

$$\gamma = \varepsilon K^{-1}(U_1 \sin K\lambda - U_2 \cos K\lambda) + \varepsilon L^{-1}(V_1e^{L\lambda} - V_2e^{-L\lambda}) \qquad (77)$$

$$\tau = \varepsilon[K^{-2}U_1 - L^{-2}(V_1 + V_2) - K^{-2}U(\lambda) + L^{-2}V(\lambda)] \equiv \Gamma(\lambda) \qquad (78)$$

To obtain Z from (71) and (41) requires

$$J = \int_o^\lambda \varepsilon S\gamma(\lambda') \sin \mu(\lambda - \lambda')d\lambda'$$

for the $\gamma(\lambda)$ in (77). Evaluation of J involves nothing new and we find that

$$J = J_1 \cos \mu\lambda + J_2 \sin \mu\lambda + J_3 \cos K\lambda + J_4 \sin K\lambda + J_5e^{L\lambda} + J_6e^{-L\lambda} \qquad (79)$$

$$J_1 = \varepsilon^2\mu S[U_2/K(\mu^2 - K^2) - (V_1 - V_2)/L(\mu^2 + L^2)] = -Z_o$$
$$J_2 = -\varepsilon^2S[U_1/(\mu^2 - K^2) + (V_1 + V_2)/(\mu^2 + L^2)] = -Z_o'/\mu$$
$$J_3 = -\varepsilon^2\mu SU_2/K(\mu^2 - K^2), \quad J_4 = \varepsilon^2\mu SU_1/K(\mu^2 - K^2)$$
$$J_5 = \varepsilon^2\mu SV_1/L(\mu^2 + L^2), \quad J_6 = -\varepsilon^2\mu SV_2/L(\mu^2 + L^2) \quad .$$

Hence from (41),

$$Z = (J_2 + B_2)\sin \mu\lambda + J_3 \cos K\lambda + J_4 \sin K\lambda + J_5 e^{L\lambda} + J_6 e^{-L\lambda}$$

and substitution in (69) shows that $B_2 = -J_2$.

Thus we obtain

$$Z = J_3 \cos K\lambda + J_4 \sin K\lambda + J_5 e^{L\lambda} + J_6 e^{-L\lambda} \quad (80)$$

and by integration,

$$z - \zeta_o = \varepsilon^2 \mu S[-U(\lambda)/K^2(\mu^2 - K^2) + V(\lambda)/L^2(\mu^2 + L^2)] \quad (81)$$

$$\zeta_o = Z_o + Z_1,\ Z_1$$

$$= \varepsilon^2 \mu S[U_1/K^2(\mu^2 - K^2) - (V_1 + V_2)/L^2(\mu^2 + L^2)]$$

$$= -Y_o/\mu C \quad .$$

APPENDIX A

Computational routines for computing, at $\tau = ct$, position $R = (x,y,z)$, direction $\psi = (a_x, a_y, a_z)$, and kinetic energy k' (MeV), given arbitrary initial values of these parameters. The configurations of II, III, IV are provided for. The case of perpendicular fields (IV) involves numerical solution of the equation $\tau = \Gamma(\lambda)$ for λ in terms of τ. This is discussed in Appendix B, only the case $\mu = \varepsilon$ being completely treated.

I. $\mathfrak{E} = (\mathcal{E},0,0)$

a. $\gamma_o = 1 + (k'_o/e')$

b. $\beta_o = (1 - \gamma_o^{-2})^{1/2}$

c. $A = \gamma_o/\varepsilon$ $\qquad (\varepsilon = \mathcal{E}'/e')$

d. $T = \tau/A$

e. $\beta_x^o = \beta_o a_x^o$, $\quad \beta_y^o = \beta_o a_y^o$, $\quad \beta_z^o = \beta_o a_z^o$

f. $R = [1 + T(2\beta_x^o + T)]^{1/2}$

g. $\Delta x = A(R - 1)$, $\quad x = x_o + \Delta x$, $\quad k' = k'_o + \mathcal{E}'\Delta x$

h. $L = \ln\,[(T + \beta_x^o + R)/(1 + \beta_x^o)]$

i. $\Delta y = Ab_y^o L$, $\quad y = y_o + \Delta y$

j. $\Delta z = Ab_z^o L$, $\quad z = z_o + \Delta z$

k. $B = [(T + \beta_x^o)^2 + (\beta_y^o)^2 + (\beta_x^o)^2]^{1/2} \equiv [\beta_o^2 + T(2\beta_x^o + T)]^{1/2}$

l. $a_x = (T + \beta_x^o)/B$, $\quad a_y = \beta_y^o/B$, $\quad a_z = \beta_z^o/B$.

