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BOUNDS FOR CREEP IN THICK SPHERICAL PRESSURE VESSELS

by

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I. INTRODUCTION

This paper concerns the creep of a hollow sphere subject to constant internal pressure. Considerable attention has been given to this problem over the years and it is the subject of a definitive monograph [1] by Hult. While Hult's work considers strain-hardening creep and thus involves a more general material description than that assumed below, the main results developed in this paper employ methods not common in such analysis and, further, should be capable of extension to more general problems of creep analysis.

In what follows, a method is derived for obtaining *a priori* upper and lower bounds for the displacement of any point in the spherical body at any time. It is well known, of course, that if elastic strains are neglected in comparison to creep strains, the sphere problem has a closed-form solution. However, in certain situations of technological importance, such an assumption cannot be made. For example, in the design of nuclear reactor components for elevated temperature service, quite restrictive limits are often placed on allowable creep deformation. In such cases, elastic response should be included in the constitutive law as is done in equation (2.4) below.

It could be argued that even with the inclusion of elastic strain, the problem could be solved directly using a general purpose finite element program. Such a procedure, however, would involve considerable time and expense and, since in preliminary design work precise answers are often not required, it is better to obtain simple bounds on deformation. Examples of such bounds for the case of displacement of the outer surface are given in inequalities (3.13) and (3.14) below.

The method by which these bounds are derived is based on various elementary principles of analysis and is different from the bounding techniques of Leckie, Martin, and Ponter [2], [3]. These authors have derived bounds for the displacement of points on the surface of bodies of fairly general geometry under creep and other inelastic deformations. Their results are quite general; however, in applying them to a specific problem, one must obtain a solution to the stress equations of equilibrium which has a prescribed singularity on the boundary of the body. The bounds (3.13), (3.14) are given entirely in terms of known quantities and their application does not involve the solution of boundary value problems.

Although the results of paper apply only to the hollow sphere, the method of derivation should be adaptable to other creep problems - especially those with a high degree of symmetry.

In the course of the derivation of the displacement bounds, two other results were obtained, both of which have counterparts in the work of Hult [1]. First a technique is indicated for representing the solution of the hollow sphere problem as a power series in time, t. Second, a formal limit has been obtained for the stress state in the body as time tends to infinity. This result is given in equation (2.32) below and agrees with that of Hult in the case of no strain-hardening. Of course, the method of derivation is quite different in that elastic strain has not been neglected in the process of obtaining the limiting behavior.

II. STATEMENT OF THE PROBLEM

We consider a hollow thick-walled sphere with inner radius $a > 0$ and outer radius $b < \infty$ described by a spherical coordinate system r, θ, ϕ whose center is at the center of the sphere. Here θ stands for latitude and ϕ for longitude.

The sphere is assumed to be under a uniform internal pressure and zero body force. Thus, the boundary conditions are

$$\sigma_{rr}\Big|_{r=a} = -p \quad , \quad \sigma_{rr}\Big|_{r=b} = 0 \quad , \; t > 0 \tag{2.1}$$

where σ_{rr} is the component of the stress normal to surfaces r = constant. It is assumed that $p > 0$.

By virtue of the symmetry of this problem together with the assumption of material isotropy we may take the displacement components u_r, u_θ, u_ϕ in the form

$$u_r = u(r,t), \; u_\theta = u_\phi = 0 \quad \text{in} \quad [a,b] \times [0,\infty) \; . \tag{2.2}$$

Using the infinitesimal strain-displacement relations in polar coordinates (see e.g.[4] page 184) together with the assumptions (2.2) we obtain

$$\epsilon_{rr} = \frac{\partial u}{\partial r} \, , \; \epsilon_{\theta\theta} = \epsilon_{\phi\phi} = \frac{u}{r} \, , \; \epsilon_{r\theta} = \epsilon_{r\phi} = \epsilon_{\theta\phi} = 0 \; . \tag{2.3}$$

We shall consider creep laws of the form

$$\epsilon_{ij} = \frac{1}{E} \left[(1+\nu)\,\sigma_{ij} - \nu\,\delta_{ij}\,\sigma_{kk} \right] + c \int_0^t s^{2m} s_{ij} \, d\tau \tag{2.4}$$

where σ_{ij} , ϵ_{ij} are components of stress and strain respectively, δ_{ij} is the Kronecker delta, indices i,j,k have the range 1,2,3 and summation over repeated indices is implied. The s_{ij} are components of the deviatoric stress defined by

$$s_{ij} = \sigma_{ij} - \frac{\delta_{ij}}{3}\sigma_{kk}$$

and

$$s^2 = \frac{3}{2} \, s_{ij} \, s_{ij} \quad .$$

It is assumed that

$$E > 0\ ,\ c > 0\ ,\ -1 < \nu < \frac{1}{2}\ ,\ m = 1,2,\ldots \quad . \tag{2.5}$$

