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AN ENGINEERING APPROACH TO MULTIAXIAL PLASTICITY

J. G. Merkle

OAK RIDGE NATIONAL LABORATORY
operated by
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Correction to ORNL-4138

1. On page 32. Delete everything below Eq. (83). Change the comma following Eq. (83) to a period. Then substitute:

Solving Eq. (26) for the case of $df = d\sigma_3 = 0$ gives

$$\frac{\frac{\partial f}{\partial \sigma_1}}{\frac{\partial f}{\partial \sigma_2}} = -\frac{d\sigma_2}{d\sigma_1} . \qquad (84)$$

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2. On page 33. Delete everything down to and including Eq. (86). Then substitute:

Combining Eqs. (84) and (83) gives

$$\frac{d\epsilon_1^P}{d\epsilon_2^P} = -\frac{d\sigma_2}{d\sigma_1} . \qquad (85)$$

Let us define an angle θ such that

$$\frac{d\epsilon_1^P}{d\epsilon_2^P} = -\frac{d\sigma_2}{d\sigma_1} = \tan \theta . \qquad (86)$$

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JULY 1967

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CONTENTS

ABSTRACT

Multiaxial plastic stress analysis techniques will become more widely used by engineers once a straightforward derivation of the basic equations of plasticity is available. The objective of this report is to present such a derivation. The basic equations of plasticity are derived by using only calculus and vector algebra; tensor notation is not used. All assumptions are explicitly stated. The flow rule is derived by two different methods. In one method the area under the effective stress-strain curve is assumed to equal the plastic work. For the other method the plastic material is assumed to be "stable." In both derivations the ratios of the principal plastic strain increments are uniquely determined by the state of stress at any point on the yield surface, except at a discontinuity. Plastic volume changes are related to the effects of hydrostatic pressure on the yield function.

The equations of Hill's yield function and the Mohr-Coulomb yield function are examined. For anisotropic materials that obey Hill's yield function, the uniaxial stress-plastic strain curves in the principal directions must plot parallel to each other on log-log paper; this leads to a general method for determining the coefficients of anisotropy. The plastic volume changes associated with the Mohr-Coulomb yield function are examined. The tensile and compressive stress-plastic strain curves for Mohr-Coulomb material must also plot parallel to each other on log-log paper. Furthermore, the flow rules associated with the Mohr-Coulomb and the Von Mises yield functions can both be derived by assuming that slip causes no plastic strain in the direction of slip. Two example problems are solved: one uses Hill's yield function and the other the Mohr-Coulomb yield function. Finally, present limitations and future extensions of the theory are discussed.

NOMENCLATURE

A	Arbitrary constant, dimensionless
A_1, A_2	Stress-plastic strain parameters, psi
a	Scaling factor for anisotropic stress-plastic strain curves, dimensionless
B	Plastic strain parameter, psi^{-1}
b	Scaling factor for anisotropic stress-plastic strain curves, dimensionless
C, C_1, C_2, C_3	Stress-plastic strain parameters, psi
c	Cohesion, psi
D	Arbitrary constant, psi
d	Width of a prismatic test specimen, in.
E	Modulus of elasticity, psi
E_p	Plastic secant modulus, psi
$\overline{e_u}, \overline{e_v}, \overline{e_n}$	Unit vectors along the σ_u, σ_v, and σ_n axes in stress space, dimensionless
F'	Plastic tangent modulus, psi
f, f'	Yield functions, psi
$\overline{\nabla}f$	Gradient of the yield function, dimensionless
G	Derived function of α_{12}, α_{23}, and α_{31}, dimensionless
g	General functional relationship
H, H_1, H_2, H_3	Strain-hardening parameters for linear strain hardening, dimensionless
h	Height of a prismatic test specimen, in.
$\overline{i}, \overline{j}, \overline{k}$	Unit vectors along the principal axes, dimensionless
k	Stress-plastic strain parameter, psi^{-1}
l	Distance between two points on a line drawn parallel to a slip plane in a prismatic test specimen, in.
m	Parameter in the Mohr-Coulomb yield function, dimensionless
n	Strain-hardening exponent for power-law strain hardening, dimensionless
n	Octahedral normal
n'	Line parallel to the octahedral normal in stress space
P	Point on a yield surface in stress space
p	Arbitrary constant, psi^{-2}

Q	Point on a yield surface in stress space
q, q'	Arbitrary constants, dimensionless
R	Point on a yield surface in stress space
S	Point on a corner of a yield surface in stress space
S_u, S_v	Normalized stresses, dimensionless
T	Temperature, °F
t	Thickness of a prismatic test specimen, in.
$\overline{U}$	Stress vector along the σ_u axis in stress space, psi
dV_p	Differential plastic volume change
ΔV_p	Plastic volume change, dimensionless
dW_p	Plastic work increment per unit volume, psi
dw_p	Plastic work increment per unit volume performed by a set of stress increments, psi
x, y, z	Scalar variables (stress or dimensionless, as indicated by equations in which used); also subscripts indicating principal directions
α	Coefficient of thermal expansion, $°F^{-1}$
$\alpha_{12}, \alpha_{23}, \alpha_{31}$	Coefficients of anisotropy in Hill's yield function, dimensionless
β	Parameter that defines the shape of the Mohr-Coulomb yield surface, dimensionless
γ_p	Slip-plane plastic shear strain, dimensionless
$\epsilon_1, \epsilon_2, \epsilon_3$	Principal total strains, dimensionless
$\epsilon_1^E, \epsilon_2^E, \epsilon_3^E$	Principal elastic strains, dimensionless
$\epsilon_1^P, \epsilon_2^P, \epsilon_3^P$	Principal plastic strains, dimensionless
$\epsilon_A^P, \epsilon_B^P$	Effective plastic strains associated with the components of a plastic strain vector at the corner of a piecewise linear yield surface, dimensionless
ϵ_{eff}	Effective total strain, dimensionless
ϵ_{eff}^P	Effective plastic strain, dimensionless
$\overline{d\epsilon^P}$	Plastic strain increment vector in stress space, dimensionless
η	Reciprocal of the strain-hardening exponent, dimensionless
λ, λ'	Factors of proportionality, dimensionless
ν	Poisson's ratio, dimensionless
$\sigma_1, \sigma_2, \sigma_3$	Principal stresses, psi

$\sigma_x, \sigma_y, \sigma_z$	Principal stresses, psi
σ_c	Yield stress in compression, psi
σ_{eff}	Effective stress, psi
σ_m	Hydrostatic stress, psi
σ_N	Normal stress on a slip plane, psi
$\sigma_0, \sigma_{01}, \sigma_{02}, \sigma_{03}$	Initial yield stresses for linear strain-hardening material, psi
σ_t	Yield stress in tension, psi
$\sigma_u, \sigma_v, \sigma_n$	Coordinates of a point in stress space referred to an octahedral coordinate system, psi
σ_{u1}, σ_{u2}	Stress components in the octahedral plane, psi
$\overline{\sigma}$	Total stress vector in stress space, psi
$\overline{\sigma}_n$	Component of the total stress vector acting normal to the octahedral plane in stress space, psi
$\overline{\tau}$	Component of the total stress vector acting tangential to the octahedral plane in stress space, psi
τ_f	Shear stress on a slip plane, psi
τ_{oct}	Octahedral shear stress, psi
ϕ	Angle of internal friction, deg
ψ	Functional relationship
1,2,3	Subscripts indicating the principal directions
$\overline{}$	Overbar indicating a vector quantity
$\lvert \ldots \rvert$	Brackets indicating absolute magnitude (length) of a vector
$\perp$	Symbol denoting perpendicular to
$\times$	Indicates the cross product of two vectors
$\cdot$	Indicates the dot product of two vectors

DEFINITIONS

Angle of internal friction — the angle with a tangent equal to the coefficient of friction of a material slipping on itself.

Anisotropic material — a material with properties that vary with direction.

Associated flow rule — a flow rule applicable to a particular yield function.

Cohesion — the shear stress on a slip plane when there is zero normal stress acting on that plane.

Deformation theory — a set of equations relating stress to total plastic strain. Deformation theory is a special case of incremental theory.

Elastic strain — the strains related to the stresses by Hooke's law. Elastic strains are recoverable by unloading.

Effective plastic strain increment — a function of the principal plastic strain increments, the value of which can be determined from the effective stress-strain curve. The product of the effective stress and the effective plastic strain increment equals the plastic work increment.

Effective stress — a known function of the principal stresses that uniquely determines the amount of plastic work required to attain a particular state of stress.

Effective stress-strain relation — a unique relation between the effective stress and the effective plastic strain.

Effective total strain — a function of the principal total strains that is uniquely related to the effective stress for isotropic deformation theory.

Flow rule — a set of three partial differential equations relating the principal plastic strain increments to the stresses and the stress increments via the yield function.

Hydrostatic stress — the average of the three principal stresses; also the normal stress on the octahedral plane.

Ideally plastic material — material that does not strain harden; therefore the effective stress is a constant in the plastic range.

Incremental theory – a set of equations in which the stresses are related to the plastic strain increments.

Integrated flow rule – a set of stress-strain equations relating the stresses to the total plastic strains.

Isotropic material – material with properties equal in all directions.

Modulus of elasticity – the initial slope of the uniaxial stress-strain curve.

Mohr-Coulomb yield function – a yield function which specifies that the shear stress on a slip plane equals cohesion plus the product of the normal stress times the tangent of the angle of internal friction.

Octahedral coordinate system – a set of coordinates in stress space having one axis colinear with the octahedral normal and the other two axes lying in the octahedral plane.

Octahedral plane – a plane equally inclined to three mutually perpendicular directions.

Octahedral shear stress – the shear stress on the octahedral plane in a unit element.

Plastic strain – the difference between total strain and elastic strain.

Plastic strain increment vector – the vector in stress space whose components are the principal plastic strain increments.

Plastic volume change condition – the equation resulting from summing the principal plastic strain increments.

Plastic work – the work done by the total stresses on the plastic strains.

Poisson's ratio – the negative of the ratio of transverse elastic strain to elastic strain in the direction of stress in a uniaxial test.

Principal strains – the normal strains in the three mutually perpendicular directions between which there is no shear strain.

Principal stresses – the normal stresses on the three mutually perpendicular planes on which there is no shear stress.

Slip planes – planes of discontinuity formed by sliding during yielding.

Stability – the assumption that plastic material cannot do any net work on its loads or their increments but must always have some work done on itself during yielding.

Strain-hardening material — material for which the derivative of effective stress with respect to effective plastic strain is positive.

Stress space — a set of cartesian coordinates in which the unit vectors apply to the stresses or strains in the principal directions.

Transversely isotropic material — material with equal properties in any direction within a plane but another set of properties in the direction perpendicular to that plane.

Tresca yield function — the difference between the algebraically largest and smallest principal stresses. According to this criterion, the maximum shear stress controls yielding.

Ultimate strength analysis — the calculation of the actual strength of a structure. The strength of a structure is the load that causes failure.

Von Mises yield function — a constant times the octahedral shear stress.

Yield function — an algebraic function of the principal stresses having the dimensions of a stress and controlling the ratios of the principal plastic strain increments via the flow rule.

Yield stress — the value of stress at which yielding first occurs in a uniaxial test.

Yield surface — a plot of the yield function in stress space.

AN ENGINEERING APPROACH TO MULTIAXIAL PLASTICITY

J. G. Merkle

Within the past 15 years, failure criteria based on plastic stress analysis[1] have become an important part of most structural design codes. Previously, structures were assumed to behave elastically until failure occurred; that is, no strength beyond yielding was recognized. In fact, failure was usually defined as the occurrence of yielding (barring buckling, fatigue, or brittle fracture). This criterion frequently led to an uneconomical use of material. Therefore it was inevitable that plastic analysis should be developed and applied to the design of structures.

More recently, another set of problems in plastic analysis has arisen with regard to the design of structures for nuclear power plants. Among these problems are the ultimate strength analysis of reactor containment shells and pressure vessels and the ultimate strength analysis, for thermal loading, of reactor fuel elements. The need for practical solutions to these problems is now providing a strong motivation for the development of plastic theory along practical lines.

The fact that plastic analysis has become a widely accepted method of structural analysis for buildings is due primarily to two factors. The first is the thorough program of analytical and experimental research undertaken to develop and verify the theory. The second is the concurrent and equally important effort undertaken to explain the theory in its simplest terms.

Since beams, columns, and frames are usually assumed to carry only uniaxial stress, the uniaxial stress-strain curve is usually sufficient for their analysis. An exception occurs in the case of shear-moment interaction in beams. However, for structures involving multiaxial stress, such as reactor containment shells, pressure vessels, and fuel elements, all the principles of the general theory of plasticity are required for analysis. In most papers on multiaxial plastic stress analysis, however, the basic principles of the theory are stated without proof, with reference usually being made to a few books or papers in which the basic theory is developed. While it is true that collectively these references contain the basic theory, their presentations are generally rather abstract. A few attempts have been made to present the basic principles of plasticity in a less abstract form, notably by Dorn and his co-workers[2] in 1945 and by Hill[3] in 1950; however, in most of the more recent works on plasticity, only a part of the theory is presented and some derivations are omitted. It seems that only a few attempts have ever been made to simplify the derivations of the basic equations of plasticity without abbreviating, and not many attempts have been made to correlate the derivations of the equations of plasticity with the procedures actually used for problem solving. This paper constitutes such an attempt.

It may be argued that engineers do not need to know the derivations of the equations they use but need only have available the final equations in a ready-to-use form. The answer to this argument is that this philosophy may be a workable expedient for solving routine problems, but it is not an adequate basis for solving new problems. This is because new problems can be solved only through understanding, and understanding is gained only by following derivations. An engineer who is not familiar with the derivations of the equations he is now using is in no position to derive new equations because he has no place to start. Therefore the most important parts of this paper are not necessarily the final equations, most of which can be found elsewhere (although not all in one place), but the subject matter outline, the statements of basic principles, and the unabbreviated derivations, many of which cannot be found elsewhere.

It is not the purpose of this paper to develop a new theory but, rather, by progressing from the simple to the complex, to present a more easily understood explanation of an existing theory. By so doing, it is hoped that the many fine solutions to multiaxial stress problems in plasticity that already exist will become more understandable and more useful to students and engineers in practice.

STATEMENT OF THE BASIC PROBLEM

The behavior of material in the plastic range is best described in terms of the stress-strain curve and other experimental observations of actual inelastic behavior. Such a description is given in Ref. 4, which also contains the definitions of many terms encountered in the field of inelastic analysis. For analytical purposes, certain assumptions concerning plastic behavior are made in order to render an analysis tractable. Reference 4 points out the degree to which these assumptions are approximations and discusses the limitations of present methods of plastic analysis.

The three most important assumed characteristics of plastic behavior are nonlinearity, independence of time, and permanence of plastic strains. For analytical simplicity, the uniaxial stress-strain curve for the loading and unloading of a plastic material is assumed to be of the form shown in Fig. 1. Loading is nonlinear, but unloading is assumed to be linear.

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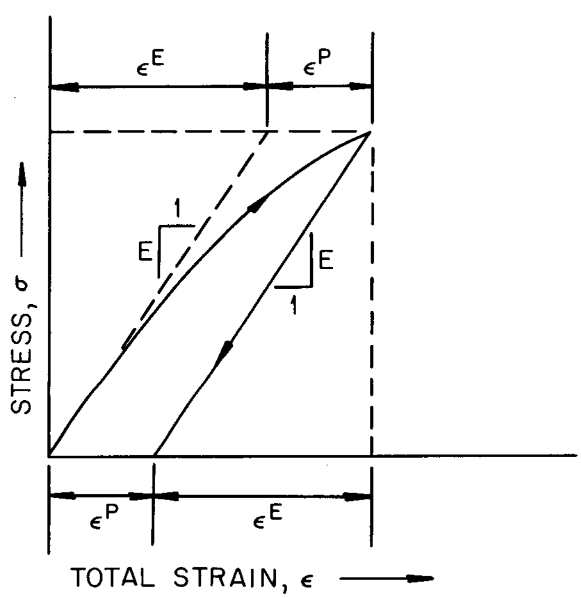

Fig. 1. Typical Uniaxial Elastic-Plastic Stress-Strain Curve.

The slope of the unloading curve is assumed to be the initial slope of the loading curve. The elastic strain is defined as the linear recoverable portion of the total strain. The plastic strain is defined as the nonlinear irrecoverable portion of the total strain. Therefore, by definition,

$$\epsilon = \epsilon^E + \epsilon^P , \quad (1)$$

where ϵ is the total strain, ϵ^E is the elastic strain, and ϵ^P is the plastic strain. Thus, for the purpose of this discussion, yielding is assumed to begin at the true proportional limit of the material rather than at some arbitrarily defined yield point. Whether or not Eq. (1) holds for a real material must be determined by experiment. This determination is logically the first step in investigating the applicability of plastic analysis to a real material, especially if cyclic loading is involved.

The basic problem in multiaxial plastic stress analysis can be defined by considering the unit cube of material shown in Fig. 2. This cube

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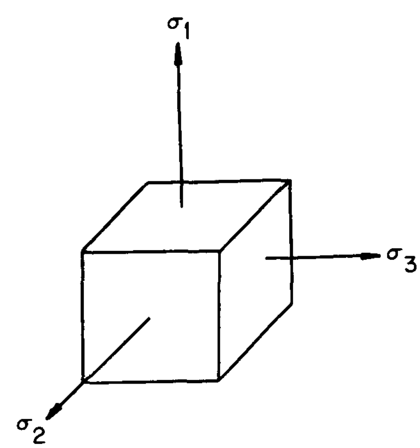

Fig. 2. Unit Cube of Material Loaded into the Plastic Range.

of material is assumed to be loaded into the plastic range by a set of principal stresses that act normal to its faces. The magnitudes of the principal stresses are given, and the problem is to find the principal strains. While there are only three principal total strains, each principal total strain has two components, an elastic component and a plastic component. Since the two components of total strain are physically separate, there are six unknowns in the problem. However, if it is assumed that the elastic strains can still be determined by Hooke's law, only three unknowns remain in the problem without equations for their solution. These unknowns are the three principal plastic strains. It follows that plastic theory must provide three new equations for computing the three principal plastic strains. In addition, if any new unknowns are introduced into the problem, there must be one additional equation for each new unknown. Since the elastic and plastic strains are physically independent of each other, the elastic strains should not appear in the equations for the plastic strains unless they are substituted for the stresses according to Hooke's law.

Although the final objective of plastic analysis is to determine the stresses and the total plastic strains, in many cases no unique algebraic relationship between the two sets of variables will exist. This is because of the basic nonlinear nature of plastic deformation, which can manifest itself by creating stress interaction terms in the differential equations relating stress to plastic strain. These stress interaction terms lead to indefinite integrals in the algebraic equations relating stress to total plastic strain. These indefinite integrals can be evaluated only by knowing the continuous relationship between the principal stresses. For example, a differential equation of the form

$$d\epsilon_1^P = p\sigma_1 \, d\sigma_1 \tag{2}$$

has as its integral

$$\epsilon_1^P = p \frac{\sigma_1^2}{2} + q \, , \tag{3}$$

which is always the same algebraic function, regardless of its nonlinear form. However, the differential equation

$$d\epsilon_1^P = p(\sigma_1 + \sigma_2) \, d\sigma_1 \tag{4}$$

has as its integral

$$\epsilon_1^P = p \frac{\sigma_1^2}{2} + p \int \sigma_2 \, d\sigma_1 + q' \, , \tag{5}$$

which is not a unique algebraic function because of the interaction term, $p\sigma_2 \, d\sigma_1$, in the differential equation. If interaction terms are assumed to exist in the differential equations relating the principal stresses to the principal plastic strains, the three stress-plastic strain equations must be derived in differential form. Thus the three unknowns in the problem for which definite equations can be written are the three principal plastic strain increments. Therefore in a design analysis the total plastic strains must be determined by a process of integration that considers the continuous relationship between the principal stresses.

Plastic theories in which the effects of stress interaction are recognized and the total plastic strains are determined by integration are known as "incremental" theories. Plastic theories in which the effects of stress interaction are ignored and a unique algebraic relationship between the stresses and the total plastic strains is assumed are known as "deformation" theories. Deformation theories are exact only for the one assumed relationship between the principal stresses, which can be used to derive them from incremental theory (usually constant stress ratios). Otherwise, they are approximate, although often convenient. In a few cases the differential equations of incremental theory contain no interaction terms and can therefore be directly integrated. All three types of equations are derived and discussed in this report.

BASIC EQUATIONS OF PLASTICITY

Assumption of Fixed Axes (A Simplifying Assumption for This Report)

In many plasticity problems of immediate interest to design engineers, the principal axes of stress and strain are assumed to coincide and to have fixed directions. Although perfect generality is lost by assuming such conditions, a considerable amount of simplicity and clarity is gained. Therefore, such conditions are assumed for all the following discussion.

Number and Types of Equations

In general, there are seven independent equations involved in a plastic stress analysis. The first four equations permit the evaluation of the four auxiliary variables introduced into the problem. The remaining three equations specify the relative values of the three principal plastic strain increments. The first equation is the definition of a function of the principal stresses and is called the yield function, or the plastic potential. The yield function is assumed to control the initiation and progression of yielding by controlling the ratios of the principal plastic strain increments. The second equation is a so-called "effective, generalized, or universal stress-strain relation."[2-5] This equation relates an effective stress to an effective plastic strain. The third and fourth equations are the definitions of the effective stress and the effective plastic strain. The last three equations are a set of equations known as a flow rule. The flow rule is a set of three linear partial differential equations that specify the relative values of the principal plastic strain increments in terms of the principal stresses and the effective plastic strain increment. An eighth equation, the plastic volume change condition, although usually introduced as an independent condition, can always be derived by taking the sum of the three flow rule equations. Or if the plastic volume change condition is specified as an independent condition, the definition of the yield function becomes a dependent condition. Each of these equations is discussed in detail in the following sections.

