Quantization of Fields Obeying Non-Linear Constraints: The Minimally Interacting Spin-3/2 Field*

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Abstract

A system consisting of an electromagnetic field interacting minimally with a massive Rarita-Schwinger spin-3/2 field is quantized to all orders in the coupling constant. The generators of the Poincaré group are identified and the field (anti)commutators are used to explicitly verify that the independent fields transform in accordance with the action principle. Because we have quantized the electromagnetic field in the (non-manifestly covariant) Coulomb gauge this strongly suggests, but does not prove covariance. The spin-3/2 field anticommutator is not positive definite and is in fact identical to the expression obtained when the electromagnetic field is taken to be external. While the non-linear constraints which appear are relatively simple in comparison with many systems obeying such constraints, the constraint structure is sufficiently rich to illustrate some of the basic techniques required to quantize the fields and construct the Poincaré generators.

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DISCLAIMER

Introduction

The original interest in a massive spin-3/2 field interacting minimally with an electromagnetic field resulted from the work of Johnson and Sudarshan.[1] They considered the case where the electromagnetic field was external, quantized the spin-3/2 field, and discovered the anticommutator was not positive definite. The major advantage in taking the electromagnetic field to be an external field is that all the constraints are then linear in the second quantized fields. But then it is difficult to determine whether or not the fields transform covariantly under a Poincaré transformation because the external electromagnetic field is not transformed while only that part of the spin-3/2 field which is independent of the electromagnetic field is transformed. Recently, however, Johnson and Sudarshan's quantization has been shown to lead to covariant transformation properties for the spin-3/2 field.[2]

A more direct approach is to quantize both the electromagnetic and spin-3/2 fields. Using the non-linear constraints to eliminate the dependent fields, Gupta and Repko[3] rewrote the Lagrangian as a power series in the electromagnetic coupling constant e. They then canonically quantized the system to second order in e. Kimel and Nath[4] carried out a similar program to second order using the Yang-Feldman quantization formalism, and Soo[5] extended their work to fourth order in e. A shortcoming of this approach is that it is difficult to find out if the spin-3/2 field anticommutator is positive definite since only the first few terms in an infinite expansion in e are known.

Here we quantize both fields to all orders in the coupling constant. We find that the indefinite metric found by Johnson and Sudarshan[1] persists in the fully quantized theory.[6] In fact, the spin-3/2 field anticommutator is the same regardless of whether the electromagnetic field is quantized or external, a result anticipated from the fourth order calculation.[5] As a check on the consistency of our quantization we identify the generators of the Poincaré group and explicitly verify that the transformation properties of the independent fields are consistent with the action principle.

If a second quantized field is to carry a representation of the Poincaré group, a delicate interdependent relationship must exist between the field equations, the expressions for the generators of the Poincaré group, and the field quantization conditions. Dirac[7] discovered that if the classical field equations and definitions of the generators of the Poincaré group are retained, the usual prescription for quantizing the fields, namely by making the substitution

$$\begin{aligned} \{A,B\}_{D.B.} &\to -i[A,B] && \text{for integral spin} \\ &\to -i\{A,B\} && \text{for half integral spin} \qquad (1.1) \end{aligned}$$

does not always lead to a set of quantized fields which carry representation of the Poincaré group. [In (1.1), $\{A,B\}_{D.B.}$, $[A,B]$ and $\{A,B\}$ are respectively the Dirac bracket, commutator, and anticommutator.] It is possible to see what goes wrong by examining, for example, Heisenberg's equations of motion. If we use the Dirac bracket for classical fields, in general we find that

$$\{ \phi^i , P^\nu \}_{D.B.} = \partial^\nu \phi^i \qquad + \text{ commutators} \qquad (1.2)$$

Since all classical fields commute, Heisenberg's equation of motion are satisfied. If we calculated $-i[\phi^i, P^\nu]$ for fields quantized according to (1.1), we would again obtain the result (1.2) but we would have no guarantee that the additional commutators summed to zero. If we were unlucky and they did not, the classical equations of motion and classical expressions for Poincaré generators in combination with the usual quantization procedure would not provide quantized fields which obey Heisenberg's equations of motion.

A major problem in making the transition from a classical to a quantum field theory is determining the arrangement of products of fields appearing in the field equations, field (anti)commutators, and Poincaré generators. The ordering of fields in classical expressions is arbitrary as all the fields commute; thus it is arbitrary to take the classical field equations and Poincaré generators with some particular ordering of the fields to be the quantum expressions. If the fields are quantized according to (1.1), one ordering of fields in the field equations, field (anti)commutators, and Poincaré generators may result in the quantized fields carrying a representation of the Poincaré group while other orderings may not. When fields obey linear constraints there is generally little ambiguity in the ordering of fields in the equations of motion and generators after the dependent fields have been eliminated. Only when non-linear constraints occur does the question of ordering become a serious problem. As we shall show, for the spin-3/2 field interacting minimally with an electromagnetic field, there exists an ordering of the fields for the classical Poincaré generators such that when the generators are taken to be the quantum expressions, the field transformations are consistent with the action principle.[8]

We will quantize the fields using the action principle[8] rather than the Dirac procedure (1.1) although both lead to the same result. The advantage of the action principle is that it also determines what the commutators of the Poincaré generators with the second quantized fields should be. It is then possible to use the quantization conditions and expressions for the Poincaré generators to explicitly calculate the commutators of the generators with the fields and verify that they are consistent with the relations demanded by the action principle.

