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HYDRAULIC INSTABILITY OF REACTOR PARALLEL-PLATE FUEL ASSEMBLIES

by

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ABSTRACT

The work of D.R. Miller* on the hydraulic collapse or instability of flat plates has been extended by considering flow redistribution and the effect of unequal friction drops in the deflected region of the channels. A general formula for the pressure distribution over a plate as a function of the plate deflection is derived. From this general formula, linearized formulas for small deflections (less than about 30% channel area change, and less than about one-half the plate thickness) are derived for the pressure distribution and the critical velocity. Graphs of pressure distribution for various assumed deflection curves are presented. Formulas and curves are given for the magnification of initial deflections as a function of approach to the critical velocity

*Critical Flow Velocities for Collapse of Reactor Parallel-Plate Fuel Assemblies, to be published in an ASME Journal.

TABLE OF CONTENTS

I. INTRODUCTION

In Miller's original work on the instability problem, it is assumed that (1) the length of the plates is large compared to the length of the deflected region, (2) that the friction drop in the deflected region is small, and (3) the plate stiffness is equal to that of a wide beam subject to a uniform load.

In many reactor plate designs the length of the plates is of the order of 6 to 15 times the plate width. Any local deflection of the plates must extend over a length of the order of a plate width or more, or there would be a large increase in the stiffness of the plates for this deformation because of axial bending. Therefore, in many cases the deflected region is an appreciable portion of the total length, and some redistribution of flow is to be expected.

As a local region deflects, the friction drop in the constricted channel will be more rapid than in the expanded channel. This causes a pressure difference over the deflected region which is in addition to the Venturi effect considered by Miller. This pressure difference adds to the Venturi effect for deflections near the inlet, and subtracts for deflections near the exit.

If the deflected region extends over an axial length less than about three plate spans, axial bending stresses are not negligible compared to the transverse bending stresses. This increases the stiffness of the plate.

In the analysis presented here these effects are considered. A general expression for the pressure distribution caused by an arbitrary plate deflection is obtained. This expression is linearized by considering deflections corresponding to about 30% channel area change or less. Using this linearized expression, critical velocities can be found for various assumed positions of a deflected region.

In the design of plate fuel assemblies, large deflections are to be avoided. Large deflections will cause flow redistribution, resulting in overheating and/or departure from nucleate boiling in the adjacent plates. The exact limitation on deflection will depend on the particular design, but deflections equal to 30% of the channel thickness or channel area changes of 30% would be excessive. If the discussion is limited to such deflections, the linearized theory presented here will be reasonably accurate. In addition, with the linearized theory, magnification of initial deflections by the hydraulic forces becomes a simple function of the approach to the critical velocity. Then one can design with respect to the critical velocity by setting some limit on permissible deflection, estimating the expected initial deflections, and calculating the factor by which these deflections are multiplied as a function of the velocity.

The problem of large deflections is much more complex, and will not be considered. Such factors enter as: (1) stiffening of the plates by membrane stresses; (2) variation of flow across the span of the plates, (at the mid-span where the plates nearly touch, flow may be choked off); (3) the pressure as a function of deflection becomes highly non-linear. Such effects make interpretation of experimental data very difficult at velocities near and above the critical.

Assumptions

1. The pressure drops thru all channels are equal. That is, the sum of the exit and entrance loss plus the integral of the local friction drop over the length of channel, is the same for all channels.
2. The static pressure distribution across the span of a plate is uniform at each axial location.
3. Small deflection elastic theory holds for the plates; membrane stresses are negligible.
4. The water is incompressible and no voids or bubbles exist.

II. NOMENCLATURE

y -	distance along plate length - in.
L -	plate length - in.
$z = y/L$ -	dimensionless distance along plate.
A -	channel area for undeflected plates - $in.^2$
$\delta A(z)$ -	change in channel area as a function of distance along channel - $in.^2$.
$g(z) = \frac{\delta A}{A}$ -	dimensionless channel area change.
p_s -	static pressure - psi.
p_d -	dynamic pressure - psi.
p_t -	total pressure - psi.
Δp_L -	pressure drop across fuel assembly.
V -	local velocity in channel - in./sec.
V_{avg} -	average velocity thru channels, based on original area.
V_{cr} -	critical velocity.
V_{crm} -	critical velocity as derived by Miller.
Q -	flow rate - $in.^3/sec.$
ρ -	fluid mass density - lb. $sec.^2$ $in.^{-4}$
f -	friction factor.
h -	channel thickness - in.
t -	plate thickness - in.
b -	plate span - in.
k_i -	inlet loss coefficient.
k_e -	exit loss coefficient.
K_∞ -	stiffness of plate considered as a wide beam subject to uniform pressure - psi./in. of deflection at midspan.
α -	parameter which corrects for axial bending stiffness of the plates ("short plate effects").

$\delta(z)$	-	mid-span deflection of plate.
C	=	ratio of plate deflection averaged over span of plate to mid-span deflection.
n	-	number of plates deflecting into a channel.
ℓ	-	axial length of deflected region.
ℓ_1	-	axial distance from inlet to start of deflected region.
ℓ_2	-	axial distance from inlet to middle of deflected region.
E	-	modulus of elasticity - psi.
ν	-	Poisson's ratio.

III. RESULTS

Pressure Distributions

The general expression for axial variation of the static pressure acting on the plates, as a function of the deflection curve g(z) is

$$\frac{\Delta p_s}{\Delta p_L} = \frac{1 + k_i + \frac{fL}{2h} z + \frac{3fL}{2h}\int_0^z g(z)\,dz + 2g(z)}{k_i + k_e + \frac{fL}{2h} + 3\frac{fL}{2h}\int_0^1 g(z)\,dz} - \frac{1 + k_i + \frac{fL}{2h} z - 3\frac{fL}{2h}\int_0^z g(z)\,dz - 2g(z)}{k_i + k_e + \frac{fL}{2h} - 3\frac{fL}{2h}\int_0^1 g(z)\,dz} \qquad (1)$$

The pressure is assumed to be uniform over the plate width. For small deflections this reduces to

$$\Delta p_s = \tfrac{1}{2}\rho V_{avg}^2 \left\{ 4g(z) + 6\frac{fL}{2h}\int_0^z g(z)\,dz - \frac{6}{1+R}\int_0^1 g(z)\,dz \left[1 + k_i + \frac{fL}{2h} z\right] \right\} \qquad (2)$$

where $$R = \frac{\left(\frac{fL}{2h}\right)}{k_i + k_e} \qquad (3)$$

Figures I thru IV show plots of this relation for various assumed deflections. The deflections are assumed to be unmodified by the pressure distribution. In Figure I a sinusoidal deflection curve with an integral number of waves over the full length of the plate was assumed, so that the two channels are identical with respect to total friction drop. In this case there is no flow redistribution.

The following parameters were used in the calculations for Figures I thru IV:

$$k_i = 0.10 \qquad k_e = 0.36 \qquad \ell/b = 2.0$$

$$\frac{fL}{2h} = 3.9 \qquad L/b = 8.75$$

In Figure II a deflection curve of the form $\delta = \frac{1}{2}\delta_m\left(1-\cos\frac{2\pi x}{\ell}\right)$ was used, and ℓ was taken as two plate spans. The deflection starts just beyond the inlet. Figure III shows the results for the same deflection curve, but starting near the middle of the plate length. Figure IV considers a deflection starting at the inlet.

Plate Stiffness

If a uniform pressure acts over length of the plate which is large compared to the plate span, the plate acts as a wide beam and its stiffness is given by

$$K_\infty = \frac{384D}{b^4} \qquad \text{for a clamped beam.} \qquad (4)$$

$$K_\infty = \frac{384D}{5b^4} \qquad \text{for a simply supported beam.} \qquad (5)$$

where D is the flexural rigidity of the plate. If the properties of the plate are homogeneous and isotropic throughout the thickness, $D = \frac{Et^3}{12(1-\nu^2)}$. If these assumptions do not apply, the flexural rigidity of the plate must be evaluated on the basis of the specific materials used and service environment. There is some experimental evidence which indicates that fuel may exhibit very low creep strength during irradiation.

If the deflected region is not long, there is an additional stiffness due to axial bending. This is accounted for in an approximate manner by defining a parameter α. The local stiffness of a deflected region of length (ℓ) is approximately

$$K = K_{\infty}(1+\alpha)^2 \quad * \qquad (6)$$

where α has the values listed below

ℓ/b	α
1	0.61
1.5	0.21
2.0	0.10
2.5	0.06
3.0	0.04

In deriving the values for α, it was assumed that the central deflection over the length (ℓ) varied in smooth manner, being zero at each end. It was also assumed that the local pressure was approximately proportional to the deflection. The values of α were found by analogy to the problem of determining the natural frequencies of plates.

Critical Velocity

The condition for instability is the following: at the critical velocity, the pressure caused by a small plate deflection is just equal to the restoring force of the plate at that deflection.

*The local stiffness is defined as the maximum Δp divided by maximum deflection.