The law (2.4) has the form

$$\epsilon_{ij} = \epsilon_{ij}^{e} + \epsilon_{ij}^{c}$$

where ϵ_{ij}^{e} is the elastic strain and depends linearly on the stress, and ϵ_{ij}^{c} is the creep strain, whose dependance on the stress takes the form of Norton's law as generalized to the case of triaxial stress by Odqvist [5].

By virtue of the symmetry of the problem,

$$\sigma_{r\theta} = \sigma_{r\phi} = \sigma_{\theta\phi} = 0\ . \tag{2.6}$$

Also, it follows from (2.3) and (2.4) that

$$0 = \frac{(1+\nu)}{E}\left[\sigma_{\theta\theta}(r,t) - \sigma_{\phi\phi}(r,t)\right] + c\int_0^t s^{2m}(r,\tau)(\sigma_{\theta\theta} - \sigma_{\phi\phi})\, d\tau\ . \tag{2.7}$$

Since (2.7) has the form of a homogeneous linear Volterra integral equation of the second kind in $\sigma_{\theta\theta} - \sigma_{\phi\phi}$, it follows that

$$\sigma_{\phi\phi} = \sigma_{\theta\theta} \quad \text{in } [a,b] \times [0,\infty)\ . \tag{2.8}$$

Due to (2.6) and (2.8),

$$s^2 = (\sigma_{rr} - \sigma_{\theta\theta})^2\ . \tag{2.9}$$

It is convenient to make the abbreviations

$$\sigma \equiv \sigma_{rr} - \sigma_{\theta\theta}\ , \tag{2.10}$$

$$\Sigma \equiv \frac{c}{3}\int_0^t s^{2m}\,(\sigma_{rr} - \sigma_{\theta\theta})\, d\tau = \frac{c}{3}\int_0^t \sigma^{2m+1}\,(r,\tau)\, d\tau\ . \tag{2.11}$$

With this notation, it follows from (2.3), (2.4), (2.8), and (2.9) that the displacement-stress relations take the form

$$\frac{\partial u}{\partial r} = \epsilon_{rr} = \frac{1}{E}\left[\sigma_{rr} - 2\nu\,\sigma_{\theta\theta}\right] + 2\Sigma \tag{2.12}$$

$$\frac{u}{r} = \epsilon_{\theta\theta} = \frac{1}{E}\left[(1-\nu)\sigma_{\theta\theta} - \nu\sigma_{rr}\right] - \Sigma\quad . \tag{2.13}$$

In the absence of body force, the only non trivial equation of equilibrium (or quasistatic equation of motion) is

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{2}{r}\sigma = 0 \ . \tag{2.14}$$

Equations (2.12), (2.13), and (2.14) together with boundary conditions (2.1) form a complete statement of the problem. We now use them to derive an equation in σ alone. To this end, we first eliminate u from (2.12), (2.13) and obtain the stress equation of compatibility

$$\frac{r}{E}\left[(\nu - 1)\frac{\partial\sigma}{\partial r} + (1 - 2\nu)\frac{\partial\sigma_{rr}}{\partial r}\right] - r\frac{\partial\Sigma}{\partial r} = \frac{(1 + \nu)}{E}\sigma + 3\Sigma \quad . \tag{2.15}$$

Elimination of the term $\frac{\partial\sigma_{rr}}{\partial r}$ between (2.14) and (2.15) yields the desired equation in σ :

$$\frac{(\nu - 1)}{E}\left[r\frac{\partial\sigma}{\partial r} + 3\sigma\right] = r\frac{\partial\Sigma}{\partial r} + 3\Sigma \quad . \tag{2.16}$$

Multiplying (2.16) through by r^2 and integrating, we get

$$\frac{(\nu - 1)}{E}\sigma(r,t) = \Sigma(r,t) + \frac{f(t)}{r^3} \ . \tag{2.17}$$