We wish to emphasize that the action principle, like the Dirac procedure, at best does not provide a fully specified procedure for making the transition from classical field theory to quantum field theory. The quantities we obtain from the action principle are classical: (classical) Dirac brackets, classical expressions for the Poincaré generators, and classical equations of motion. In general, if the quantized fields are to carry a representation of the Poincaré group, the classical fields must be reordered on a trial-and-error basis on the right-hand-side of the Dirac brackets, in the expressions for the poincaré generators, and in the field equations before the fields are quantized according to the prescription (1.1). [While (1.1) is implicit in the action principle it is not generally stated explicitly.]

Our notation is that of Bjorken and Drell.[9] The space-time coordinates are denoted $x^\mu = (t, x^1, x^2, x^3)$ and we use the metric tensor $g^{\mu\nu}$ where $g^{oo} = -g^{11} = -g^{22} = -g^{33} = 1$. The Dirac gamma matrices γ^μ satisfy $\gamma_o^\dagger = \gamma_o$, $\gamma^{i\dagger} = -\gamma^i$. Greek indices range from 0 through 3, Roman indices range from 1 through 3, and all repeated indices are summed over the range of the index.

II. Classical Equations of Motion

The spin-3/2 field can be conveniently represented by the Rarita-Schwinger[10] vector spinor ψ^λ which, in the presence of a minimal electromagnetic interaction, obeys field equation that can be obtained from the Lagrangian[3]

$$\mathcal{L} = \bar{\Psi}_\mu \left[(D_\sigma \gamma^\sigma + m) g^{\mu\nu} - (D^\nu \gamma^\mu + D^\mu \gamma^\nu) + \gamma^\mu (D_\rho \gamma^\rho - m) \gamma^\nu \right] \Psi_\nu - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{2.1}$$

where $D^\mu = -i\partial^\mu + eA^\mu$, $F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}$ and e is the charge of the spin-3/2 field. The classical field equations obtained from (2.1) are

$$(D_\sigma \gamma^\sigma + m)\Psi^\mu - (D^\nu \gamma^\mu + D^\mu \gamma^\nu)\Psi_\nu + \gamma^\mu (D_\rho \gamma^\rho - m)\gamma^\nu \Psi_\nu = 0 \tag{2.2}$$

and

$$\partial_\beta F^{\beta\sigma} = J^\sigma \tag{2.3}$$

where

$$J^\sigma = e(\bar{\Psi}_\mu \gamma^\sigma \Psi^\mu - \bar{\Psi}_\mu \gamma^\mu \Psi^\sigma - \bar{\Psi}^\sigma \gamma^\mu \Psi_\mu + \bar{\Psi}_\mu \gamma^\mu \gamma^\sigma \gamma^\nu \Psi_\nu). \tag{2.4}$$

Before quantizing the fields we must determine all the constraints obeyed by the system and choose a set of independent fields. We obtain one of the two primary constraints

$$D_i \Psi^i - D^j \gamma_j \gamma_i \Psi^i + m \gamma_i \Psi^i = 0 \tag{2.5}$$

by taking $\mu = 0$ in (2.2). The secondary constraint

$$\left(\gamma^\nu + \frac{ie}{3m^2} F_{\rho\mu} \gamma^\rho \gamma^\nu \gamma^\mu\right) \Psi_\nu = 0 \tag{2.6}$$

is found by left-multiplying (2.2) by γ_μ and D_μ and combining the two resulting equations. The second primary constraint

$$\partial_i \partial^i A_0 = -J^0 \tag{2.7}$$

follows from (2.3) when we take $\sigma = 0$ and use the Coulomb gauge condition

$$\partial_i A^i = 0, \tag{2.8}$$

the gauge in which we choose to quantize the electromagnetic field.

The quantization conditions take a particularly simple form when they are expressed in terms of the spin-3/2 field ϕ_j and the spin-1/2 field χ where

$$\phi_j = P_{jk} \psi^k \tag{2.9}$$

$$\chi = \gamma_j \psi^j \tag{2.10}$$

and P_{jk} is a projection operator defined by

$$P_{jk} = g_{jk} - \frac{1}{3}\gamma_j \gamma_k. \tag{2.11}$$

We would now like to rewrite our constraint equations and equations of motion in terms of the field variables (2.9) and (2.10), but before we do this it is convenient to define the following quantities which occur regularly throughout the remainder of the paper.

$$S = \frac{3}{2} m - D^i \gamma_i \tag{2.12a}$$

$$T = \frac{3}{2} m + D^i \gamma_i \tag{2.12b}$$

$$U = \frac{3}{2} m^2 - \frac{e}{2} F^{ij} \sigma_{ij} \tag{2.12c}$$

$$V = \frac{9}{4} m^2 - D^i D_i - \frac{e}{2} F^{ij} \sigma_{ij} = ST = TS \tag{2.12d}$$

$$\eta = -ie U^{-1} [\gamma^i F_{ij} \phi^j - \frac{i}{3} F^{ij} \sigma_{ij} \chi] \tag{2.12e}$$

$$\begin{aligned} \partial_0 \tilde{\phi}_r = & -i\gamma_0 (D_i \gamma^i + m) \phi_r + i\gamma_0 P_{ri} D^i \eta \\ & + \frac{2i}{3} \gamma_0 (-D_r + \gamma_r D_i \gamma^i - m\gamma_r) \chi \end{aligned} \tag{2.12f}$$

In terms of the newly defined quantities, the constraints (2.5)-(2.7) can be rewritten respectively as

$$\chi = -\frac{3}{2} S^{-1} D^k \phi_k \tag{2.13}$$

$$\Psi_0 = \gamma_0 (\eta - \chi) + ie U^{-1} F_{j0} (\phi^j - \frac{2}{3} \gamma^j \chi) \tag{2.14}$$

$$A_0 = -e (\partial_k \partial^k)^{-1} [\phi_i^\dagger \phi^i + \frac{2}{3} \chi^\dagger \chi] \tag{2.15}$$