The Yield Function

In uniaxial and biaxial tests, on at least some materials, yielding is observed to begin at a certain definite combination of the principal stresses. Furthermore, if after yielding, the load is reduced, the plastic strains do not decrease but remain permanently. In other words, unloading from the plastic range is observed to be elastic. In addition, in a uniaxial test at some load, if an increase in axial strain corresponds to a decrease in true stress, it is observed that behavior is not plastic but involves some form of separation, such as cracking. Therefore it is assumed that for multiaxial loading, there is some function of the principal stresses, called the yield function, f, or the plastic potential, that has the dimensions of a stress and either stays constant or increases when yielding occurs. In other words, for behavior to be plastic,

$$df \geqslant 0 . \tag{6}$$

The Effective Stress-Strain Relation

The effective stress-strain relation is a single algebraic or graphical relation between some function of the principal stresses and some function of the principal plastic strains of the general form

$$\sigma_{eff} = g(\epsilon^P_{eff}) ,$$

where g indicates a functional relationship, which is assumed to be always satisfied in the plastic range under any state of stress.[2-5] Whether the existence of the effective stress-strain relation is assumed before or after the flow rule is derived depends on the method of deriving the flow rule, as will be shown subsequently. In the effective stress-strain relation, the principal stress function is called the effective stress, and the principal plastic strain function is called the effective plastic strain. The effective stress has the dimensions of a stress, and the effective plastic strain has the dimensions of a plastic strain. The rate of strain hardening is specified by the slope of the effective stress-strain curve. If the rate of strain hardening is zero, the effective stress-strain curve is a horizontal straight line; this indicates that the effective stress is a constant that is independent of the plastic strains. Such an effective stress-strain curve is characteristic of mild steel for plastic strains less than about 1.4% (Ref. 6) and has been used extensively for analysis. A uniaxial stress-strain curve for a material without strain hardening is shown schematically in Fig. 3. Such a stress-strain curve is called an "elastic-ideally plastic" or a "flat-top" stress-strain curve.

The effective stress is always defined as an algebraic function of the three principal stresses, and the effective plastic strain increment is always defined as an algebraic function of the three principal plastic strain increments. In incremental theory, the effective plastic strain

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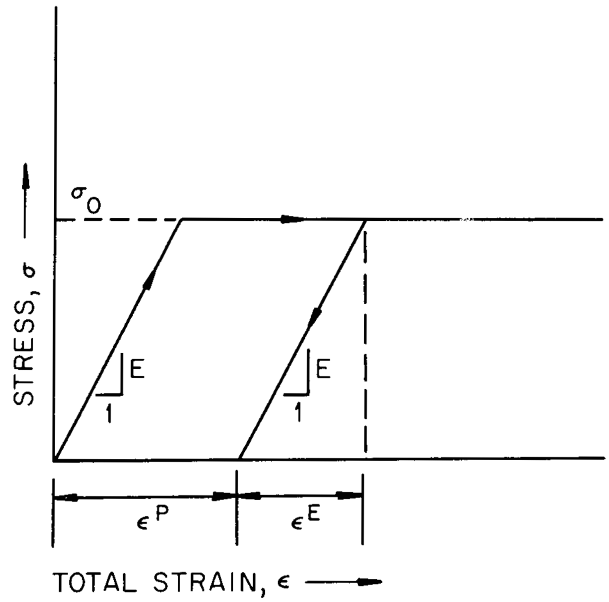

Fig. 3. Elastic-Ideally Plastic Uniaxial Stress-Strain Curve.

is determined by integration. However, regardless of the algebraic expression for the effective plastic strain, its numerical value can always be determined from the effective stress-strain curve once the numerical value of the effective stress is known. The effective stress-strain curve itself is usually determined from a conventional uniaxial tensile test.

The algebraic definition of the effective plastic strain increment is an independently specified condition. However, it is usually chosen as the algebraic form determined by the form of the effective stress function such that the area under the effective stress-strain curve equals the plastic work. In deformation theory, the effective plastic strain is defined algebraically as the integral of the effective plastic strain increment for constant plastic strain increment ratios.

The effective stress-strain curve may be utilized numerically, in its original form, or it may be fit with an empirical equation. Two of the most commonly used empirical equations for strain-hardening materials are the power law and the linear strain-hardening law. The power law is expressed by the equation

$$\sigma_{eff} = C(\epsilon^P_{eff})^n \ , \tag{7}$$

and the linear strain-hardening law, by the equation

$$\sigma_{eff} = \sigma_0(1 + H\epsilon^P_{eff}) \ , \tag{8}$$

where σ_{eff} is the effective stress, ϵ^P_{eff} is the effective plastic strain, and C, n, σ_0, and H are constants. The choice of an effective stress-strain equation is independent of the other conditions in plasticity. However, this choice may have a strong influence on the difficulty of obtaining a closed-form mathematical solution for a given problem. Of course, in principle, computer solutions can be obtained with any stress-strain curve.

The Flow Rule

Since the yield function involves only a function of the principal stresses, Eq. (6) does not, by itself, provide a complete basis for computing the principal plastic strain increments. Some other relationship involving the principal stresses and the principal plastic strain increments is needed. In order to derive such a relationship, it is necessary to make some additional assumptions.

Two partially different approaches to deriving the flow rule can be taken, but both essentially lead to the same conclusion. One approach is based on the assumed existence of an effective stress-strain curve, the area under which equals plastic work. The other approach is based on the assumption of stability. Both derivations utilize the assumption that the ratios of the principal plastic strain increments are uniquely determined by the state of stress.[3] Both derivations are shown below.

Derivation Based on the Assumed Existence of an Effective Stress-Strain Relation

Referring to Fig. 2, consider a unit element of volume of strain-hardening material at equilibrium in the plastic range under the action of a set of principal stresses acting normal to its faces. If a set of plastic strain increments occurs, the increment in plastic work performed by the existing stresses is defined as[7]

$$dW_p = \sigma_1 \, d\epsilon_1^P + \sigma_2 \, d\epsilon_2^P + \sigma_3 \, d\epsilon_3^P \ . \tag{9}$$

Now we will assume that there does exist an effective stress-strain relation in the plastic range and that the incremental area under the effective stress-strain curve equals the increment in plastic work,[3] that is,

$$dW_p = \sigma_{eff} \, d\epsilon_{eff}^P \ . \tag{10}$$

Combining Eqs. (9) and (10) then gives

$$\sigma_{eff} \, d\epsilon_{eff}^P = \sigma_1 \, d\epsilon_1^P + \sigma_2 \, d\epsilon_2^P + \sigma_3 \, d\epsilon_3^P \ . \tag{11}$$

Dividing both sides of Eq. (11) by $d\epsilon_{eff}^P$ gives

$$\sigma_{eff} = \sigma_1 \frac{d\epsilon_1^P}{d\epsilon_{eff}^P} + \sigma_2 \frac{d\epsilon_2^P}{d\epsilon_{eff}^P} + \sigma_3 \frac{d\epsilon_3^P}{d\epsilon_{eff}^P} \ . \tag{12}$$

For simplicity, we use the following substitutions:

$$x = \frac{d\epsilon_1^P}{d\epsilon_{eff}^P} \ , \qquad y = \frac{d\epsilon_2^P}{d\epsilon_{eff}^P} \ , \qquad z = \frac{d\epsilon_3^P}{d\epsilon_{eff}^P} \ . \tag{13}$$

Then combining Eqs. (12) and (13) gives

$$\sigma_{eff} = \sigma_1 x + \sigma_2 y + \sigma_3 z \ . \tag{14}$$

A basic assumption in plastic theory is that the ratios of the plastic strain increments are uniquely determined by the state of stress, independently of the ratio of the stress increments.[3] Therefore, the ratios of the plastic strain increments are the same for all incremental loading paths into the plastic range that originate at the same state of stress. It follows that if the plastic strain increment ratios can be determined for any particular incremental loading path from a given state of stress, they are determined in general for that state of stress.

Now we assume that from every state of stress in the plastic range there is some incremental loading path along which the plastic strain increment ratios stay constant. If the rule for determining the plastic strain increment ratios can be found for this condition, the general rule has been found by the previous argument.

Because $d\epsilon_{eff}^{P}$ can be determined from the effective stress-strain relation, the numerical value of the effective plastic strain increment must be independent of the loading path. Since different total plastic strains may exist at the same state of stress, depending on the loading path, the effective plastic strain increment cannot be affected by the total plastic strains.

Therefore, the effective plastic strain increment must be a function only of the plastic strain increments. Furthermore, since the effective plastic strain increment has the dimensions of a plastic strain increment, x, y, and z should be functions only of the plastic strain increment ratios. Therefore, x, y, and z should remain constant when the plastic strain increment ratios remain constant.

By the chain rule, the total differential of σ_{eff} is given by

$$d\sigma_{eff} = \frac{\partial\sigma_{eff}}{\partial\sigma_1} d\sigma_1 + \frac{\partial\sigma_{eff}}{\partial\sigma_2} d\sigma_2 + \frac{\partial\sigma_{eff}}{\partial\sigma_3} d\sigma_3 \ . \tag{15}$$

Substituting Eq. (14) into Eq. (15) and performing the indicated partial differentiation gives

$$\begin{aligned} d\sigma_{eff} = {} & \left(x + \sigma_1 \frac{\partial x}{\partial\sigma_1} + \sigma_2 \frac{\partial y}{\partial\sigma_1} + \sigma_3 \frac{\partial z}{\partial\sigma_1}\right) d\sigma_1 \\ & + \left(y + \sigma_1 \frac{\partial x}{\partial\sigma_2} + \sigma_2 \frac{\partial y}{\partial\sigma_2} + \sigma_3 \frac{\partial z}{\partial\sigma_2}\right) d\sigma_2 \\ & + \left(z + \sigma_1 \frac{\partial x}{\partial\sigma_3} + \sigma_2 \frac{\partial y}{\partial\sigma_3} + \sigma_3 \frac{\partial z}{\partial\sigma_3}\right) d\sigma_3 \ . \end{aligned} \tag{16}$$

Rearranging terms in Eq. (16) gives

$$d\sigma_{eff} = x\,d\sigma_1 + y\,d\sigma_2 + z\,d\sigma_3$$

$$+ \left(\frac{\partial x}{\partial\sigma_1}\,d\sigma_1 + \frac{\partial x}{\partial\sigma_2}\,d\sigma_2 + \frac{\partial x}{\partial\sigma_3}\,d\sigma_3\right)\sigma_1$$

$$+ \left(\frac{\partial y}{\partial\sigma_1}\,d\sigma_1 + \frac{\partial y}{\partial\sigma_2}\,d\sigma_2 + \frac{\partial y}{\partial\sigma_3}\,d\sigma_3\right)\sigma_2$$

$$+ \left(\frac{\partial z}{\partial\sigma_1}\,d\sigma_1 + \frac{\partial z}{\partial\sigma_2}\,d\sigma_2 + \frac{\partial z}{\partial\sigma_3}\,d\sigma_3\right)\sigma_3 \ . \qquad (17)$$

However, by the chain rule, Eq. (17) reduces to

$$d\sigma_{eff} = x\,d\sigma_1 + y\,d\sigma_2 + z\,d\sigma_3 + \sigma_1\,dx + \sigma_2\,dy + \sigma_3\,dz \ . \qquad (18)$$

Since x, y, and z were assumed to remain constant,

$$dx = dy = dz = 0 \ , \qquad (19)$$

and Eq. (18) reduces to

$$d\sigma_{eff} = x\,d\sigma_1 + y\,d\sigma_2 + z\,d\sigma_3 \ . \qquad (20)$$

Equating the right-hand sides of Eqs. (15) and (20) now gives

$$x\,d\sigma_1 + y\,d\sigma_2 + z\,d\sigma_3 = \frac{\partial\sigma_{eff}}{\partial\sigma_1}\,d\sigma_1 + \frac{\partial\sigma_{eff}}{\partial\sigma_2}\,d\sigma_2 + \frac{\partial\sigma_{eff}}{\partial\sigma_3}\,d\sigma_3 \ . \qquad (21)$$

Collecting terms and dividing by $d\sigma_1$ then gives

$$\left(x - \frac{\partial\sigma_{eff}}{\partial\sigma_1}\right) + \left(y - \frac{\partial\sigma_{eff}}{\partial\sigma_2}\right)\frac{d\sigma_2}{d\sigma_1} + \left(z - \frac{\partial\sigma_{eff}}{\partial\sigma_3}\right)\frac{d\sigma_3}{d\sigma_1} = 0 \ . \qquad (22)$$

The terms in parentheses are independent of the stress increment ratios. Therefore, by taking the partial derivatives of the left-hand side of Eq. (22) with respect to the ratios $d\sigma_2/d\sigma_1$ and $d\sigma_3/d\sigma_1$, it follows that

$$y - \frac{\partial \sigma_{eff}}{\partial \sigma_2} = 0 \tag{23a}$$

and

$$z - \frac{\partial \sigma_{eff}}{\partial \sigma_3} = 0 , \tag{23b}$$

and by substituting Eqs. (23a) and (23b) into Eq. (22),

$$x - \frac{\partial \sigma_{eff}}{\partial \sigma_1} = 0 . \tag{23c}$$

Consequently, by substituting Eq. (13) into Eq. (23) and rearranging,

$$\begin{aligned} d\epsilon_1^P &= d\epsilon_{eff}^P \frac{\partial \sigma_{eff}}{\partial \sigma_1} , \\ d\epsilon_2^P &= d\epsilon_{eff}^P \frac{\partial \sigma_{eff}}{\partial \sigma_2} , \\ d\epsilon_3^P &= d\epsilon_{eff}^P \frac{\partial \sigma_{eff}}{\partial \sigma_3} . \end{aligned} \tag{24}$$

Equation (24) is an "incremental" flow rule based on the assumed existence of an effective stress-strain curve, the area under which equals plastic work. Note that the factor of proportionality, $d\epsilon_{eff}^P$, is of differential magnitude and has known physical significance. From Eq. (24) it can be seen that the values of the plastic strain increments are independent of the loading history.

Evidently, there are two ways to investigate the foregoing theory. One is to experimentally test the existence of an effective stress-strain relation that determines the plastic work, and the other is to derive the flow rule without assuming the existence of an effective stress-strain relation. Both these approaches to plastic theory have been under investigation for the past several years, but the relationship between them has not always been clear. In the next section the flow rule is derived without the assumption of an effective stress-strain relation.

Derivation Based on the Assumption of Stability

The basic assumption underlying this approach to plastic theory is that plastic material always satisfies the condition of stability.[8-10] This condition specifies that a unit volume of plastic material cannot do any net plastic work upon its loads or their increments but must always have some plastic work done upon it during loading. The first part of this condition is expressed by the requirement that, during yielding, the work done by the existing stresses on a set of plastic strain increments must be positive. In other words,

$$dW_p = \sigma_1 \, d\epsilon_1^P + \sigma_2 \, d\epsilon_2^P + \sigma_3 \, d\epsilon_3^P > 0 \ . \tag{25}$$

There seems to be ample experimental evidence to justify this assumption.

The second part of the stability postulate seems to be based partly on experimental evidence and partly on mathematical intuition. If we write Eq. (6) in the form

$$df = \frac{\partial f}{\partial \sigma_1} d\sigma_1 + \frac{\partial f}{\partial \sigma_2} d\sigma_2 + \frac{\partial f}{\partial \sigma_3} d\sigma_3 \geqslant 0 \ , \tag{26}$$

a similarity in form between the right-hand sides of Eqs. (25) and (26) becomes evident. However, in Eq. (25), the independent variables are the stresses, and in Eq. (26) the independent variables are the stress increments. In order to obtain two equations involving the stress increments, we are led, quite naturally, to consider the plastic work done by the stress increments due to an increment in the applied loads. In uniaxial loading a negative tangent modulus indicates nonplastic behavior. Furthermore there is apparently no energy available for doing work on the stress increments. Therefore, although it cannot be fully justified thermodynamically, it seems reasonable to assume that dw_p, the plastic work done by the stress increments on a set of plastic strain increments, must be zero or positive. In other words,

$$dw_p = \frac{1}{2} d\sigma_1 \, d\epsilon_1^P + \frac{1}{2} d\sigma_2 \, d\epsilon_2^P + \frac{1}{2} d\sigma_3 \, d\epsilon_3^P \geq 0 \ . \qquad (27)$$

Because dw_p and df are both scalars, it is possible to assume a general relationship between them of the form

$$dw_p = \frac{\lambda}{2} df \ , \qquad (28)$$

where λ is some unknown function.

From dimensional analysis and the fact that both dw_p and df have been assumed positive, it follows that λ has the dimensions of a plastic strain increment and a value greater than zero. Using Eqs. (26) and (27), Eq. (28) may be rewritten in the form

$$d\epsilon_1^P \, d\sigma_1 + d\epsilon_2^P \, d\sigma_2 + d\epsilon_3^P \, d\sigma_3 = \lambda \frac{\partial f}{\partial \sigma_1} d\sigma_1 + \lambda \frac{\partial f}{\partial \sigma_2} d\sigma_2 + \lambda \frac{\partial f}{\partial \sigma_3} d\sigma_3 \ . \qquad (29)$$

Combining terms, Eq. (29) becomes

$$\left(d\epsilon_1^P - \lambda \frac{\partial f}{\partial \sigma_1}\right) d\sigma_1 + \left(d\epsilon_2^P - \lambda \frac{\partial f}{\partial \sigma_2}\right) d\sigma_2 + \left(d\epsilon_3^P - \lambda \frac{\partial f}{\partial \sigma_3}\right) d\sigma_3 = 0 \ . \qquad (30)$$

Dividing both sides of Eq. (30) by the product $\lambda \, d\sigma_1$ then gives

$$\left(\frac{d\epsilon_1^P}{\lambda} - \frac{\partial f}{\partial \sigma_1}\right) + \left(\frac{d\epsilon_2^P}{\lambda} - \frac{\partial f}{\partial \sigma_2}\right) \frac{d\sigma_2}{d\sigma_1} + \left(\frac{d\epsilon_3^P}{\lambda} - \frac{\partial f}{\partial \sigma_3}\right) \frac{d\sigma_3}{d\sigma_1} = 0 \ . \qquad (31)$$

Since λ has the dimensions of a plastic strain increment, it is reasonable to assume that λ is a function only of the principal plastic strain increments. Therefore, the terms $d\epsilon_i^P/\lambda$ are functions only of the principal plastic strain increment ratios. If these ratios are assumed independent of the stress increment ratios, then the terms in parentheses are independent of the stress increment ratios. Consequently, Eq. (31) has the same solution as Eq. (22), and the plastic strain increments are given by

$$d\epsilon_1^P = \lambda \frac{\partial f}{\partial \sigma_1} ,$$

$$d\epsilon_2^P = \lambda \frac{\partial f}{\partial \sigma_2} , \qquad (32)$$

$$d\epsilon_3^P = \lambda \frac{\partial f}{\partial \sigma_3} .$$

The assumption made in plasticity that the ratios of the plastic strain increments are uniquely determined by the state of stress and are independent of the ratios of the stress increments is analogous to the assumption in creep analysis that creep strain rates are independent of stress rates. From Eq. (32) it can be seen that the reason f is also called the plastic potential is that the plastic strain increments are proportional to its partial derivatives. It can also be seen that the real effect of assuming dw_p to be positive was to prevent a possible ambiguity in the sign of λ and hence in the sign of the plastic strain increments.

Because Eq. (32) is a set of three equations in four unknowns, one more equation relating the principal plastic strain increments to the principal stresses is needed. For strain-hardening materials, this equation is obtained by assuming the existence of an effective stress-strain relation, as discussed previously. For ideally plastic materials, the effective plastic strain can have any value that satisfies compatibility, but the effective stress must have a constant value. For both types of materials, an analysis based on the plastic volume change condition is often used, as explained below.

The Plastic Volume Change Condition

The plastic volume change condition can be derived from the flow rule by adding the three principal plastic strain increments. Therefore, by adding the three expressions in Eq. (32), the differential plastic volume change, dV_p, is given by the equation

$$dV_p = d\epsilon_1^P + d\epsilon_2^P + d\epsilon_3^P = \lambda\left(\frac{\partial f}{\partial \sigma_1} + \frac{\partial f}{\partial \sigma_2} + \frac{\partial f}{\partial \sigma_3}\right) . \qquad (33)$$

It can be seen from Eq. (33) that the plastic volume change condition depends upon the yield function. Therefore, the two functions are not independent of each other.