In order to evaluate f(t), we integrate (2.14) and apply the boundary conditions (2.1) to get

$$-\frac{p}{2} = \int_a^b \sigma(\xi,t)\frac{d\xi}{\xi} \ . \tag{2.18}$$

If we then multiply (2.17) through by r^{-1} and integrate with respect to r from a to b, we obtain an expression for f(t), which, when inserted back into (2.17), yields the equation

$$\sigma(r,t) = -\frac{\beta_1 p}{2r^3} - \left(\frac{E}{1 - \nu}\right)\Sigma(r,t) + \frac{\beta_1}{r^3}\left(\frac{E}{1 - \nu}\right)\int_a^b \Sigma(\xi, t)\frac{d\xi}{\xi} \tag{2.19}$$

where

$$\beta_1^{-1} = \int_a^b \frac{d\xi}{\xi^4} = \frac{1}{3}(a^{-3} - b^{-3}) \ . \tag{2.20}$$

Defining

$$\mu_1 = \frac{c\ E}{3(1 - \nu)} \tag{2.21}$$

and using the definition (2.11) of Σ we express (2.19) in the form

$$\sigma(r,t) = -\frac{\beta_1\ p}{2\ r^3} - \mu_1 \int_0^t \sigma^{2m+1}(r,\tau)d\tau + \frac{\beta_1\mu_1}{r^3} \int_0^t\int_a^b \sigma^{2m+1}(\xi,\tau)\frac{d\xi}{\xi}\ d\tau\ . \tag{2.22}$$

This integral equation completely determines σ . In order to calculate u, given σ , we use (2.13), (2.14) and 2.1) to obtain

$$u(r,t) = -\frac{r}{E}\left[(1 - \nu)\sigma(r,t) + (1 - 2\nu)p + 2(1 - 2\nu)\int_a^r \sigma(\xi,t)\frac{d\xi}{\xi}\right] - r\ \ \Sigma\ (r,t)\ . \tag{2.23}$$

Notice that (2.22) immediately yields a power series expansion in t. In fact,

$$\sigma(r,0) = -\frac{\beta_1\ p}{2\ r^3} \tag{2.24}$$

which is the initial elastic response, and

$$\dot\sigma(r,t) = -\mu_1\ \sigma^{2m+1}(r,t) + \frac{\beta_1\mu_1}{r^3}\int_a^b \sigma^{2m+1}(\xi,t)\frac{d\xi}{\xi}\ . \tag{2.25}$$

Setting t = 0 in (2.25) and substituting from (2.24) yields the second coefficient in the expansion:

$$\dot\sigma(r,0) = \frac{\mu_1}{r^3}\left(\frac{\beta_1\ p}{2}\right)^{2m+1}\left[\frac{1}{r^{6m}} - \frac{1}{6^{m+3}}\left(\frac{1}{a^{6m+3}} - \frac{1}{b^{6m+3}}\right)\right]\ . \tag{2.26}$$

Equations (2.25) and (2.26) are recorded above not only because of their role in the power series expansion but also because they are important in the theory developed below.

Using (2.20), we put (2.26) in the form

$$\dot\sigma(r,0) = \frac{\mu_1}{r^3}\left(\frac{\beta_1\ p}{2}\right)^{2m+1}\left[\frac{1}{r^{6m}} - \frac{1}{(2m+1)a^{6m}}\sum_{k=0}^{2m}\left(\frac{a^3}{b^3}\right)^k\right]. \tag{2.27}$$

From (2.27) and the assumption $a < b$, it follows immediately that

$$\dot{\sigma}(a,0) > 0 \ , \ \dot{\sigma}(b,0) < 0 \ . \tag{2.28}$$

Equations (2.18) and (2.25) lead to the formal asymptotic limit of $\sigma(r,t)$ as $t \to \infty$. We assume that

$$\lim_{t \to \infty} \sigma(r,t) = \psi(r) \ , \ \lim_{t \to \infty} \dot{\sigma}(r,t) = 0 \ . \tag{2.29}$$

Then, in the limit, (2.25) becomes

$$\psi^{2m+1}(r) = \frac{\beta_1}{r^3} \int_a^b \psi^{2m+1}(\xi)\frac{d\xi}{\xi} \ . \tag{2.30}$$

Function $\psi(r)$ is a solution of (2.30) if and only if

$$\psi^{2m+1}(r) = \frac{k}{r^3} \ , \tag{2.31}$$

where k is arbitrary. In order to evaluate k, we substitute $\psi(r)$ for σ in (2.18) on the theory that, since (2.18) holds for all $\sigma(r,t)$ it should hold in the limit as well. This leads to the formula

$$\psi(r) = \frac{-3\,p}{4m+2} \ \frac{r^{-\frac{3}{2m+1}}}{a^{-\frac{3}{2m+1}} - b^{-\frac{3}{2m+1}}} \ . \tag{2.32}$$