From the definition of the spin-3/2 field, it satisfies the constraint

$$\gamma_i \phi^i = 0. \tag{2.16}$$

To write the equation of motion for ϕ_r in a convenient form, we take $\mu = i$ in (2.2) and left-multiply by P_{ri} with the result

$$\partial_0 \phi_r = \partial_0 \tilde{\phi}_r - ie A_0 \phi_r - e P_{ri} D^i U^{-1} F_{j0} (\phi^j - \tfrac{2}{3}\gamma^j \chi). \qquad (2.17)$$

At first sight, the manner in which (2.14) and (2.17) are split might seem arbitrary. As we shall show, $[\phi^i(\bar{x},t), A^j(\bar{x}',t)] = 0$ with the result that the ordering of fields in η and $\partial_o\tilde{\phi}_r$ is of no consequence. However, neither ϕ^i nor A^j commutes with F_{ko} and it is just such terms involving non-commuting fields that can cause unwanted commutators of the type appearing in (1.2). We group the terms that do not cause problems into η and $\partial_o\tilde{\phi}_r$ and thereby isolate the potential trouble makers.

To summarize the constraint structure, we see from (2.13)-(2.15) that χ, ψ_o, and A_o are dependent fields. The spin-3/2 field ϕ^i obeys one constraint (2.16), the electromagnetic field A^i obeys one constraint (2.8), and the time derivative of the electromagnetic field $A_{i,o}$ satisfies the Coulomb gauge constraint.

III. The Action Principle

In this section we give the Poincaré transformation properties and the canonical (anti)commutation relations for the independent fields. Since the derivation of these equations from the action principle appears in the literature for various systems,[11,12] we merely state the results.

Under a Poincaré transformation the action principle requires that the independent fields transform in the following manner:

$$-i[A^k, P^\mu] = \partial^\mu A^k \qquad (3.1a)$$

$$-i[A^{k,0}, P^\mu] = \partial^\mu A^{k,0} \qquad (3.1b)$$

$$-i[\phi^k, P^\mu] = \partial^\mu \phi \tag{3.1c}$$

$$-i[A^k, J_{ij}] = (x_i \partial_j - x_j \partial_i) A^k + \hat{I}^{k\beta}_{ij} A_\beta \tag{3.2a}$$

$$-i[A^{k,o}, J_{ij}] = (x_i \partial_j - x_j \partial_i) A^{k,o} + \hat{I}^{k\beta}_{ij} A_\beta{}^{,o} \tag{3.2b}$$

$$-i[\phi^k, J_{ij}] = P^k{}_r[(x_i \partial_j - x_j \partial_i)\Psi^r + I^{r\beta}_{ij} \Psi_\beta] \tag{3.2c}$$

$$-i[A^k, J_{oa}] = (x_o \partial_a - x_a \partial_o) A^k + \hat{I}^{k\beta}_{oa} A_\beta + W_a{}^{,k} \tag{3.3a}$$

$$-i[A^{k,o}, J_{oa}] = \partial^o[(x_o \partial_a - x_a \partial_o) A^k + \hat{I}^{k\beta}_{oa} A_\beta + W_a{}^{,k}] \tag{3.3b}$$

$$-i[\phi^k, J_{oa}] = P^k{}_r[(x_o \partial_a - x_a \partial_o)\Psi^r + I^{r\beta}_{oa} \Psi_\beta] - \frac{ie}{2} W_a \phi^k - \frac{ie}{2} \phi^k W_a \tag{3.3c}$$

where

$$I^{\alpha\beta}_{\mu\nu} = \frac{1}{4}[\gamma_\mu, \gamma_\nu] g^{\alpha\beta} + \hat{I}^{\alpha\beta}_{\mu\nu} \tag{3.4a}$$

$$\hat{I}^{\alpha\beta}_{\mu\nu} = g^\alpha{}_\mu g^\beta{}_\nu - g^\alpha{}_\nu g^\beta{}_\mu \tag{3.4b}$$

$$W_a = (\partial_b \partial^b)^{-1} (A_{a,o} + A_{o,a}) \tag{3.4c}$$

With the exception of the terms proportional to W_a, the relations (3.1)-(3.3) are the usual relations.[13] The term $W_a{}^{,k}$ in (3.3a) is the well known term that occurs when the Coulomb gauge is used and must be present for the equation to be compatible with the (non-manifestly covariant) Coulomb gauge condition.[9] The terms proportional to W_a in (3.3c) are required for (3.3a) and (3.3c) to be compatible with the constraint (2.13). In (3.3c) W_a and ϕ_k do not commute so the action principle does not uniquely specify that commutator for quantum fields. Motivated by past experience with quantum fields, we have symmetrized the non-commuting terms. If we had introduced an indefinite metric for the electromagnetic field by quantizing in the (manifestly covariant) Lorentz gauge the terms proportional to W_a in (3.3) would be missing and the electromagnetic field would transform in a manifestly covariant manner. But then it would be difficult to distinguish between the indefinite metric inherent in the minimally interacting Rarita-Schwinger field and that introduced to quantize the electromagnetic field.