Equation (33) may be rewritten in a simpler form by considering the definition of the hydrostatic component of stress, which is defined by the equation

$$\sigma_m = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} . \qquad (34)$$

Since the effects of hydrostatic stress are equal in all directions, it follows that

$$\frac{\partial \sigma_1}{\partial \sigma_m} = \frac{\partial \sigma_2}{\partial \sigma_m} = \frac{\partial \sigma_3}{\partial \sigma_m} . \qquad (35)$$

Consequently, taking partial derivatives on both sides of Eq. (34) and using Eq. (35) gives

$$\frac{\partial \sigma_1}{\partial \sigma_m} = \frac{\partial \sigma_2}{\partial \sigma_m} = \frac{\partial \sigma_3}{\partial \sigma_m} = 1 . \qquad (36)$$

Therefore, Eq. (33) may be rewritten in the form

$$dV_p = \lambda\left(\frac{\partial f}{\partial \sigma_1}\frac{\partial \sigma_1}{\partial \sigma_m} + \frac{\partial f}{\partial \sigma_2}\frac{\partial \sigma_2}{\partial \sigma_m} + \frac{\partial f}{\partial \sigma_3}\frac{\partial \sigma_3}{\partial \sigma_m}\right) , \qquad (37)$$

which, by the chain rule, reduces to

$$dV_p = \lambda \frac{\partial f}{\partial \sigma_m} . \qquad (38)$$

Experimental observation has shown that for many metals the plastic volume change is zero.[2,3,5,7] Therefore it is of interest to determine what general conditions, if any, lead to a zero plastic volume change. Since λ is assumed to be nonzero for any incremental loading into the plastic range the plastic volume change will be zero if, and only if,

$$\frac{\partial f}{\partial \sigma_m} = 0 \ . \tag{39}$$

Therefore the plastic volume change will be zero if, and only if, the yield function is independent of the hydrostatic component of stress. Furthermore, it is easily shown that for the yield function to be independent of the hydrostatic component of stress it must be a function only of the principal stress differences $\sigma_1 - \sigma_2$, $\sigma_2 - \sigma_3$, and $\sigma_3 - \sigma_1$.

If the plastic volume change is zero, Eq. (33) can be used in either its differential or its integrated form without knowing the value of λ. For zero plastic volume change the integrated form of Eq. (33) is simply

$$\epsilon_1^P + \epsilon_2^P + \epsilon_3^P = 0 \ , \tag{40}$$

where the constant of integration has been set equal to zero because the plastic volume change is zero when all the plastic strains are zero. The plastic strains in Eq. (40) can always be expressed in terms of the total strains and the elastic strains according to Eq. (1). Then by using the flow rule, the strain displacement relations, Hooke's law, the equilibrium equations, and the effective stress-strain relation it is often possible to reduce Eq. (40) to a form that can be directly integrated.

Relationship Between the Yield Function and the Effective Stress

By comparing Eqs. (24) and (32) it can be seen that if the values of the plastic strain increments are to be the same for both derivations of the flow rule,

$$\lambda \frac{\partial f}{\partial \sigma_1} = d\epsilon^P_{eff} \frac{\partial \sigma_{eff}}{\partial \sigma_1} ,$$

$$\lambda \frac{\partial f}{\partial \sigma_2} = d\epsilon^P_{eff} \frac{\partial \sigma_{eff}}{\partial \sigma_2} , \tag{41}$$

$$\lambda \frac{\partial f}{\partial \sigma_3} = d\epsilon^P_{eff} \frac{\partial \sigma_{eff}}{\partial \sigma_3} .$$

By multiplying each of the expressions in Eq. (41) by the corresponding stress increment and adding the results, Eq. (41) becomes

$$\lambda \left(\frac{\partial f}{\partial \sigma_1} d\sigma_1 + \frac{\partial f}{\partial \sigma_2} d\sigma_2 + \frac{\partial f}{\partial \sigma_3} d\sigma_3 \right) =$$

$$d\epsilon^P_{eff} \left(\frac{\partial \sigma_{eff}}{\partial \sigma_1} d\sigma_1 + \frac{\partial \sigma_{eff}}{\partial \sigma_2} d\sigma_2 + \frac{\partial \sigma_{eff}}{\partial \sigma_3} d\sigma_3 \right) . \tag{42}$$

By using the chain rule and noting Eq. (28), Eq. (42) can be rewritten as

$$dw_p = \frac{\lambda}{2} df = \frac{d\sigma_{eff}\, d\epsilon^P_{eff}}{2} . \tag{43}$$

In order to solve Eq. (43), another expression involving the same quantities must first be obtained. Substituting Eq. (32) into Eq. (9) gives

$$dW_p = \lambda \left(\sigma_1 \frac{\partial f}{\partial \sigma_1} + \sigma_2 \frac{\partial f}{\partial \sigma_2} + \sigma_3 \frac{\partial f}{\partial \sigma_3} \right) . \tag{44}$$

To evaluate the expression in parentheses in Eq. (44), Eq. (26) is rewritten in the form

$$df = \left(\frac{\partial f}{\partial \sigma_1} + \frac{\partial f}{\partial \sigma_2} \frac{d\sigma_2}{d\sigma_1} + \frac{\partial f}{\partial \sigma_3} \frac{d\sigma_3}{d\sigma_1} \right) d\sigma_1 . \tag{45}$$

Since f has the dimensions of a stress, the partial derivatives of f are dimensionless, and they must be either functions of the stress ratios or constants. Therefore, if the stress ratios remain constant, the partial derivatives of f must remain constant, but if the stress ratios remain constant, the ratios of the stress increments must also remain constant. Since the yield function is specified algebraically, its value is the same for a given state of stress regardless of how that state of stress is reached. Therefore, Eq. (45) can be integrated over any path, including the path along which the stress ratios remain constant. Performing that integration gives

$$f = \left(\sigma_1 \frac{\partial f}{\partial \sigma_1} + \sigma_2 \frac{\partial f}{\partial \sigma_2} + \sigma_3 \frac{\partial f}{\partial \sigma_3}\right) - D \ . \tag{46}$$

If

$$f(0,0,0) = 0 \ , \tag{47}$$

it follows that

$$D = 0 \ , \tag{48}$$

and

$$\sigma_1 \frac{\partial f}{\partial \sigma_1} + \sigma_2 \frac{\partial f}{\partial \sigma_2} + \sigma_3 \frac{\partial f}{\partial \sigma_3} = f \ . \tag{49}$$

Therefore, substituting Eq. (49) into Eq. (44) gives

$$dW_p = f\lambda \ . \tag{50}$$

Of particular interest is the fact that Eq. (50) is independent of any assumptions regarding an effective stress-strain relation. Furthermore, since both dW_p and f are known to be positive, λ must be positive. Thus, the assumption that dw_p is positive seems to have been unnecessary, since its only purpose was to make λ positive.

By combining Eqs. (50) and (10), it can be seen that

$$dW_p = f\lambda = \sigma_{eff} \, d\epsilon^P_{eff} \ . \tag{51}$$

Thus, since Eqs. (43) and (51) are general conditions, the plastic work increments dW_p and dw_p can always be represented as incremental areas under the effective stress-strain curve, as shown in Fig. 4.

ORNL-DWG 67-3374

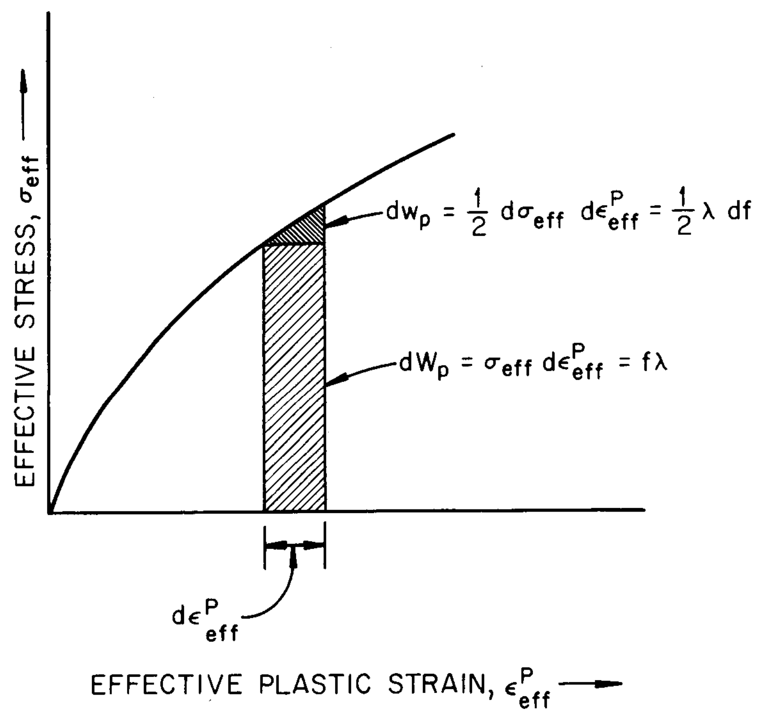

Fig. 4. Representation of the Plastic Work Increments as Incremental Areas Under the Effective Stress-Strain Curve.

Combining Eqs. (43) and (51) also leads to the desired relationship between f and σ_{eff}. Dividing Eq. (43) by Eq. (51) and multiplying by 2 gives

$$\frac{df}{f} = \frac{d\sigma_{eff}}{\sigma_{eff}} , \qquad (52)$$

which can be rewritten as

$$d(\log f) = d(\log \sigma_{eff}) \ . \qquad (53)$$

Therefore, by direct integration,

$$\log f = \log \sigma_{eff} + \log A , \tag{54}$$

where A is an arbitrary constant. Thus

$$f = A\sigma_{eff} , \tag{55}$$

and by substituting Eq. (55) into Eq. (51),

$$A\lambda = d\epsilon^{P}_{eff} . \tag{56}$$

The factor of proportionality, λ, is thus defined simply by the equation

$$\lambda = \frac{d\epsilon^{P}_{eff}}{A} . \tag{57}$$

Therefore, λ is always a constant times the effective plastic strain increment, regardless of the yield function. Since the value of A is arbitrary, it is usually taken equal to unity, in which case

$$\lambda = d\epsilon^{P}_{eff} \tag{58}$$

and

$$f = \sigma_{eff} . \tag{59}$$

In general, the stress function on which the flow rule is based is called the plastic potential, and the stress function on which the effective stress-strain relation is based is called the effective stress. Although it is mathematically possible to obtain solutions to plasticity problems if Eqs. (55) and (56) are not satisfied, in any such case, the condition of stability might not always be satisfied. In most of the literature on plasticity, the plastic potential and the effective stress are taken to be the same function. Therefore, the terms yield stress, yield function, plastic potential, and effective stress have come to be regarded as synonyms. However, in a trade of accuracy for simplicity it is worth remembering that the plastic potential and the effective stress do not always have to satisfy Eq. (55). Steele[11] gives a good example of such a compromise.

The Effective Plastic Strain Increment in Terms of the Stresses and the Stress Increments

If the tangent modulus of the effective stress-strain curve is defined as[12]

$$F' = \frac{d\sigma_{eff}}{d\epsilon^{P}_{eff}} = \frac{df}{\lambda} , \qquad (60)$$

it follows that

$$\lambda = \frac{df}{F'} . \qquad (61)$$

Equation (61) can be substituted into Eq. (32) to obtain the principal plastic strain increments as functions of the principal stresses and the principal stress increments.[12] For instance, if power-law strain hardening, as given by Eq. (7), is assumed,

$$F' = nC(\epsilon^{P}_{eff})^{n-1} , \qquad (62)$$

but from Eqs. (7) and (59),

$$\epsilon^{P}_{eff} = C^{-1/n} f^{1/n} , \qquad (63)$$

and

$$(\epsilon^{P}_{eff})^{n-1} = C^{(1-n)/n} f^{(n-1)/n} . \qquad (64)$$

Therefore

$$F' = nC^{1/n} f^{(n-1)/n} \qquad (65)$$

and

$$\lambda = \frac{df}{F'} = \frac{df}{nC^{1/n} f^{(n-1)/n}} . \qquad (66)$$

If linear strain hardening, as given by Eq. (8), is assumed,

$$F' = \sigma_0 H \tag{67}$$

and

$$\lambda = \frac{df}{F'} = \frac{df}{\sigma_0 H} \ . \tag{68}$$

It will be noted that if there is no strain hardening, F' is zero, and Eq. (61) becomes indeterminate. This results from the plastic strain increments not being uniquely determined by the stresses and the stress increments for an ideally plastic material.

Associated and Integrated Flow Rules

When a particular algebraic yield function is entered into the flow rule and the indicated partial differentiation performed, the flow rule becomes an associated flow rule (i.e., associated with that particular yield function). If the partial derivatives of f remain constant during loading and the principal axes do not rotate, the flow rule equations can be integrated directly. The partial derivatives of f will remain constant during loading if f is a linear function of the principal stresses or if the principal stress ratios remain constant. The latter condition is referred to as radial loading. Whenever the flow rule equations are written in integrated form, in terms of the total plastic strains, they are known as an integrated flow rule.

CHARACTERISTICS OF STRESS SPACE

Basic Equations

Since the yield function is a scalar function of the three principal stresses, it is convenient to represent it as a yield surface in stress space. Stress space is simply a three-dimensional coordinate system in which the values of the principal stresses (or strains) are the coordinates.[2,7] Such a set of coordinates is shown in Fig. 5. Although the relative magnitudes of the principal stresses may vary, it is assumed

ORNL-DWG 67-3375

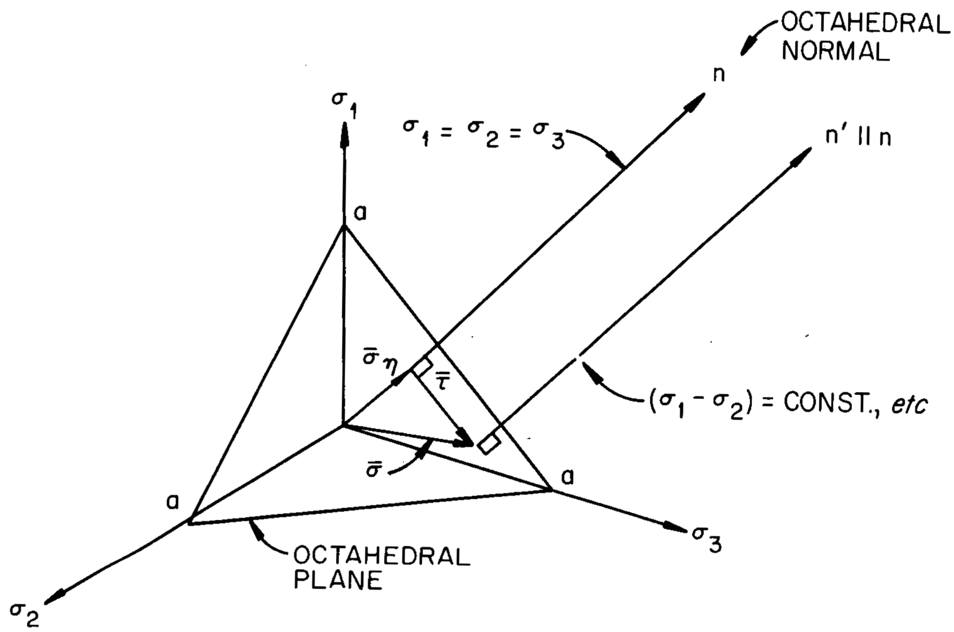

Fig. 5. Basic Characteristics of Stress Space.

for this discussion that the principal stresses continue to act in the 1, 2, and 3 directions, as shown in Fig. 2. Therefore, in this discussion, contrary to the usual convention, the subscripts 1, 2, and 3 do not signify the relative magnitudes of the principal stresses but only indicate their directions.

Referring to Fig. 5, the total state of stress at a point in a body can be represented by a vector $\bar{\sigma}$ in stress space. The vector $\bar{\sigma}$ is defined by the equation

$$\bar{\sigma} = \sigma_1 \bar{i} + \sigma_2 \bar{j} + \sigma_3 \bar{k} \ , \tag{69}$$

where the vectors $\bar{i}$, $\bar{j}$, and $\bar{k}$ are the unit vectors acting in the coordinate directions. It is also possible to resolve the vector $\bar{\sigma}$ into two components such that

$$\bar{\sigma} = \bar{\sigma}_n + \bar{\tau} \ , \tag{70}$$

where $\bar{\sigma}_n$ acts along the line n, which is equally inclined to all three coordinate axes, and $\bar{\tau}$ is perpendicular to $\bar{\sigma}_n$. The plane containing $\bar{\tau}$ is called the octahedral plane,* and the line n is called the octahedral normal. Since the direction cosines of n are all $1/\sqrt{3}$, it follows that

$$|\bar{\sigma}_n| = \bar{\sigma} \cdot \frac{\bar{i} + \bar{j} + \bar{k}}{\sqrt{3}} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{\sqrt{3}} = \sqrt{3}\,\sigma_m , \tag{71}$$

where σ_m is the hydrostatic component of stress and is defined by[13]

$$\sigma_m = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} . \tag{72}$$

By referring to Eq. (71) it can be seen that

$$\bar{\sigma}_n = \sigma_m(\bar{i} + \bar{j} + \bar{k}) . \tag{73}$$

Substituting Eqs. (69) and (73) into Eq. (70) and solving for $\bar{\tau}$ gives

$$\bar{\tau} = (\sigma_1 - \sigma_m)\bar{i} + (\sigma_2 - \sigma_m)\bar{j} + (\sigma_3 - \sigma_m)\bar{k} . \tag{74}$$

Consequently

$$|\bar{\tau}| = \left[(\sigma_1 - \sigma_m)^2 + (\sigma_2 - \sigma_m)^2 + (\sigma_3 - \sigma_m)^2\right]^{1/2} , \tag{75}$$

and by using Eq. (72) it can be shown that

$$|\bar{\tau}| = \frac{1}{\sqrt{3}} \left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2} = \sqrt{3}\,\tau_{oct} , \tag{76}$$

where τ_{oct} is the octahedral shear stress and is defined by[13]

$$\tau_{oct} = \frac{1}{3} \left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2} . \tag{77}$$

In any octahedral plane in stress space, the hydrostatic component of stress is a constant. Furthermore, at any given radius perpendicular

*Hill[3] calls this plane the π plane.

to the octahedral normal, the octahedral shear stress is a constant. Since all three principal stresses are equal along the octahedral normal, by a change of coordinates it can be shown that along any line n′ parallel to the octahedral normal, all the principal stress differences are constant.

The Plastic Strain Increment Vector

It is now convenient to define a plastic strain increment vector in stress space such that

$$d\bar{\epsilon}^P = d\epsilon_1^P \, \bar{i} + d\epsilon_2^P \, \bar{j} + d\epsilon_3^P \, \bar{k} \; . \qquad (78)$$

By using Eq. (32), Eq. (78) can be rewritten in the form

$$d\bar{\epsilon}^P = \lambda \, \bar{\nabla} f \; , \qquad (79)$$

where the gradient of f, $\bar{\nabla} f$, is defined by the equation

$$\bar{\nabla} f = \frac{\partial f}{\partial \sigma_1} \, \bar{i} + \frac{\partial f}{\partial \sigma_2} \, \bar{j} + \frac{\partial f}{\partial \sigma_3} \, \bar{k} \; . \qquad (80)$$

It is known that $\bar{\nabla} f$ is perpendicular to the yield surface and points in the direction of increasing f, which is outward. Since λ is positive, it follows that the plastic strain increment vector is directed along the outward normal to the yield surface as shown in Fig. 6.

ORNL-DWG 67-3376

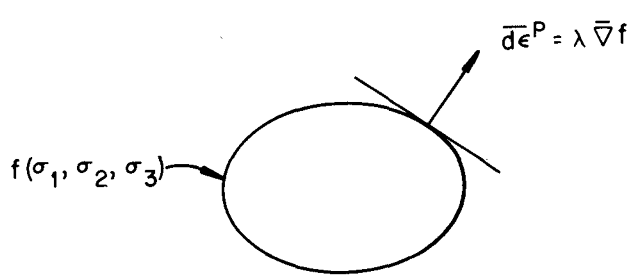

Fig. 6. Section of a Yield Surface Showing the Normality of the Plastic Strain Increment Vector.

Characteristics of Yield Surfaces

Based on the definitions of $\overline{d\epsilon}^P$ and $\overline{\nabla}f$, it is possible to determine two important characteristics of yield surfaces. The first characteristic is general and applies to all yield surfaces. The second characteristic applies whenever the condition of zero plastic volume change is known or assumed to hold.

The first commonly accepted characteristic of yield surfaces is that they must be flat or convex outward. The proof of this rule will not be given here, but one consequence of its violation will be discussed. Consider a set of strain measurements made on the surface of a structure that has undergone large plastic deformations under biaxial stress. If the biaxial yield locus of the shell material is allowed to have inflection points, there will be three stress ratios for which the plastic strain increment ratios will be the same. This condition is shown in Fig. 7.

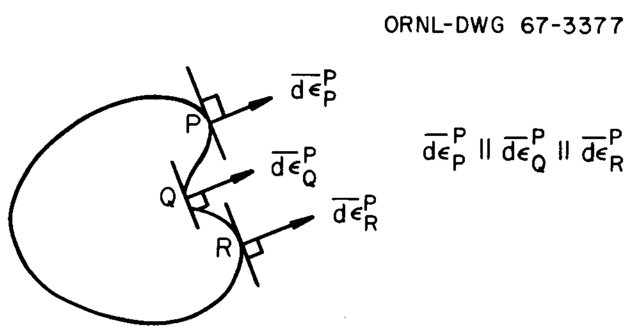

Fig. 7. Cross Section of a Yield Surface Having Both Convex Outward and Concave Outward Portions (Not Permitted by Theory).

