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Contents

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Abstract

In this paper we describe a method for determining the eigenvalues for the scalar Helmholtz equation in the elliptic cylinder coordinate system. Complex expansion parameters are considered so that numerical results for physical problems involving energy dissipation as well as energy flow can be obtained. Sample numerical results are presented for the eigenvalues, radial eigenfunctions, and angle eigenfunctions. Applications of these results are discussed.

Introduction

Elliptic cylinder functions are encountered in a large number of problems in mathematical physics.[1-16] Examples of problems involving elliptic cylinder functions are solutions of the diffusion equation and the scalar wave equation in the elliptic cylinder coordinate system. Problems in acoustics, thermal flow, gravitation, electromagnetics, and optics (among others), can thus involve elliptic cylinder functions.

The damped scalar wave equation for ϕ can be expressed as

$$\nabla^2\phi = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} + \frac{1}{h^2}\frac{\partial\phi}{\partial t} \tag{1}$$

where in electric problems, ϕ represents the electric potential (volts); in magnetic problems, ϕ is the magnetic scalar potential (ampere-turns); in thermal problems, ϕ represents the temperature (degrees Kelvin); in gravitation problems, ϕ is the gravitational potential (joule/kilogram); in vibration problems, ϕ represents the displacement (meters); and in hydrodynamics and acoustics, ϕ is the velocity potential (meter^2/second).

The diffusion equation

$$\nabla^2\phi = \frac{1}{h^2}\frac{\partial\phi}{\partial t} \tag{2}$$

and the undamped scalar wave equation

$$\nabla^2\phi = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} \tag{3}$$

are of course special cases of the damped scalar wave equation [Eq. (1)]. The quantities c^2 and h^2 are physical quantities dependent upon the properties of the medium.

By assuming ϕ is of the form

$$\phi = R(\text{space})\, T(\text{time}), \tag{4}$$

substitution of ϕ into Eq. (1) yields

$$\nabla^2 R + \gamma^2 R = 0, \tag{5}$$

and

$$\frac{d^2T}{dt^2} + \frac{c^2}{h^2}\frac{dT}{dt} + \gamma^2 c^2 T = 0 \tag{6}$$

where γ^2 is a separation constant. Using the general solution of Eq. (6), we can express ϕ via Eq. (4) as

$$\phi = R(\text{space}) \exp\left[-\frac{c^2}{2h^2}t \pm \left(\sqrt{\frac{c^4}{4h^4} - \gamma^2 c^2}\right)^{1/2} t\right]. \tag{7}$$

For the special case of the diffusion equation,

$$\phi = R(\text{space}) \exp\left[-\gamma^2 h^2 t\right] \tag{8}$$

and for the undamped wave equation, Eq.(3),

$$\phi = R(\text{space}) \exp\left[\pm j\gamma ct\right]. \tag{9}$$

It is thus seen from Eqs. (7) through (9) that solutions of the diffusion, the undamped scalar wave, and the damped scalar wave equations can be expressed as a product of a spatial variation R(space) and a characteristic time variation T(time).

For the transient response, the separation constant γ is determined by spatial boundary conditions. For problems where the interest is directed towards the steady-state response to a forced sinusoidal variation of exp [+jwt], the separation constant γ is determined by the temporal variation, Eq. (6). This leads to the results

$$\gamma^2 = \frac{\omega^2}{c^2} - \frac{j\omega}{h^2} \tag{10}$$

for the damped scalar wave equation,

$$\gamma^2 = -\frac{j\omega}{h^2} \tag{11}$$

for the diffusion equation, and

$$\gamma^2 = \frac{\omega^2}{c^2} \tag{12}$$

for the undamped scalar wave equation. From Eqs. (10) through (12), it is seen that γ^2 can be either pure real, pure imaginary, or a complex number, dependent upon the particular problem of interest.

It is helpful to provide examples of the above equations applied to physical problems. In heat flow thermal problems, the diffusion equation is pertinent, and the parameter h^2 is defined as

$$h^2 = \frac{k_t}{c_s \delta} \tag{13}$$

where h^2 is the diffusivity (m^2/sec), k_t is the thermal conductivity (watt/meter/degree), c_s is the specific heat (joule/kg/degree), and δ is the density (kg/m^3) of the medium.

In acoustical problems, the undamped scalar wave equation is pertinent, and the parameter c^2 is defined as

$$c^2 = p\frac{\gamma_t}{\delta} \tag{14}$$

where c is the velocity of propagation (m/sec), p is the equilibrium pressure (newton/m^2), γ_t is the ratio of the thermitivity (joule) at constant pressure to the thermitivity (joule) at constant volume, and δ is the equilibrium density (kg/m^3) of the medium.

In electromagnetic wave propagation problems, the damped scalar wave equation is sometimes applicable. The parameter c^2 is then defined as

$$c^2 = \frac{1}{\mu\epsilon} \tag{15}$$

where c is the velocity of propagation (m/sec), μ is the permeability of the medium (henries/m), and ϵ is the permittivity of the medium (farads/m). The parameter h^2 is defined via

$$h^2 = \frac{1}{\mu\sigma} \tag{16}$$

where h^2 is the diffusivity of the medium (m^2/sec), and σ is the conductivity of the medium (mho/m).

The temporal response of ϕ for both transient and forced sinusoidal responses has been discussed above. For those interested in more detailed examples, representative references have been listed. The remainder of this report is concerned with the spatial response, R(space), governed by Eq. (5). This equation is the scalar Helmholtz equation.

Solutions of the Scalar Helmholtz Equation

This section is concerned with solutions of the scalar Helmholtz equation [Eq. (5)] in the elliptic coordinate system. The elliptical coordinate system consists of:

- The system of confocal hyperbolas and ellipses comprising the two-dimensional elliptic coordinate system (see Fig. 1),
- The cylindrical coordinate system obtained by translating the elliptic coordinate system perpendicular to its plane of definition.

We assume that the reader is familiar with this coordinate system; if not, refer to the list of references.

The spatial variable R in the scalar Helmholtz equation [Eq. (5)] can be represented as

$$R(\text{space}) = U(\xi)V(\eta)Z(z) \tag{17}$$

where ξ, η, and z are the elliptic coordinate system coordinates. Substituting Eq. (17) into Eq. (5) yields

$$\frac{d^2Z}{dZ^2} + k_z^2\, Z = 0 \tag{18}$$

$$(\xi^2 - 1)\frac{d^2U}{d\xi^2} + \xi\frac{dU}{d\xi} - \left[b - \ell^2\left(\gamma^2 - k_z^2\right)\xi^2\right]U = 0 \tag{19}$$

$$(1 - \eta^2)\frac{d^2V}{d\eta^2} - \eta\frac{dV}{d\eta} + \left[b - \ell^2\left(\gamma^2 - k_z^2\right)\eta^2\right]V = 0. \tag{20}$$

The separation constants, k_z^2 and b, are associated with the z and elliptic cylinder variations, respectively.