To quantize the fields using the action principle, we assume that the field variations commute with the fields. Then, with the help of Lagrange multipliers, it is a straightforward task to construct the canonical (anti)commutation relations for the independent fields.[6,11]

$$\{\phi^i(\bar{x},t),\ \phi^k(\bar{x}',t)\} = [\phi^i(\bar{x},t),\ A^k(\bar{x}',t)] = [A^i(\bar{x},t),\ A^k(\bar{x}',t)] = 0 \tag{3.5}$$

$$\{\phi^j(\bar{x},t),\ \phi^{\dagger i}(\bar{x}',t)\} = -P^{jr}[P_{rk} - D_r U^{-1} D_k]\, P^{ki}\, \delta^3(\bar{x}-\bar{x}') \tag{3.6}$$

$$[A^{k,0}(\bar{x},t),\ A^i(\bar{x}',t)] = i\,[g^{ik} - \partial^i\partial^k(\partial_a\partial^a)^{-1}]\, \delta^3(\bar{x}-\bar{x}') \tag{3.7}$$

$$\begin{aligned}[A^{k,0}(\bar{x},t),\ \phi^{\dagger i}(\bar{x}',t)] = -ie\,[g^{kr} - \partial^k\partial^r(\partial_a\partial^a)^{-1}]\,\times \\ (\phi_r^\dagger + \tfrac{2}{3}\chi^\dagger\gamma_r)\, U^{-1} D_s P^{si}\, \delta^3(\bar{x}-\bar{x}')\end{aligned} \tag{3.8}$$

Using the expressions (2.13) and (2.15) which express χ and A_o in terms of independent fields, we readily calculate the additional useful (anti)commutators:

$$\{\chi(\bar{x},t), \phi^{\dagger i}(\bar{x}',t)\} = TU^{-1}D_r P^{ri}\,\delta^3(\bar{x}-\bar{x}') \tag{3.9}$$

$$[A^o(\bar{x},t), \phi^{\dagger i}(\bar{x}',t)] = -e(\partial_a\partial^a)^{-1}[\phi^\dagger_j(P^{ji} - P^{jr}D_r U^{-1}D_k P^{ki}) - \tfrac{2}{3}\chi^\dagger TU^{-1}D_r P^{ri}]\,\delta^3(\bar{x}-\bar{x}') \tag{3.10}$$

$$[F^{ok}(\bar{x},t), A^i(\bar{x}',t)] = -i\,[g^{ki} - \partial^k\partial^i(\partial_a\partial^a)^{-1}]\,\delta^3(\bar{x}-\bar{x}') \tag{3.11}$$

From (3.5) we note that all fields on the right-hand-side of (3.6)-(3.11) commute; therefore, for this system the action principle uniquely specifies the (anti)commutators. The problems created by non-linear constraints appear when we attempt to construct the Poincaré generators.

As a check on the internal consistency of the quantization conditions and the field equations, we calculate the commutator $[A_{j,o}(\bar{x},t), A_{k,o}(\bar{x}',t)]$ to verify that $A_{j,o}$ commutes with itself. Since the commutator (3.7) is time-independent

$$\begin{aligned} 0 &= \partial_o[A_{j,o}(\bar{x},t), A_k(\bar{x}',t)] \\ &= [A_{j,oo}(\bar{x},t), A_k(\bar{x}',t)] + [A_{j,o}(\bar{x},t), A_{k,o}(\bar{x}',t)] \end{aligned} \tag{3.12}$$

or

$$[A_{j,o}(\bar{x},t), A_k(\bar{x}',t)] = [A_k(\bar{x}',t), A_{j,oo}(\bar{x},t)]. \tag{3.13}$$

Using the equation of motion for $A_{j,o}$ we are able to evaluate the commutator on the right-hand-side of (3.13) with the result

$$[A_{j,0}(\bar{x},t), A_{k,0}(\bar{x}',t)] = e^2[g_{js} - \partial_j\partial_s(\partial_a\partial^a)^{-1}] \times$$
$$\{(\phi^{\dagger s} + \tfrac{2}{3}\chi^\dagger\gamma^s)U^{-1}(\phi^r - \tfrac{2}{3}\gamma^r\chi) - (\phi^{\dagger r} + \tfrac{2}{3}\chi^\dagger\gamma^r)U^{-1}(\phi^s - \tfrac{2}{3}\gamma^s\chi)\} \times$$
$$[g_{rk} - \partial'_r\partial'_k(\partial'_b\partial'^b)^{-1}]\,\delta^3(\bar{x}-\bar{x}'). \qquad (3.14)$$

The left-hand-side of (3.14) has the obvious symmetry property that it goes into the negative of itself under the transformation $\bar{x}\leftrightarrow\bar{x}'$, $j\leftrightarrow k$. By Fourier transforming the right-hand-side of (3.14), it is not difficult to ascertain that it also possesses this symmetry. Thus the right-hand-side of (3.14) vanishes if $j = k$ and $\bar{x} = \bar{x}'$, verifying that $A_{j,o}$ commutes with itself as required.

IV. Translational Invariance

To demonstrate that the theory is translationally invariant it is sufficient to show that relations (3.1) are satisfied. Using the fact that A^k commutes with the time derivative of P_μ, (3.1b) follows from (3.1a) so we need only explicitly verify (3.1a) and (3.1c). To construct the generators of Poincaré transformations we begin with the classical expressions which follow from the action principle. In the limit that all fields commute, the quantum expressions must reduce to the classical ones so we go from the classical to the quantum expressions by commuting fields on a trial and error basis as if they were classical until we obtain generators satisfying (3.1). For the system under study we have succeeded in constructing the generators by using the constraints to eliminate as many terms as possible in the classical expressions. Then, guided by intuition we symmetrize some of the non-commuting fields.