A deflected region of length (ℓ), at a distance ℓ_1 from the inlet having the shape shown on Figures II and III was assumed. Application of the instability criterion gives:

$$V_{cR} = (1+\alpha)\sqrt{\frac{2h\,K_{\infty}}{\rho n C\left\{4 + 3/2\,\frac{f\ell}{2h} - \frac{3}{1+R}\,\frac{\ell}{L}\left[1 + k_i + \frac{f(\ell_1 + \frac{1}{2}\ell)}{2h}\right]\right\}}} \quad (7)$$

For a clamped plate, C = 8/15; for a simply-supported plate C = 16/25. If only one plate deflects into a channel, n = 1. If an assembly of plates deflects such that alternate channels are opened and closed at the same axial position, then n = 2. This is the lowest mode of instability.

The critical velocity found by Miller is

$$V_{crm} = \sqrt{\frac{15 E t^3 h}{\rho b^4 (1-\nu^2)}\,\frac{2}{n}} \qquad \text{for fixed edges} \quad (8)$$

$$V_{crm} = \sqrt{\frac{5 E t^3 h}{2\rho b^4 (1-\nu^2)}\,\frac{2}{n}} \qquad \text{for simply supported edges} \quad (9)$$

The ratio of this critical velocity to the one found by Miller is

$$\frac{V_{cr}}{V_{crm}} = \frac{1+\alpha}{\sqrt{1 + 3/8\,\frac{f\ell}{2h} - \frac{3}{4(1+R)}\,\frac{\ell}{L}\left[1 + k_i + \frac{f(\ell_1 + \frac{1}{2}\ell)}{2h}\right]}} \quad (10)$$

This ratio is independent of the plate edge conditions and the mode of instability.

The above relations for critical velocity apply for instability downstream of the entrance. A relation for entrance instability was obtained using the deflection curve of Figure IV. The result is

$$\frac{V_{cr}}{V_{crm}} = \frac{1}{\sqrt{1 + 1.23\frac{fl}{2h} - \frac{3}{2(1+k)}\frac{l}{L}\left[1 + k_i + \frac{1}{2}\frac{fl}{2h}\right]}} \qquad (11)$$

The critical velocity ratios are plotted on Figure **V** as a function of the distance of the middle of the deflected region to the inlet. Curves are given for several values of the ratio (l/b).

If one assumes a sinusoidal deflection curve over the full length of the plate, as in Figure I, a slightly different critical velocity is obtained. There is no flow redistribution and the critical velocity is the same as Miller's critical velocity, except for axial bending effects

$$\frac{V_{cr}}{V_{crm}} = 1 + \alpha \qquad (12)$$

Effect of Flow on Initial Deflections

If initial deflections extending over an axial distance of about 1/3 the plate length or less are present, they are magnified by the hydraulic forces.

If all the plates in an assembly have initial deflections at the same axial location, such that alternate channels are increased and decreased in area,

$$\frac{\delta}{\delta_0} = \frac{1}{1-\left(\frac{V}{V_{cr}}\right)^2} \tag{13}$$

This relation was given by Miller in Reference (1)

If two adjacent plates are deflected towards each other at the same axial location,

$$\frac{\delta}{\delta_0} = \frac{1}{1-\frac{1+\beta}{2}\left(\frac{V}{V_{cr}}\right)^2} \tag{14}$$

where $$\beta = 2\left(\frac{V_{cr}}{V}\right)^2\left[1-\sqrt{1-\left(\frac{V}{V_{cr}}\right)^2}\right]-1 \tag{15}$$

If one plate in an assembly has an initial deflection

$$\frac{\delta}{\delta_0} = \frac{1}{1-\left(\frac{3+\beta}{4}\right)\left(\frac{V}{V_{cr}}\right)^2} \tag{16}$$

If a single plate centered in a rigid duct has an initial deflection, equation (13) applies. The critical velocity is, however, $\sqrt{2}$ times the critical velocity for an assembly of plates

These relations are given in graphical form on Figure VI. These relations apply at any axial location, and for any length of deflected region (up to about 1/3 the plate length).*

These relations also apply to a sinusoidal initial deflection over the full plate length. If a uniform deflection exists over the full plate length, the hydraulic force tends to suppress this deflection.

*The effect of axial position and length of deflected region are taken into account by selecting the appropriate critical velocity, using equation (10) or (11).

Effect of In-Plane Compressive Loads.

If the plates are subject to an in-plane compressive load, the critical velocity is reduced. Such loading could result from residual stresses associated with welding of plate assemblies, differences in average temperatures of adjacent plates, and differential irradiation growth of fuel plates relative to each other and to poison and non-fuel bearing plates. The condition for instability is

$$\left(\frac{P}{P_{cr}}\right) + \left(\frac{V}{V_{cr}}\right)^2 = 1 \qquad (17)$$

where P_{cr} is the critical load for column buckling alone, and V_{cr} is the critical velocity for flow with no compressive load. This relation is plotted on Fig. VII for the following cases: (a) all plates in an assembly under compression; (b) one plate in an assembly under compression; (c) one plate bisecting a rigid duct.

The above relation and Fig. VII were presented by Miller in Reference (1). They are included here for completeness.

IV. DISCUSSION

Pressure Distributions

Figures I thru IV were derived by considering an initial deflection and calculating the pressure distribution caused by this deflection. If the plates were rigid, this would be the final pressure distribution. With non-rigid plates, this pressure will cause a further deflection, and this deflection will generate an increment of pressure. If the velocity is below the critical this process converges, and if the velocity is at or above the critical the process diverges to large deflections.

Examination of Figures I thru V points out several interesting effects. The pressure distribution of Figure I can be thought of as a sinusoidal distribution plus a uniform distribution. The uniform pressure tends to bias the entire plate to one side. However, as soon as the uniform deflection begins, the flow redistributes and stabilizes this component of the deflection. The sinusoidal component of the pressure, however, will magnify the sinusoidal deflection shape.

In Figure II the pressure over the entire plate is in the same direction as the deflection except at the very end. Thus, there is a tendency to spread the deflected region downstream. Such a motion would cause further flow redistribution, and a decrease in the pressure. This effect was not considered in the analysis. In Figure III the pressure upstream of the deflection acts in the opposite direction to the deflection. In the deflected region, and downstream it acts in the same direction.

Again there is a tendency to spread the deflected region downstream, and perhaps to move the deflected region downstream.

Critical Velocities

Figure V shows that the critical velocity is lowest at the inlet. The different rates of friction pressure drop in the deflected and opened channels causes the critical velocity to increase with distance from the inlet. Comparing Figures II and III, it is seen that a deflection near the inlet generates a larger pressure difference than one downstream. At the inlet there is a further reduction in critical velocity because of the reduced plate stiffness with respect to a pressure load at the inlet.

Effect of Inlet Support Comb

The addition of a support comb at the inlet will greatly increase the local stiffness there, and raise the local critical velocity by a factor of three to four. Then the most critical region moves to about one or two spans beyond the inlet. For the particular geometry used in the calculations, the addition of an inlet support comb raises the minimum critical velocity by approximately 20%. Without the support comb the minimum critical velocity for this geometry is about 80% of Miller's value, and with the support comb it is approximately equal to Miller's value.

Two other factors contribute to making the inlet the most critical region: (1) lack of perfect flow distribution imposes pressure loads at the inlet; (2) the vena contracta between the plates at the inlet causes the channel thickness to be effectively smaller. This lowers the critical velocity. Streamlining of the leading edges minimizes this effect.

Exit Plenum

In the calculations for the figures, an exit loss coefficient of $k_e = 0.36$ was used. This is based on flow discharge from the plates to a duct of the same dimensions as the plate assembly. If the discharge is to atmosphere, as might occur in some experimental work on plate instability, then there can be no static pressure differences between the channels at the exit. This effect can be accounted for by setting $k_e = 1$. The quantity $R = \frac{\left(\frac{fL}{2h}\right)}{k_i + k_e}$ becomes larger. Examination of the critical velocity formulas then shows that the critical velocity is increased by the lack of an exit plenum.

Experimental Investigations

It is recommended that future experimental work be directed toward determining the deflection of plates as a function of the ratio $\left(V/V_{cr}\right)$. Both the plate deflection and the static pressure difference should be measured along the plate length. A knowledge of the initial deflection (deviations from flatness) of the plates as assembled in the test fixture is necessary to interpret the data.

Investigations should concentrate on deflections less than about one-half the plate thickness and channel area changes of less than about 30% to 40%, for the following reasons: (1) heat transfer considerations require that deflections of reactor fuel plates be limited; (2) the additional complications which occur with large deflections (as discussed in the Introduction) will make interpretation of data extremely difficult.