The solution of Eq. (18) is

$$Z(z) = \exp\,[\pm jk_z Z]. \tag{21}$$

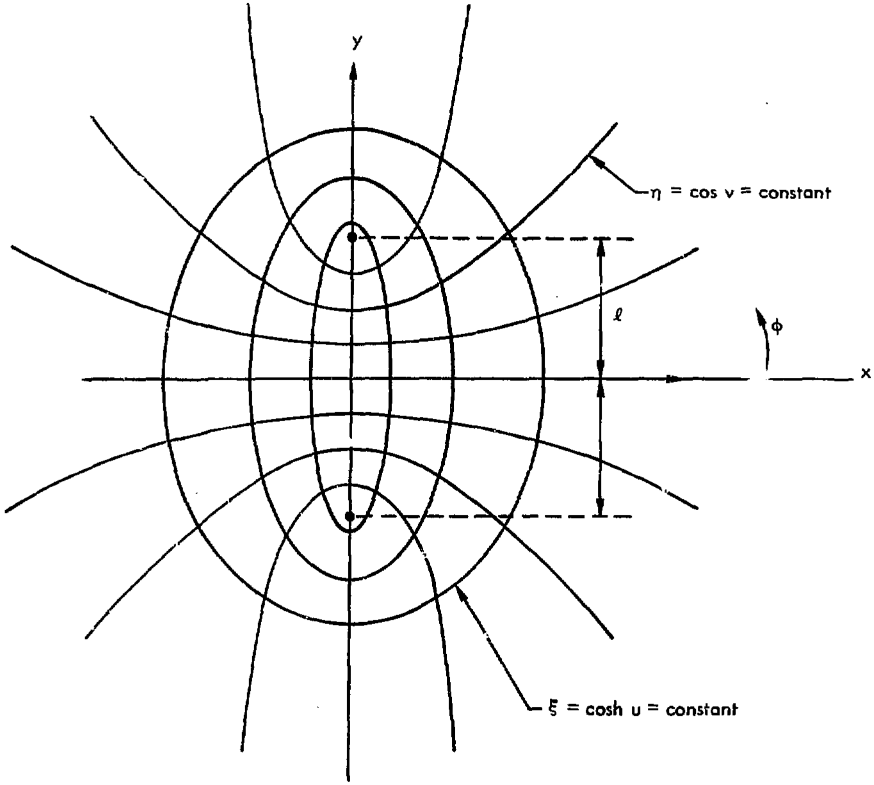

Fig. 1. The elliptic coordinate system ($x = \ell\xi\eta = \ell \cosh u \cos v$, $y = \ell\,[(\xi^2 - 1)(1 - \eta^2)]^{1/2} = \ell \sinh u \sin v$, z = constant, ℓ = semifocal length). Note: ξ describes the system of confocal hyperbolas. $\eta = \cos v$, $\xi = \cosh u$, $-1 \leq \eta \leq +1$, $0 \leq v \leq 2\pi$, $\xi \geq 1$, $u \geq 0$.

Solutions of Eqs. (19) and (20) are solutions of Mathieu's equation, as they are both of the form (letting $m = \xi$ or η and $y = U$ or V)

$$(1 - m^2)\frac{d^2y}{dm^2} - m\frac{dy}{dm} + [a + 2q - 4qm^2]y = 0 \tag{22}$$

where

$$4q = \ell^2\left(\gamma^2 - k_z^2\right) \tag{23}$$

$$b = a + 2q. \tag{24}$$

The quantity a in Eq. (22) is commonly denoted as the characteristic value (eigenvalue) of differential Eq. (22), and q is the expansion parameter of this equation. The quantity k_z in Eq. (21) is determined by the forcing function governing the $Z(z)$ variation of the physical phenomena.

The sequence by which we determine the eigenvalues and eigenfunctions for a steady-state problem (the approach followed for a transient problem is not discussed further herein) is as follows:

1. determine k_z for the Z(z) variation,
2. determine γ^2 from Eqs. (10) through (12),
3. determine q from Eq. (23),
4. determine the characteristic value (eigenvalue) a, as described below,
5. determine b from Eq. (24),
6. generate the eigenfunctions [the angle function $V(\eta)$ and the radial function $U(\xi)$] for the values η and ξ of interest.

Determination of Eigenvalues and Eigenfunctions

The eigenvalue a, the angle eigenfunction $V(\eta)$, and the radial eigenfunction $U(\xi)$ are intimately connected as can be seen by examining Eqs. (19), (20), (22), and (24). The angle function $V(\eta)$, where $\eta = \cos v$ (see Fig. 1), can be expressed as the sum of an even function in η as well as an odd function in η. The variable η (see Fig. 1) satisfies $-1 \le \eta \le 1$. Because $\eta = \cos v$, the variable v can have a period of $0 \le v \le \pi$ or $0 \le v \le 2\pi$.

This means that there are four sets of solutions to the angle differential equation [Eq. (20)]. These four solutions are:

- Even solutions of period π in v,
- Even solutions of period 2π in v,
- Odd solutions of period π in v,
- Odd solutions of period 2π in v.

These four cases each have their own set of eigenvalues. For each of the four cases of eigenvalues there are associated angle and radial functions. That is, there are four cases of angle functions and an associated set of four radial eigenfunctions.

Methods exist for determining the eigenvalues, but they are complicated to apply except for small values of q. If q is small, power series expansions have been determined for the eigenvalues.

For larger values of q, no simple and accurate method has been available for determining the eigenvalues. A simple and accurate (although approximate) method by which we can determine the eigenvalues for larger values of q follows. Note that the quantity q is in general complex, as can be seen by referring to Eqs. (10)-(12), and (23).

There is an infinite number of eigenvalues for each of the four angle functions. The sequence of eigenvalues is denoted by the integer number r, where $r \ge 0$. The notation used for the eigenvalues is a_r—to represent the even periodic solutions in v (with r = 0,1,2,...), and b_r—to represent the odd periodic solutions in v (with r = 1,2,...). This notation is consistent with that used in Ref. 7.