The classical generator of translations $P^{\nu}_{(c)}$ is given by

$$P^{\nu}_{(c)} = \mathrm{Re}\int d^3x \left\{ -i\left(\phi^{\dagger j}\partial^{\nu}\phi_j + \tfrac{2}{3}\chi^{\dagger}\partial^{\nu}\chi\right) + F^{0k}\partial^{\nu}A_k + \tfrac{1}{4}\, g^{0\nu} F_{\alpha\beta}F^{\alpha\beta} \right\} \tag{4.1}$$

Taking $\nu = k$ and eliminating $\partial^k\chi$ with the help of (2.13) we find

$$\begin{aligned} P^k = \int d^3x \Big\{ & -i\phi^{\dagger s}\left[g_{sj} + \tfrac{3}{2} D_s V^{-1} D_j \right]\partial^k \phi^j \\ & - \tfrac{3ie}{2}\,\phi^{\dagger s} D_s V^{-1}\left(\phi^j - \tfrac{2}{3}\gamma^j\chi\right) A_j{}^{,k} \\ & - \tfrac{1}{2}\left(A_{j,0}A^{j,k} + A^{j,k}A_{j,0} \right) \Big\} \end{aligned} \tag{4.2}$$

There is no ambiguity as to where $A_{j,k}$ should appear in the second term as it commutes with A^i and ϕ^i, and in the last term the two non-commuting fields are symmetrized. We take (4.2) to be the quantum expression for P^k. A short, simple calculation verifies that P^k transforms A^i and ϕ^i as shown in (3.1a) and (3.1c).

Before we construct the quantum expression for P^0, we would like to explicitly show what goes wrong if we take P^0 to be given by the classical expression (4.1). Using the classical equation of motion (2.17) and the canonical (anti)commutation relations we find

$$-i\left[P^{o}_{(c)}, \phi^{i}(\bar{x},t)\right] = \partial_0 \phi^{i}(\bar{x},t) - \frac{ie}{2}\left[A_0(\bar{x},t), \phi^{i}(\bar{x},t)\right]$$
$$-\frac{e}{2} P^{ia} D_a U^{-1}(\bar{x},t)\left[\phi_r(\bar{x},t) - \frac{2}{3}\gamma_r \chi(\bar{x},t), A^{0,r}(\bar{x},t) + A^{r,0}(\bar{x},t)\right]$$
$$+\int d^3x' \left\{-\frac{e}{2}\left[A_{k,0}(\bar{x}',t), \delta^3(\bar{x}-\bar{x}') P^{ia} \overleftarrow{D}'_a U^{-1}(\bar{x}',t)\right]\left(\phi^k(\bar{x}',t) - \frac{2}{3}\gamma^k \chi(\bar{x}',t)\right)\right\}$$

where (4.3)

$$\overleftarrow{D}'_a = i \frac{\overleftarrow{\partial}}{\partial x'_a} + e A_a(\bar{x}',t) \tag{4.4}$$

The commutators appearing on the right hand side of (4.3) are those discussed in the introduction and their presence stops us from identifying $P^{o}_{(c)}$ in (4.1) as the quantum generator of time translations.

Starting with (4.1) we will now construct the quantum expression for P^o by first eliminating as many terms as possible that are functions of the dependent fields. Taking $\nu = 0$ in (4.1) and using (2.13) to rewrite $\partial^o\chi$

$$P^{o}_{(c)} = \mathrm{Re}\int d^3x \left\{-i\phi^{\dagger s}\left(g_{sk} + \frac{3}{2} D_s V^{-1} D_k\right)\partial^0 \phi^k\right.$$
$$-\frac{3ie}{2}\phi^{\dagger s} D_s V^{-1} A_{k,0}\left(\phi^k - \frac{2}{3}\gamma^k\chi\right)$$
$$\left.+\frac{1}{2}\left(A_{0,j}A^{0,j} - A^{j,0}A_{j,0} + A_{j,s}A^{j,s}\right)\right\} \tag{4.5}$$

The first term in (4.5) can be written in a more convenient form by employing the expression (2.17) for $\partial^o\phi^k$ and simplifying one of the resulting quantities using the identity

$$P^{ij} D_i D_j = U - \frac{2}{3} V \tag{4.6}$$

with the result

$$\int d^3x \{-i\phi^{\dagger s}(g_{sk} + \tfrac{3}{2} D_s V^{-1} D_k)\,\partial^0 \phi^k\}$$
$$= \int d^3x \{-i\phi^{\dagger s}(g_{sk} + \tfrac{3}{2} D_s V^{-1} D_k)(\partial^0 \tilde{\phi}^k - ieA_0\phi^k)$$
$$- \tfrac{3ie}{2}\,\phi^{\dagger s} D_s V^{-1} F_{0k}\,(\phi^k - \tfrac{2}{3}\gamma^k\chi)\}. \tag{4.7}$$

Combining (4.5) and (4.7)

$$P^0_{(c)} = \mathrm{Re}\int d^3x \{-i\phi^{\dagger s}(g_{sk} + \tfrac{3}{2} D_s V^{-1} D_k)\,\partial^0\tilde{\phi}^k - e\phi^{\dagger s}(g_{sk} + \tfrac{3}{2} D_s V^{-1} D_k) A_0 \phi^k$$
$$- \tfrac{3ie}{2}\,\phi^{\dagger s} D_s V^{-1} A_{0,k}\,(\phi^k - \tfrac{2}{3}\gamma^k\chi)$$
$$+ \tfrac{1}{2}(A_{0,j}A^{0,j} - A^{j,0}A_{j,0} + A_{j,s}A^{j,s})\}. \tag{4.8}$$

The second term in (4.8) is simplified by writing $V^{-1}D_k = T^{-1}S^{-1}D_k$ and commuting A_0 to the left of $S^{-1}D_k$. At this point we are still treating all the fields as if they were classical but A_0 neither commutes with D_k nor S^{-1} because of the derivatives. With the above operation the second and third terms in 4.8 combine as follows:

$$\int d^3x \{-e\phi^{\dagger s}(g_{sk} + \tfrac{3}{2} D_s V^{-1} D_k) A_0 \phi^k - \tfrac{3ie}{2}\,\phi^{\dagger s} D_s V^{-1} A_{0,k}\,(\phi^k - \tfrac{2}{3}\gamma^k\chi)$$
$$= \int d^3x\,[-e(\phi^{\dagger s} A_0 \phi_s + \tfrac{2}{3}\chi^\dagger A_0 \chi)] \tag{4.9}$$