Under the assumption of an integrated flow rule (deformation theory) and rigid plasticity (elastic strains neglected) the solution for stresses will not be unique. The only way to avoid this ambiguity is to prevent its occurrence by preventing the yield locus from having concave outward portions. Thus the permissible characteristics of a yield surface are shown in Fig. 8. If the yield surface consists of flat pieces (planes), ambiguity is avoided by avoiding concave outward angles. Consequently,

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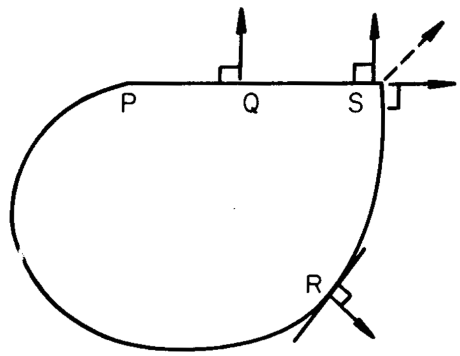

Fig. 8. Cross Section of a Yield Surface with Characteristics Permitted by Theory.

only one set of assumptions regarding the directions of the major and minor principal axes* will result in a solution satisfying all conditions.

It is mathematically possible to have an outward pointing corner in the yield surface, such as the point S shown in Fig. 8. In this case, $d\bar{\epsilon}^P$ is confined to the plane normal to the edge line through S, but if all directions of $d\bar{\epsilon}^P$ are assumed possible, then $d\bar{\epsilon}^P$ can act in any direction between the normals to the adjacent surfaces. Although the direction of $d\bar{\epsilon}^P$ is not uniquely determined by the state of stress at a corner, ambiguities do not arise in the solution of problems because the plastic strain increment ratios in "corner" regions are determined by compatibility.

If the condition of zero plastic volume change is known or assumed to hold, then Eq. (39) holds at every point on the yield surface. By using Eqs. (80) and (36), Eq. (39) can be rewritten in vector form as

$$\frac{\partial f}{\partial \sigma_m} = \bar{\nabla} f \cdot (\bar{i} + \bar{j} + \bar{k}) = 0 \ , \qquad (81)$$

which implies that

$$\bar{\nabla} f \perp (\bar{i} + \bar{j} + \bar{k}) \qquad (82)$$

---

*The major axis is the direction of σ_1. The minor axis is the direction of σ_3.

everywhere on the yield surface. Therefore, if there is no plastic volume change associated with a given yield function, the yield surface for that yield function must be a prism of constant cross section in stress space, with its generators all parallel to the octahedral normal. Such a yield surface is therefore completely defined by its lines of intersection with any plane or set of planes in stress space. For example, two commonly used yield functions that are independent of the hydrostatic component of stress are the Von Mises yield function and the Tresca yield function. The Von Mises yield function is a constant times the octahedral shear stress. Therefore, as shown in Fig. 9, its line of intersection with the

ORNL-DWG 67-3379

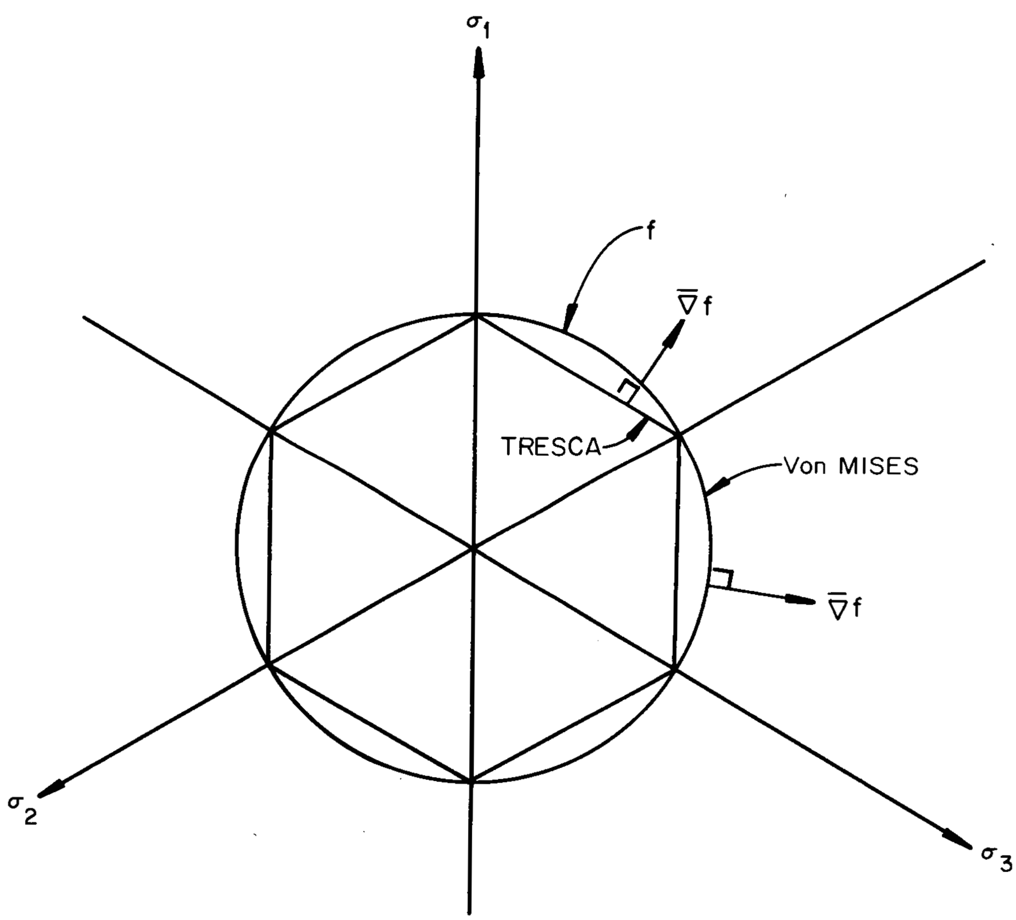

Fig. 9. Intersection Lines of the Tresca and Von Mises Yield Surfaces with an Octahedral Plane in Stress Space.

octahedral plane is a circle; its line of intersection with a coordinate plane is an ellipse. The Tresca yield function is a constant times the maximum shear stress. Therefore, its lines of intersection with the coordinate planes are either rectangles or single straight lines and, as shown in Fig. 9, its line of intersection with the octahedral plane is a regular hexagon. If the two yield functions are made equivalent for uniaxial tension, they are equivalent whenever two principal stresses are equal. Under these conditions, the corners of the Tresca hexagon coincide with the Von Mises circle, as shown in Fig. 9. It can be seen that the Von Mises yield surface is convex outward everywhere and that the Tresca yield surface is piecewise linear, with only outward pointing corners.

Determination of the Yield Locus for a State of Biaxial Stress

The intersection of a yield surface with any one coordinate plane in stress space, say the $\sigma_3 = 0$ plane, can be determined from a biaxial test in which the stresses and the plastic strain increments in the 1 and 2 directions are measured.

From Eq. (32), the ratio $d\epsilon_1^P/d\epsilon_2^P$ is given by

$$\frac{d\epsilon_1^P}{d\epsilon_2^P} = \frac{\dfrac{\partial f}{\partial \sigma_1}}{\dfrac{\partial f}{\partial \sigma_2}} , \tag{83}$$

but by the chain rule,

$$\frac{\partial f}{\partial \sigma_1} = \frac{\partial f}{\partial \sigma_2}\,\frac{\partial \sigma_2}{\partial \sigma_1} , \tag{84}$$

where $\partial\sigma_2/\partial\sigma_1$ is the rate of change of σ_2 with respect to σ_1 with f and σ_3 held constant. Substituting Eq. (84) into Eq. (83) gives

$$\frac{d\epsilon_1^P}{d\epsilon_2^P} = \frac{\partial\sigma_2}{\partial\sigma_1} , \qquad (85)$$

where $\sigma_2 = \sigma_2(\sigma_1, \sigma_3, f)$. Let us define an angle θ such that

$$\frac{d\epsilon_1^P}{d\epsilon_2^P} = \frac{\partial\sigma_2}{\partial\sigma_1} = \tan\theta . \qquad (86)$$

At an arbitrary point in the σ_1, σ_2 plane, say at the point B in Fig. 10, the angle ABC will be $(90° - \theta)$, and the angle CBD will be θ. The sum of these two angles, ABD, will always be 90°. Therefore, since BD is always tangent to the yield locus, AB is always perpendicular to the yield locus.

The yield locus can also be determined by connecting all the points in the σ_1, σ_2 plane at which the value of the plastic work is the same. This method has the advantage of depending upon integration rather than

ORNL-DWG 67-3380

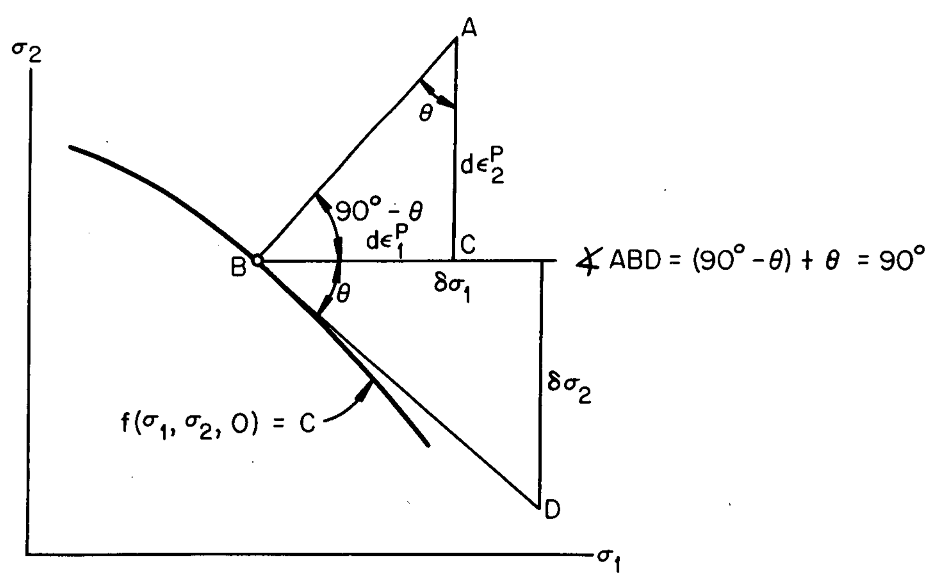

Fig. 10. Determination of the Yield Locus for Biaxial Stress.

differentiation. Therefore it should be more accurate numerically. Of course, theoretically, the two methods should give the same result. Therefore they can be used in combination with each other.

Transformation of Yield Surface Equations into an Octahedral Coordinate System

The surfaces of most commonly used yield functions are defined by their octahedral cross sections and by the variation in size of these cross sections with hydrostatic stress. Therefore it is often convenient to refer the equation of a yield surface to a set of axes, two of which lie in the octahedral plane $\sigma_m = 0$, and the third of which lies along the octahedral normal.[3,14] This transformation of coordinates is easily made by using vector notation. Referring to Fig. 11, let the σ_u axis be the projection of the σ_1 axis onto the octahedral plane, the σ_v axis be perpendicular to the u axis in the octahedral plane, and the σ_n axis be the octahedral normal. Since all projection lines to the octahedral plane

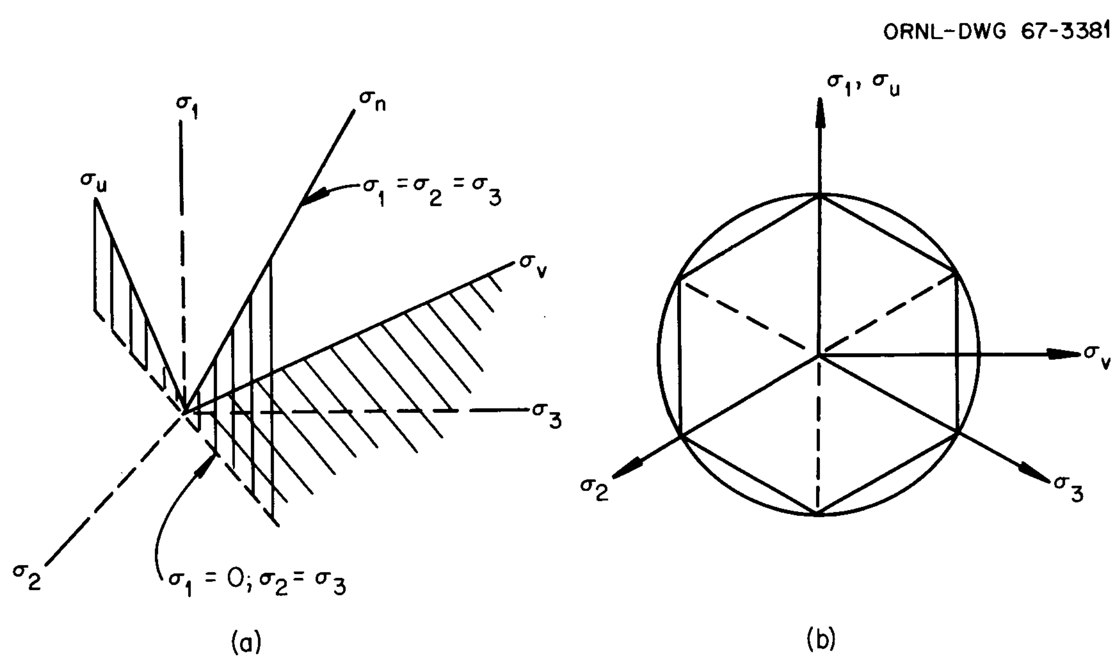

Fig. 11. Octahedral Coordinate Axes in Stress-Space (Adapted from Ref. 14).

are parallel to the σ_n axis, any point in the σ_1, σ_n plane will be projected onto the σ_u axis. The line of intersection of the σ_1, σ_n plane with the σ_2, σ_3 plane has the equation

$$\sigma_2 = \sigma_3 \ . \tag{87}$$

The octahedral plane passing through the origin has the equation

$$\sigma_m = 0 \ . \tag{88}$$

Therefore, if a vector $\overline{U}$ is defined by the equation

$$\overline{U} = x\overline{i} + y\overline{j} + z\overline{k} \tag{89}$$

and is also assumed to lie along the σ_u axis in the $\sigma_m = 0$ plane, its components must satisfy the relations

$$x + y + z = 0 \tag{90}$$

and

$$y = z \ . \tag{91}$$

Therefore

$$x = -2y \ , \tag{92}$$

and

$$\overline{U} = (-y)(2\overline{i} - \overline{j} - \overline{k}) \ . \tag{93}$$

If

$$(-y) = 1 \ , \tag{94}$$

it follows that

$$\overline{U} = 2\overline{i} - \overline{j} - \overline{k} \ . \tag{95}$$

Therefore the unit vector in the u direction, denoted by $\overline{e}_u$, is given by

$$\overline{e}_u = \frac{2\overline{i} - \overline{j} - \overline{k}}{\sqrt{6}} \ . \tag{96}$$

It is well known that the unit vector along the octahedral normal, denoted here by $\overline{e}_n$, has the equation

$$\bar{e}_n = \frac{\bar{i} + \bar{j} + \bar{k}}{\sqrt{3}} \ . \tag{97}$$

If the transformed coordinate system is to be right handed, its unit vectors must satisfy the relation

$$\bar{e}_u \times \bar{e}_n = \bar{e}_v \ . \tag{98}$$

Substituting Eqs. (96) and (97) into Eq. (98) gives

$$\bar{e}_v = \frac{\bar{k} - \bar{j}}{\sqrt{2}} \ . \tag{99}$$

Therefore the projected lengths of any stress vector onto the octahedral axes are given by

$$\sigma_u = \bar{\sigma} \cdot \bar{e}_u \ , \tag{100a}$$

$$\sigma_v = \bar{\sigma} \cdot \bar{e}_v \ , \tag{100b}$$

$$\sigma_n = \bar{\sigma} \cdot \bar{e}_n \ . \tag{100c}$$

Since

$$\bar{\sigma} = \sigma_1 \bar{i} + \sigma_2 \bar{j} + \sigma_3 \bar{k} \ , \tag{69}$$

it follows that[3,14]

$$\sigma_u = \frac{1}{\sqrt{6}} (2\sigma_1 - \sigma_2 - \sigma_3) \ , \tag{101a}$$

$$\sigma_v = \frac{1}{\sqrt{2}} (\sigma_3 - \sigma_2), \tag{101b}$$

$$\sigma_n = \frac{1}{\sqrt{3}} (\sigma_1 + \sigma_2 + \sigma_3) \ . \tag{101c}$$

Furthermore, it follows from the Pythagorean theorem and Eq. (76) that

$$\sigma_u^2 + \sigma_v^2 = |\bar{\tau}|^2 = 3\tau_{oct}^2 \ . \tag{102}$$

APPLICATIONS

The foregoing principles will now be applied to two rather general yield functions. The first yield function is the generalized Von Mises yield function proposed by Hill[3,15] for use with anisotropic metals. The second yield function is the well-known Mohr-Coulomb yield function used in soil mechanics.

The Generalized Von Mises Yield Function of Hill

Equations of Anisotropic Incremental Theory

Assuming that the characteristic axes of anisotropy coincide with the principal directions, Hill's yield function[3,15,16] is given by

$$f = \left[\alpha_{12}(\sigma_1 - \sigma_2)^2 + \alpha_{23}(\sigma_2 - \sigma_3)^2 + \alpha_{31}(\sigma_3 - \sigma_1)^2\right]^{1/2} , \tag{103}$$

where the α_{ij}'s are constants. Since

$$\frac{\partial f^2}{\partial \sigma_i} = 2f \frac{\partial f}{\partial \sigma_i} , \tag{104}$$

it follows that

$$\frac{\partial f}{\partial \sigma_i} = \frac{\dfrac{\partial f^2}{\partial \sigma_i}}{2f} \ . \tag{105}$$

Since

$$f^2 = \alpha_{12}(\sigma_1 - \sigma_2)^2 + \alpha_{23}(\sigma_2 - \sigma_3)^2 + \alpha_{31}(\sigma_3 - \sigma_1)^2 , \tag{106}$$

$$\frac{\partial f^2}{\partial \sigma_1} = 2\alpha_{12}(\sigma_1 - \sigma_2) - 2\alpha_{31}(\sigma_3 - \sigma_1) , \tag{107}$$

and

$$\frac{\partial f}{\partial \sigma_1} = \frac{\alpha_{12}(\sigma_1 - \sigma_2) - \alpha_{31}(\sigma_3 - \sigma_1)}{f} . \tag{108a}$$

Similarly,

$$\frac{\partial f}{\partial \sigma_2} = \frac{\alpha_{23}(\sigma_2 - \sigma_3) - \alpha_{12}(\sigma_1 - \sigma_2)}{f} \tag{108b}$$

and

$$\frac{\partial f}{\partial \sigma_3} = \frac{\alpha_{31}(\sigma_3 - \sigma_1) - \alpha_{23}(\sigma_2 - \sigma_3)}{f} . \tag{108c}$$

Consequently, from Eq. (32), the associated incremental flow rule becomes

$$d\epsilon_1^P = \frac{\lambda}{f} \left[\alpha_{12}(\sigma_1 - \sigma_2) - \alpha_{31}(\sigma_3 - \sigma_1)\right] , \tag{109a}$$

$$d\epsilon_2^P = \frac{\lambda}{f} \left[\alpha_{23}(\sigma_2 - \sigma_3) - \alpha_{12}(\sigma_1 - \sigma_2)\right] , \tag{109b}$$

$$d\epsilon_3^P = \frac{\lambda}{f} \left[\alpha_{31}(\sigma_3 - \sigma_1) - \alpha_{23}(\sigma_2 - \sigma_3)\right] . \tag{109c}$$

Since f is a function only of the principal stress differences, it follows that there will be no plastic volume change, which is verified by summing Eqs. (109).