For each eigenvalue a_r and b_r, there is an associated angle function $V_r(\eta)$, where

$$V_r(\eta) = ce_r(v,q) = \sum_{m=0}^{\infty}{}' A_m^r \cos mv \tag{25}$$

$$V_r(\eta) = se_r(v,q) = \sum_{m=0}^{\infty}{}' B_m^r \sin mv \tag{26}$$

for solutions of period π in v (r and m even);

$$V_r(\eta) = ce_r(v,q) = \sum_{m=1}^{\infty}{}' A_m^r \cos mv \qquad (27)$$

$$V_r(\eta) = se_r(v,q) = \sum_{m=1}^{\infty}{}' B_m^r \sin mv \qquad (28)$$

for solutions of period 2π in v (r and m odd). The interrelationship between the eigenvalues a_r and expansion coefficients A_m^r, and also the eigenvalues b_r and expansion coefficients B_m^r, is determined by substituting Eqs. (25) through (28) into Eq. (20). The results are:

$$a_r A_0^r - q A_2^r = 0$$

$$(a_r - 4)A_2^r - q\left(2A_0^r + A_4^r\right) = 0$$

$$(a_r - m^2)A_m^r - q\left(A_{m-2}^r + A_{m+2}^r\right) = 0 \text{ for } m \geq 4 \qquad (29)$$

for even solutions of period π in v (r and m even);

$$(b_r - 4)B_2^r - q B_4^r = 0$$

$$\left(b_r - m^2\right)B_m^r - q\left(B_{m-2}^r + B_{m+2}^r\right) = 0 \text{ for } m \geq 4 \qquad (30)$$

for odd solutions of period π in v (r and m even);

$$(a_r - 1)A_1^r - q\left(A_1^r + A_3^r\right) = 0$$

$$\left(a_r - m^2\right)A_m^r - q\left(A_{m-2}^r + A_{m+2}^r\right) = 0 \text{ for } m \geq 3 \qquad (31)$$

for even solutions of period 2π in v (r and m odd); and

$$(b_r - 1)B_1^r + q\left(B_1^r - B_3^r\right) = 0$$

$$(b_r - m^2)\, B_m^r - q\left(B_{m-2}^r + B_{m+2}^r\right) = 0 \text{ for } m \geq 3 \qquad (32)$$

for odd solutions of period 2π in v (r and m odd).

Alternately, Eqs. (29) through (32) can be represented in matrix form as infinite tridiagonal matrices as below:

Even solutions of period π (r even)

$$\begin{pmatrix} a & -q & 0 & 0 & \cdot & \cdot & \cdot & \cdot \\ -2q & a-4 & -q & 0 & & & & \\ 0 & -q & a-16 & -q & & & & \\ 0 & 0 & -q & a-36 & & & & \\ \cdot & & & & & & & \\ \cdot & & & & & & & \\ \cdot & & & & & & & \\ \cdot & & & & & & & \end{pmatrix} \begin{pmatrix} A_0 \\ A_2 \\ A_4 \\ A_6 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} = 0 \qquad (33)$$

Odd solutions of period π (r even)

$$\begin{pmatrix} b-4 & -q & 0 & 0 & \cdot & \cdot \\ -q & b-16 & -q & 0 & & \\ 0 & -q & b-36 & -q & & \\ 0 & 0 & -q & b-64 & & \\ \cdot & & & & & \\ \cdot & & & & & \end{pmatrix} \begin{pmatrix} B_2 \\ B_4 \\ B_6 \\ B_8 \\ \\ \\ \end{pmatrix} = 0 \qquad (34)$$

Even solutions of period 2π (r odd)

$$\begin{pmatrix} a-1-q & -q & 0 & 0 & \cdot & \cdot & \cdot \\ -q & a-9 & -q & 0 & & & \\ 0 & -q & a-25 & -q & & & \\ 0 & 0 & -q & a-49 & & & \\ \cdot & & & & & & \\ \cdot & & & & & & \\ \cdot & & & & & & \end{pmatrix} \begin{pmatrix} A_1 \\ A_3 \\ A_5 \\ A_7 \\ \\ \\ \\ \end{pmatrix} = 0 \qquad (35)$$

Odd solutions of period 2π

$$\begin{pmatrix} b-1+q & -q & 0 & 0 & \cdot & \cdot & \cdot \\ -q & b-9 & -q & 0 & & & \\ 0 & -q & b-25 & -q & & & \\ 0 & 0 & -q & b-49 & & & \\ \cdot & & & & & & \\ \cdot & & & & & & \\ \cdot & & & & & & \end{pmatrix} \begin{pmatrix} B_1 \\ B_3 \\ B_5 \\ B_7 \\ \\ \\ \\ \end{pmatrix} = 0 \qquad (36)$$

Equations (34) through (36) are infinite, symmetric, tridiagonal matrices whereas Eq. (33) is an infinite, nonsymmetric, tridiagonal matrix. The eigenvalues of symmetric, tridiagonal, finite matrices are easy to generate. The eigenvalues of nonsymmetric, tridiagonal, finite matrices are more complicated to generate. However, numerical solutions are feasible[10] for each of the Eqs. (33) through (36), as long as one truncates the infinite matrix to finite order.

The truncation process, of course, introduces errors into the solutions for a_r and b_r for infinite matrices. However, this error can be made very small. It can be shown that for small q and large r,

$$\begin{matrix} a_r \\ b_r \end{matrix} = r^2 + \frac{q^2}{2(r^2-1)} + \frac{(5r^2+7)q^4}{32(r^2-1)^3(r^2-4)} + \ldots \qquad (37)$$

Thus, for $r^2 >>> |q|$, an approximate value for a_r and b_r is r^2. This is just the term on the diagonal of Eqs. (33) through (36) for large r. Thus, for this large r, one may assume that there is very little coupling of the diagonal term of the matrix with any off-diagonal terms.

This means that the higher order terms are decoupled from the lower order terms, and thus one can truncate the matrices to a large but finite size without significantly affecting the numerical results for a_r and b_r. The criterion for generating accurate results for the lower order a_r and b_r is that the finite matrices include terms up to $r^2 >>> |q|$.

After using matrix eigenvalue methods to evaluate a_r and b_r, one can determine the relative values of the expansion parameters A_m^r and B_m^r through the use of Eqs. (33) through (36). The normalization of the A_m^r and B^r coefficients is governed by

$$2\left(A_0^r\right)^2 + \left(A_2^r\right)^2 + \left(A_4^r\right)^2 + \ldots = 1,$$

$$\sum_{s=0}^{\infty}{}' A_s^r > 0$$

$$\left(A_1^r\right)^2 + \left(A_3^r\right)^2 + \ldots = 1, \quad \sum_{s=1}^{\infty}{}' A_s^r > 0$$

$$\left(B_1^r\right)^2 + \left(B_3^r\right)^2 + \ldots = 1, \quad \sum_{s=1}^{\infty}{}' sB_s^r > 0$$

$$\left(B_2^r\right)^2 + \left(B_4^r\right)^2 + \ldots = 1, \quad \sum_{s=2}^{\infty}{}' sB_s^r > 0 \, . \qquad (38)$$

By using these expansion parameters, we can now generate the angle functions of Eqs. (25) through (28). For each angle function, there is an associated radial function $U(\xi)$ governed by Eq. (19). Radial functions $U_r(\xi)$, for the r<u>th</u> eigenvalue, analogous to the angle functions of Eqs. (25) through (28) are

$$U_r(\xi) = Ce_r(u,q) = \sum_{m=0}^{\infty}{}' A_m^r \cosh mu \qquad (39)$$

$$U_r(\xi) = Se_r(u,q) = \sum_{m=2}^{\infty}{}' B_m^r \sinh mu \qquad (40)$$

for solutions of period π in v (r and m even);

$$U_r(\xi) = Ce_r(u,q) = \sum_{m=1}^{\infty}{}' A_m^r \cosh mu \qquad (41)$$

$$U_r(\xi) = Se_r(u,q) = \sum_{m=1}^{\infty}{}' B_m^r \sinh mu \qquad (42)$$

for solutions of period 2π in v (r and m odd), where $\xi = \cosh u$ and $\mu = \cosh^{-1}\xi = \ln[\xi + (\xi^2 - 1)^{1/2}]$.