II. $\mathfrak{H} = (-H,0,0)$

a. $\gamma_o = 1 + (k'_o/e')$, $\beta_o = (1 - \gamma_o^{-2})^{1/2}$, $\omega = \mu/\gamma_o$ $\qquad (\mu = 3 \times 10^{-4}\ H/e')$

b. $\beta_x^o = \beta_o a_x^o$, $\quad \beta_y^o = \beta_o a_y^o$, $\quad \beta_z^o = \beta_o a_z^o$

c. $C = \cos \omega\tau$, $\quad S = \sin \omega\tau$

d. $\Delta x = \beta_x^o \tau$, $\quad x = x_o + \Delta x$

e. $\Delta y = [\beta_y^o S - \beta_z^o (1 - C)]/\omega$, $\quad y = y_o + \Delta y$

f. $\Delta z = [\beta_z^o S + \beta_y^o (1 - C)]/\omega$, $\quad z = z_o + \Delta z$

g. $a_x = a_x^o$, $a_y = a_y^o C - a_z^o S$, $a_z = a_z^o C + a_y^o S$

h. $k' = k'_o$

III. $\mathfrak{H} = (-H,0,0)$, $\mathfrak{E} = (\mathcal{E},0,0)$

a. $\gamma_o = 1 + (k'_o/e')$, $\beta_o = (1 - \gamma_o^{-2})^{1/2}$, $\omega = \mu/\gamma_o$ $\qquad (\mu = 3 \times 10^{-4}\ H/e')$

b. $A = \gamma_o/\varepsilon$, $T = \tau/A$ $\qquad (\varepsilon = \mathcal{E}'/e')$

c. $\beta_x^o = \beta_o a_x^o$, $\beta_y^o = \beta_o a_y^o$, $\beta_z^o = \beta_o a_z^o$

d. $R = [1 + T(2\beta_x^o + T)]^{1/2}$

e. $\Delta x = A(R - 1)$, $x = x_o + \Delta x$, $k' = k'_o + \mathcal{E}'\Delta x$

f. $\lambda = \varepsilon^{-1} \ln[(T + \beta_x^o + R]/(1 + \beta_x^o)]$

g. $C = \cos \mu\lambda$, $S = \sin \mu\lambda$

h. $\Delta y = [\beta_y^o S - \beta_z^o(1 - C)]/\omega$, $y = y_o + \Delta y$

i. $\Delta z = [\beta_z^o S + \beta_y^o(1 - C)]/\omega$, $z = z_o + \Delta z$

j. $B = [\beta_o^2 + T(2\beta_x^o + T)]^{1/2}$

k. $a_x = (T = \beta_x^o)/B$, $a_y = (\beta_y^o C - \beta_z^o S)/B$

$a_z = (\beta_z^o C + \beta_y^o S)/B$

IV. $\mathfrak{h} = (0,H,0)$ $\quad \mathfrak{e} = (\mathcal{E},0,0)$

$H = \mathcal{E}$

a. $\gamma_o = 1 + (k_o'/e')$, $\beta_o = (1 - \gamma_o^{-2})^{1/2}$

b. $\beta_x^o = \beta_o a_x^o$, $\beta_y^o = \beta_o a_y^o$, $\beta_z^o = \beta_o a_z^o$

c. $\lambda = \Gamma^{-1}(\tau)$ (Eq. (47) cf. APP. B)

d. $\Delta x = \gamma_o \lambda[\beta_x^o + \mu(1 - \beta_z^o)\lambda/2]$, $x = x_o + \Delta x$,

$k' = k_o' + \mathcal{E}' \Delta x$

e. $\Delta y = \gamma_o \beta_y^o \lambda$, $y = y_o + \Delta y$

f. $\Delta z = \tau - \gamma_o(1 - \beta_z^o)\lambda$, $z = z_o + \Delta z$

g. $B_x = \beta_x^o + \mu(1 - \beta_z^o)\lambda$

$B_z = \beta_z^o + \mu\lambda[\beta_x^o + \mu\lambda(1 - \beta_z^o)/2]$

h. $B = [B_x^2 + (\beta_y^o)^2 + B_z^2]^{1/2}$

i. $a_x = B_x/B$, $a_y = \beta_y^o/B$, $a_z = B_z/B$.