Symmetrizing (4.9)

$$= \int d^3x\,[-\tfrac{e}{2}(\phi^{\dagger s}\phi_s + \tfrac{2}{3}\chi^\dagger\chi)A_0 - \tfrac{e}{2}A_0(\phi^{\dagger s}\phi_s + \tfrac{2}{3}\chi^\dagger\chi)] \tag{4.10}$$

The constraint (2.15) permits (4.10) to be rewritten in the simple form

$$= \int d^3x\, (-A_{0,k}A^{0,k}) \tag{4.11}$$

Using (4.11), (4.8) becomes

$$\begin{aligned} P^0 = \int d^3x\, \Big[& -\tfrac{i}{2}\phi^{\dagger s}(g_{sk} + \tfrac{2}{3} D_s V^{-1} D_k)\partial_0 \tilde{\phi}^k \\ & + \tfrac{i}{2}(\partial_0 \tilde{\phi}^k)^\dagger (g_{ks} + \tfrac{3}{2}\overleftarrow{D}_k V^{-1} \overleftarrow{D}_s)\phi^s \\ & - \tfrac{1}{2}(A_{0,k}A^{0,k} + A_{k,0}A^{k,0} - A_{j,s}A^{j,s})\Big]. \end{aligned} \tag{4.12}$$

We take (4.12) to be the quantum generator of time translations. The calculation of the commutator $-i[A^i, P^0]$ is simple with the result agreeing with (3.1a) as required. The evaluation of $-i[\phi^i, P^0]$ is more difficult. From (4.12)

$$\begin{aligned} -i[\phi^i(\bar{x}',t), P^0] = \int d^3x \Big\{ & \Big[-\tfrac{1}{2}\{\phi^i(\bar{x}',t), \phi^{\dagger s}\}(g_{sk} + \tfrac{3}{2} D_s V^{-1} D_k)\partial^0 \tilde{\phi}^k\Big]_{\mathrm{I}} \\ & + \Big[\tfrac{1}{2}\{\phi^i(\bar{x}',t), (\partial_0 \tilde{\phi}^k)^\dagger\}(g_{ks} + \tfrac{3}{2} D_k V^{-1} D_s)\phi^s\Big]_{\mathrm{II}} \\ & + \Big\{\tfrac{i}{2}[\phi^i(\bar{x}',t), A_{0,k}]A^{0,k} + \tfrac{i}{2}A^{0,k}[\phi^i(\bar{x}',t), A_{0,k}]\Big\}_{\mathrm{III}} \\ & + \Big\{\tfrac{i}{2}[\phi^i(\bar{x}',t), A_{k,0}]A^{k,0} + \tfrac{i}{2}A^{k,0}[\phi^i(\bar{x}',t), A_{k,0}]\Big\}_{\mathrm{IV}} \end{aligned} \tag{4.13}$$

where we have suppressed the coordinates $\bar{x}$ and t $\left(\phi^{\dagger s} \equiv \phi^{\dagger s}(\bar{x},t)\right)$. The four brackets respectively simplify so that (4.13) can be rewritten as

$$
\begin{aligned}
-i[\phi^i(\bar{x}',t),P^0] = \int d^3x \Big[& \left[\tfrac{1}{2}\delta^3(\bar{x}-\bar{x}')\partial_0\tilde{\phi}^i\right]_{\mathrm{I}} + \left[\tfrac{1}{2}\delta^3(\bar{x}-\bar{x}')\partial_0\tilde{\phi}^i\right]_{\mathrm{II}} \\
& + \Big\{ -\tfrac{ie}{2}\delta^3(\bar{x}-\bar{x}')[\phi^i A^0 + A^0\phi^i] + \tfrac{e}{2}\delta^3(\bar{x}-\bar{x}')P^{ia}\overleftarrow{D}_a U^{-1}(\phi^r - \tfrac{2}{3}\gamma^r\chi)A_{o,r} \\
& \quad + \tfrac{e}{2}A_{o,r}\delta^3(\bar{x}-\bar{x}')P^{ia}\overleftarrow{D}_a U^{-1}(\phi^r - \tfrac{2}{3}\gamma^r\chi)\Big\}_{\mathrm{III}} \\
& + \Big\{ -\tfrac{e}{2}\delta^3(\bar{x}-\bar{x}')P^{ia}D_a U^{-1}(\phi^r - \tfrac{2}{3}\gamma^r\chi)A_{r,o} \\
& \quad - \tfrac{e}{2}A_{r,o}\delta^3(\bar{x}-\bar{x}')P^{ia}\overleftarrow{D}_a U^{-1}(\phi^r - \tfrac{2}{3}\gamma^r\chi)\Big\}_{\mathrm{IV}} \Big]. \qquad (4.14)
\end{aligned}
$$

Only the contents of the second bracket in (4.13) are difficult to simplify and that calculation is outlined in the appendix. Combining terms in (4.14) we obtain the desired result:

$$
\begin{aligned}
-i[\phi^i(\bar{x}',t),P^0] = \int d^3x \Big[& \delta^3(\bar{x}-\bar{x}')\left(\partial_0\tilde{\phi}^i - \tfrac{ie}{2}\phi^i A_0 - \tfrac{ie}{2}A_0\phi^i\right) \\
& + \tfrac{e}{2}\delta^3(\bar{x}-\bar{x}')P^{ia}D_a U^{-1}(\phi^r - \tfrac{2}{3}\gamma^r\chi)F_{or} \\
& + \tfrac{e}{2}F_{or}\delta^3(\bar{x}-\bar{x}')P^{ia}\overleftarrow{D}_a U^{-1}(\phi^r - \tfrac{2}{3}\gamma^r\chi)\Big]
\end{aligned}
\qquad (4.15)
$$

The derivatives that act to the left do not operate on any fields to the left of the delta function. The right hand side of (4.15) is the quantum expression for $\partial_o\phi^i(\bar{x}',t)$ and differs from the classical equation (2.17) in that the non-commuting fields have been symmetrized in a particular manner. Translational invariance has now been established so we turn our attention to (homogeneous) Lorentz invariance.