A general solution of the scalar Helmholtz equation expressed in terms of the ce_r, Ce_r, se_r, and Se_r eigenfunctions is

$$R(\text{space}) = Z(z)\Bigg\{\sum_{r=0}^{\infty}{}' [E_r Ce_r(u,q)ce_r(v,q) + F_r Se_r(u,q)se_r(v,q)] + \sum_{r=1}^{\infty}{}' [G_r Ce_r(u,q)ce_r(v,q) + H_r Se_r(u,q)se_r(v,q)]\Bigg\} \qquad (4\_)$$

where the coefficients E_r, F_r, G_r, and H_r are determined by the spatial boundary conditions of the physical problem of interest.

The radial function expansions expressed in Eqs. (39) through (42) are useful for problems of interest where $\xi \leq 5$ and q is of small or moderate size ($|q| \lesssim 2$). If the physical problem involves radiation to infinity ($\xi \to \infty$), a choice of radial eigenfunctions $U_r(\xi)$ other than Eqs. (39) through (42) is desired. The reason is that the asymptotic expansion of the Ce_r and Se_r functions do not individually follow a radiation field behavior for $\xi \to \infty$ of $\exp(-j\kappa\xi)/(\kappa\xi)^{1/2}$.

A set of linearly independent radial eigenfunctions $U_r(\xi)$ that can satisfy the radiation field behavior for $\xi \to \infty$ is the set $U_r(\xi) = Mc_r(u,q)$ associated with even angle functions, and $U_r(\xi) = Ms_r(u,q)$, associated with odd angle functions. By definition,

$$U_r(\xi) = Mc_{2r}^{(j)}(u,q) = \sum_{k=0}^{\infty} (-1)^{r+k} \times \frac{A_{2k}^{2r}}{A_2^{2r}} \left[J_{k-1}(u_1)Z_{k+1}^{(j)}(u_2) + J_{k+1}(u_1)Z_{k-1}^{(j)}(u_2)\right] \qquad (44)$$

$$U_r(\xi) = Ms_{2r}^{(j)}(u,q) = \sum_{k=0}^{\infty} (-1)^{r+k} \times \frac{B_{2k}^{2r}}{B_2^{2r}} \left[J_{k-1}(u_1)Z_{k+1}^{(j)}(u_2) - J_{k+1}(u_1)Z_{k-1}^{(j)}(u_2)\right] \qquad (45)$$

for solutions associated with angle variations of period π in v;

$$U_r(\xi) = Mc^{(j)}_{2r+1}(u,q) = \sum_{k=0}^{\infty} (-1)^{r+k} \times \frac{A^{2r+1}_{2k+1}}{A^{2r+1}_{1}} \Big[J_k(u_1)Z^{(j)}_{k+1}(u_2) + J_{k+1}(u_1)Z^{(j)}_{k}(u_2)\Big] \quad (46)$$

$$U_r(\xi) = Ms^{(j)}_{2r+1}(u,q) = \sum_{k=0}^{\infty} (-1)^{r+k} \times \frac{B^{2r+1}_{2k+1}}{B^{2r+1}_{1}} \Big[J_k(u_1)Z^{(j)}_{k+1}(u_2) - J_{k+1}(u_1)Z^{(j)}_{k}(u_2)\Big] \quad (47)$$

for solutions associated with angle variations of period 2π in v, where $\xi = \cosh u$, $u_1 = \sqrt{q}\exp(-u)$, $u_2 = \sqrt{q}\exp(+u)$, $Z^{(1)}_p = J_p$, $Z^{(2)}_p = Y_p$, $Z^{(3)}_p = H^{(1)}_p = J_p + jY_p$, $Z^{(4)}_p = H^{(2)}_p = J_p - jY_p$. J_p and Y_p are cylindrical Bessel and Neumann functions of order p.

For a time variation of exp $(+j\omega t)$, use of the above radial eigenfunctions with $j = 4$ yields a radial variation (as $\xi \to \infty$) of $\exp(-j\kappa\xi)/\sqrt{\kappa\xi}$, which is the required radiation condition for $\xi \to \infty$. A general solution of the scalar Helmholtz equation in the elliptic cylinder coordinate system can thus [in addition to the Eq. (43) expansion] be represented as

$$R(\text{space}) = Z(z) \Bigg\{ \sum_{r=0}^{\infty}{}' \times \Big[E_r Mc^{(j)}_r(u,q)ce_r(v,q) + F_r Ms^{(j)}_r(u,q)se_r(v,q)\Big] + \sum_{r=1}^{\infty}{}' \Big[G_r Mc^{(j)}_r(u,q)ce_r(v,q) + H_r Ms^{(j)}_r(u,q)se_r(v,q)\Big]\Bigg\}, \quad (48)$$

where the coefficients E_r, F_r, G_r and H_r are determined by the spatial boundary conditions of the problem of interest.

Sample Numerical Results

From the general expression for q given in Eq. (20), it is seen that physical problems can lead to a wide variety of q values of interest. The quantity q can be small, moderate, or large in magnitude, and can be a complex number if the medium is dissipative. Some sample numerical results for the eigenvalues a_r and b_r (obtained using the matrix method described herein) are given in Table 1. Sample numerical results for the expansion coefficients A^r_m and B^r_m are given in Table 2. Three-dimensional plots of the angle and radial eigenfunctions are shown in Fig. 2. These results have been compared with available tabular and qualitative results.[3, 4, 7, 16]