Although any positive value of λ will satisfy the flow rule, it is of interest to determine its algebraic form. By rearranging Eqs. (109),

$$\frac{f}{\lambda} d\epsilon_1^P = \sigma_1(\alpha_{12} + \alpha_{31}) - \sigma_2\alpha_{12} - \sigma_3\alpha_{31} , \tag{110a}$$

$$\frac{f}{\lambda} d\epsilon_2^P = -\sigma_1\alpha_{12} + \sigma_2(\alpha_{23} + \alpha_{12}) - \sigma_3\alpha_{23} , \tag{110b}$$

$$\frac{f}{\lambda}\, d\epsilon_3^P = -\sigma_1\alpha_{31} - \sigma_2\alpha_{23} + \sigma_3(\alpha_{31} + \alpha_{23}) \;, \tag{110c}$$

σ_3 may be eliminated from Eqs. (110a) and (110b) by subtraction. Subtracting

$$\alpha_{31}\, \frac{f}{\lambda}\, d\epsilon_2^P = -\sigma_1\alpha_{12}\alpha_{31} + \sigma_2(\alpha_{23}\alpha_{31} + \alpha_{12}\alpha_{31}) - \sigma_3\alpha_{23}\alpha_{31}$$

from

$$\alpha_{23}\, \frac{f}{\lambda}\, d\epsilon_1^P = \sigma_1(\alpha_{12}\alpha_{23} + \alpha_{23}\alpha_{31}) - \sigma_2\alpha_{12}\alpha_{23} - \sigma_3\alpha_{23}\alpha_{31}$$

gives

$$\frac{f}{\lambda}\,(\alpha_{23}\, d\epsilon_1^P - \alpha_{31}\, d\epsilon_2^P) = \sigma_1(\alpha_{12}\alpha_{23} + \alpha_{23}\alpha_{31} + \alpha_{31}\alpha_{12}) - \sigma_2(\alpha_{12}\alpha_{23} + \alpha_{23}\alpha_{31} + \alpha_{31}\alpha_{12}) \;. \tag{111}$$

Now, noting that σ_1 and σ_2 are both multiplied by the same coefficient, we let

$$\alpha_{12}\alpha_{23} + \alpha_{23}\alpha_{31} + \alpha_{31}\alpha_{12} = G \;. \tag{112}$$

Then, by rearranging Eq. (111),

$$\sigma_1 - \sigma_2 = \frac{f}{\lambda}\, \frac{\alpha_{23}\, d\epsilon_1^P - \alpha_{31}\, d\epsilon_2^P}{G} \;, \tag{113a}$$

and noting the pattern of subscripts on the right-hand side of Eq. (113a),

$$\sigma_2 - \sigma_3 = \frac{f}{\lambda}\, \frac{\alpha_{31}\, d\epsilon_2^P - \alpha_{12}\, d\epsilon_3^P}{G} \tag{113b}$$

and

$$\sigma_3 - \sigma_1 = \frac{f}{\lambda}\, \frac{\alpha_{12}\, d\epsilon_3^P - \alpha_{23}\, d\epsilon_1^P}{G} \;. \tag{113c}$$

Consequently, substituting Eq. (113) into Eq. (103) gives

$$\lambda = \frac{1}{G}\left[\alpha_{12}(\alpha_{23}\ d\epsilon_1^P - \alpha_{31}\ d\epsilon_2^P)^2 + \alpha_{23}(\alpha_{31}\ d\epsilon_2^P - \alpha_{12}\ d\epsilon_3^P)^2 + \alpha_{31}(\alpha_{12}\ d\epsilon_3^P - \alpha_{23}\ d\epsilon_1^P)^2\right]^{1/2} . \qquad (114)$$

Therefore λ represents a function of the principal plastic strain increments, the form of which is determined by the form of the yield function. In addition, the three expressions in Eq. (113) can be combined to yield the general expression

$$\frac{\alpha_{23}\ d\epsilon_1^P - \alpha_{31}\ d\epsilon_2^P}{\sigma_1 - \sigma_2} = \frac{\alpha_{31}\ d\epsilon_2^P - \alpha_{12}\ d\epsilon_3^P}{\sigma_2 - \sigma_3} = \frac{\alpha_{12}\ d\epsilon_3^P - \alpha_{23}\ d\epsilon_1^P}{\sigma_3 - \sigma_1} = G\,\frac{\lambda}{f} , \qquad (115)$$

which is an alternate form of the incremental flow rule.

So far, there has been no condition invoked that would fix the algebraic form of the effective plastic strain increment. Suppose it is assumed that

$$\sigma_{eff} = f \qquad (116)$$

and that[2,3]

$$dW_p = f\ d\epsilon_{eff}^P . \qquad (117)$$

These assumptions are logical, since they require that not only should there be a consistent relationship between the effective stress and the effective plastic strain but also that the area under the effective stress-strain curve should equal plastic work. Substituting Eq. (109) into Eq. (9) gives

$$dW_p = \frac{\lambda}{f}\left[\alpha_{12}(\sigma_1 - \sigma_2)^2 + \alpha_{23}(\sigma_2 - \sigma_3)^2 + \alpha_{31}(\sigma_3 - \sigma_1)^2\right] , \qquad (118)$$

and substituting Eq. (106) into Eq. (118) then gives

$$dW_p = f\lambda . \qquad (119)$$

Therefore, by combining Eqs. (117) and (119),

$$d\epsilon^P_{eff} = \lambda \ , \tag{120}$$

a result which could have been anticipated by referring to Eq. (58).

Equations of Anisotropic Deformation Theory

If the ratios of the principal stresses are assumed to remain constant during loading, the ratios of the principal plastic strain increments will also remain constant during loading. (These conditions are fulfilled under uniaxial loading.) Under these conditions, Eqs. (114), (110), and (113) can be integrated directly to give a set of deformation theory equations. Integrating Eq. (114) and using Eq. (120) gives

$$\epsilon^P_{eff} = \frac{1}{G}\left[\alpha_{12}(\alpha_{23}\epsilon^P_1 - \alpha_{31}\epsilon^P_2)^2 + \alpha_{23}(\alpha_{31}\epsilon^P_2 - \alpha_{12}\epsilon^P_3)^2 + \alpha_{31}(\alpha_{12}\epsilon^P_3 - \alpha_{23}\epsilon^P_1)^2\right]^{1/2} \ . \tag{121}$$

If the secant modulus of the effective stress-strain curve is defined as

$$E_p = \frac{\sigma_{eff}}{\epsilon^P_{eff}} = \frac{f}{\epsilon^P_{eff}} \ , \tag{122}$$

then by using Eq. (120), the integral of Eq. (110) is

$$\epsilon^P_1 = \frac{1}{E_p}\left[\sigma_1(\alpha_{12} + \alpha_{31}) - \sigma_2\alpha_{12} - \sigma_3\alpha_{31}\right] , \tag{123a}$$

$$\epsilon^P_2 = \frac{1}{E_p}\left[-\sigma_1\alpha_{12} + \sigma_2(\alpha_{23} + \alpha_{12}) - \sigma_3\alpha_{23}\right] , \tag{123b}$$

$$\epsilon^P_3 = \frac{1}{E_p}\left[-\sigma_1\alpha_{31} - \sigma_2\alpha_{23} + \sigma_3(\alpha_{31} + \alpha_{23})\right] . \tag{123c}$$

By using Eqs. (115) and (120), the stress-plastic strain relations of anisotropic deformation theory can also be written in the alternate form

$$\frac{\alpha_{23}\epsilon^P_1 - \alpha_{31}\epsilon^P_2}{\sigma_1 - \sigma_2} = \frac{\alpha_{31}\epsilon^P_2 - \alpha_{12}\epsilon^P_3}{\sigma_2 - \sigma_3} = \frac{\alpha_{12}\epsilon^P_3 - \alpha_{23}\epsilon^P_1}{\sigma_3 - \sigma_1} = G\,\frac{\epsilon^P_{eff}}{f} \ . \tag{124}$$

It may be noted immediately that Eq. (123) is a set of linear stress-plastic strain relations that can be combined with Hooke's law, according to Eq. (1), to produce a set of linear stress-total strain relations. Of course, E_p is actually a variable that must be determined as a function of position. If power-law strain hardening is assumed, Eq. (63) gives

$$E_p = \left(\frac{C}{f}\right)^{1/n} f \ . \tag{125}$$

If linear strain hardening is assumed, Eqs. (8) and (122) give

$$\epsilon^P_{eff} = \frac{\dfrac{f}{\sigma_0} - 1}{H} \tag{126}$$

and

$$E_p = \frac{Hf}{\dfrac{f}{f_0} - 1} \ . \tag{127}$$

Again, if there is no strain hardening, E_p is either undefined or indeterminate, since the plastic strains are not uniquely determined by the stresses for ideally plastic material.

The special case of deformation theory applied to an elastic-ideally plastic material presents an interesting problem from another point of view. For ideally plastic material, f remains constant, and for deformation theory, solutions are exact only when the stress ratios remain constant. These three conditions are sufficient to determine all three principal stresses. Therefore, no elastic-ideally plastic deformation-theory solution will ever be exact unless all three principal stresses stay constant after yielding. In graphical terms, the only set of principal stresses for which an exact solution exists is the intersection point of a radial line in stress space with the yield surface.

The Coefficients of Anisotropy, α_{12}, α_{23}, and α_{31}

The three coefficients of anisotropy, α_{12}, α_{23}, and α_{31}, can be evaluated by comparing the uniaxial stress-plastic strain curves in the 1, 2, and 3 directions.

For uniaxial loading in the 1 direction, $\sigma_2 = \sigma_3 = 0$, and the result of combining Eqs. (122) and (123a) is that

$$\epsilon_1^P = \frac{\epsilon_{eff}^P}{f} (\alpha_{12} + \alpha_{31})\sigma_1 , \tag{128}$$

but from Eq. (103),

$$f = \sigma_1(\alpha_{12} + \alpha_{31})^{1/2} . \tag{129}$$

Therefore substituting Eq. (129) into Eq. (128) gives

$$\epsilon_1^P = (\alpha_{12} + \alpha_{31})^{1/2} \epsilon_{eff}^P , \tag{130}$$

and from Eq. (129),

$$\sigma_1 = (\alpha_{12} + \alpha_{31})^{-1/2} f . \tag{131}$$

Because of algebraic symmetry, it follows that for uniaxial loading in the 2 direction, $\sigma_1 = \sigma_3 = 0$, and

$$\epsilon_2^P = (\alpha_{23} + \alpha_{12})^{1/2} \epsilon_{eff}^P , \tag{132}$$

$$\sigma_2 = (\alpha_{23} + \alpha_{12})^{-1/2} f . \tag{133}$$

Consequently ϵ_{eff}^P and f may be eliminated to give

$$\epsilon_2^P = \left(\frac{\alpha_{23} + \alpha_{12}}{\alpha_{12} + \alpha_{31}}\right)^{1/2} \epsilon_1^P \tag{134}$$

and

$$\sigma_2 = \left(\frac{\alpha_{23} + \alpha_{12}}{\alpha_{12} + \alpha_{31}}\right)^{-1/2} \sigma_1 . \tag{135}$$

Using the substitution

$$\left(\frac{\alpha_{23} + \alpha_{12}}{\alpha_{12} + \alpha_{31}}\right)^{1/2} = a \tag{136}$$

and taking logarithms on both sides of Eqs. (134) and (135) gives

$$\log \epsilon_2^P = \log \epsilon_1^P + \log a \tag{137}$$

and

$$\log \sigma_2 = \log \sigma_1 - \log a \ . \tag{138}$$

Consequently, the stress-plastic strain curves in the principal directions must plot parallel to each other on log-log paper, as shown in Fig. 12.

ORNL-DWG 67-3382

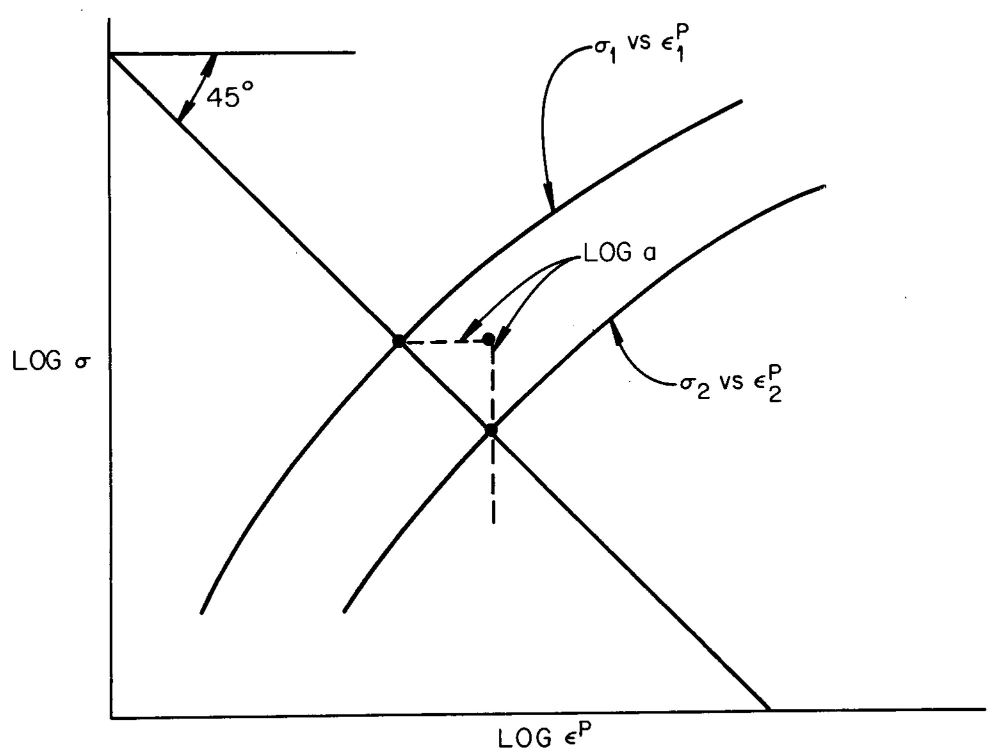

Fig. 12. Parallel Relationship on Log-Log Plot of Stress-Plastic Strain Curves for Anisotropic Material Obeying Hill's Yield Function.

The factor log a equals the logarithmic coordinate differences between the points of intersection of the two stress-plastic strain curves with a straight line having a slope of minus one.

From algebraic symmetry [interchanging the subscripts 2 and 3 in Eq. (136)] it follows that the scale factor between the 1 and 3 directions is defined by the equation

$$b = \left(\frac{\alpha_{23} + \alpha_{31}}{\alpha_{12} + \alpha_{31}}\right)^{1/2} . \tag{139}$$

Furthermore, b can be determined graphically from the equations

$$\log \epsilon_3^P = \log \epsilon_1^P + \log b \tag{140}$$

and

$$\log \sigma_3 = \log \sigma_1 - \log b . \tag{141}$$

For a given material it would be convenient to define the coefficients α_{12}, α_{23}, and α_{31} in such a way that under uniaxial loading in a specified direction, the effective stress-strain curve would coincide with the stress-plastic strain curve. If it is specified that when $\sigma_2 = \sigma_3 = 0$,

$$f = \sigma_1 , \tag{142}$$

from Eq. (129)

$$(\alpha_{12} + \alpha_{31}) = 1 . \tag{143}$$

Substituting Eq. (143) into Eq. (130) then gives

$$\epsilon_{eff}^P = \epsilon_1^P . \tag{144}$$

Three conditions are required for determining the three coefficients α_{12}, α_{23}, and α_{31}. The first condition, which is expressed by Eq. (143) is arbitrary. The other two conditions result from comparing the stress-plastic strain curves in the 2 and 3 directions with the stress-plastic strain curve in the 1 direction,[2] as explained below.

If the effective stress-strain curve is the stress-plastic strain curve in the 1 direction, Eq. (143) holds. Substituting Eq. (143) into Eqs. (136) and (139) gives

$$a = (\alpha_{23} + \alpha_{12})^{1/2} \tag{145}$$

and

$$b = (\alpha_{23} + \alpha_{31})^{1/2} . \tag{146}$$

Squaring both sides of Eqs. (145) and (146) and adding gives

$$a^2 + b^2 = 2\alpha_{23} + \alpha_{12} + \alpha_{31} . \tag{147}$$

Substituting Eq. (143) into Eq. (147) and rearranging then gives the following equations for <u>generally anisotropic material</u>:

$$\alpha_{23} = \frac{a^2 + b^2 - 1}{2} , \tag{148}$$

and substituting Eq. (148) into Eq. (145), squaring both sides, and rearranging gives

$$\alpha_{12} = \frac{a^2 - b^2 + 1}{2} . \tag{149}$$

From Eq. (143) it follows that

$$\alpha_{31} = \frac{b^2 - a^2 + 1}{2} . \tag{150}$$

For <u>transversely isotropic material</u>,

$$a = b , \tag{151}$$

and consequently

$$\alpha_{12} = 1/2 , \tag{152a}$$

$$\alpha_{23} = a^2 - 1/2 , \tag{152b}$$

$$\alpha_{31} = 1/2 . \tag{152c}$$

Of course, the existence of a consistent theory does not constitute sufficient evidence to prove that all materials must obey it. Therefore, if the stress-plastic strain curves for a certain anisotropic material do not plot parallel on log-log paper, it cannot be concluded that either there is something wrong with the theory or with the material. If the curves are almost parallel, they can probably be fit by a set of parallel curves. If they are nowhere near parallel, it must be remembered that the α terms in the expression for f will have partial derivatives with respect to the principal stresses and will thereby affect both the flow rule and the definition of the effective plastic strain.

If the effective stress-strain curve can be fit by either the power law or the linear strain-hardening law, an analytical determination of the factors "a" and "b" is possible. Taking antilogs on both sides of Eqs. (137) and (138) gives

$$\epsilon_2^P = a\epsilon_1^P \tag{153}$$

and

$$\sigma_2 = \frac{\sigma_1}{a} \, . \tag{154}$$

For power-law strain hardening, the stress-plastic strain curves in the 1 and 2 directions will be given by equations of the same form as Eq. (7). Therefore

$$\sigma_1 = C_1(\epsilon_1^P)^n \tag{155a}$$

and

$$\sigma_2 = C_2(\epsilon_2^P)^n \, . \tag{155b}$$

It should be noted that the strain-hardening exponent, n, must be the same in both directions. Substituting Eqs. (153) and (154) into Eq. (155b) gives

$$\frac{\sigma_1}{a} = C_2(a\epsilon_1^P)^n \, . \tag{156}$$

Substituting Eq. (155a) into Eq. (156) then gives

$$\frac{C_1(\epsilon_1^P)^n}{a} = C_2(a\epsilon_1^P)^n \quad . \tag{157}$$

Consequently

$$\frac{C_1}{a} = C_2 a^n \quad , \tag{158}$$

$$\frac{C_1}{C_2} = a^{n+1} \quad , \tag{159}$$

$$a = \left(\frac{C_1}{C_2}\right)^{1/(n+1)} \quad . \tag{160}$$

By the same argument,

$$b = \left(\frac{C_1}{C_3}\right)^{1/(n+1)} \quad . \tag{161}$$

For linear strain hardening, the stress-plastic strain curves in the 1 and 2 directions will be given by equations of the same form as Eq. (8). Therefore

$$\sigma_1 = \sigma_{01}(1 + H_1\epsilon_1^P) \tag{162a}$$

and

$$\sigma_2 = \sigma_{02}(1 + H_2\epsilon_2^P) \quad , \tag{162b}$$

where σ_{01} and σ_{02} are the initial uniaxial yield stresses in the 1 and 2 directions. Substituting Eqs. (153) and (154) into (162b) gives

$$\frac{\sigma_1}{a} = \sigma_{02}(1 + aH_2\epsilon_1^P) \quad . \tag{163}$$

Substituting Eq. (162a) into Eq. (163) then gives

$$\frac{\sigma_{01}(1 + H_1\epsilon_1^P)}{a} = \sigma_{02}(1 + aH_2\epsilon_1^P) \ . \qquad (164)$$

Consequently, at $\epsilon_1^P = 0$,

$$\frac{\sigma_{01}}{a} = \sigma_{02} \qquad (165)$$

and

$$a = \frac{\sigma_{01}}{\sigma_{02}} \ . \qquad (166)$$

However, Eq. (166) must hold at all values of ϵ_1^P. Therefore, substituting Eq. (165) into the left-hand side of Eq. (164) gives

$$\sigma_{02}(1 + H_1\epsilon_1^P) = \sigma_{02}(1 + aH_2\epsilon_1^P) \ , \qquad (167)$$

which reduces directly to

$$H_1 = aH_2 \ , \qquad (168)$$

and thus

$$a = \frac{H_1}{H_2} = \frac{\sigma_{01}}{\sigma_{02}} \ . \qquad (169)$$

By the same argument,

$$b = \frac{H_1}{H_3} = \frac{\sigma_{01}}{\sigma_{03}} \ . \qquad (170)$$

Equations for Isotropic Material

For isotropic material, a = 1, and it follows from Eq. (152) that when a = 1,

$$\alpha_{12} = \alpha_{23} = \alpha_{31} = 1/2 \ . \qquad (171)$$

Consequently, substituting Eq. (171) into Eq. (103) gives

$$f = \frac{1}{\sqrt{2}} \left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]^{1/2} . \qquad (172)$$

By comparing Eqs. (77) and (172) it can be seen that for isotropic material,

$$f = \frac{3}{\sqrt{2}} \tau_{oct} . \qquad (173)$$

Substituting Eq. (171) into Eq. (112) gives

$$G = \frac{3}{4} . \qquad (174)$$

<u>For incremental theory</u>, substituting Eqs. (171) and (174) into Eq. (114) and substituting Eq. (120) on the left-hand side gives

$$d\epsilon^P_{eff} = \frac{\sqrt{2}}{3} \left[(d\epsilon^P_1 - d\epsilon^P_2)^2 + (d\epsilon^P_2 - d\epsilon^P_3)^2 + (d\epsilon^P_3 - d\epsilon^P_1)^2 \right]^{1/2} . \qquad (175)$$

In addition, applying Eqs. (171), (174), and (120) to Eq. (115) gives

$$\frac{d\epsilon^P_1 - d\epsilon^P_2}{\sigma_1 - \sigma_2} = \frac{d\epsilon^P_2 - d\epsilon^P_3}{\sigma_2 - \sigma_3} = \frac{d\epsilon^P_3 - d\epsilon^P_1}{\sigma_3 - \sigma_1} = \frac{3}{2} \frac{d\epsilon^P_{eff}}{f} . \qquad (176)$$

<u>For deformation theory</u>, Eqs. (175) and (176) are integrated to give

$$\epsilon^P_{eff} = \frac{\sqrt{2}}{3} \left[(\epsilon^P_1 - \epsilon^P_2)^2 + (\epsilon^P_2 - \epsilon^P_3)^2 + (\epsilon^P_3 - \epsilon^P_1)^2 \right]^{1/2} \qquad (177)$$

and

$$\frac{\epsilon^P_1 - \epsilon^P_2}{\sigma_1 - \sigma_2} = \frac{\epsilon^P_2 - \epsilon^P_3}{\sigma_2 - \sigma_3} = \frac{\epsilon^P_3 - \epsilon^P_1}{\sigma_3 - \sigma_1} = \frac{3}{2} \frac{\epsilon^P_{eff}}{f} . \qquad (178)$$

Equations (176) and (178) are known as the Von Mises incremental and deformation theory flow rules, respectively. By using Eq. (40), Eq. (177) can be rewritten in terms of any two principal plastic strains. For example, eliminating ϵ_3^P gives

$$\epsilon_{eff}^P = \frac{2}{\sqrt{3}} \left[(\epsilon_1^P)^2 + \epsilon_1^P \epsilon_2^P + (\epsilon_2^P)^2 \right]^{1/2} . \tag{179}$$

Of course, Eq. (179) has its incremental counterpart, the same as Eqs. (177) and (178).