APPENDIX I

Derivation of Static Pressure Acting on the Plate as a Function of the Plate Deflection

Consider a single plate in the middle of a channel bounded by two rigid plates. Let the plate have an arbitrary deflection shape. Let the change in cross-sectional area of Channel 1 as a function of distance from the inlet be expressed as

$$\frac{\delta A_1}{A_o} = g(y/L) = g(z) \qquad \text{where } z = y/L \tag{18}$$

In Channel 2, the change in area is then

$$\frac{\delta A_2}{A_o} = -g(z) \tag{19}$$

At any point along the length of the plate, the static pressure acting across the plate is given by

$$\Delta p_s = p_{1t} - p_{2t} - (p_{1d} - p_{2d}) \tag{20}$$

The dynamic pressures are simply

$$p_{1d} = 1/2 \rho V_1^2 \qquad p_{2d} = 1/2 \rho V_2^2$$

where V_1 and V_2 are the local velocities. Then (20) can be written as

$$\Delta p_s = p_{1t} - p_{2t} - 1/2 \rho (V_1^2 - V_2^2) \tag{21}$$

p_{1t} and p_{2t} are equal upstream of the plate and downstream of the plate. They are not equal along the plate, except in the case of no plate deflection. At any point along the plate, the total pressure is equal to the total pressure just ahead of the plate, minus the losses up to the point considered . In Channel 1

$$p_{1t} = p_{0t} - k_i \tfrac{1}{2}\rho V_{1i}^2 - \int_0^z \tfrac{1}{2}\rho V_1^2 \frac{fL}{2h}\left(1+\frac{\delta A_1}{A_0}\right)^{-1} dz \qquad (22)$$

The second term on the right is the inlet loss. The integral represents the friction drop. The factor $\left(1+\frac{\delta A_1}{A_0}\right)^{-1}$ corrects the hydraulic diameter from the undeformed channel to the deformed channel. In Channel 2,

$$p_{2t} = p_{0t} - k_i \tfrac{1}{2}\rho V_{2i}^2 - \int_0^z \tfrac{1}{2}\rho V_2^2 \frac{fL}{2h}\left(1-\frac{\delta A_1}{A_0}\right)^{-1} dz \qquad (23)$$

Substituting (22) and (23) into (21) gives

$$\Delta p_s = k_i \tfrac{1}{2}\rho\left(V_{2i}^2 - V_{1i}^2\right) + \int_0^z \tfrac{1}{2}\rho V_2^2 \frac{fL}{2h}\left(1-\frac{\delta A_1}{A_0}\right)^{-1} dz$$
$$- \int_0^z \tfrac{1}{2}\rho V_1^2 \frac{fL}{2h}\left(1+\frac{\delta A_1}{A_0}\right)^{-1} dz - \tfrac{1}{2}\rho\left(V_1^2 - V_2^2\right) \qquad (24)$$

The local velocities can be expressed in terms of the flow thru each channel and the local area change.

$$V_1\left(1+\frac{\delta A_1}{A_0}\right)A_0 = Q_1$$

$$V_2\left(1-\frac{\delta A_1}{A_0}\right)A_0 = Q_2$$

Substituting these expressions into (24) gives

$$\Delta p_s = -k_i \tfrac{1}{2}\rho\left(\frac{Q_1}{A_0}\right)^2\left(1+\frac{\delta A_{1i}}{A_0}\right)^{-2} + k_i \tfrac{1}{2}\rho\left(\frac{Q_2}{A_0}\right)^2\left(1-\frac{\delta A_{1i}}{A_0}\right)^{-2}$$
$$\tfrac{1}{2}\rho\left(\frac{Q_2}{A_0}\right)^2\int_0^z \frac{fL}{2h}\left(1-\frac{\delta A_1}{A_0}\right)^{-3} dz - \tfrac{1}{2}\rho\left(\frac{Q_1}{A_0}\right)^2\int_0^z \frac{fL}{2h}\left(1+\frac{\delta A_1}{A_0}\right)^{-3} dz$$
$$-\tfrac{1}{2}\rho\left[\left(\frac{Q_1}{A_0}\right)^2\left(1+\frac{\delta A_1}{A_0}\right)^{-2} + \left(\frac{Q_2}{A_0}\right)^2\left(1-\frac{\delta A_1}{A_0}\right)^{-2}\right]$$

Let Q_0 be the flow thru each channel with an undeflected plate. Multiply thru by $\left(\frac{A_0}{Q_0}\right)^2 \times \frac{2}{\rho}$

$$\Delta p_s\left(\frac{A_0}{Q_0}\right)^2\left(\frac{2}{\rho}\right) = -k_i\left(\frac{Q_1}{Q_0}\right)^2\left(1+\frac{\delta A_{1i}}{A_0}\right)^{-2} + k_i\left(\frac{Q_2}{Q_0}\right)^2\left(1-\frac{\delta A_{1i}}{A_0}\right)^{-2}$$
$$+\left(\frac{Q_2}{Q_0}\right)^2\frac{fL}{2h}\int_0^z\left(1-\frac{\delta A_1}{A_0}\right)^{-3}dz - \left(\frac{Q_1}{Q_0}\right)^2\frac{fL}{2h}\int_0^z\left(1+\frac{\delta A_1}{A_0}\right)^{-3}dz \qquad (25)$$
$$-\left(\frac{Q_1}{Q_0}\right)^2\left(1+\frac{\delta A_1}{A_0}\right)^{-2} + \left(\frac{Q_2}{Q_0}\right)^2\left(1-\frac{\delta A_1}{A_0}\right)^{-2}$$

Now we determine the ratios of the Q's by imposing the condition that the pressure drops thru the two channels must be equal. Let Δp_L be the pressure drop thru the channels. For Channel 1

$$k_i \tfrac{1}{2}\rho V_{1i}^2 + \frac{fL}{2h}\int_0^1 \tfrac{1}{2}\rho V_1^2\left(1+\frac{\delta A_1}{A_0}\right)^{-1}dz + k_e \tfrac{1}{2}\rho V_{1e}^2 = \Delta p_L$$

Substituting for V_1 in terms of Q_1 and multiplying by $\left(\frac{A_0}{Q_0}\right)^2\left(\frac{2}{\rho}\right)$ gives

$$k_i\left(\frac{Q_1}{Q_0}\right)^2\left(1+\frac{\delta A_{1i}}{A_0}\right)^{-2} + \left(\frac{Q_1}{Q_0}\right)^2\frac{fL}{2h}\int_0^1\left(1+\frac{\delta A_1}{A_0}\right)^{-3}dz + k_e\left(\frac{Q_1}{Q_0}\right)^2\left(1+\frac{\delta A_{1e}}{A_0}\right)^{-2} = \Delta p_L\left(\frac{A_0}{Q_0}\right)^2\left(\frac{2}{\rho}\right)$$

Solving for $\left(\frac{Q_1}{Q_0}\right)^2$ gives

$$\left(\frac{Q_1}{Q_0}\right)^2 = \frac{\Delta p_L\left(\frac{A_0}{Q_0}\right)^2\left(\frac{2}{\rho}\right)}{k_i\left(1+\frac{\delta A_{1i}}{A_0}\right)^{-2} + k_e\left(1+\frac{\delta A_{1e}}{A_0}\right)^{-2} + \frac{fL}{2h}\int_0^1\left(1+\frac{\delta A_1}{A_0}\right)^{-3}dz}$$

In order to simplify the analysis, we will neglect the effect of area changes at the exit and inlet on the exit and inlet losses.

That is, we set $k_i \delta A_{1i} = k_e \delta A_{1e} = 0$ However, deflections at the inlet and exit are still accounted for in the friction drop terms.

$$\left(\frac{Q_1}{Q_0}\right)^2 = \frac{\Delta p_L \left(\frac{A_0}{Q_0}\right)^2 \left(\frac{2}{\rho}\right)}{k_i + k_e + \frac{fL}{2h}\int_0^1 \left(1 + \frac{\delta A_1}{A_0}\right)^{-3} dz} \tag{26}$$

Similarly, the flow in Channel 2 is

$$\left(\frac{Q_2}{Q_0}\right)^2 = \frac{\Delta p_L \left(\frac{A_0}{Q_0}\right)^2 \left(\frac{2}{\rho}\right)}{k_i + k_e + \frac{fL}{2h}\int_0^1 \left(1 - \frac{\delta A_1}{A_0}\right)^{-3} dz}$$

Substitute for Q_1 and Q_2 from (26) and (27) into (25). In (25) set $\delta A_{1i} = \delta A_{2i} = 0$

$$\frac{\Delta p_s}{\Delta p_L} = -\frac{k_i + \frac{fL}{2h}\int_0^z \left(1 + \frac{\delta A_1}{A_0}\right)^{-3} dz + \left(1 + \frac{\delta A_1}{A_0}\right)^{-2}}{k_i + k_e + \frac{fL}{2h}\int_0^1 \left(1 + \frac{\delta A_1}{A_0}\right)^{-3} dz} \tag{28}$$

$$+ \frac{k_i + \frac{fL}{2h}\int_0^z \left(1 - \frac{\delta A_1}{A_0}\right)^{-3} dz + \left(1 - \frac{\delta A_1}{A_0}\right)^{-2}}{k_i + k_e + \frac{fL}{2h}\int_0^1 \left(1 - \frac{\delta A_1}{A_0}\right)^{-3} dz}$$