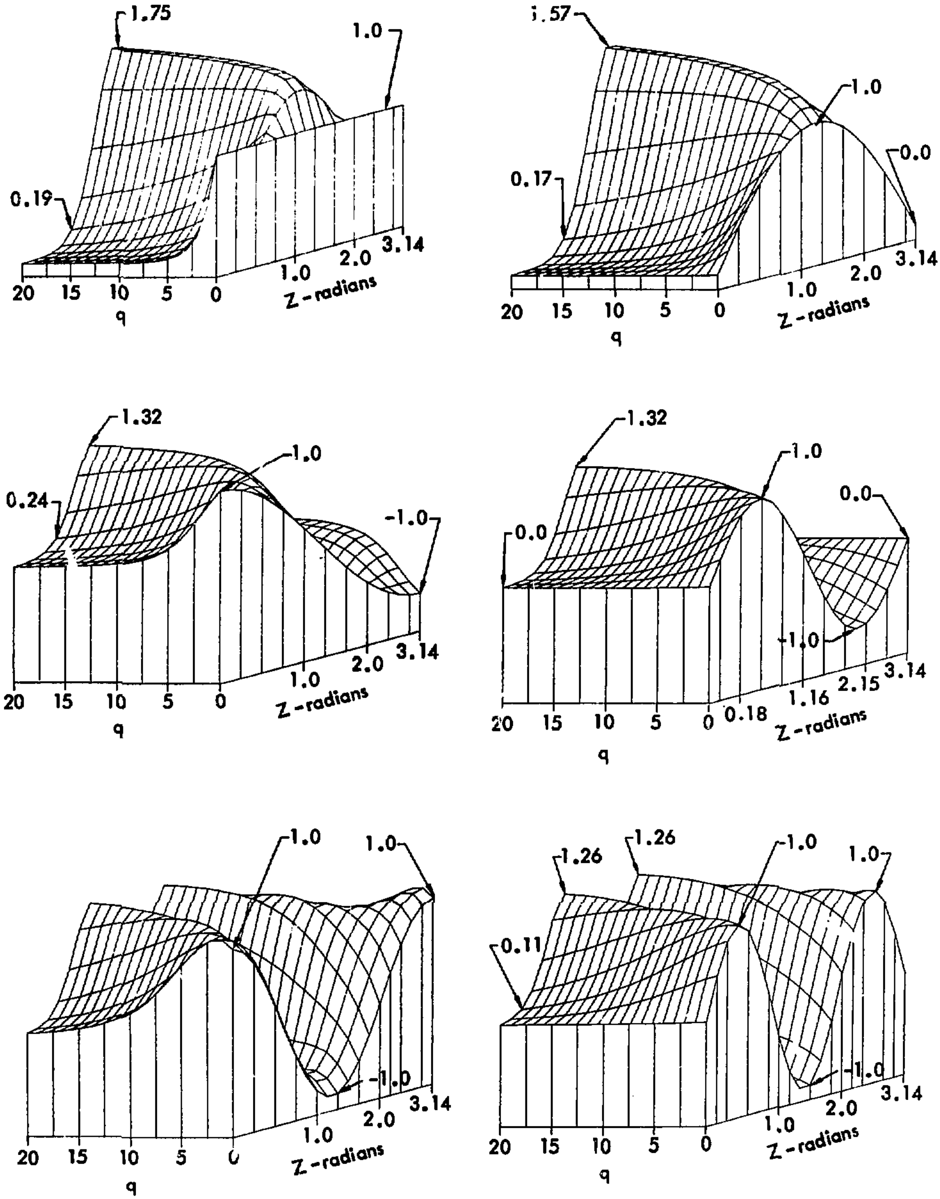

Fig. 2. Three-dimensional plots of angle and radial eigenfunctions.

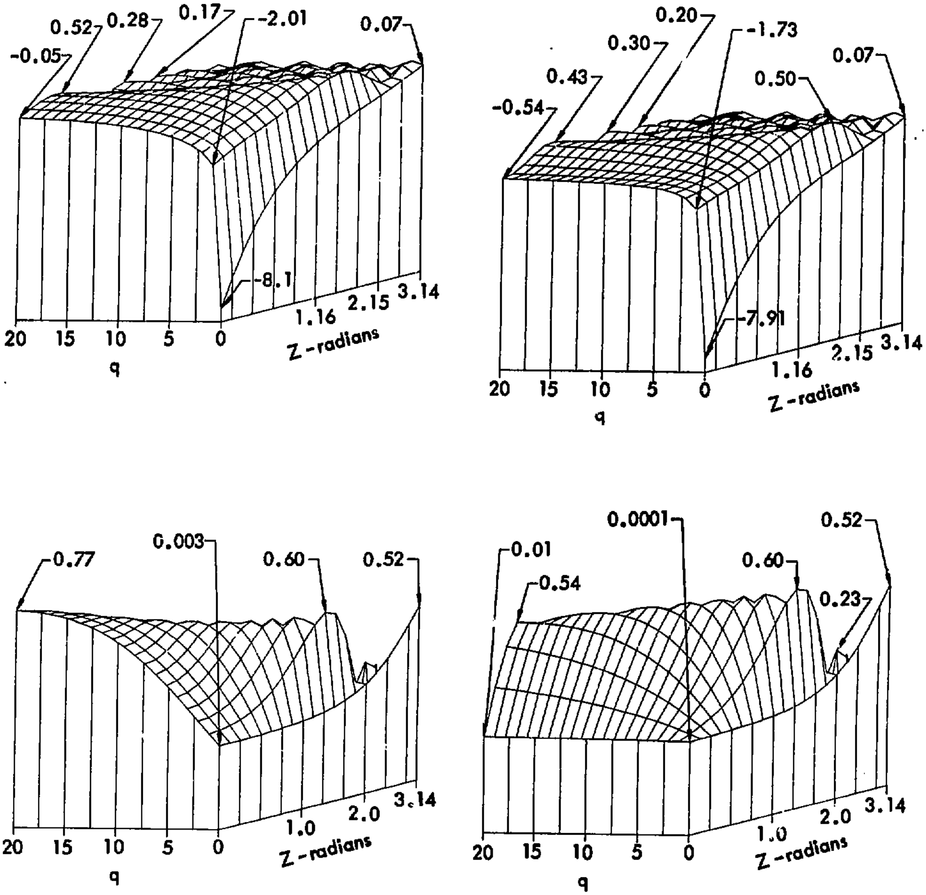

Fig. 2 (continued)

Table 1. Eigenvalue Results.

q = 0.1

r	a_r	b_r
1	1.0987343130	0.8987655570
3	9.0006406853	9.0006094414
5	25.0002083348	25.0002083347
7	49.0001041668	49.0001041668
9	81.0000625000	81.0000625000
11	121.0000416667	121.0000416667
13	169.0000297619	169.0000297619
15	225.0000223214	225.0000223214
17	289.0000173611	289.0000173611
19	361.0000138889	361.0000138889

q = 0.1

r	a_r	b_r
0	-0.0049945438	0.
2	4.0041611598	3.9991667028
4	16.0003333834	16.0003332966
6	36.0001428575	36.0001428576
8	64.0000793651	64.0000793651
10	100.0000505050	100.0000505051
12	144.0000349650	144.0000349650
14	196.0000256410	196.0000256410
16	256.0000196077	256.0000196078
18	324.0000154798	324.0000154799
20	400.0000125312	400.0000125313

q = 1.0

r	a_r	b_r
1	1.8591080725	-0.1102488170
3	9.0783688472	9.0477392508
5	25.0208543454	25.0208408233
7	49.0104182504	49.0104182494
9	81.0062503266	81.0062503266
11	121.0041667613	121.0041667613
13	169.0029762245	169.0029762245
15	225.0022321571	225.0022321571
17	289.0017361178	289.0017361178
19	361.0013888923	361.0013888923