Equations Relating the Total Strains to the Stresses for Isotropic Material

If the applicability of deformation theory is assumed, it is possible to develop a set of equations relating the stresses to the total strains. This is the approach taken by Mendelson and Manson[17] in their work on thermal stresses. For isotropic material, Hooke's law states that

$$\epsilon_1^E = \frac{\sigma_1}{E} - \frac{\nu}{E} (\sigma_2 + \sigma_3) + \alpha T , \tag{180a}$$

$$\epsilon_2^E = \frac{\sigma_2}{E} - \frac{\nu}{E} (\sigma_1 + \sigma_3) + \alpha T , \tag{180b}$$

$$\epsilon_3^E = \frac{\sigma_3}{E} - \frac{\nu}{E} (\sigma_1 + \sigma_2) + \alpha T . \tag{180c}$$

Furthermore, for isotropic material, Eq. (123) reduces to

$$\epsilon_1^P = \frac{\sigma_1}{E_p} - \frac{0.5}{E_p} (\sigma_2 + \sigma_3) , \tag{181a}$$

$$\epsilon_2^P = \frac{\sigma_2}{E_p} - \frac{0.5}{E_p} (\sigma_1 + \sigma_3) , \tag{181b}$$

$$\epsilon_3^P = \frac{\sigma_3}{E_p} - \frac{0.5}{E_p}(\sigma_1 + \sigma_2) . \tag{181c}$$

Adding the elastic and plastic components of strain, as given by Eqs. (180) and (181), gives

$$\epsilon_1 = \sigma_1 \left(\frac{1}{E} + \frac{1}{E_p}\right) - \left(\frac{\nu}{E} + \frac{0.5}{E_p}\right)(\sigma_2 + \sigma_3) + \alpha T , \tag{182a}$$

$$\epsilon_2 = \sigma_2 \left(\frac{1}{E} + \frac{1}{E_p}\right) - \left(\frac{\nu}{E} + \frac{0.5}{E_p}\right)(\sigma_1 + \sigma_3) + \alpha T , \tag{182b}$$

$$\epsilon_3 = \sigma_3 \left(\frac{1}{E} + \frac{1}{E_p}\right) - \left(\frac{\nu}{E} + \frac{0.5}{E_p}\right)(\sigma_1 + \sigma_2) + \alpha T . \tag{182c}$$

Subtracting Eqs. (182) by pairs gives

$$\epsilon_1 - \epsilon_2 = (\sigma_1 - \sigma_2)\left(\frac{1 + \nu}{E} + \frac{1.5}{E_p}\right) , \tag{183a}$$

$$\epsilon_2 - \epsilon_3 = (\sigma_2 - \sigma_3)\left(\frac{1 + \nu}{E} + \frac{1.5}{E_p}\right) , \tag{183b}$$

$$\epsilon_3 - \epsilon_1 = (\sigma_3 - \sigma_1)\left(\frac{1 + \nu}{E} + \frac{1.5}{E_p}\right) . \tag{183c}$$

It should be noted that the αT terms have been eliminated by subtraction. Equation (183) can be rewritten in the form

$$\frac{\epsilon_1 - \epsilon_2}{\sigma_1 - \sigma_2} = \frac{\epsilon_2 - \epsilon_3}{\sigma_2 - \sigma_3} = \frac{\epsilon_3 - \epsilon_1}{\sigma_3 - \sigma_1} = \frac{1 + \nu}{E} + \frac{1.5}{E_p} . \tag{184}$$

By squaring both sides of each expression in Eq. (183), and adding the results,

$$\left(\frac{3}{\sqrt{2}} \epsilon_{eff}\right)^2 = (\sqrt{2}\, f)^2 \left(\frac{1 + \nu}{E} + \frac{1.5}{E_p}\right)^2 , \tag{185}$$

where, by definition,

$$\epsilon_{eff} = \frac{\sqrt{2}}{3}\left[(\epsilon_1 - \epsilon_2)^2 + (\epsilon_2 - \epsilon_3)^2 + (\epsilon_3 - \epsilon_1)^2\right]^{1/2} . \qquad (186)$$

Here ϵ_{eff} is called the effective total strain. Taking square roots on both sides of Eq. (185) and dividing by $\sqrt{2}$ gives

$$\frac{3}{2}\,\epsilon_{eff} = f\left(\frac{1 + \nu}{E} + \frac{1.5}{E_p}\right) . \qquad (187)$$

Substituting Eq. (122) into Eq. (187) gives

$$\frac{3}{2}\,\epsilon_{eff} = f\,\frac{1 + \nu}{E} + \frac{3}{2}\,\epsilon^P_{eff} . \qquad (188)$$

Since, according to the effective stress-strain relation,

$$f = g\,(\epsilon^P_{eff}) , \qquad (189)$$

Eq. (188) is a direct relationship between the effective total strain and the effective plastic strain. Equation (188) can be inverted to give the effective plastic strain in terms of the effective total strain. Consequently the equation

$$\epsilon^P_{eff} = \psi(\epsilon_{eff}) , \qquad (190)$$

where ψ indicates a functional relationship, is a valid equation that can be used for analysis. It should be noted that Eq. (190) results from combining Hooke's law with the Von Mises condition, and therefore Eq. (190) is not a redundant relationship between the stresses and the elastic strains. It follows directly from Eq. (188) that

$$f = \frac{3E}{2(1 + \nu)}\,(\epsilon_{eff} - \epsilon^P_{eff}) . \qquad (191)$$

The Equation of Hill's Yield Function for Transversely Isotropic Material in the Octahedral Coordinate System

One important special form of Hill's yield function is the form applicable to a material such as graphite,* which has one set of properties "with the grain" and another set of properties in the plane perpendicular to the grain. Such a material is called a transversely isotropic material. For transversely isotropic material, according to Eq. (152), the constants α_{12} and α_{31} may be taken equal to one-half. Under these conditions, the 2,3 plane is considered the plane of isotropy, and the 1 axis is considered the axis of anisotropy. The effective stress-strain curve is the stress-plastic strain curve in the 1 direction. Accordingly, the expression for f from Eq. (103) is given by

$$f = \left[\frac{1}{2}(\sigma_1 - \sigma_2)^2 + \alpha_{23}(\sigma_2 - \sigma_3)^2 + \frac{1}{2}(\sigma_3 - \sigma_1)^2\right]^{1/2} \tag{192}$$

and

$$f^2 = \frac{1}{2}(\sigma_1 - \sigma_2)^2 + \alpha_{23}(\sigma_2 - \sigma_3)^2 + \frac{1}{2}(\sigma_3 - \sigma_1)^2 \ . \tag{193}$$

Now, by squaring both sides of Eq. (77) and multiplying by 9/2,

$$\frac{9}{2}\tau_{oct}^2 = \frac{1}{2}(\sigma_1 - \sigma_2)^2 + \frac{1}{2}(\sigma_2 - \sigma_3)^2 + \frac{1}{2}(\sigma_3 - \sigma_1)^2 \ . \tag{194}$$

Then, by subtracting Eq. (194) from Eq. (193),

$$f^2 - \frac{9}{2}\tau_{oct}^2 = \left(\alpha_{23} - \frac{1}{2}\right)(\sigma_2 - \sigma_3)^2 \ . \tag{195}$$

Equation (195) can be transformed into the octahedral coordinate system by applying Eq. (102) on the left-hand side and Eq. (101b) on the right-hand side. Performing these substitutions gives

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*Graphite does not actually exhibit a zero plastic volume change. However, until a more appropriate yield function can be derived, this assumption serves as an expedient.

$$f^2 - \frac{3}{2}\left(\sigma_u^2 + \sigma_v^2\right) = \left(\alpha_{23} - \frac{1}{2}\right) 2\sigma_v^2 . \tag{196}$$

Therefore

$$\frac{3}{2}\sigma_u^2 + \frac{4\alpha_{23} + 1}{2}\sigma_v^2 = f^2 , \tag{197}$$

which is the equation of an ellipse in the octahedral plane. If α_{23} is taken equal to one-half, Eq. (197) reduces to

$$\sigma_u^2 + \sigma_v^2 = \frac{2f^2}{3} , \tag{198}$$

which represents the Von Mises circle.

The Mohr-Coulomb Yield Function

Derivation of the Yield Function

The principal reasons for discussing the Mohr-Coulomb yield function are that (1) the Mohr-Coulomb yield function is the only common example of a yield function that leads to a plastic volume change and (2) the equations for the Mohr-Coulomb yield function reveal the basic facts that slip is two dimensional and that plastic volume changes are caused by the occurrence of slip on planes other than the 45° planes of maximum shear stress. Plastic volume changes are the rule rather than the exception in the case of nonmetallic materials such as soil, concrete, and graphite. The derivation of the equations for the Mohr-Coulomb yield function demonstrates that plastic volume changes can be considered in plastic analysis by directly applying the principles contained in this report.

Not all the predictions of the Mohr-Coulomb yield theory have been observed experimentally. For instance, the occurrence of "corners" in the yield surface and the absence of any plastic strain in the direction of the intermediate principal stress are still conditions more hypothetical than real. Nevertheless, the theory is being presented here because it has already been used extensively for analysis and because it still constitutes a very useful simplifying approximation for obtaining closed-form solutions to problems.

The Mohr-Coulomb yield function is based on the concept that yielding is controled by internal friction plus cohesion.[18,19] According to this theory, as shown in Fig. 13, the point of tangency of the outer Mohr stress circle and a linear envelope gives the state of stress on, and the inclination of, the plane of yielding. For the determination of ultimate bearing capacities and limiting lateral earth pressures, soil is usually treated as a rigid ideally plastic material. However, there is nothing to prevent the Mohr-Coulomb yield function from being applied to a strain-hardening material. In fact, investigations along these lines have been conducted by Drucker, Gibson, and Henkel,[20] and by Haythornthwaite.[21]

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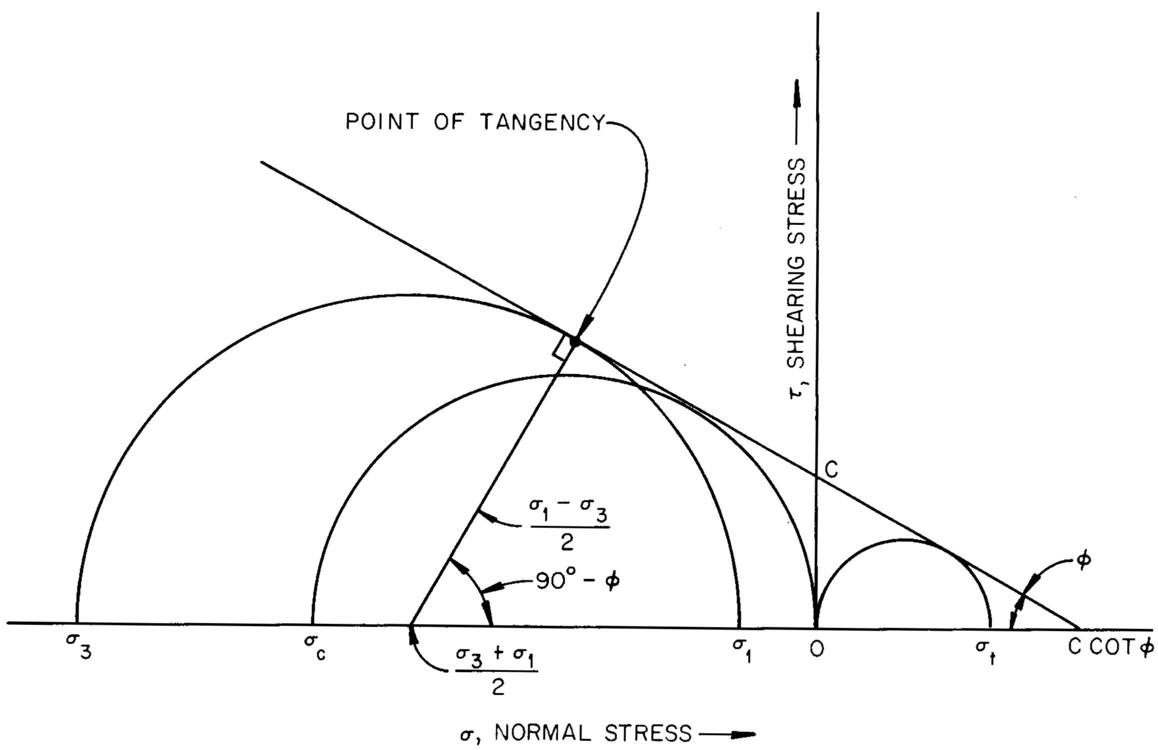

Fig. 13. Mohr's Circle Diagram for the Mohr-Coulomb Yield Criterion.

From the diagram shown in Fig. 13, it can be seen that [18,19]

$$\frac{\sigma_1 - \sigma_3}{2} = \left[\frac{-(\sigma_3 + \sigma_1)}{2} + c \cot \phi\right] \sin \phi \ , \qquad (199)$$

where, in contrast to the previous discussions, σ_1 is the maximum principal tensile stress, σ_3 is the minimum principal tensile stress, c is cohesion, and ϕ is the angle whose tangent equals the coefficient of internal friction; ϕ is called the angle of internal friction. Clearing fractions, Eq. (199) becomes

$$\sigma_1 - \sigma_3 = -(\sigma_3 + \sigma_1) \sin \phi + 2c \cos \phi \ . \qquad (200)$$

Consequently

$$\sigma_1(1 + \sin \phi) = \sigma_3(1 - \sin \phi) + 2c \cos \phi \qquad (201)$$

and

$$\sigma_1 \frac{1 + \sin \phi}{1 - \sin \phi} - \sigma_3 = \frac{2c \cos \phi}{1 - \sin \phi} \ . \qquad (202)$$

If

$$\frac{1 + \sin \phi}{1 - \sin \phi} = m \ , \qquad (203)$$

it follows that

$$\frac{2c \cos \phi}{1 - \sin \phi} = 2c \left(\frac{1 + \sin \phi}{1 - \sin \phi}\right)^{1/2} = 2cm^{1/2} \ . \qquad (204)$$

Substituting Eqs. (203) and (204) into Eq. (202) gives

$$m\sigma_1 - \sigma_3 = 2cm^{1/2} \ . \qquad (205)$$

When $\sigma_1 = 0$, $\sigma_3 = -\sigma_c$, and consequently

$$2cm^{1/2} = \sigma_c \ . \qquad (206)$$

Therefore

$$m\sigma_1 - \sigma_3 = \sigma_c \ . \qquad (207)$$

When $\sigma_3 = 0$, $\sigma_1 = \sigma_t$, and consequently

$$m\sigma_t = \sigma_c \tag{208}$$

and

$$m = \frac{\sigma_c}{\sigma_t} \, . \tag{209}$$

For analysis, the yield function, f, is defined as

$$f = m\sigma_1 - \sigma_3 \, . \tag{210}$$

It can now be seen that the Mohr-Coulomb yield function represents a linear relation between the major and the minor principal stresses. In addition, if m is set equal to unity, the Mohr-Coulomb yield function reduces to the Tresca yield function. Furthermore, by adding a constant σ_m to both principal stresses, it can be seen that, in general, f is not independent of σ_m. Therefore, a plastic volume change is to be expected. Such volume changes are observed in granular soil.[20]

The Incremental and Integrated Flow Rules

According to the flow rule given by Eq. (32),

$$d\epsilon_1^P = \lambda \frac{\partial f}{\partial \sigma_1} = m\lambda \, , \tag{211a}$$

$$d\epsilon_2^P = \lambda \frac{\partial f}{\partial \sigma_2} = 0 \, , \tag{211b}$$

$$d\epsilon_3^P = \lambda \frac{\partial f}{\partial \sigma_3} = -\lambda \, . \tag{211c}$$

The plastic strain increment in the direction of the intermediate principal stress is zero. Therefore, Hooke's law will give the total strain in that direction for an elastic-plastic material. Because f is a linear function of the principal stresses, the plastic strain increment ratios are constant, even if the stress ratios are not, providing σ_1 and σ_3 continue to act in the same direction. Therefore, Eq. (211) can always be integrated directly to give the ratios of the total plastic strains. Accordingly

$$\frac{d\epsilon_1^P}{d\epsilon_3^P} = -m \tag{212}$$

and

$$\epsilon_1^P = -m\epsilon_3^P \ . \tag{213}$$

<u>Stress-Plastic Strain Relationships When the Three Principal Stresses Are Unequal</u>

If Eqs. (10) and (59) are again assumed to hold, it follows that in multiaxial loading, with $\sigma_1 > \sigma_2 > \sigma_3$,

$$dW_p = f\ d\epsilon_{eff}^P = \sigma_1 m\lambda - \sigma_3\lambda = \lambda(m\sigma_1 - \sigma_3) = f\lambda \ . \tag{214}$$

Therefore

$$d\epsilon_{eff}^P = \lambda \tag{215}$$

and

$$\int \lambda = \epsilon_{eff}^P \ . \tag{216}$$

Consequently, integrating Eqs. (211) gives

$$\epsilon_1^P = m\epsilon_{eff}^P \ , \tag{217a}$$

$$\epsilon_2^P = 0 \ , \tag{217b}$$

$$\epsilon_3^P = -\epsilon_{eff}^P \ . \tag{217c}$$

From Eqs. (217) it may be seen that the plastic volume change, also called the dilatation, is given by

$$\Delta V_p = (m - 1)\ \epsilon_{eff}^P \ , \tag{218}$$

and, using Eq. (217), that

$$\Delta V_p = (m - 1)(-\epsilon_3^P) = \frac{m - 1}{m} \epsilon_1^P . \quad (219)$$

The plastic volume change is an expansion.

Stress-Plastic Strain Relationships When Two Principal Stresses Are Equal

If two principal stresses are equal, the plastic strains in the directions of the two equal stresses are not uniquely determined by the flow rule. However, as shown in Fig. 14, the total plastic strain vector (the vector integral of $d\bar{\epsilon}^P$) at a corner can always be resolved into two components, one normal to each of the two adjacent sides. The properties of the total plastic strain vector can then be deduced from the known properties of its two components.