This expression gives the pressure distribution over the plate as a function of the channel area change, and as a function of the pressure drop Δp_L thru the channels. If the total flow is fixed, rather than the pressure drop, then it is necessary to determine Δp_L in terms of the total flow, or average velocity. The total flow is simply $Q_1 + Q_2$. From (26) and (27)

$$Q_1 + Q_2 = \sqrt{\Delta p_L\left(\frac{2}{\rho}\right)} \left\{ \frac{A_0}{\left[k_i + k_e + \frac{fL}{2h}\int_0^1 \left(1+\frac{\delta A_1}{A_0}\right)^{-3} dz\right]^{1/2}} + \frac{A_0}{\left[k_i + k_e + \frac{fL}{2h}\int_0^1 \left(1-\frac{\delta A_1}{A_0}\right)^{-3} dz\right]^{1/2}} \right\}$$

The average velocity is

$$V_{avg} = \frac{Q_1 + Q_2}{2A_0}$$

Solving for Δp_L gives

$$\Delta p_L = \tfrac{1}{2}\rho V_{avg}^2 \left\{ \frac{1}{\frac{1}{2}\left[k_i + k_e + \frac{fL}{2h}\int_0^1 \left(1+\frac{\delta A_1}{A_0}\right)^{-3} dz\right]^{-1/2} + \frac{1}{2}\left[k_i + k_e + \frac{fL}{2h}\int_0^1 \left(1-\frac{\delta A_1}{A_1}\right)^{-3} dz\right]^{-1/2}} \right\} \qquad (29)$$

When $\delta A_1 = 0$, (12) reduces to

$$\Delta p_L = \frac{1}{2}\rho V_{avg}^2 \left[k_i + k_e + \frac{fL}{2h}\right] \qquad (30)$$

In many cases it may be sufficiently accurate to use the expression and neglect the increase in pressure drop due to the deflection.

APPENDIX II

Plate Stiffness - Effect of Length of Deflected Region

It is assumed that the pressure distribution is uniform over the span of the plate at any axial position. If the deflected region is long compared to the plate span and if the transition from the deflected region to the undeflected region is smooth, axial bending can be neglected. Then the deflection at any axial position is related to the pressure acting at that position by

$$\delta(z) = \delta_0(z) + \frac{1-\nu^2}{384}\frac{b^4}{EI}\Delta p_s(z) \qquad (31)$$

where $\delta_0(z)$ is the initial deflection of the plate. Let

$$K_\infty = \frac{384\,EI}{(1-\nu^2)\,b^4} \qquad (32)$$

Then (18) can be written as

$$\delta(z) = \delta_0(z) + \frac{\Delta p_s(z)}{K_\infty} \qquad (33)$$

Now we consider how K must be modified to account for axial bending in the case of short deflected regions. If the pressure acting on the plate were proportional to the plate deflection, the modification could be made very easily by an analogy to the problem of determining frequencies of plates.

With small friction drops per inch, and small deflected regions (or a plate deflection curve having nearly equal deflections in each direction, such that $\int_0^1 g(z)dz << 1$), Equation (16) becomes

$$\Delta p_s(z) \simeq \tfrac{1}{2}\rho V_{avg}^2 \cdot 4g(z)$$

Then the pressure becomes very nearly proportional to the deflection; the curve of pressure versus length has nearly the same shape as the curve of deflection versus length.

In the actual case the friction drop in the deflected region shifts the pressure curve downstream slightly, and causes a pressure in the opposite direction upstream, and in the same direction downstream. Also the friction drop and the flow redistribution affect the average pressure over the deflected region, increasing the pressure for a deflection near the inlet and decreasing the pressure for deflections away from the inlet.

To a first approximation we neglect the shift of the curve downstream. The higher pressure downstream of the middle of the deflected region compensates for the lower pressure upstream. Then the pressure at each point is considered to be proportional to the deflection.

This loading is the same as the inertia loading in a vibration problem. The plate stiffness in the pressure loading problem is modified by axial bending in precisely the same way as it is in the vibration problem. We consider a deformation shape of the form

$$\delta = \tfrac{1}{2}\delta_m \left(1 - \cos \frac{2\pi y}{\ell}\right)$$

This is the deflection curve used in obtaining Figures I thru III. The deflected region is considered as a plate clamped on all four edges, of span (b) and length ℓ. Then the ratio of the stiffness of this plate to one of infinite length is given by

$$\frac{K}{K_\infty} = \left(\frac{\omega}{\omega_\infty}\right)^2$$

where ω = angular frequency of plate of length

and ω = angular frequency of plate of infinite length.

We introduce a parameter α, defined by

$$\omega = \omega_\infty (1+\alpha)$$

Then $$\frac{K}{K_\infty} = (1+\alpha)^2$$

Using values of $\left(\frac{\omega}{\omega_\infty}\right)$ for various values of l/b,* the table of α versus l/b in the Results section is derived.

Now (33) can be modified to account for axial bending by substituting K for K_∞

$$\delta(z) = \delta_0(z) + \frac{\Delta p_s(z)}{K_\infty (1+\alpha)^2} \tag{34}$$

If the deflection occurs at the inlet, and no inlet comb is present, the change in stiffness is somewhat different. Here we must consider a plate clamped on three sides and free on the fourth side. Frequency calculations for the combination are not readily available. However, for the same l/b, the stiffening effect must be smaller than for the fully clamped plate. Therefore, to be conservative we take no credit for increased stiffness, and use $\alpha = 0$ for entrance deflection.

*Young, Dana, "Vibration of Rectangular Plates by the Ritz Method", Journal of Applied Mechanics, Vol. 73, June, 1951, p. 229.

Finally we introduce a constant C which gives the average deflection over the span in terms of the deflection at mid-span

$$C = \frac{1}{b}\int_0^b \delta(u)\,du$$

The change in channel area caused by **plate deflection is then**

$$\frac{\delta A}{A} = n\,\frac{C\,\delta\,b}{h\,b} = n\,C\,\delta/h$$

n = 1 for a single plate

n = 2 for two plates deflecting toward each other

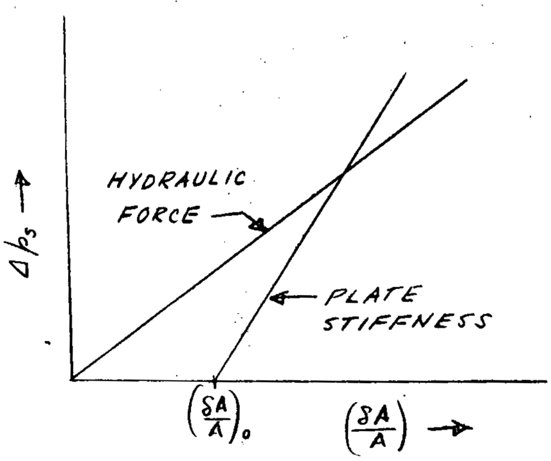
Substituting in (34) for δ gives

$$\frac{\delta A}{A} = \left(\frac{\delta A}{A}\right)_0 + \frac{nC}{h}\cdot\frac{\Delta p_s}{K_\infty(1+\alpha)^2} \qquad (35)$$

APPENDIX III

Magnification of Initial Deflections as a Function of the Approach to the Critical Velocity

Consider plots of Δp_s versus channel area change. Plot the hydraulic force curve and the plate stiffness curve on the same graph. Allow some initial channel area change. The plate stiffness curve then starts at $\left(\frac{\delta A}{A}\right)_0$, whereas the hydraulic force curve starts at the origin. These curves are approximately straight lines for the range of interest.



The plate will reach equilibrium at the point where the two lines intersect.

The plate stiffness line is given by

$$\Delta p_s = \frac{K_\infty(1+\alpha)^2}{C}\left[\frac{\delta A}{A} - \left(\frac{\delta A}{A}\right)_1\right]$$

At the critical velocity the slopes of the two lines are equal. Then at a velocity V, the hydraulic force line is

$$\Delta p_s = \frac{K_\infty(1+\alpha)^2}{C}\left(\frac{V}{V_{cr}}\right)^2\left(\frac{\delta A}{A}\right)$$

Setting the pressure equal gives

$$\frac{K_\infty(1+\alpha)^2}{C}\left[\frac{\delta A}{A} - \left(\frac{\delta A}{A}\right)_0\right] = \frac{K_\infty(1+\alpha)^2}{C}\left(\frac{V}{V_{cr}}\right)^2\left(\frac{\delta A}{A}\right)$$

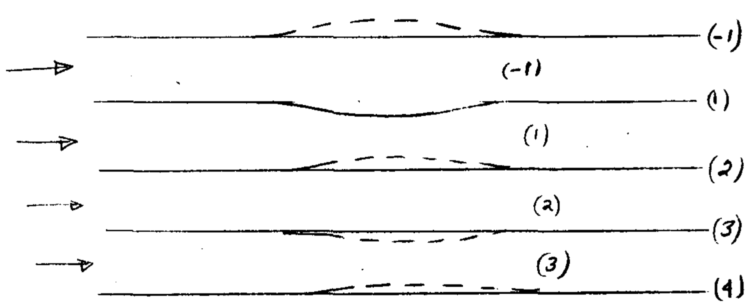
Divide thru by δA_o

$$\frac{\delta A}{\delta A_o} - 1 = \left(\frac{V}{V_{cr}}\right)^2 \frac{\delta A}{\delta A_o}$$

$$\frac{\delta A}{\delta A_o} = \frac{1}{1 - \left(\frac{V}{V_{cr}}\right)^2}$$

The above derivation applies to either a single plate bisecting a rigid duct, or to an assembly of plates in which all the plates at an axial position are deflected so as to give alternately opened and closed channels.