q = 1.0

r	a_r	b_r
0	-0.4551386041	0.
2	4.3713009827	3.9170247730
4	16.0338323403	16.0329700814
6	36.0142900460	36.0142899106
8	64.0079371892	64.0079371892
10	100.0050506751	100.0050506752
12	144.0034965589	144.0034965590
14	196.0025641241	196.0025641242
16	256.0019607937	256.0019607939
18	324.0015479922	324.0015479923
20	400.0012531352	400.0012531353

q = 5

r	a_r	b_r
1	1.8581875415	-5.7900805986
3	11.5488320363	9.2363277137
5	25.5499717500	25.5108160463
7	49.2614549086	49.2613831113
9	81.1564549921	81.1564549559
11	121.1042258933	121.1042258933
13	169.0744260499	169.0744260499
15	225.0558124767	225.0558124767
17	289.0434069445	289.0434069445
19	361.0347243474	361.0347243474

q = 5

r	a_r	b_r
0	-5.8000460208	0.
2	7.4491097395	2.0994604455
4	17.0965816843	16.6482199372
6	36.3608999793	36.3588668480
8	64.1988423870	64.1988405393
10	100.1263692161	100.1263692156
12	144.0874473134	144.0874473135
14	196.0641161133	196.0641161135
16	256.0490256255	256.0490256258
18	324.0387026377	324.0387026379
20	400.0313299792	400.0313298794

q = 10

r	a_r	b_r
1	-2.3991424000	-13.9365524792
3	15.5027843697	7.9860691447
5	27.7037687339	26.7664263605
7	50.0626715473	50.0541572136
9	81.6283311587	81.6283131844
11	121.4176193567	121.4176193426
13	169.2979605465	169.2979605465
15	225.2233569750	225.2233569750
17	289.1736778363	289.1736778363
19	361.1389229098	361.1389229098

q = 10

r	a_r	b_r
0	-13.9369799566	0.
2	7.7173698498	-2.3821582360
4	21.1046337086	17.3813806786
6	37.5336063380	37.4198587767
8	64.8008910104	64.8004402930
10	100.5067700246	100.5067694629
12	144.3502080087	144.3502080085
14	196.2566274613	196.2566274615
16	256.1961748237	256.1961748240
18	324.1548459537	324.1548459540
20	400.1253382311	400.1253382313

Table 1. (Continued)

	q = 20			q = 20	
r	a_r	b_r	r	a_r	b_r
1	-14.4913014252	-31.3133861669	0	-31.3133900703	0.
3	15.3958109128	1.1607056792	2	1.1542828852	-14.4910632560
5	36.6449897341	28.4682213251	4	27.5945781546	15.4939775770
7	53.8644587232	53.1064751308	6	44.0629486460	40.5896640505
9	83.3602284647	83.5521975422	8	67.3458752414	67.2522407940
11	122.6822608110	122.6822337507	10	102.0489160243	102.0483928609
13	170.1959983656	170.1959983252	12	145.4076588968	145.4076577532
15	225.8951534162	225.8951534162	14	197.0291432786	197.0291432775
17	289.6955157232	289.6955157232	16	256.7858628380	256.7858628382
19	361.5561010752	361.5561010752	18	324.6199521768	324.6199521770
			20	400.5016530065	400.5016530068

Table 2. Expansion Coefficient Results.

q = 0.1

A_m^r

r	m = 0	m = 2	m = 4	m = 6	m = 8	m = 10
0	0.9987550068	0.0249653683	0.0000520838	0.0000000434	0.0000000000	-0.0000000001
2	-0.0498832563	0.9996535808	0.0083335763	0.0000156250	0.0000000122	-0.0000001157
4	0.0003116784	-0.0083335534	0.9999527737	0.0049999478	0.0000073335	-0.0001110418
6	-0.0000008657	0.0000260459	-0.0049998993	0.9999811225	0.0035714081	0.0000041668
8	0.0000000014	-0.0000000434	0.0000104166	-0.0035713993	0.9999897644	0.0027777675
10	-0.0000000000	0.0000000000	-0.0000000124	0.0000055803	-0.0027777651	0.9999935532

q = 0.1

A_m^r

r	m = 1	m = 3	m = 5	m = 7	m = 9	m = 11
1	0.9999199100	0.0126559404	0.0000261500	0.0000000217	-0.0000000023	-0.0000029185
3	-0.0126558567	0.9999003790	0.0062499057	0.0000104151	-0.0000018562	-0.0034992393
5	0.0000529510	-0.0062497253	0.9999717882	0.0041666355	0.0000055770	-0.0000036414
7	-0.0000001105	0.0000156246	-0.0041666163	0.9999864367	0.0031249863	0.0000034671
9	0.0000000001	-0.0000000217	0.0000074404	-0.0031249814	0.9999319922	0.0024999775
11	-0.0000000000	0.0000000000	-0.0000000078	0.0000043403	-0.0024999910	0.9999885825

q = 0.1

B_m^r

r	m = 0	m = 2	m = 4	m = 6	m = 8	m = 10
0	0.	0.	0.	0.	0.	0.
2	0.	0.9999652823	0.0083327084	0.0000156248	0.0000000124	-0.0000000022
4	0.	-0.0083326824	0.9999527823	0.0049999480	0.0000074379	-0.0000020766
6	0.	0.0000260391	-0.0049998993	0.9999811225	0.0035714085	0.0000043370
8	0.	-0.0000000434	0.0000104166	-0.0035713993	0.9999897644	0.0027777680
10	0.	0.0000000000	-0.0000000124	0.0000055803	-0.0027777651	0.9999935593

q = 0.1

B_m^r

r	m = 1	m = 3	m = 5	m = 7	m = 9	m = 11
1	0.9999238149	0.0123435686	0.0000259330	0.0000000217	-0.0000000023	-0.0000029136
3	-0.0123434896	0.9999042838	0.0062499044	0.0000104151	-0.0000018562	-0.0034992393
5	0.0000512156	-0.0062497375	0.9999717882	0.0041666355	0.0000055770	-0.0000036414
7	-0.0000001065	0.0000156245	-0.0041666163	0.9999864367	0.0031249863	0.0000034671
9	0.0000000001	-0.0000000217	0.0000074404	-0.0031249814	0.9999919922	0.0024999775
11	-0.0000000000	0.0000000000	-0.0000000078	0.0000043403	-0.0024999910	0.9999885825

q = 1.0

A_m^r

r	m = 0	m = 2	m = 4	m = 6	m = 8	m = 10
0	0.9098735393	0.2222195229	0.0052120752	0.0000433964	0.0000001937	0.0000000005
2	-0.4141185726	0.9713884187	0.0835695398	0.0015628927	0.0000124009	0.0000000523
4	0.0252085608	-0.0837615713	0.9952376807	0.0499480920	0.0007437625	0.0000050236
6	-0.0006917899	0.0026496827	-0.0498983199	0.9981127251	0.0356941822	0.0004339200
8	0.0000107345	-0.0000444442	0.0010405399	-0.0356850160	0.9989766513	0.0277680472
10	-0.0000001069	0.0000004648	-0.0000123935	0.0005577836	-0.0277651155	0.9993560245

q = 1.0

A_m^r

r	m = 1	m = 3	m = 5	m = 7	m = 9	m = 11
1	0.9902020594	0.1396156546	0.0027110963	0.0000221476	0.0000000981	-0.0000000020
3	-0.1395114768	0.9882511001	0.0624117524	0.0010411694	0.0000077473	-0.0000002345
5	0.0060343187	-0.0621675551	0.9971784986	0.0416354746	0.0005578674	0.0000036136
7	-0.0001280404	0.0015577825	-0.0416162475	0.9986439892	0.0312363120	0.0003471529
9	0.0000016161	-0.0000216621	0.0007435625	-0.0312313515	0.9991993551	0.0249928391
11	-0.0000000136	0.0000001936	-0.0000077477	0.0004339818	-0.0249909897	0.9994705498