V. $\mathfrak{h} = (0,H,0)$ $\quad \mathfrak{e} = (\mathcal{E},0,0)$

$H > \mathcal{E}$

Store: $\delta = (\mu^2 - \varepsilon^2)^{1/2}$, $\varepsilon_1 = \varepsilon/\delta$, $\mu_1 = \mu/\delta$,

$\mu_{11} = \mu/\delta^2$

a. $\gamma_o = 1 + (k_o'/e')$, $\beta_o = (1 - \gamma_o^{-2})^{1/2}$

b. $\beta_x^o = \beta_o a_x^o$, $\beta_y^o = \beta_o a_y^o$, $\beta_z^o = \beta_o a_z^o$, $B_{11} = \varepsilon_1 - \mu_1 \beta_z^o$

c. $\lambda = \Gamma^{-1}(\tau)$ (Eq. (55). Cf. APP. B)

d. $C = \cos \delta\lambda$, $S = \sin \delta\lambda$

e. $\Delta x = \gamma_o[\beta_x^o S + B_{11}(1 - C)]/\delta$,

$x = x_o + \Delta x$, $k' = k_o' + \mathcal{E}'\Delta x$

f. $\Delta y = \gamma_o \beta_y^o \lambda$, $y = y_o + \Delta y$

g. $B_{12} = \varepsilon_1(\mu_1 - \varepsilon_1 \beta_z^o)$,

$\Delta z = \gamma_o\{B_{12}\lambda + \mu_{11}[\beta_x^o(1 - C) - B_{11}S]\}$

$z = z + \Delta z$

h. $B_x = \beta_x^o C + B_{11}S$, $B_z = B_{12} - \mu_1(B_{11}C - \beta_x^o S)$

i. $B = [B_x^2 + (\beta_y^o)^2 + B_z^2]^{1/2}$

j. $a_x = B_x/B$, $a_y = \beta_y^o/B$, $a_z = B_z/B$

VI. $\mathfrak{h} = (0,H,0)$ $\quad \mathfrak{e} = (\mathcal{E},0,0)$

$H < \mathcal{E}$

Store: $\delta = (\varepsilon^2 - \mu^2)^{1/2}$, $\varepsilon_1 = \varepsilon/\delta$, $\mu_1 = \mu/\delta$,

$\mu_{11} = \mu/\delta^2$

a. $\gamma_o = 1 + (k_o'/e')$, $\beta_o = (1 - \gamma_o^{-2})^{1/2}$

b. $\beta_x^o = \beta_o a_x^o$, $\beta_y^o = \beta_o a_y^o$, $\beta_z^o = \beta_o a_z^o$, $D_{11} = \varepsilon_1 - \mu_1 \beta_z^o$

$A_{11} = (\beta_x^o + D_{11})/2$, $B_{11} = (\beta_x^o - D_{11})/2$

c. $\lambda = \Gamma^{-1}(\tau)$ (Eq. (65). Cf. APP. B)

d. $C = e^{\delta\lambda}$, $S = 1/C$

e. $\Delta x = \gamma_o(-D_{11} + A_{11}C - B_{11}S)/\delta$

$x = x_o + \Delta x$, $k' = k_o' + \mathcal{E}'\Delta x$

f. $\Delta y = \gamma_o \beta_y^o \lambda$, $y = y_o + \Delta y$

g. $D_{12} = \varepsilon_1(\varepsilon_1 \beta_z^o - \mu_1)$,

$\Delta z = \gamma_o[D_{12}\lambda + \mu_{11}(A_{11}C + B_{11}S - \beta_x^o)]$,

$z = z_o + \Delta z$

h. $B_x = A_{11}C + B_{11}S$, $B_z = D_{12} + \mu_1(A_{11}C - B_{11}S)$

i. $B = [B_x^2 + (\beta_y^o)^2 + B_z^2]^{1/2}$

j. $a_x = B_x/B$, $a_y = \beta_y^o/B$, $a_z = B_z/B$

APPENDIX B

$$\lambda = \Gamma^{-1}(\tau)$$

The routines for the perpendicular case require solution of an equation

$$\tau = \Gamma(\lambda); \quad \lambda \geq 0$$

for λ in terms of τ, the function $\Gamma(\lambda)$ being strictly increasing, with $\Gamma'(\lambda) = \gamma \geqq 1$, and $\Gamma(0) = 0$. This can be done explicitly in

Case I. The equation (47),

$$\tau = \gamma_o\lambda + \mu X_o\lambda^2/2 + \mu B_1\lambda^3/6; \quad B_1 = \mu(\gamma_o - Z_o)$$

may be written in the form

$$\xi^3 + b\xi^2 + c\xi + d = 0; \quad \xi \equiv \mu\lambda$$

$$b = 3\alpha/\beta, \quad c = 6/\beta, \quad d = -6T/\beta$$

$$\alpha \equiv \beta_x^o, \quad \beta \equiv 1 - \beta_z^o > 0, \quad T \equiv \mu\tau/\gamma_o \quad .$$