If only one plate in an assembly has an initial deflection, the situation is somewhat different and must be considered in more detail. Consider a multiple plate assembly with one plate initially deflected. A portion of such an assembly is shown in the figure. Plate #1 has an initial deflection δ_o. This deflection causes the other plates to deflect as shown by the dashed lines. The pressure acting on plate #1 is



$$\Delta p_{s1} = \tfrac{1}{2}\rho\left(V_1^2 - V_{-1}^2\right) \qquad (36)$$

In the following derivation we go back to the assumptions of no flow distribution and small friction drop. Then the velocities are related to the velocity for the undeformed channel by

$$V_1\left[1 - c\,\delta_1/h - c\,\delta_2/h\right] = V_0 \tag{37}$$

$$V_{-1}\left[1 + c\,\delta_1/h + c\,\delta_2/h\right] = V_0 \tag{38}$$

Substituting into (25) and using the series approximation gives

$$\Delta p_{s1} = \tfrac{1}{2}\rho V_1^2 \frac{c}{h}\left(4\delta_1 + 4\delta_2\right) \tag{39}$$

The plate stiffness provides a restoring force

$$\Delta p_{s1} = K(\delta_1 - \delta_0)$$

Setting these pressures equal and collecting terms gives

$$\delta_1\left[1 - 4\frac{\tfrac{1}{2}\rho V_0^2 c}{hK}\right] - 4\frac{\tfrac{1}{2}\rho V_0^2 c}{hK}\delta_2 = \delta_0 \tag{40}$$

Now we consider the pressure on plate #2. Proceeding in the same manner, we obtain

$$\Delta p_{s2} = \tfrac{1}{2}\rho V_0^2 \frac{c}{h}\left[2\delta_1 + 4\delta_2 + 2\delta_3\right]$$

The plate stiffness provides a restoring force

$$\Delta p_{s2} = K\delta_2$$

Setting the pressures equal gives

$$\delta_2\left[1 - 4\frac{\tfrac{1}{2}\rho V_0^2 c}{hK}\right] - 2\frac{\tfrac{1}{2}\rho V_0^2 c}{hK}\delta_1 - 2\frac{\tfrac{1}{2}\rho V_0^2 c}{hK}\delta_3 = 0 \tag{41}$$

Applying the same procedure to the other plates we obtain the general relation

$$-2\frac{\frac{1}{2}\rho V_0^2 C}{hK}\delta_i + \left[1 - 4\frac{\frac{1}{2}\rho V_0^2 C}{hK}\right]\delta_{i+1} - 2\frac{\frac{1}{2}\rho V_0^2 C}{hK}\delta_{i+2} = 0 \qquad (42)$$

The lowest mode of collapse for the assembly occurs when all the plates deflect an equal magnitude (but alternate in directions). In this case $\delta_2 = \delta_1$ in (39). Substituting $\delta_2 = \delta_1$ and setting the hydraulic force equal to the plate restoring force with no initial deflection gives the critical velocity

$$\frac{1}{2}\rho V_{cr}^2 = \frac{hK}{8C} \qquad (43)$$

Using this relation, we write (42) in terms of the ratio of the velocity to the critical velocity.

$$-\frac{1}{4}\left(\frac{V}{V_{cr}}\right)^2\delta_i + \left[1 - \frac{1}{2}\left(\frac{V}{V_{cr}}\right)^2\right]\delta_{i+1} - \frac{1}{4}\left(\frac{V}{V_{cr}}\right)^2\delta_{i+2} = 0$$

A general solution of the set of difference equations is

$$\delta_i = k\beta^i \qquad (44)$$

Substituting into (33) and multiplying by $\left(\frac{V_{cr}}{V}\right)^2$ gives

$$\beta^2 - \left[4\left(\frac{V_{cr}}{V}\right)^2 - 2\right]\beta + 1 = 0$$

$$\beta = 2\left(\frac{V_{cr}}{V}\right)^2 - 1 \pm \sqrt{4\left(\frac{V_{cr}}{V}\right)^4 - 4\left(\frac{V_{cr}}{V}\right)^2}$$

As i becomes large (the plate considered is far from the initially deflected plate), the deflection $k\beta^i$ must become small. Then β must be less than one. This requires that we use the minus sign in front of the square root.

$$\beta = 2\left(\frac{V_{cr}}{V}\right)^2\left[1 - \sqrt{1-\left(\frac{V}{V_{cr}}\right)^2}\,\right] - 1 \tag{45}$$

When $\left(\frac{V}{V_{cr}}\right) \to 0$, $\beta \to 0$. This means that the plate with the initial deflection causes very small deflections of the other plates at low flow. When $\left(\frac{V}{V_{cr}}\right) \to 1$, $\beta \to 1$. Thus, at the critical velocity, all the plates deflect equally. As the critical velocity is approached, the initial deflection of the one plate affects plates further away.

Writing (40) in terms of the critical velocity gives

$$\delta_1\left[1 - \tfrac{1}{2}\left(\frac{V}{V_{cr}}\right)^2\right] - \tfrac{1}{2}\left(\frac{V}{V_{cr}}\right)^2 \delta_2 = \delta_0 \tag{46}$$

From our solution (44)

$$\delta_1 = k\beta \qquad \delta_2 = k\beta^2$$

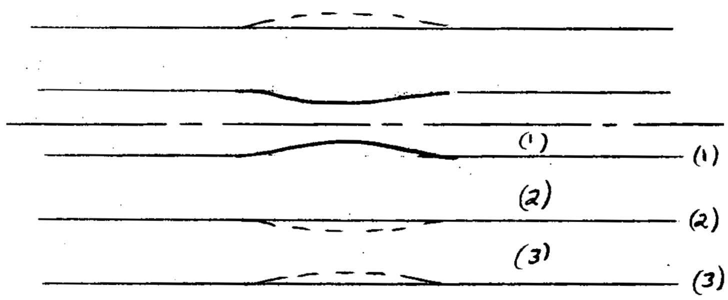
Then

$$\delta_2 = \delta_1 \beta$$

Substituting for δ_2 in (46) and solving for δ_1/δ_0 gives

$$\delta_1/\delta_0 = \frac{1}{1 - \left(\frac{1+\beta}{2}\right)\left(\frac{V}{V_{cr}}\right)^2} \tag{47}$$

Next we consider the case of two adjacent plates deflecting towards each other. For this case



$$\Delta p_{s1} = \tfrac{1}{2}\rho\left(V_1^2 - V_2^2\right)$$

$$V_1\left[1 - 2C\,\delta_1/h\right] = V_0$$

$$V_2\left[1 + C\,\delta_1/h + C\,\delta_2/h\right] = V_0$$

$$\Delta p_{s1} = \tfrac{1}{2}\rho V_0^2 \frac{C}{h}\left[6\delta_1 + 2\delta_2\right] \tag{48}$$

Setting this equal to the plate restoring pressure gives

$$\delta_1\left[1 - 6\,\frac{\frac{1}{2}\rho V_0^2 C}{hK}\right] - 2\,\frac{\frac{1}{2}\rho V_0^2 C}{hK}\,\delta_2 = \delta_0 \tag{49}$$

The relations for the other plates are the same as in the previous case. Then we again obtain

$$\delta_1' = k\beta' \quad \text{and} \quad \delta_2 = \beta\,\delta_1$$

Substituting the latter relation into (49) gives

$$\delta_1 \left[1 - 6\,\frac{\tfrac{1}{2}\rho V_0^2 C}{hK} - 2\beta\,\frac{\tfrac{1}{2}\rho V_0^2 C}{hK} \right] = \delta_0$$

In terms of the critical velocity this becomes

$$\delta_1/\delta_0 = \frac{1}{1 - \left(\frac{3+\beta}{4}\right)\left(\frac{V}{V_{cr}}\right)^2} \qquad (50)$$