Table 2. (Continued)

$q = 1.0$

B_m^r

r	m = 0	m = 2	m = 4	m = 6	m = 8	m = 10
0	0.	0.	0.	0.	0.	0.
2	0.	0.9965719156	0.0827162783	0.0015601770	0.0000123944	0.0000000538
4	0.	-0.0826907809	0.9953225020	0.0499479581	0.0007437640	0.0000051641
6	0.	0.0025787418	-0.0499004144	0.9981127370	0.0356941823	0.0004339222
8	0.	-0.0000429271	0.0010405649	-0.0356850163	0.9989766513	0.0277680472
10	0.	0.0000004468	-0.0000123937	0.0005577836	-0.0277651155	0.9993560249

$q = 1.0$

B_m^r

r	m = 1	m = 3	m = 5	m = 7	m = 9	m = 11
1	0.9939679614	0.1096424733	0.0024938544	0.0000212434	0.0000000956	-0.0000000019
3	-0.1095837919	0.9920165102	0.0623983329	0.0010411467	0.0000077473	-0.0000002345
5	0.0043676409	-0.0622843394	0.9971799039	0.0416354736	0.0005570674	0.00000[illegible]136
7	-0.0000889579	0.0015595117	-0.0416162827	0.9986437893	0.0312363120	0.0003471529
9	0.0000010969	-0.0000216769	0.0007435630	-0.0312312515	0.9991393551	0.0249928391
11	-0.0000000091	0.0000001936	-0.0000077477	0.0004338318	-0.0249909897	0.9994705498

$q = 5$

A_m^r

r	m = 0	m = 2	m = 4	m = 6	m = 8	m = 10
0	0.6426133238	0.4882368471	0.1243133778	0.0054072967	0.0001207521	0.0000015788
2	-0.7454373703	0.7273859705	0.4250667636	0.0393228351	0.0015504284	0.0000336194
4	0.1758374594	-0.4747068870	0.8647575626	0.2436898734	0.0184252946	0.0006429350
6	-0.0212155710	0.0844473289	-0.2354113027	0.9510261927	0.1760651469	0.0107048072
8	0.0015249055	-0.0075023969	0.0252581037	-0.1749004469	0.9745413710	0.1376751206
10	-0.0000721796	0.0004061161	-0.0015269776	0.0137919059	-0.1373091204	0.9839556403

$q = 5$

A_m^r

r	m = 1	m = 3	m = 5	m = 7	m = 9	m = 11
1	0.7624636873	0.6423435017	0.0776857977	0.0029772523	0.0000641247	0.0000008743
3	-0.6315963198	0.7128512402	0.3037510302	0.0257600531	0.0009638771	0.0000201277
5	0.1396848058	-0.2789558861	0.9277283962	0.2044501910	0.0138458660	0.0004504179
7	-0.0149155962	0.0376052556	-0.2017061483	0.9662917648	0.1545430732	0.0086372844
9	0.0009448418	-0.0027162626	0.0182745792	-0.1539218460	0.9800656098	0.1241065237
11	-0.0000397019	0.0001242660	-0.0009590377	0.0107593250	-0.1238758418	0.9868019280

$q = 5$

B_m^r

r	m = 0	m = 2	m = 4	m = 6	m = 8	m = 10
0	0.	0	0.	0.	0.	0.
2	0.	0.9334294415	0.3567820594	0.0375931648	0.0015301923	0.0000334443
4	0.	-0.3548039150	0.9025315914	0.2432332734	0.0184231607	0.0006429762
6	0.	0.0529637294	-0.2397742651	0.9531987779	0.1760648049	0.0107848065
8	0.	-0.0042958855	0.0254801772	-0.1749189852	0.9745415108	0.1376751205
10	0.	0.0002197964	-0.0015320845	0.0137930144	-0.1373091330	0.9839556404

$q = 5$

B_m^r

r	m = 1	m = 3	m = 5	m = 7	m = 9	m = 11
1	0.9400190217	0.3373722391	0.0503824620	0.0024115254	0.0000565635	0.0000008043
3	-0.3365419626	0.8931139037	0.2973655134	0.0256882361	0.0009633498	0.0000201248
5	0.0554775287	-0.2951587257	0.9315669962	0.2044372579	0.0138458185	0.0004504177
7	-0.0050895533	0.0374431813	-0.2021936395	0.9662978910	0.1545430664	0.0086372844
9	0.0002938790	-0.0026169523	0.0183057221	-0.1539224680	0.9900656125	0.1241065237
11	-0.0000116023	0.0001172395	-0.0009602770	0.0107593577	-0.1238758420	0.9868019280

Table 2. (Continued)

q = 10

A_m^r

r	m = 0	m = 2	m = 4	m = 6	m = 8	m = 10
0	0.5587543782	0.4488731290	0.3193743638	0.0426353061	0.0019159361	0.0000534271
2	-0.7787348569	0.3464119952	0.6740278965	0.1600256796	0.0124154365	0.0005369783
4	0.2793063957	-0.7689721074	0.5141513002	0.4513532021	0.0716550883	0.0050753501
6	-0.0574241400	0.2904991609	-0.4115724907	0.8119005377	0.3372677788	0.0423531673
8	0.0074524172	-0.0526359253	0.0989010002	-0.3268396210	0.8997061657	0.2681312525
10	-0.0006577487	0.0057496706	-0.0126669723	0.0531260697	-0.2652111208	0.9365074299

q = 10

A_m^r

r	m = 1	m = 3	m = 5	m = 7	m = 9	m = 11
1	0.5693674530	0.7552688875	0.3236196051	0.0256671596	0.0010762912	0.0000289359
3	-0.7629035580	0.3406812942	0.5405667041	0.1002627824	0.0076016653	0.0003195032
5	0.3002771766	-0.5341213551	0.6874438568	0.3860386105	0.0541333351	0.0035628433
7	-0.0598301540	0.1671852740	-0.3546977834	0.8672531076	0.2989463772	0.0340325833
9	0.0072446841	-0.0259027624	0.0679287438	-0.2938780903	0.9212788043	0.2428930233
11	-0.0005898897	0.0024706074	-0.0073368313	0.0419271930	-0.2410595593	0.9476831925