Unlike Hult [1] in the case of strain-hardening creep, we are able to obtain a formula for the limiting stress distribution without having to neglect the elastic strain. Equation (2.32) agrees exactly with Hult's equations (33) and (34) in the absence of strain-hardening. Notice that (2.23) and the assumptions (2.29) lead to the following formula for the limiting displacement rate:

$$\lim_{t \to \infty} \dot{u}(r,t) = -\frac{r\,c}{3} \ \psi^{2m+1}(r) \ . \tag{2.33}$$

III. BOUNDS ON $\sigma(r,t)$ AND $u(r,t)$.

We assume throughout that there exists a solution $\sigma(r,t)$ which is C^2 in $[a,b] \times [0,\infty)$.

Theorem 3.1. For all (r,t) in $[a,b] \times [0,\infty)$, $\sigma(r,t) < 0$.

Proof. Suppose that the theorem is false and let

$$\mathcal{J} = \left\{ t \text{ in } [0,\infty) \text{: there exists } r \text{ in } [a,b] \text{ such that } \sigma(r,t) \doteq 0 \right\}.$$

Let $t_1 = \text{g.l.b.}\mathcal{J}$. By the Bolzano-Weierstrass Theorem, there exists r_1 in $[a,b]$ such that $\sigma(r_1,t_1) \geqq 0$. Therefore (2.24) implies that $t_1 > 0$. Since $\sigma < 0$ for all t in $[0,t_1)$ it follows from the continuity of σ that

$$\sigma(r_1\ ,t_1) = 0\ ,\ \dot{\sigma}(r_1\ ,t_1) \geqq 0\ . \tag{3.1}$$

Applying (3.1) to (2.25) we see that

$$0 \leqslant \frac{\beta_1\mu_1}{r_1^3} \int_a^b \sigma^{2m+1}(\xi,t_1)\frac{d\xi}{\xi}\ . \tag{3.2}$$

Since, by the continuity of $\sigma(r,t)$, we must have $\sigma(\xi,t_1) \leqq 0$ for all ξ in $[a,b]$, it follows from (3.2) that $\sigma(\xi,t_1) \equiv 0$ in $[a,b]$. But, for $p > 0$, this contradicts (2.18).

Theorem 3.2. For all (r,t) in $[a,b] \times [0,\infty)$,

$$\text{(a) } \frac{\partial\sigma}{\partial r} > 0\ ,\ \text{(b) } \frac{\partial}{\partial r}\,(r^3\sigma) < 0\ . \tag{3.3}$$

Proof. Notice that (2.24) implies that

$$\frac{\partial\sigma}{\partial r}\,(r,0) > 0\ ,\ \frac{\partial}{\partial r}\,(r^3\sigma)\ (r,0) = 0\ . \tag{3.4}$$

Now suppose a function w on $[0,\infty)$ satisfies an equation of the form

$$\dot{w} + kw = g\ , \tag{3.5}$$

where k and g are continuous on $[0,\infty)$.

Since (3.5) implies that w has the representation

$$\exp\left[\int_0^t k(\xi)d\xi\right] w(t) = w(0) + \int_0^t \exp\left[\int_0^\tau k(\lambda)d\lambda\right] g(\tau)d\tau \; , \tag{3.6}$$

it follows that if $g > 0$ (resp. $g < 0$) in $(0,\infty)$ and $w(0) \geqq 0$ (resp. $w(0) \leqq 0$) then $w > 0$ (resp. $w < 0$) on $[0,\infty)$. We can apply this fact to both $\frac{\partial\sigma}{\partial r}$ and $\frac{\partial}{\partial r}(r^3\sigma)$. Differentiation of (2.25) with respect to r yields

$$\frac{\partial\dot{\sigma}}{\partial r} + (2m+1)\,\mu_1\,\sigma^{2m}\frac{\partial\sigma}{\partial r} = -\,3\,\frac{\beta_1\mu_1}{r^4}\int_a^b \sigma^{2m+1}(\xi,t)\frac{d\xi}{\xi} \quad . \tag{3.7}$$

This equation has the same form as (3.5) if we take $w = \frac{\partial\sigma}{\partial r}$, etc., and its right-hand side is strictly positive by Theorem 1. Inequality (3.4) furnishes the positive initial condition. This proves (3.3) (a). In order to get (3.3) (b), we multiply (2.25) through by r^3 and differentiate with respect to r :

$$\frac{\partial}{\partial r}(r^3\dot{\sigma}) + \mu_1\,\sigma^{2m}\frac{\partial}{\partial r}(r^3\sigma) = -\,2m\,\mu_1\,r^3\,\sigma^{2m}\frac{\partial\sigma}{\partial r} \quad . \tag{3.8}$$

This is another equation of the form (3.5), only this time the right-hand side is strictly negative by (3.3) (a) . This fact and the initial condition (3.4) imply (3.3) (b) .