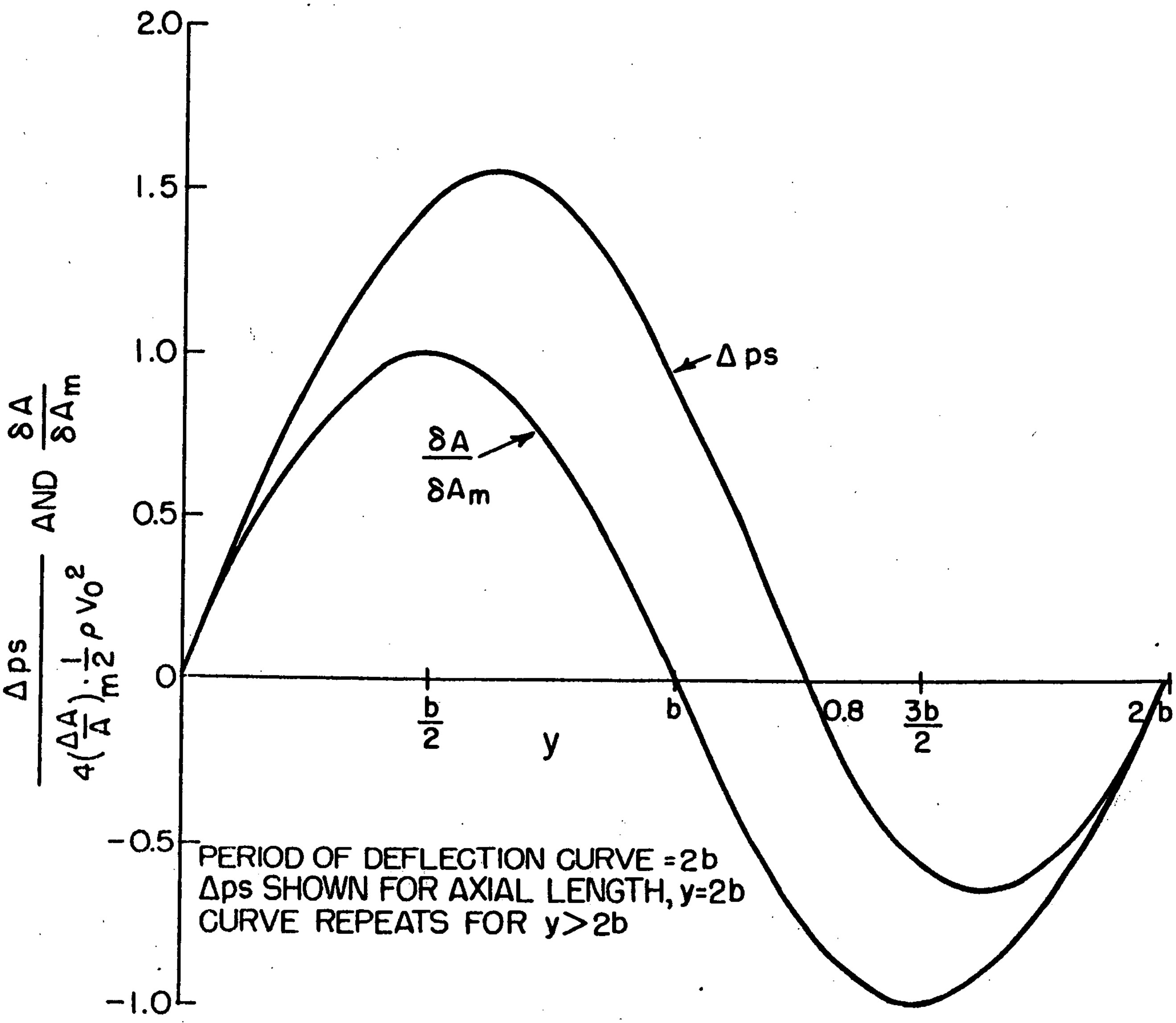


FIGURE I Δp_s GENERATED BY A SINUSOIDAL DEFLECTION CURVE

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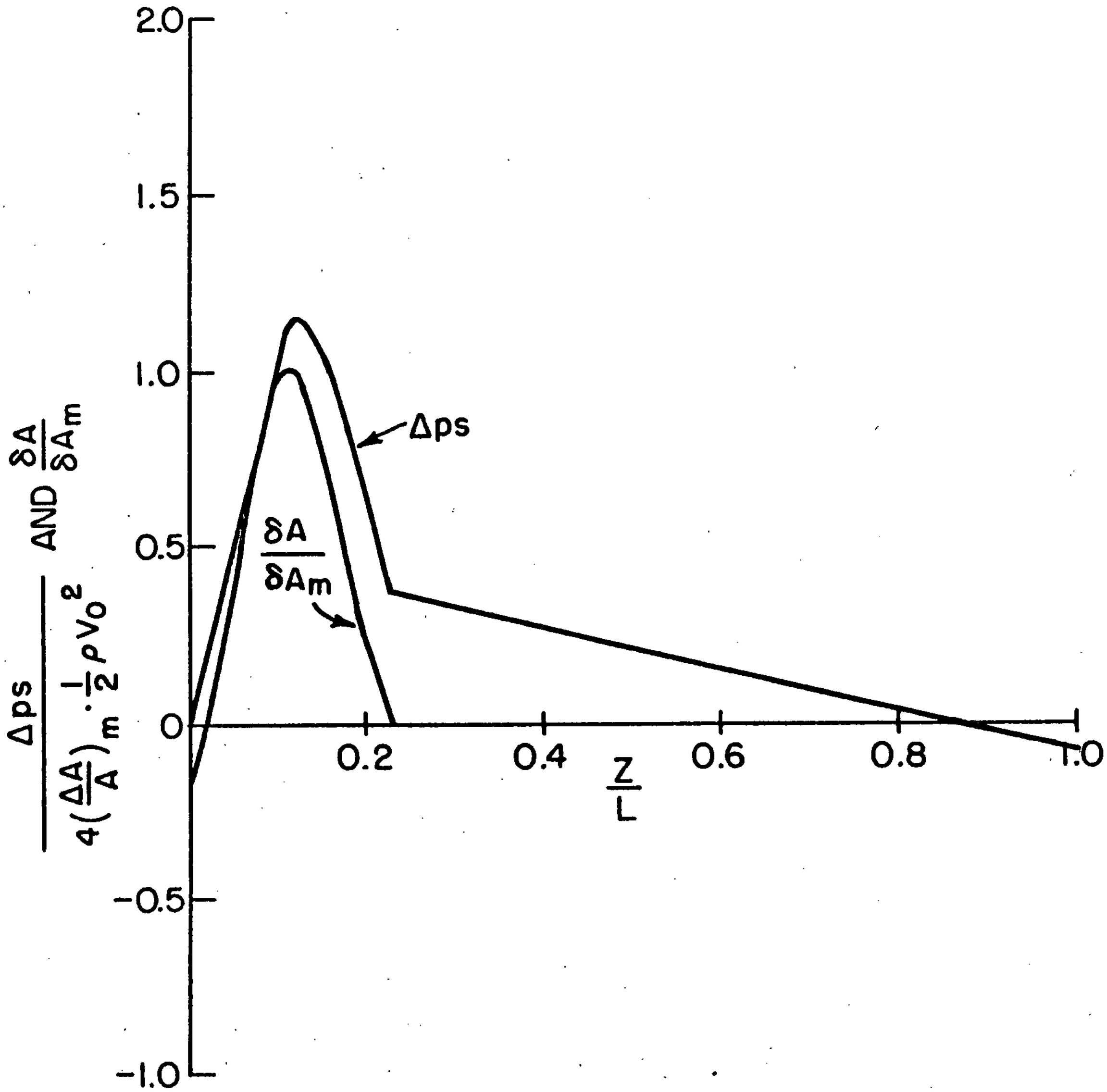

FIGURE II Δps GENERATED BY A DEFLECTION NEAR THE INLET

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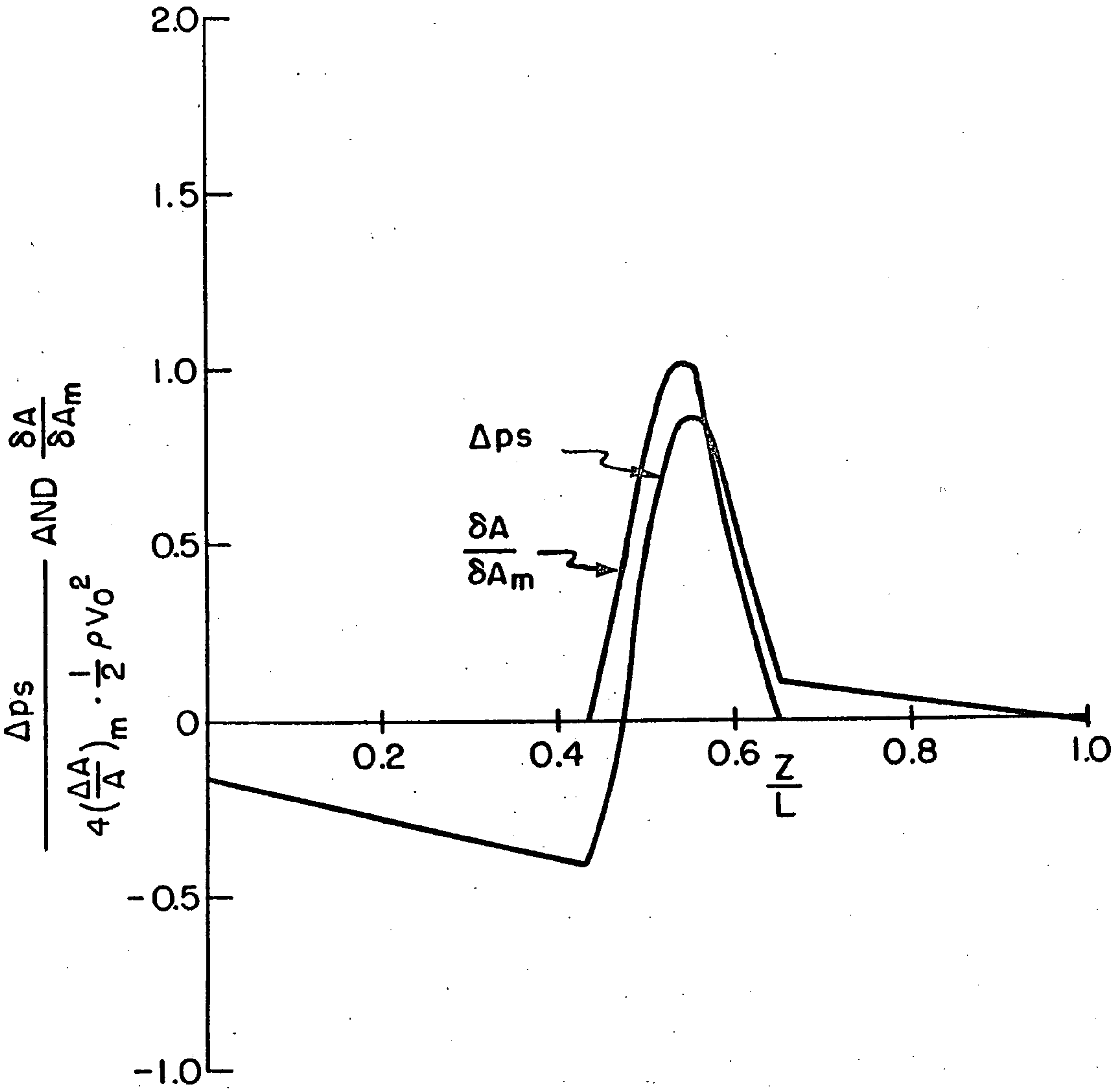