q = 10

B_m^r

r	m = 0	m = 2	m = 4	m = 6	m = 8	m = 10
0	0	0.	0.	0.	0.	0.
2	0.	0.8339073560	0.5354554495	0.1332191982	0.0117623066	0.0005257660
4	0.	-0.5322128700	0.7165133206	0.4452166791	0.0715153422	0.0050739979
6	0.	0.1444147632	-0.4364776836	0.8204286409	0.3372357122	0.0423529511
8	0.	-0.0220821592	0.0961478631	-0.3287273985	0.8997383572	0.2681312074
10	0.	0.0021713753	-0.0117503789	0.0533334266	-0.2652170288	0.9365074665

q = 10

B_m^r

r	m = 1	m = 3	m = 5	m = 7	m = 9	m = 11
1	0.8962865333	0.4323950133	0.1419460131	0.0165853111	0.0008350136	0.0000244677
3	-0.4394946193	0.7344691595	0.5076901625	0.0979431569	0.0075675871	0.0003191020
5	0.1177626067	-0.5068651077	0.7600379755	0.3855120652	0.0541270950	0.0035628010
7	-0.0190323724	0.1279076300	-0.3734350510	0.8679248320	0.2989450215	0.0340325773
9	0.0020205838	-0.0177343616	0.0702415951	-0.2940191430	0.9212800835	0.2428930221
11	-0.0001503537	0.0015779154	-0.0075102212	0.0419421854	-0.2410597792	0.9476831934

q = 20

A_m^r

r	m = 0	m = 2	m = 4	m = 6	m = 8	m = 10
0	0.4940158162	0.3691394076	-0.3843275820	0.2396717615	0.0296684691	0.0016730519
2	-0.7734654976	0.0213045650	-0.5302678749	0.5280322260	0.0999024509	0.0085366569
4	0.3776528089	-0.7413101535	0.1430828231	0.5783828747	0.2570834714	0.0385043937
6	-0.1199362352	0.5289594767	0.6132171237	0.2835242194	0.5601063415	0.1571264101
8	0.0260129203	-0.1802884609	-0.4008002535	-0.4640808136	0.6207677038	0.4803970596
10	-0.0040327457	0.0375584040	0.1163479515	0.1790959312	-0.4562557770	0.7568029528

q = 20

A_m^r

r	m = 1	m = 3	m = 5	m = 7	m = 9	m = 11
1	0.4277390314	-0.5791210090	0.6584943820	0.2184031892	0.0185045899	0.0009759944
3	-0.7590507447	0.1622751820	0.5151068924	0.3588851298	0.0578825685	0.0049620659
5	0.4638154606	0.6310150786	0.0535118555	0.5866561655	0.1972822868	0.0272289490
7	-0.1567830634	-0.4652945883	-0.4839496420	0.4877905038	0.5197622207	0.1280271992
9	0.0339025763	0.1507772882	0.2454482842	-0.4680143269	0.7008727680	0.4444377251
11	-0.0050869930	-0.0292864981	-0.0603934163	0.1472006005	-0.4300425002	0.7982312593

Table 2. (Continued)

q = 20

	r	m = 0	m = 2	m = 4	m = 6	m = 8	m = 10
B_m^r	0	0.	0.	0.	0.	0.	0.
	2	0.	0.7010247403	0.6214050277	0.3407965465	0.0780448220	0.0078174893
	4	0	-0 6481346408	0.3571207727	0.6234815574	0.2493555834	0.0383246130
	6	0.	0.2070909763	-0.6304405836	0.4257635553	0.5601567981	0.1570710783
	8	0.	-0.0766417914	0.2892706645	-0.5257759732	0.6259521735	0.4803900015
	10	0.	0.0136939087	-0 0711278833	0.1896660530	-0.4583694384	0.7568322968

q = 20

	r	m = 1	m = 3	m = 5	m = 7	m = 9	m = 11
B_m^r	1	0.8329837590	0.4759658677	0.2692495399	0.0858000811	0.0108495563	0.0006931470
	3	-0.5128425347	0.4793871745	0.6390398375	0.3093370706	0.0556322918	0.0049103311
	5	0.2007371983	-0.6634687255	0.3527989097	0.5963083096	0.1965259241	0.0272177234
	7	-0.0523670336	0.3114441364	-0.5778606024	0.5287815889	0.5197189448	0.1280240698
	9	0.0095514915	-0.0814946599	0 2404263899	-0.4878168873	0.7013456582	0.4444372483
	11	-0.0012709838	0.0138796704	-0.0536406928	0.1515650350	-0.4302203115	0.7982327939

Remarks

While this technique of generating Mathieu eigenvalues was being implemented, the authors became aware (through R. L. Pexton) of two reports concerning elliptic cylinder eigenfunctions and eigenvalues. One of these[9] presents an eigenvalue determination procedure based upon similar ideas as discussed herein. The other report[8] presents another method of generating eigenvalues. This method is valid for all values of q. It is especially useful for large values of q (i.e., $q > 10^4$).

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Appendix A
Useful Representations

Because many problems concern either plane wave excitations or line source excitations, it may be helpful to present the elliptic cylinder coordinate system representation of the fields of a plane wave and of a line source.

Plane Wave Representation

$$\exp[+j\vec{k}\cdot\vec{\rho}] = \sum_{m=0}^{\infty} \epsilon_m j^m \Big[ce_m(v,q)ce_m(v',q)Mc_m^{(1)}(u,q) + se_m(v,q)se_m(v',q)Ms_m^{(1)}(u,q)\Big]$$

with

$$\epsilon_m = \begin{cases} 1 & \text{if } m = 0 \\ 2 & \text{if } m \geq 1. \end{cases}$$

The wave is assumed to be incident from the angle ϕ' (with $v' = \cos\phi'$).

Line Source Representation

The solution of the equation

$$\nabla^2 G + k^2 G = -\delta(|\vec{\rho} - \vec{\rho}'|)$$

is

$$G = -\frac{i}{4} H_0^{(2)}(k|\vec{\rho} - \vec{\rho}'|) = -\frac{i}{2}\left\{\sum_m ce_m(v,q)ce_m(v',q)\begin{bmatrix} Mc_m^{(1)}(u',q) & Mc_m^{(4)}(u,q) \\ Mc_m^{(4)}(u',q) & Mc_m^{(1)}(u,q)\end{bmatrix} + \sum_m se_m(v,q)se_m(v',q)\begin{bmatrix} Ms_m^{(1)}(u',q) & Ms_m^{(4)}(u,q) \\ Ms_m^{(4)}(u',q) & Ms_m^{(1)}(u,q)\end{bmatrix}\right\} \text{ for } \begin{bmatrix} u \geq u' \\ u \leq u' \end{bmatrix}.$$

The source location is at $\xi' = \cosh u'$ and v'. The observation location is at $\xi = \cosh u$ and v.