Notice that if we allowed the linear case $m = 0$, then (3.4) and (3.8) would imply that σ is always proportional to r^{-3} .

<u>Theorem</u> 3.3. <u>For</u> $t > 0$,

$$\sigma(a,t) \geqq \sigma(a,0) = -\,\frac{\beta_1\,p}{2\,a^3}\,,\ \text{and} \tag{3.9}$$

$$\sigma(b,t) \leqq \sigma(b,0) = -\,\frac{\beta_1\,p}{2\,b^3} \quad . \tag{3.10}$$

<u>Proof</u>. We shall only prove (3.9), since the proof of (3.10) is very similar. It follows from (2.25) that

$$\dot{\sigma}(a,t) = -\,\mu_1\,\sigma^{2m+1}(a,t) + \frac{\beta_1\mu_1}{a^3}\int_a^b \sigma^{2m+1}(\xi,t)\frac{d\xi}{\xi} \quad .$$

Therefore, by Theorem 3.1 and (3.3) (a) , together with (2.18) and (2.24) ,

$$\dot{\sigma}(a,t) \geq -\mu_1 \sigma^{2m+1}(a,t) + \frac{\beta_1\mu_1}{a^3} \sigma^{2m}(a,t) \int_a^b \sigma(\xi,t)\frac{d\xi}{\xi}$$

$$\geq -\mu_1 \sigma^{2m}(a,t) \left[\sigma(a,t) + \frac{\beta_1 p}{2 a^3} \right]$$

$$\dot{\sigma}(a,t) \geq -\mu_1 \sigma^{2m}(a,t) \left[\sigma(a,t) - \sigma(a,0) \right] . \tag{3.11}$$

Inequality (3.11) may be thought of as a first order differential inequality in $\sigma(a,t)$ which when multiplied by the appropriate integrating factor and integrated yields

$$\exp\left[\mu_1 \int_0^t \sigma^{2m}(a,\tau)d\tau\right]\sigma(a,t) - \sigma(a,0) \geq \left(\exp\left[\mu_1 \int_0^t \sigma^{2m}(a,\tau)d\tau\right] - 1\right)\sigma(a,0) .$$

This proves (3.9).

It is immediate from (3.3) (a) and Theorem 3.3 that

$$-\frac{\beta_1 p}{2 a^3} = \sigma(a,0) \leq \sigma(r,t) \leq \sigma(b,0) = -\frac{\beta_1 p}{2 b^3} \tag{3.12}$$

for all (r,t) in $[a,b] \times [0,\infty)$. That is, we have explicit bounds on σ at any point in the shell for any time $t \geq 0$. But then by (2.23) we have bounds on the radial displacement $u(r,t)$.

For instance, suppose we want bounds on the displacement at a given time t of the outer surface of the spherical shell. Setting $r = b$ in (2.23) and appealing to (2.11) and (2.18) we get

$$u(b,t) = -b\left[\frac{(1-\nu)}{E} \sigma(b,t) + \frac{c}{3} \int_0^t \sigma^{2m+1}(b,\tau)d\tau \right] .$$

Together with (3.12), this leads to the bounds

$$u(b,t) \leq \frac{(1-\nu)}{E}\left(\frac{\beta_1 p}{2 a^3}\right) b + \frac{cb}{3}\left(\frac{\beta_1 p}{2 a^3}\right)^{2m+1} t , \tag{3.13}$$

$$u(b,t) \geqq \frac{(1-\nu)}{E}\left(\frac{\beta_1\ p}{2\ b^3}\right) b + \frac{cb}{3}\left(\frac{\beta_1\ p}{2\ b^3}\right)^{2m+1} t \quad . \tag{3.14}$$

Theorems 3.2 and 3.3 also furnish bounds for $\frac{\partial\sigma}{\partial r}$. In fact, (3.3) (a) and (b) and (3.12) imply that

$$0 < \frac{\partial\sigma}{\partial r} < -\frac{3\sigma}{r} < \frac{3\beta_1\ p}{2\ a^4} \quad . \tag{3.15}$$

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