For $\xi = \eta - (\alpha/\beta)$, this becomes

$$\eta^3 + p\eta + q = 0$$

$$p = 3(2\beta - \alpha^2)/\beta^2, \; q = -6T/\beta - 2\alpha(3\beta - \alpha^2)/\beta^3 \quad .$$

Note that $p > 0$. For, $2(1 - \beta_z^o) > (\beta_x^o)^2$ follows from $(\beta_x^o)^2 \leq \beta_o^2 - (\beta_z^o)^2 = -(1 - \beta_o^2) + 1 - (\beta_z^o)^2 < 1 - (\beta_z^o)^2 < 2(1 - \beta_z^o)$. Hence $W = (p/3)^3 + (q/2)^2 > 0$, and such a cubic has just one real root, namely,

$$\eta = H + J; \; H = (-q/2 + V)^{1/3}, \; V = W^{1/2}, \; J = (-p/3)/H.$$

One may therefore obtain λ from τ by the following method:

a. $\alpha = \beta_x^o \quad \beta = 1 - \beta_z^o, \; T = \mu\tau/\gamma_o$

b. $A = 8\beta - 3\alpha^2, \; B = \alpha(3\beta - \alpha^2),$

$R = [A + T(6B + 9\beta^2 T)]^{1/2}$

c. $S = (B + 3\beta^2 T + \beta R)^{1/3}, \; H = S/\beta, \; J = (\alpha^2 - 2\beta)/\beta S$

d. $\eta = H + J, \; \xi = \eta - \left(\frac{\alpha}{\beta}\right), \; \lambda = \xi/\mu \quad .$

In cases II, III, solution of $\tau = \Gamma(\lambda)$ for λ requires approximation methods not discussed here. We make only the following observations.

Case II. Equation (55) may be written in the form

$$\delta\tau/\gamma_o = \varepsilon_1\beta_x^o + \mu_1 C_{11}\xi - \varepsilon_1(\beta_x^o \cos\xi + B_{11}\sin\xi) \equiv F(\xi);$$

$$\xi = \delta\lambda, \; \varepsilon_1 = \varepsilon/\delta, \; \mu_1 = \mu/\delta, \; \mu_1^2 - \varepsilon_1^2 = 1$$

$$C_{11} = \mu_1 - \varepsilon_1\beta_z^o > 0 \quad B_{11} = \varepsilon_1 - \mu_1\beta_z^o \quad .$$

The function $F(\xi)$ is strictly increasing, with

$$F(0) = 0, \; F'(\xi) = \gamma/\gamma_o \geq 1/\gamma_o, \; F'(0) = 1,$$

$$F''(0) = \varepsilon_1\beta_x^o \quad .$$

Case III. Equation (65) may be written as

$$\delta\tau/\gamma_o = -\varepsilon_1\beta_x^o + \mu_1 C_{11}\xi + \varepsilon_1(A_{11}e^{\xi} + B_{11}e^{-\xi}) \equiv F(\xi),$$

$$\xi = \delta\lambda, \; \varepsilon_1 = \varepsilon/\delta, \; \mu_1 = \mu/\delta \quad \varepsilon_1^2 - \mu_1^2 = 1,$$

$$C_{11} = \varepsilon_1\beta_z^o - \mu_1$$

$$A_{11} = (\beta_x^o + D_{11})/2, \quad B_{11} = (\beta_x^o - D_{11})/2,$$

$$D_{11} = \varepsilon_1 - \mu_1\beta_z^o \quad .$$

The function $F(\xi)$ is strictly increasing, with

$$F(0) = 0, \; F'(\xi) = \gamma/\gamma_o \geqq 1/\gamma_o, \; F'(0) = 1,$$

$$F''(0) = \varepsilon_1\beta_x^o \quad .$$

It can be shown* that $B_{11} < 0 < A_{11}$, and therefore $F''(\xi) = \varepsilon_1(A_{11}e^{\xi} + B_{11}e^{-\xi}) = 0$ for $e^{2\xi} = -B_{11}/A_{11} > 0$. Moreover, $-B_{11}/A_{11} > 1$ iff $A_{11} + B_{11} \equiv \beta_x^o < 0$. Thus $F(\xi)$ is concave up for all $\xi > 0$ if $\beta_x^o \geqq 0$, and has a single inflection point at

$$\zeta = (1/2)\,\ell n\,(-B_{11}/A_{11})$$

if $\beta_x^o < 0$.

(*) $0 < \gamma_o^{-2} + C_{11}^2 \equiv D_{11}^2 - [\beta_o^2 - (\beta_z^o)^2] \leqq D_{11}^2 - (\beta_x^o)^2.$

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