V. Lorentz Invariance

In this section we will demonstrate that (3.2) and (3.3) are satisfied. Our choice of the (non-covariant) Coulomb gauge and the resultant appearance of the terms W_a in (3.3) stop us from concluding that the relations (3.1)-(3.3) imply covariance. The action principle does not uniquely specify all the field transformation properties. In (3.3c) we have symmetrized the non-commuting field variables W_a and Φ_i, an operation motivated by past experience with quantum fields but not demanded by the action principle.

We need not directly verify all six equations (3.2a) - (3.3c) since (3.2b) and (3.3b) follow respectively from (3.2a) and (3.3a) just as (3.1a) implies (3.1b). From the action principle the classical expression for the generator of spacial rotations J_{ab} is

$$J_{ab} = \int d^3x \, [x_a (-i \phi^\dagger_k \phi^k{}_{,b} - \frac{2i}{3} \chi^\dagger \chi_{,b} - \frac{1}{2} A_{k,0} A^k{}_{,b} - \frac{1}{2} A^k{}_{,b} A_{k,0}) - \frac{i}{4} \phi^\dagger_k \gamma_a \gamma_b \phi^k - \frac{i}{6} \chi^\dagger \gamma_a \gamma_b \chi - i \phi^\dagger_a \phi_b - \frac{1}{2} A_{a,0} A_b - \frac{1}{2} A_b A_{a,0} - (a \leftrightarrow b)] \tag{5.1}$$

which we also take to be the quantum expression. The calculation of the commutators of Φ^i and A^i with J_{ab} is similar to but somewhat more lengthy than the calculation of the commutators of the fields with P_a. The calculated commutators do agree with (3.2a) and (3.2c).

The steps leading from the classical to the quantum expression for J_{oa} are essentially the same as those required to construct P_o. Thus the quantum expression for J_{oa} is as follows:

$$J_{oa} = x_o P_a + \int d^3x \{ -x_a \mathcal{H} - i\phi_a^T \gamma_o \chi + i\chi^\dagger \gamma_o \phi_a$$
$$- \tfrac{3i}{4}(\phi_a^\dagger + \tfrac{2}{3}\chi^\dagger \gamma_a) V^{-1}\gamma_o [(D_i \gamma^i + m) D^k \phi_k + \tfrac{2}{3} m D^k \gamma_k \chi]$$
$$+ \tfrac{3i}{4}[\phi_k^\dagger D^k(-D_i\gamma^i + m) - \tfrac{2}{3} m \chi^\dagger \gamma_k D^k] \gamma_o V^{-1}(\phi_a - \tfrac{2}{3}\gamma_a \chi)$$
$$+ \tfrac{1}{2}(A_{a,o} A_o + A_o A_{a,o}) \} \tag{5.2}$$

where the quantum expression (4.12) for P_o is given by $P_o = \int d^3x \mathcal{H}$. Provided we take the quantum analog of the classical constraint (2.14) to be given by the symmetrized equation

$$\psi_o = \gamma_o(\eta - \chi) + \tfrac{ie}{2} F_{jo} U^{-1}(\phi^j - \tfrac{2}{3}\gamma^j \chi)$$
$$+ \tfrac{ie}{2} U^{-1}(\phi^j - \tfrac{2}{3}\gamma^j\chi) F_{jo} \tag{5.3}$$

we find that J_{oa} as given in (5.2) satisfies (3.3a) - (3.3c).

The above results strongly suggest that the fields have been quantized correctly and the generators of the Poincaré transformation have been identified. To complete the proof of covariance in the Coulomb gauge we would need to calculate the commutators of P_μ, J_{ij}, and J_{ob} with J_{oa} and verify that the relations are identical with those of the Poincaré group. An alternative and less laborious approach would be to use (3.1)-(3.4) to calculate the commutators

$$[G_i, \text{dependent fields}] \quad , \quad G_i = \text{Poincaré generator},$$

and then employ the Jacobi identity to simplify the commutators

$$[G_i,[G_j,\eta]] \quad , \quad \eta = \text{independent field}.$$

A third approach would be to quantize the electromagnetic field in the Lorentz gauge (thereby introducing a negative metric) and verify that the analog of (3.1)-(3.3) is satisfied.