For $\sigma_1 = \sigma_2 > \sigma_3$,

$$\epsilon_1^P = m\epsilon_A^P + 0 = m\epsilon_A^P , \quad (220a)$$

$$\epsilon_2^P = 0 + m\epsilon_B^P = m\epsilon_B^P , \quad (220b)$$

ORNL-DWG 67-3384

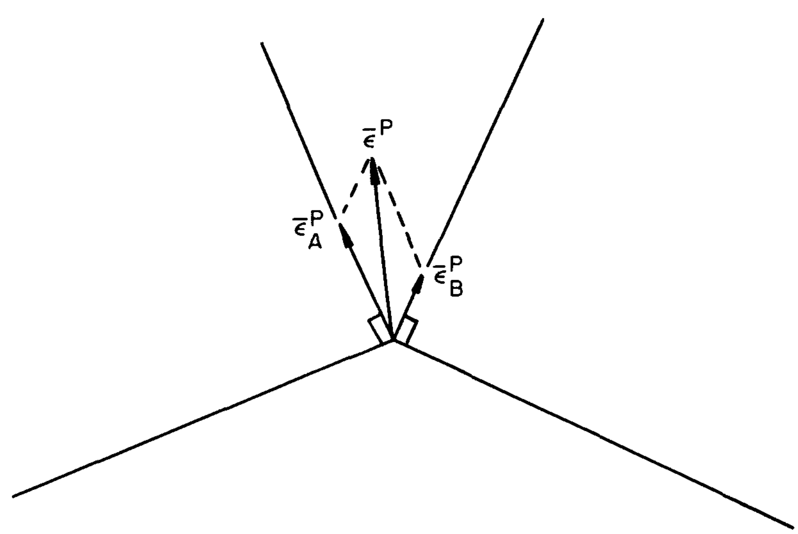

Fig. 14. Plastic Strain Vector and Its Components at a Corner of the Mohr-Coulomb Yield Surface.

$$\epsilon_3^P = -\epsilon_A^P - \epsilon_B^P = -(\epsilon_A^P + \epsilon_B^P) \ , \qquad (220c)$$

where ϵ_A^P and ϵ_B^P are the effective plastic strains associated with the two components of the total plastic strain vector. Adding Eqs. (220) gives

$$\Delta V_p = (m - 1)(\epsilon_A^P + \epsilon_B^P) \ , \qquad (221)$$

and, by using Eq. (220c),

$$\Delta V_p = (m - 1)(-\epsilon_3^P) \ . \qquad (222)$$

Furthermore, adding Eqs. (220a) and (220b) and using Eq. (220c), gives

$$\epsilon_1^P + \epsilon_2^P = -m\epsilon_3^P \ . \qquad (223)$$

Therefore

$$dW_p = f\, d\epsilon_{eff}^P = \sigma_1(-m d\epsilon_3^P) + \sigma_3 d\epsilon_3^P = (m\sigma_1 - \sigma_3)(-d\epsilon_3^P) = f(-d\epsilon_3^P) \qquad (224)$$

and

$$\epsilon_3^P = -\epsilon_{eff}^P \ . \qquad (225)$$

In uniaxial compression, when $\sigma_1 = \sigma_2 = 0$,

$$\sigma_3 = -f \ . \qquad (226)$$

For $\sigma_1 > \sigma_2 = \sigma_3$, a similar analysis gives

$$\epsilon_1^P = m\epsilon_A^P + m\epsilon_B^P = m(\epsilon_A^P + \epsilon_B^P) \ , \qquad (227a)$$

$$\epsilon_2^P = 0 - \epsilon_B^P = -\epsilon_B^P \ , \qquad (227b)$$

$$\epsilon_3^P = -\epsilon_A^P + 0 = -\epsilon_A^P \ . \qquad (227c)$$

Therefore

$$\Delta V_p = (m - 1)(\epsilon_A^P + \epsilon_B^P) \ , \qquad (228)$$

and by using Eq. (227a),

$$\Delta V_p = \frac{m - 1}{m} \epsilon_1^P . \tag{229}$$

Furthermore, adding Eqs. (227b) and (227c) and using Eq. (227a) gives

$$\epsilon_2^P + \epsilon_3^P = - \frac{\epsilon_1^P}{m} . \tag{230}$$

Therefore

$$dW_p = f \, d\epsilon_{eff}^P = \sigma_1 \, d\epsilon_1^P + \sigma_3 \left(- \frac{d\epsilon_1^P}{m} \right) = \frac{m\sigma_1 - \sigma_3}{m} d\epsilon_1^P = f \frac{d\epsilon_1^P}{m} \tag{231}$$

and

$$\epsilon_1^P = m\epsilon_{eff}^P . \tag{232}$$

In uniaxial tension, when $\sigma_2 = \sigma_3 = 0$,

$$\sigma_1 = \frac{f}{m} . \tag{233}$$

A Few Consequences of the Mohr-Coulomb Theory

The stress-plastic strain curve in uniaxial compression gives the effective stress-strain curve directly, but the tensile stress-plastic strain curve is the same curve with the stress divided by m and the plastic strain multiplied by m. The tensile stress-plastic strain curve will fall below the compressive stress-plastic strain curve for $m > 1$. Furthermore, the two curves must plot parallel to each other on log-log paper. Therefore, m can also be determined by the graphical method illustrated in Fig. 12. From Eqs. (222) and (229), it can be seen that m can also be determined by measuring the axial strain and the plastic volume change in a triaxial test. It may be noted further that when $m = 1$, the plastic volume change is always zero, and the tensile and compressive stress-plastic strain curves coincide. These conditions are always assumed to hold for the special case of the Tresca yield function, which applies to materials without internal friction ($\phi = 0°$).

Another important observation concerning the Mohr-Coulomb and the Tresca equations is that although they utilize an integrated flow rule, they are still based on incremental theory. Therefore the Mohr-Coulomb and the Tresca yield functions possess an important advantage over the Von Mises deformation theory for ideally plastic materials in that they can produce exact solutions with an integrated flow rule, whereas the Von Mises deformation theory cannot.

GEOMETRICAL DERIVATION OF THE FLOW RULES FOR THE MOHR-COULOMB AND THE VON MISES YIELD FUNCTIONS

The Mohr-Coulomb Yield Function

It is interesting to note that the flow rule associated with the Mohr-Coulomb yield function can also be derived by assuming that yielding is a process of slippage between thin parallel sections, as shown in Fig. 15, and that no plastic strain occurs in the direction of slip.[22] As shown in Figs. 13 and 16, the angle between the plane on which σ_1 acts and the slip plane is $(45° - \phi/2)$. Since no slippage occurs in the direction of σ_2, which acts normal to the paper in Fig. 16, Δt, as shown in Fig. 15, is zero, and therefore ϵ_2^P is zero. It can also be seen in Fig. 16 that the angle between the direction of ϵ_3^P and the direction of zero plastic strain, which is the slip direction, is $(45° - \phi/2)$. Consequently, the Mohr diagram of plastic strain is as shown in Fig. 17. It is correct to draw a Mohr diagram of plastic strain because Eq. (1) is linear. Therefore, if the element shown in Fig. 2 is unloaded, all the elastic strains vanish and leave the plastic strains unchanged but equal to the total strains. From Fig. 17, it may be seen that

$$-\epsilon_3^P + x = \epsilon_1^P - x \tag{234}$$

and that

$$x = \frac{\epsilon_1^P - \epsilon_3^P}{2} \sin \phi \ . \tag{235}$$

ORNL-DWG 67-3385

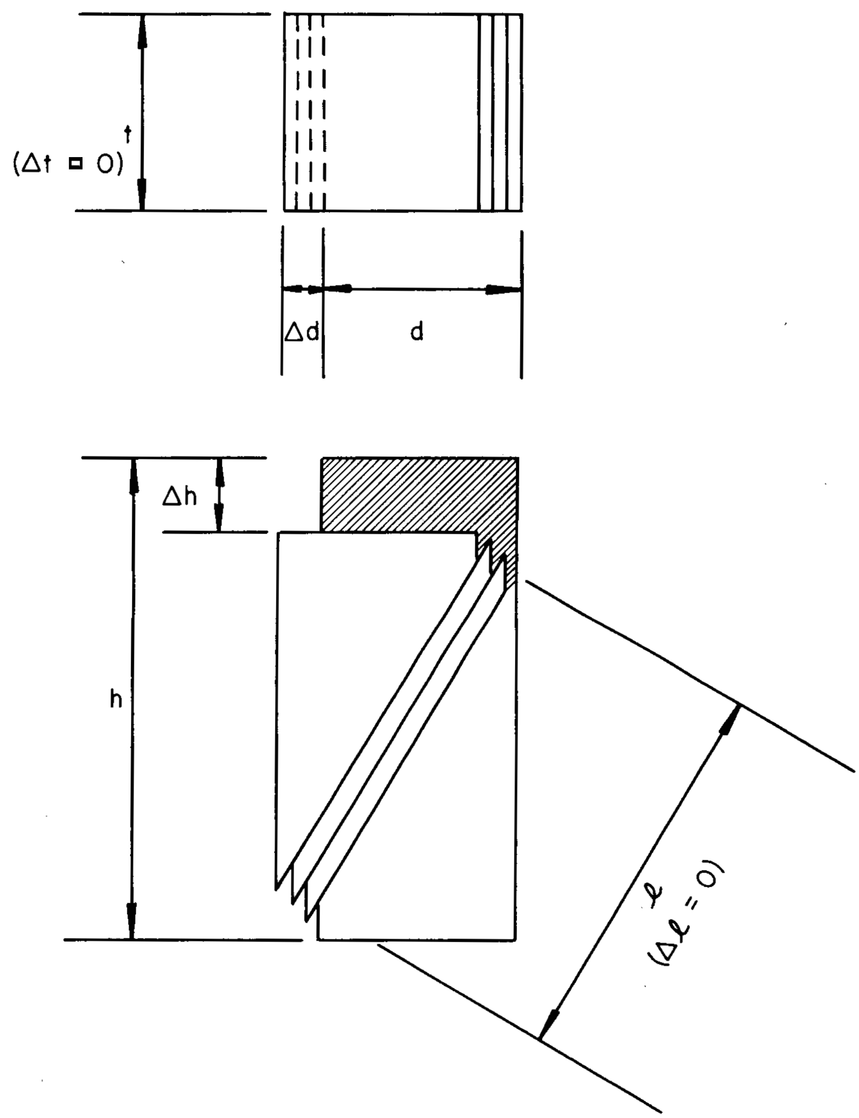

Fig. 15. Displacement Model for the Mohr-Coulomb Yield Criterion.

Substituting Eq. (235) into Eq. (234) and solving for ϵ_1^P gives

$$\epsilon_1^P = -\epsilon_3^P \frac{1 + \sin\phi}{1 - \sin\phi} . \tag{236}$$

By substituting Eq. (203) into Eq. (236), it may be seen that

$$\epsilon_1^P = -m\epsilon_3^P , \tag{237}$$

ORNL-DWG 67-3386

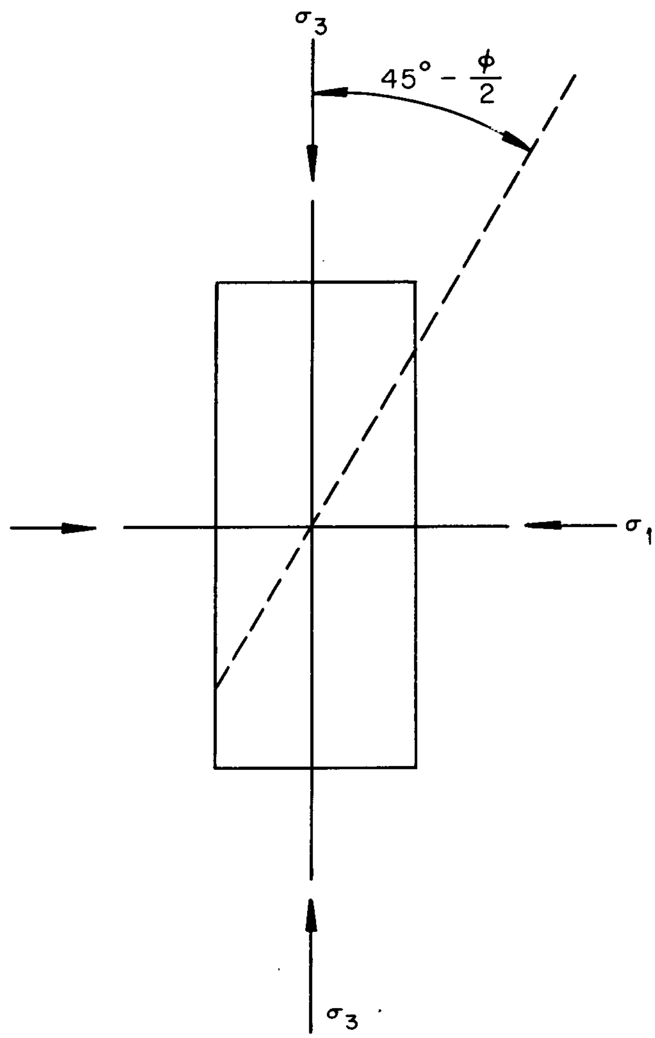

Fig. 16. Stress Diagram for the Mohr-Coulomb Yield Criterion.

which agrees with Eq. (213). From the foregoing argument, it appears that Hooke's law should apply in the direction of slip, as well as in the direction of the intermediate principal stress.

For a material that obeys the Mohr-Coulomb yield function, slip is assumed to occur in only one direction. Evidently, for such a material, the assumption that the plastic strain increment ratios are uniquely determined by the state of stress is really equivalent to the assumptions that all the slip planes have a known orientation and that the plastic

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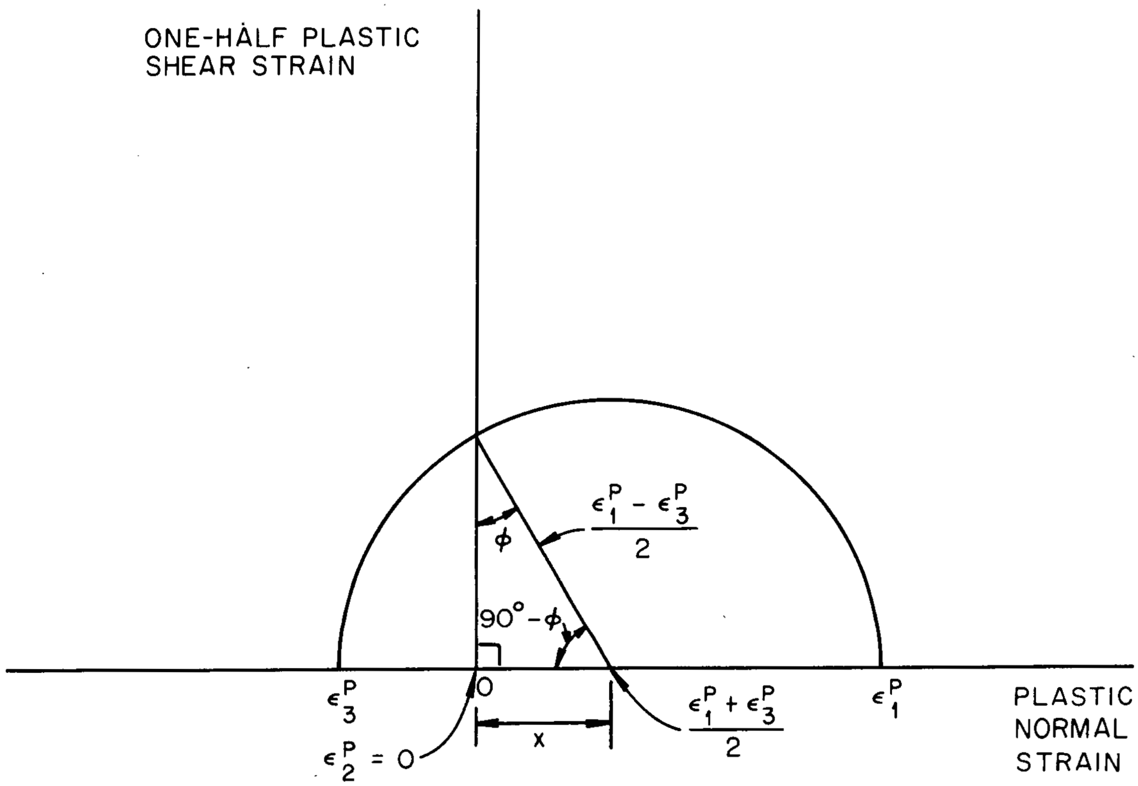

Fig. 17. Mohr Diagram of Plastic Strain for the Mohr-Coulomb Yield Criterion.

strain in the direction of slip is zero. For Mohr-Coulomb material, when $\sigma_1 > \sigma_2 > \sigma_3$, the ratios of the plastic strain increments are independent of the principal stress magnitudes and depend only on the directions of the major and minor principal axes.

From Figs. 13 and 17 it can be seen that a plastic strain vector, the components of which are the slip-plane plastic shear strain, $(\epsilon_1^P - \epsilon_3^P) \cos \phi$, and the plastic volume change, $\epsilon_1^P + \epsilon_3^P$, will be normal to the Mohr envelope in Fig. 13. This same relationship can be proven directly by using the flow rule and a slightly different expression for the yield function. Referring to Fig. 18, the yield condition can be described by the equation[20]

$$\tau_f = c - \sigma_N \tan \phi , \tag{238}$$

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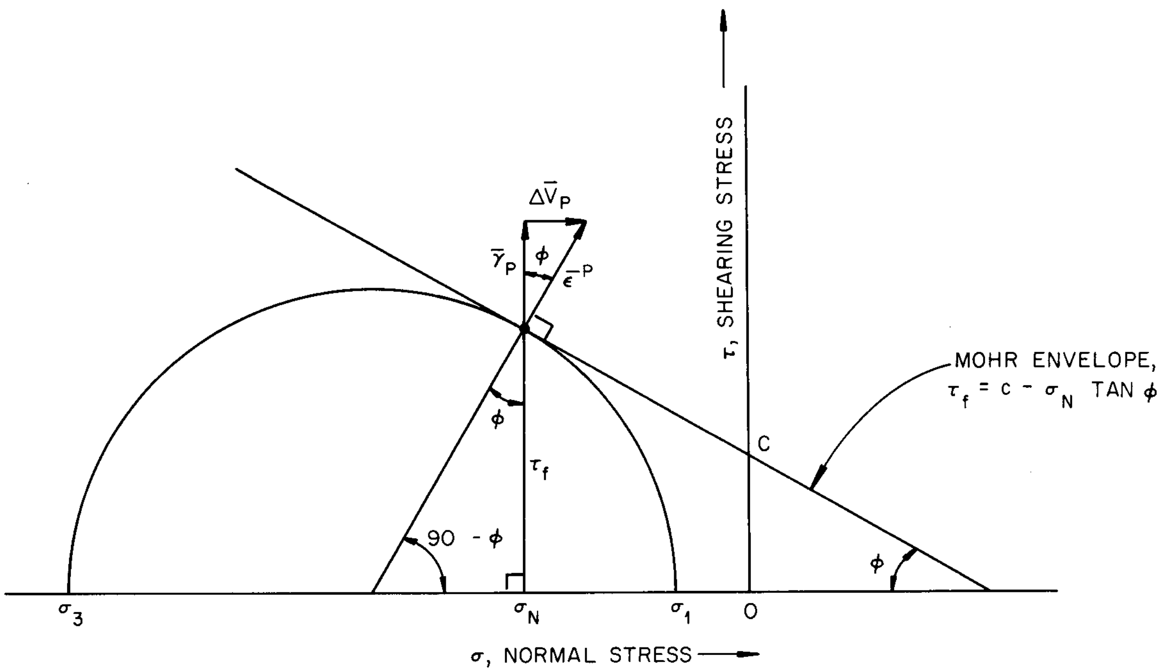

Fig. 18. The Mohr Envelope and Its Associated Plastic Strain Increment Vector.

where τ_f is the shear stress on the slip plane, c is cohesion, and σ_n is the normal stress on the slip plane. Since c is a parameter of the Mohr's circle envelope, the yield condition can be expressed by the equation,

$$f' = \tau_f + \sigma_N \tan \phi \ . \tag{239}$$

By referring to Eqs. (206), (207), and (210), it can be seen that

$$f' = \frac{f}{2m^{1/2}} \ . \tag{240}$$

According to Eq. (38) the incremental plastic volume change is given by

$$dV_p = \lambda' \frac{\partial f'}{\partial \sigma_m} \ . \tag{241}$$

The term λ' is defined by the condition

$$dW_p = f\lambda = f'\lambda' \ . \tag{242}$$

Thus

$$\lambda' = 2m^{1/2}\lambda \ . \tag{243}$$

By the chain rule,

$$dV_p = \lambda' \frac{\partial f'}{\partial \sigma_N} \frac{\partial \sigma_N}{\partial \sigma_m} \ . \tag{244}$$

If a hydrostatic stress is superimposed on the state of stress acting on the slip plane, σ_N changes by the amount of the added hydrostatic stress, but τ_f is unaffected. Therefore

$$\frac{\partial \sigma_N}{\partial \sigma_m} = 1 \ . \tag{245}$$

Substituting Eq. (245) into Eq. (244) and using Eq. (239),

$$dV_p = \lambda' \tan \phi \ . \tag{246}$$

The slip-plane plastic shear strain increment will be given by

$$d\gamma_p = \lambda' \frac{\partial f'}{\partial \tau_f} = \lambda' \ . \tag{247}$$

From Eqs. (246) and (247), it follows that

$$\frac{dV_p}{d\gamma_p} = \frac{\Delta V_p}{\gamma_p} = \tan \phi \ . \tag{248}$$

Therefore the vector whose components are γ_p and ΔV_p is normal to the Mohr's circle envelope in Fig. 18. It is also evident from Eqs. (244) and (245) and from Fig. 17 that the plastic strain normal to the slip plane is equal to the plastic volume change. (Besides considering the flow rule, it has to be, since the plastic strains in the other two mutually perpendicular directions are zero.)

The Von Mises Yield Function

The above reasoning can also be applied to a material that yields simultaneously on three planes, with each plane being inclined to a different pair of principal axes and having zero slope in the direction of the third.[23] Since the slippage along each plane causes plastic strain in the direction of only two principal axes, the plastic strain along each axis is the sum of only two components: one for each of the two slip planes inclined to that axis. For instance, considering the plastic strain in the direction of σ_1, assuming m = 1, and assuming that the individual slip displacements are proportional to the corresponding maximum shear stresses, gives[23]

$$\epsilon_{13}^{P} = \frac{k(\sigma_1 - \sigma_3)}{2} \tag{249}$$

and

$$\epsilon_{12}^{P} = \frac{k(\sigma_1 - \sigma_2)}{2} . \tag{250}$$

Adding Eqs. (249) and (250) gives

$$\epsilon_{1}^{P} = \frac{k}{2} (2\sigma_1 - \sigma_2 - \sigma_3) . \tag{251}$$

If the value of k happens to be given by

$$k = \frac{\epsilon_{eff}^{P}}{f} , \tag{252}$$

it follows that

$$\epsilon_{1}^{P} = \frac{\epsilon_{eff}^{P}}{2f} (2\sigma_1 - \sigma_2 - \sigma_3) . \tag{253}$$

Equation (253) is identical to the expression for an isotropic material obeying the Von Mises yield function and subjected to radial loading. An incremental expression can be derived by assuming that

$$k = \frac{d\epsilon^P_{eff}}{f} \; ; \tag{254}$$

whereupon

$$d\epsilon^P_1 = \frac{d\epsilon^P_{eff}}{2f} (2\sigma_1 - \sigma_2 - \sigma_3) \; . \tag{255}$$

By rotating the indexes in Eq. (253), expressions for ϵ^P_2 and ϵ^P_3 can be obtained, and by subtraction the deformation theory flow rule, as given by Eq. (178), can be derived. Then by clearing fractions in the three expressions contained in Eq. (178), squaring both sides of each, and adding,

$$(\sqrt{2}\ f)^2 \left[(\epsilon^P_1 - \epsilon^P_2)^2 + (\epsilon^P_2 - \epsilon^P_3)^2 + (\epsilon^P_3 - \epsilon^P_1)^2\right] = \left(\frac{3}{\sqrt{2}}\, \epsilon^P_{eff}\right)^2 \left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right] \; . \tag{256}$$

Therefore, if

$$f = \frac{1}{\sqrt{2}} \left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2} \; , \tag{257}$$

it follows that

$$\epsilon^P_{eff} = \frac{\sqrt{2}}{3} \left[(\epsilon^P_1 - \epsilon^P_2)^2 + (\epsilon^P_2 - \epsilon^P_3)^2 + (\epsilon^P_3 - \epsilon^P_1)^2\right]^{1/2} \; . \tag{258}$$

It is interesting to note that the geometrical derivation produces not only the Von Mises flow rule but also the definitions of the yield function and the effective plastic strain. Since m was taken equal to unity and the individual plastic strains were linearly combined, there is no plastic volume change.