FIGURE III Δps GENERATED BY A DEFLECTION NEAR THE MIDDLE OF THE PLATE LENGTH

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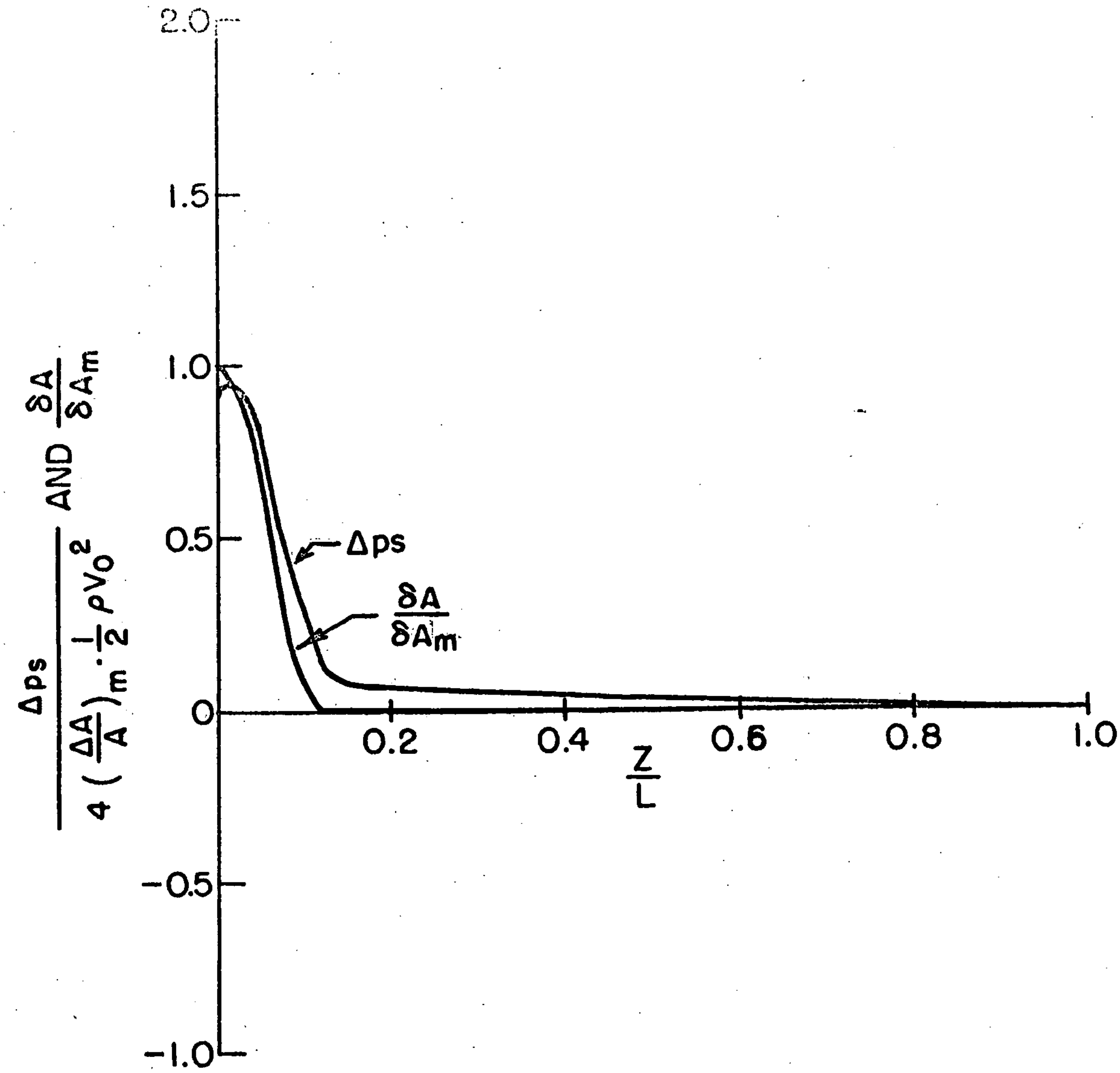

FIGURE IV Δps GENERATED BY A DEFLECTION AT THE INLET

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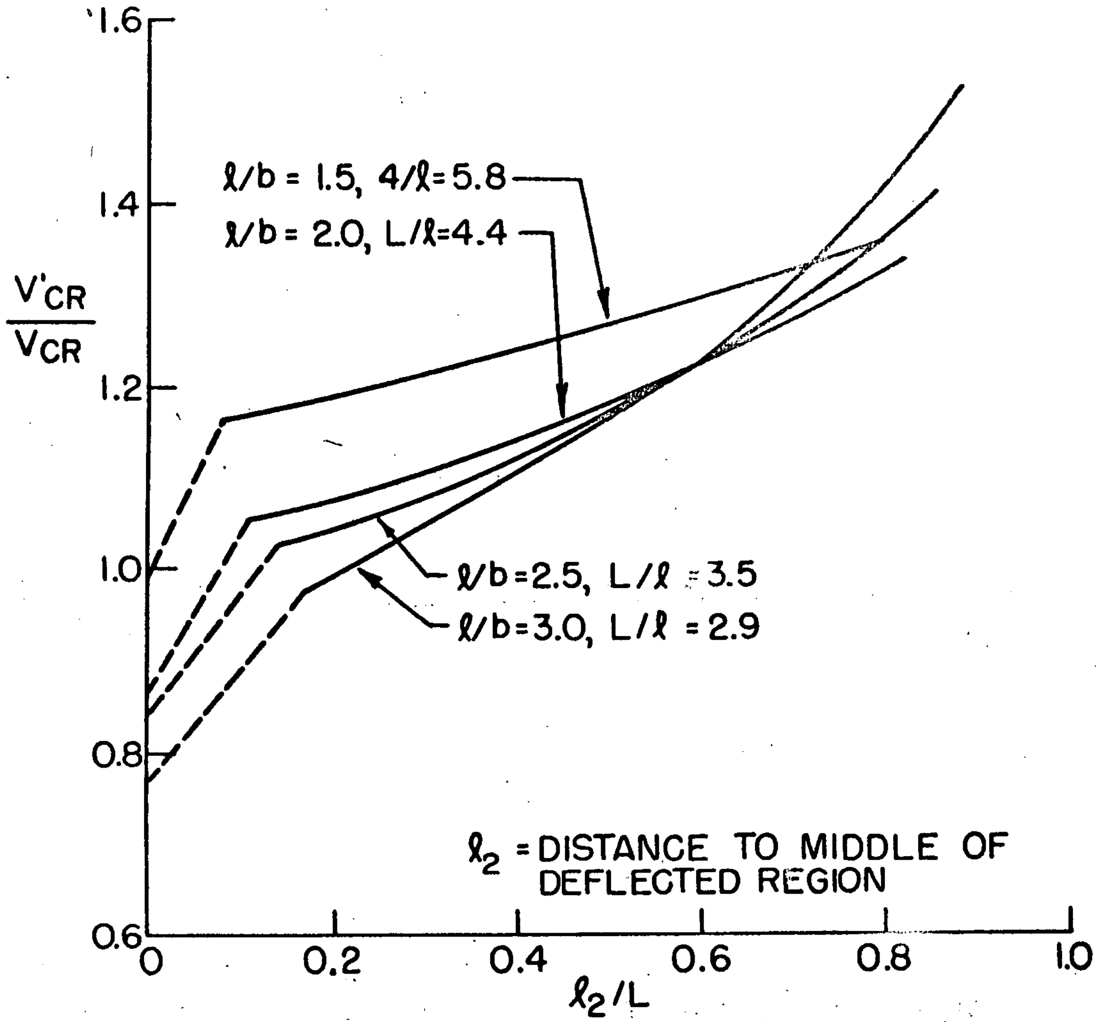