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Appendix

In this appendix we calculate and then simplify the term in the second bracket of (4.13). From the definitions (2.12 e,f) and the (anti)commutators (3.5)

$$
\begin{aligned}
&\int d^3x \tfrac{1}{2}\{\phi^i(\bar{x}',t), (\partial_0\tilde{\phi}_r)^\dagger\}(g^r{}_s + \tfrac{3}{2}D^r V^{-1} D_s)\phi^s \\
&= \tfrac{1}{2}\int d^3x [\, i\,\{\phi^i(\bar{x}',t), \phi_r^\dagger\}(-D_j\gamma^j + m)\gamma_0 \\
&\quad -\tfrac{2i}{3}\{\phi^i(\bar{x}',t), \chi^\dagger\}(-D_r + D_j\gamma^j\gamma_r + m\gamma_r)\gamma_0 \\
&\quad + e\{\phi^i(\bar{x}',t), \phi^{\dagger j}F_{jk}\gamma^k + \tfrac{i}{3}\chi^\dagger F^{jk}\sigma_{jk}\}U^{-1}D^m P_{mr}\gamma_0] \times \\
&\qquad (g^r{}_s + \tfrac{3}{2}D^r V^{-1} D_s)\phi^s
\end{aligned}
\tag{A1}
$$

Using (3.6) and (3.9) to evaluate the anticommutators

$$
\begin{aligned}
&= \tfrac{1}{2}\int d^3x\, \delta^3(\bar{x}-\bar{x}')\{[-i P^{ik}(P_{ka} - D_k U^{-1} D_a)P^a{}_r(-D_j\gamma^j + m)\gamma_0 \\
&\quad -\tfrac{2i}{3}P^{ia}D_a U^{-1} S(-D_r + D_j\gamma^j\gamma_r + m\gamma_r)\gamma_0](g^r{}_s + \tfrac{3}{2}D^r V^{-1} D_s)\phi^s \\
&\quad + \tfrac{3e}{2}[-(P^{ij} - P^{in}D_n U^{-1} D_r P^{rj})F_{jk}\gamma^k \\
&\qquad + \tfrac{i}{3}P^{ir}D_r U^{-1} S F^{jk}\sigma_{jk}]\gamma_0 V^{-1} D_s \phi^s\}
\end{aligned}
\tag{A2}
$$

The last term in (A1) has been simplified using (4.6). The first two terms in (A2) can be rewritten in a more convenient form by commuting to the left respectively the quantities $(-D_j\gamma^j + m)$ and $(-D_r + D_j\gamma^j\gamma_r + m\gamma_r)$ with the result that (A1) becomes

$$
\begin{aligned}
&= \tfrac{1}{2}\int d^3x\ \delta^3(\bar{x}-\bar{x}')\Big[\Big\{-i\gamma_0(D_j\gamma^j + m)\phi^i - \tfrac{2i}{3}\gamma^0\gamma^i(-\tfrac{2}{3}D_j\gamma^j + m)\chi \\
&\quad - \tfrac{2i}{3}\gamma_0 P^{ij} D_j U^{-1} D_r \gamma^r D^k \phi_k + e P^{ij} D_j U^{-1}\gamma_s F^{ks}\gamma_0\phi_k \\
&\quad + \tfrac{3e}{2}(P^{ij}D_j U^{-1} D^r P_{rs} - P^i{}_s) F^{as}\gamma_a V^{-1} D^k \gamma_0 \phi_k\Big\} \\
&\quad + \Big\{\tfrac{2i}{3} P^{ij} D_j U^{-1} S\gamma_0 D^k\phi_k - im P^{ij} D_j U^{-1}\gamma_0 D^k\phi_k \\
&\qquad + \tfrac{2i}{3}\gamma_0(-D^i + \tfrac{1}{3}\gamma^i D_j\gamma^j)\chi \\
&\qquad + i P^{in} D_n U^{-1}(-\tfrac{ie}{2}F^{rs,j}\sigma_{rs}\gamma_j - 2ie D_i F^{ri}\gamma_r) V^{-1}\gamma_0 D^k\phi_k\Big\} \\
&\quad + \Big\{\tfrac{3e}{2}\Big[-(P^{ij} - P^{in} D_n U^{-1} D_r P^{rj}) F_{jk}\gamma^k \\
&\qquad + \tfrac{i}{3} P^{ir} D_r U^{-1} S F^{jk}\sigma_{jk}\Big]\gamma_0 V^{-1} D_s \phi^s\Big\}\Big]
\end{aligned}
\tag{A3}
$$

Combining various terms

$$
\begin{aligned}
&= \tfrac{1}{2}\int d^3x\ \delta^3(\bar{x}-\bar{x}')\Big\{-i\gamma_0(D_j\gamma^j + m)\phi^i - \tfrac{2i}{3}\gamma^0\gamma^i(-\tfrac{2}{3}D_j\gamma^j + m)\chi \\
&\quad + \tfrac{2i}{3}\gamma_0(-D^i + \tfrac{1}{3}\gamma^i D_j\gamma^j)\chi + e\gamma_0 P^{ir} D_r U^{-1}\gamma_s F^{sk}\phi_k \\
&\quad + e\gamma_0 P^{ia} D_a U^{-1}(\tfrac{i}{2} T F^{jk}\sigma_{jk} - \tfrac{1}{2}F^{rs,j}\sigma_{rs}\gamma_j \\
&\qquad - 2 D_i F^{ri}\gamma_r) V^{-1} D^k \phi_k\Big\}
\end{aligned}
\tag{A4}
$$

The quantity in the parenthesis in the last term in (A4) equals $(i/2)F^{rs}\sigma_{rs}T$. Therefore (A4) can be rewritten as

$$= \frac{1}{2}\int d^3x\, \delta^3(\bar{x}-\bar{x}')\Big\{-i\gamma_0(D_j\gamma^j+m)\phi^i+\frac{2i}{3}\gamma_0(-D^i+\gamma^i D_j\gamma^j-m\gamma^i)\chi + i\gamma_0 P^{ir}D_r\big[-ie\,U^{-1}(\gamma_5 F^{sk}\phi_k-\frac{i}{3}F^{sk}\sigma_{sk}\chi)\big]\Big\} \quad \text{(A5)}$$

Comparing (A5) with (2.12 e,f) gives the result

$$= \frac{1}{2}\int d^3x\; \delta^3(\bar{x}-\bar{x}')\,\partial_0\tilde{\phi}^i$$

which appears in the second bracket of (4.14).

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