FIGURE V RATIO OF CRITICAL VELOCITY TO MILLER CRITICAL VELOCITY AS FUNCTION OF POSITION OF DEFLECTED REGION FOR SEVERAL VALUES OF L/ℓ AND ℓ/b

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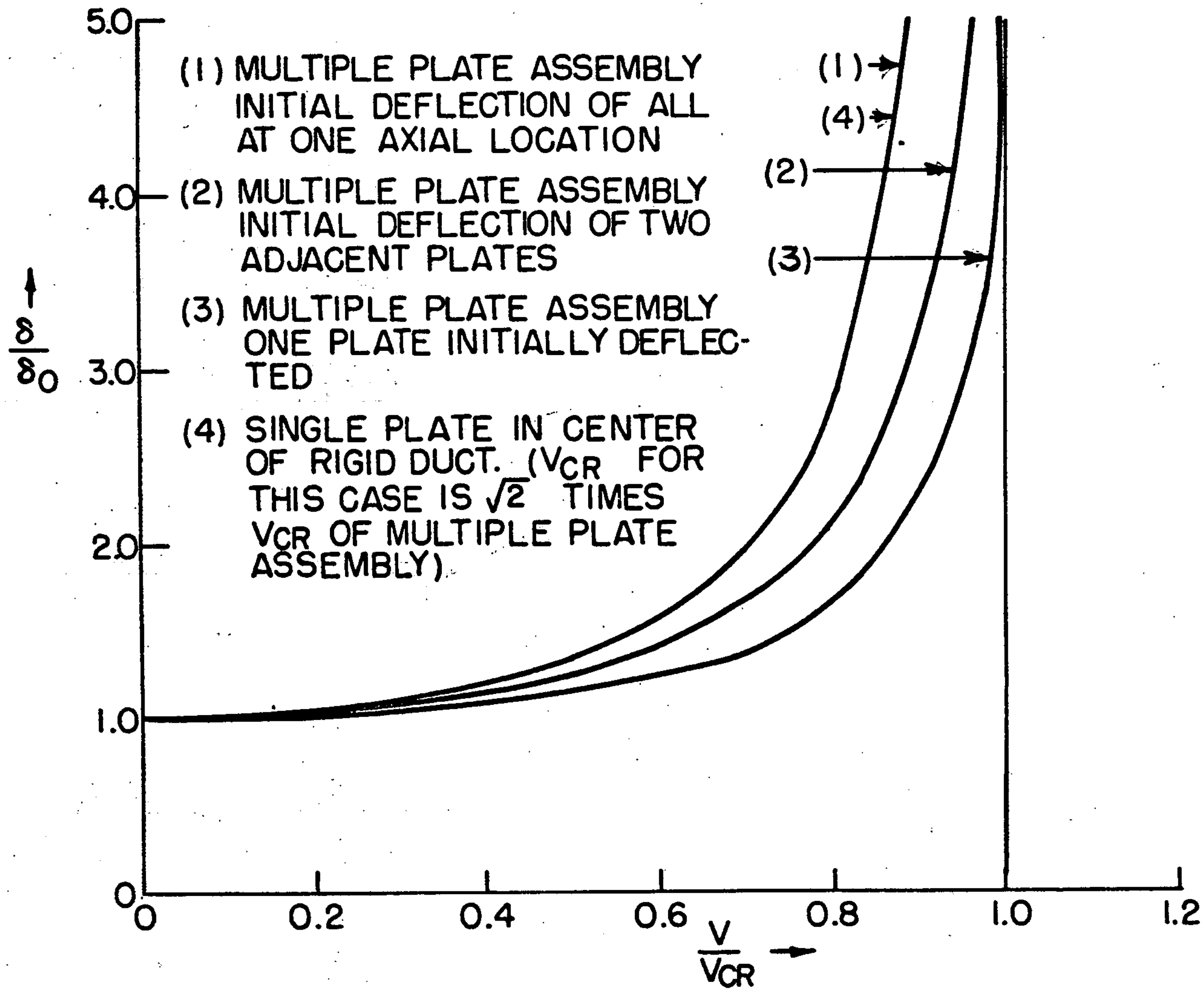

FIGURE VI MAGNIFICATION OF INITIAL DEFLECTION AS A FUNCTION OF APPROACH TO THE CRITICAL VELOCITY

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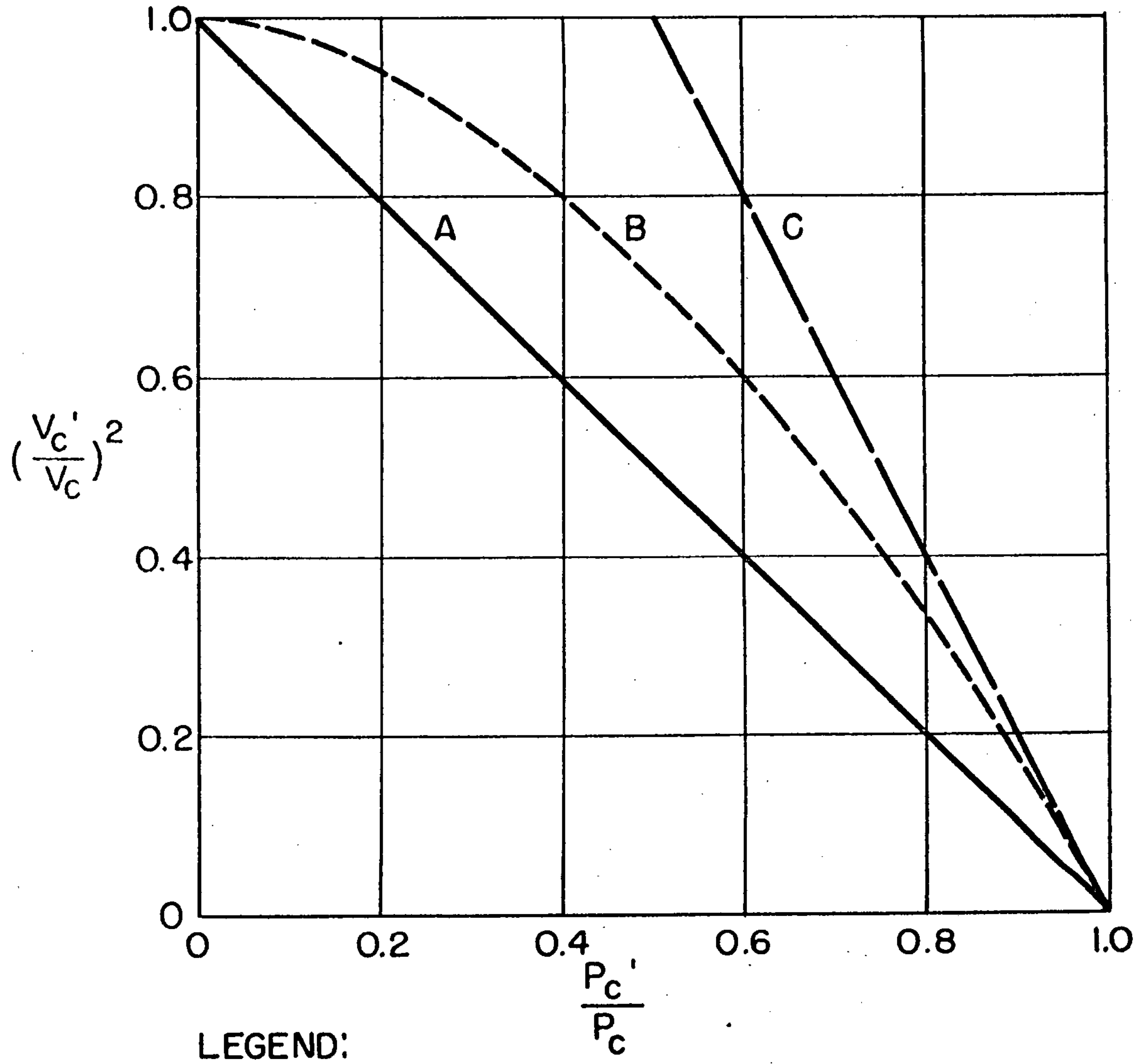

LEGEND:

A - ALL PLATES UNDER COMPRESSION

B - ONE PLATE UNDER COMPRESSION IN MULTI-PLATE ASSEMBLY

C - ONE PLATE BISECTING A RIGID DUCT.

FIGURE VII COMBINED EFFECT OF FLOW AND MEMBRANE COMPRESSION OF FLAT PLATES.

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