CROSS SECTIONS OF HILL'S YIELD SURFACE FOR TRANSVERSELY ISOTROPIC MATERIAL AND THE MOHR-COULOMB YIELD SURFACE

A useful device for displaying information about a particular yield function is the cross section of the associated yield surface in an octahedral plane. Therefore, it is appropriate to plot typical octahedral cross sections for the two general yield functions discussed in this report.

Hill's Yield Function for Transversely Isotropic Material

The case of transverse isotropy is of particular interest with regard to graphite. The equation for the cross section of this yield surface is given by Eq. (197). Dividing both sides of Eq. (197) by f^2 gives

$$\left(\frac{\sigma_u}{\sqrt{2/3}\ f}\right)^2 + \frac{4\alpha_{23} + 1}{3}\left(\frac{\sigma_v}{\sqrt{2/3}\ f}\right)^2 = 1 \ . \tag{259}$$

Let

$$\frac{\sigma_u}{\sqrt{2/3}\ f} = S_u \tag{260}$$

and

$$\frac{\sigma_v}{\sqrt{2/3}\ f} = S_v \ . \tag{261}$$

Then

$$S_u^2 + \frac{4\alpha_{23} + 1}{3} S_v^2 = 1 \ . \tag{262}$$

The stress-plastic strain curves for EGCR-type AGOT graphite have been measured by Greenstreet et al.[24] These curves can be expressed by the equations

$$\epsilon_1^P = \left(\frac{\sigma_1}{A_1}\right)^\eta \tag{263a}$$

and

$$\epsilon_2^P = \left(\frac{\sigma_2}{A_2}\right)^{\eta} . \tag{263b}$$

Since Eq. (163) can be written in the form

$$\epsilon_{eff}^P = \left(\frac{f}{C}\right)^{1/n} , \tag{264}$$

it can be seen that

$$n = 1/\eta , \tag{265a}$$

$$C_1 = A_1 , \tag{265b}$$

$$C_2 = A_2 . \tag{265c}$$

Substituting Eqs. (265) into Eq. (160) gives

$$a = \left(\frac{A_1}{A_2}\right)^{\eta/(\eta+1)} . \tag{266}$$

Consequently, using Eq. (152b),

$$\alpha_{23} = \left(\frac{A_1}{A_2}\right)^{2\eta/(\eta+1)} - \frac{1}{2} . \tag{267}$$

For EGCR-type AGOT graphite, the following average tensile values for the constants η, A_1, and A_2 can be obtained from Table 13 of Ref. 24:

$$\eta = \frac{\eta_1 + \eta_2}{2} = \frac{2.34 + 2.17}{2} = 2.25 ,$$

$$A_1 = 45{,}800 ,$$

$$A_2 = 27{,}400 .$$

Inserting these numerical values into Eq. (244) gives

$$\alpha_{23} = (1.67)^{1.39} - 0.50 = 1.54 .$$

Therefore Eq. (262) can be rearranged to read

$$S_u = \left[1 - (2.39)\ S_v^2\right]^{1/2} \ . \tag{268}$$

Based on Eq. (268), the concurrent values of S_u and S_v for EGCR-type AGOT graphite with transverse isotropy are given below:

S_v	S_u
0	1.00
0.2	0.95
0.3	0.89
0.4	0.78
0.5	0.64
0.55	0.53
0.60	0.37
0.647	0

Plotting these values of S_u and S_v gives the octahedral cross section of Hill's yield surface for EGCR-type AGOT graphite, as shown in Fig. 19.

The Mohr-Coulomb Yield Function

Because of material isotropy, the octahedral cross section of the Mohr-Coulomb yield surface is symmetrical about each of the projected principal stress axes. Since the yield surface cross section is also piecewise linear, it is completely defined by the relative distances between the origin and any two opposite corners, all of which lie on the projected principal stress axes. As shown in Fig. 20, if the state of stress at the first corner is taken to represent uniaxial tension, then the state of stress at the opposite corner will represent a cylindrical state of stress having the same mean stress as the first corner. Since the mean stresses for both cases are equal, it follows that in case 2,

$$\sigma_x + \sigma_y + \sigma_z = \sigma_t \ . \tag{269}$$

In this discussion, letter subscripts are used instead of number subscripts to prevent the ambiguity that would otherwise occur when there was a change in the relative magnitude of the principal stresses. Since in case 2,

$$\sigma_z = \sigma_y \ , \tag{270}$$

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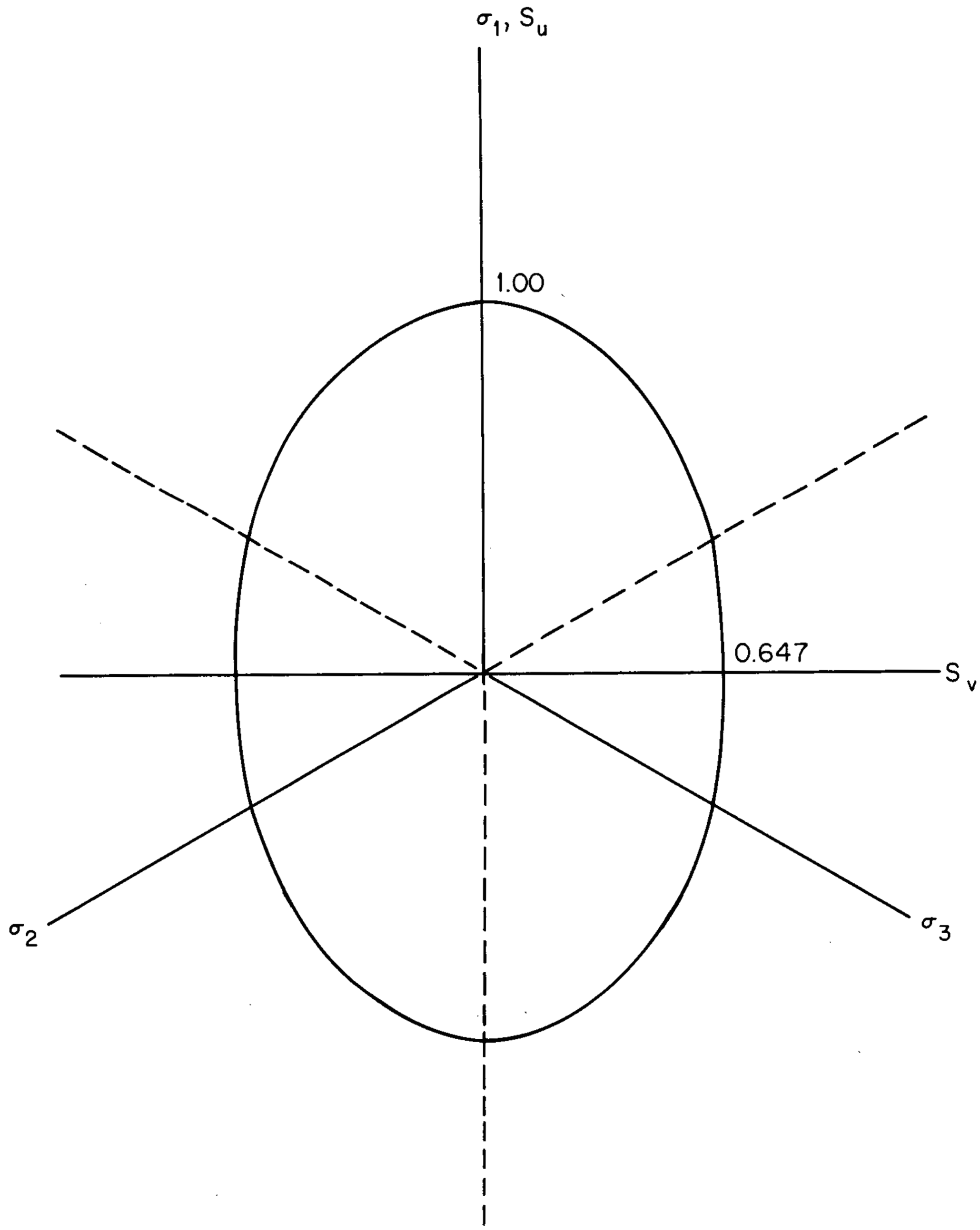

Fig. 19. Octahedral Cross Section of Hill's Yield Surface for EGCR-Type AGOT Graphite with Transverse Isotropy.

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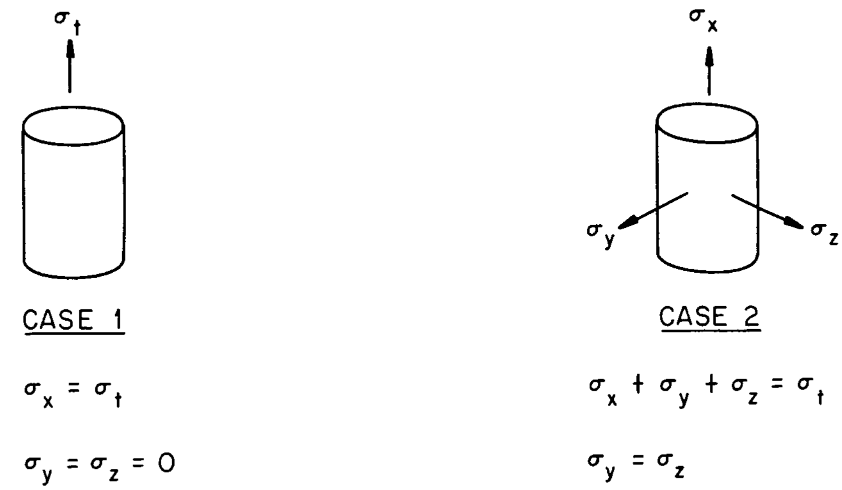

Fig. 20. Stress States Used for Defining the Cross Section of the Mohr-Coulomb Yield Surface.

combining Eqs. (270) and (269) gives

$$\sigma_x + 2\sigma_y = \sigma_t \; . \tag{271}$$

The value of the yield function is the same in both cases, so

$$f = m\sigma_t = m\sigma_y - \sigma_x \; . \tag{272}$$

Adding Eqs. (271) and (272) gives

$$(2 + m)\sigma_y = (1 + m)\sigma_t \; . \tag{273}$$

Therefore

$$\sigma_y = \frac{1 + m}{2 + m}\,\sigma_t \; . \tag{274}$$

Substituting Eq. (274) into Eq. (271) and rearranging then gives

$$\sigma_x = \frac{-m}{2 + m}\,\sigma_t \; . \tag{275}$$

For case 2, substituting Eq. (270) into Eq. (101a) and substituting the subscripts x, y, and z for 1, 2, and 3 gives

$$\sigma_{u2} = \frac{2(\sigma_x - \sigma_y)}{\sqrt{6}} \ . \tag{276}$$

For case 1,

$$\sigma_{u1} = \frac{2\sigma_t}{\sqrt{6}} \ . \tag{277}$$

If

$$\beta = -\frac{\sigma_{u2}}{\sigma_{u1}} \ , \tag{278}$$

it follows that substituting Eqs. (276) and (277) into Eq. (278) gives

$$\beta = \frac{\sigma_y - \sigma_x}{\sigma_t} \ . \tag{279}$$

Then, by substituting Eqs. (274) and (275) into Eq. (279), it is found that

$$\beta = \frac{1 + 2m}{2 + m} \ . \tag{280}$$

Substituting Eq. (203) into Eq. (280) then gives

$$\beta = \frac{3 + \sin\phi}{3 - \sin\phi} \ . \tag{281}$$

For many soils, the angle of internal friction is about 30°. For $\phi = 30°$, $\sin\phi = 1/2$, it is found that

$$\beta = 7/5 \ . \tag{282}$$

The cross section of the Mohr-Coulomb yield surface for $\phi = 30°$ is shown in Fig. 21. Since the plotting scale for this figure is a matter of

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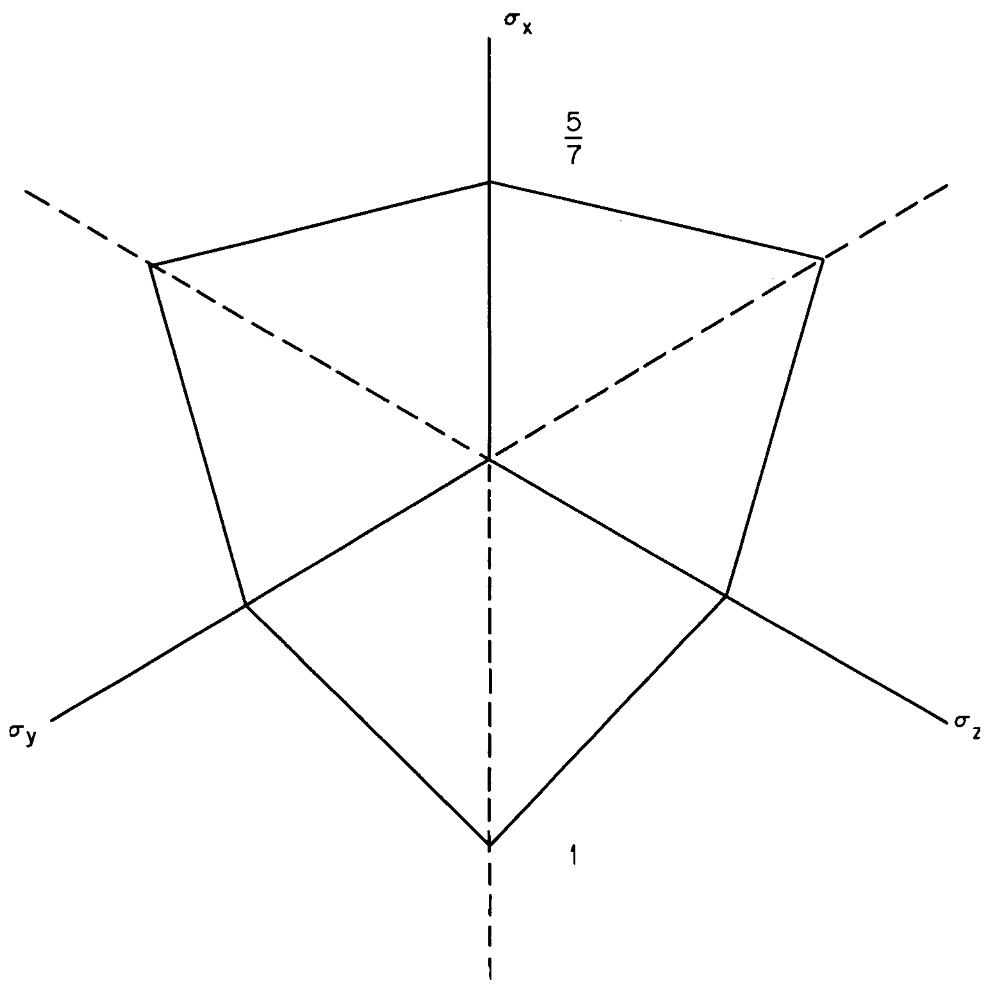

Fig. 21. Octahedral Cross Section of the Mohr-Coulomb Yield Surface for $\phi = 30°$.

convenience, σ_{u2} was taken equal to unity, and it was found that

$$\sigma_{u1} = 1/\beta = 5/7 \ .$$

EXAMPLE PROBLEMS

In most practical problems in multiaxial plasticity the stresses and the plastic strains cannot be determined independently of each other. However, in a few situations the stresses can be determined independently of the plastic strains. The following two examples are of this type.

Because of their relative simplicity, these examples provide good initial illustrations of the general procedure for solving problems in multiaxial plasticity.

Problem No. 1 — Hill's Yield Function

A 1-in.-diam cylindrical nuclear reactor fuel element is enclosed within a 0.015-in.-thick zirconium alloy cladding. For a safety analysis the plastic strains in the cladding are to be determined under a net internal pressure of 900 psi caused by fission-gas buildup. The operating temperature is 700°F.

The uniaxial stress-plastic strain curves of the cladding material at 700°F have been obtained and found to have the following equations:

$$\sigma_1 = 51{,}300\ (\epsilon_1^P)^{0.17}\ \text{psi}\ ,$$

in the circumferential direction, and

$$\sigma_2 = 33{,}800\ (\epsilon_2^P)^{0.17}\ \text{psi}\ ,$$

in the axial and radial directions.

From statics, the computed stresses in the cladding are as follows:

$$\sigma_1 = 30{,}000\ \text{psi}\ ,$$

circumferentially, and

$$\sigma_2 = 15{,}000\ \text{psi}\ ,$$

axially.

If the stress-plastic strain curve in the circumferential direction is chosen as the effective stress-strain relation, according to Eqs. (151) and (152),

$$\alpha_{12} = 0.50\ ,$$

$$\alpha_{31} = 0.50\ ,$$

$$\alpha_{23} = a^2 - 0.500\ .$$

Since power-law strain hardening has been assumed, the term "a" is determined by Eq. (160):

$$a = \left(\frac{C_1}{C_2}\right)^{1/(n+1)} = \left(\frac{51,300}{33,800}\right)^{1/1.17} = 1.43 \ .$$

Consequently

$$\alpha_{23} = (1.43)^2 - 0.500 = 1.54 \ .$$

From Eq. (103), with $\sigma_3 = 0$,

$$f = \left[0.50\ (\sigma_1 - \sigma_2)^2 + 1.54\ \sigma_2^2 + 0.50\ (-\sigma_1)^2\right]^{1/2} \ .$$

For the known values of the principal stresses, the yield function is computed to be

$$f = 30,300 \text{ psi} \ .$$

Since the stress ratios remain constant during loading, deformation theory is valid and the total plastic strains are given by Eq. (123):

$$\epsilon_1^P = \frac{1}{E_p}\left[\sigma_1(1.00) - \sigma_2(0.50)\right] \ ,$$

$$\epsilon_2^P = \frac{1}{E_p}\left[-\sigma_1(0.50) + \sigma_2(2.04)\right] \ ,$$

$$\epsilon_3^P = \frac{1}{E_p}\left[-\sigma_1(0.50) - \sigma_2(1.54)\right] \ .$$

For power-law strain hardening, the plastic secant modulus, E_p, is given by Eq. (125):

$$E_p = \left(\frac{C}{f}\right)^{1/n} f = \left(\frac{51,300}{30,300}\right)^{1/0.17} (30,300) = 667,000 \text{ psi} \ .$$

The plastic strains are therefore computed to be

$$\epsilon_1^P = 0.0337 \ ,$$

$$\epsilon_2^P = 0.0234 ,$$

$$\epsilon_3^P = -0.0571 .$$

The sum of the plastic strains is zero as expected.

Problem No. 2 — The Mohr-Coulomb Yield Function

A prestressed concrete pressure vessel is lined internally with a 1-in.-thick steel membrane liner anchored at close spacing to the concrete. The inside diameter of the vessel is 60 ft. The concrete is to have an ultimate compressive strength, f'_c, of 6000 psi. The applied prestress in the concrete at the inside surface of the vessel is to be 2700 psi circumferentially and 1350 psi axially. It is desired to determine whether the application of this much prestress to the concrete will cause general yielding in the steel liner. Since the liner is anchored to the concrete, its circumferential and axial strains will match those of the concrete. Therefore, the problem is to compute the total strains in the concrete, at the inside surface of the vessel, due to the application of the prestressing only.

The properties of the concrete can be taken as follows:

$$E = 1000\ f'_c = 6{,}000{,}000 \text{ psi} ,$$

$$\nu = 0.12 ,$$

$$\epsilon_{ult} = 0.003$$

(ultimate strain in uniaxial compression),

$$\phi = 30° .$$

The stress-strain curve of concrete in uniaxial compression can be assumed to have the equation

$$\epsilon = \frac{\sigma}{E} + \left(\frac{\sigma}{B}\right)^2 ,$$

when $\sigma = f'_c$, $\epsilon = 0.003$, and therefore, $B = 1.34 \times 10^5$ psi.

If the circumferential stress in the liner is assumed to be the yield stress, 30,000 psi, by statics the contact stress between the liner and the concrete is −83 psi (compression).

Since all three principal stresses in the concrete are unequal, Eqs. (217) apply, and

$$\epsilon_1^P = m\epsilon_{eff}^P ,$$

$$\epsilon_2^P = 0 ,$$

$$\epsilon_3^P = -\epsilon_{eff}^P .$$

From Eq. (203)

$$m = \frac{1 + \sin\phi}{1 - \sin\phi} = \frac{1 + 1/2}{1 - 1/2} = 3 .$$

From the ordering of the principal stresses

$$\sigma_1 = \sigma_r = -83 \text{ psi} ,$$

$$\sigma_2 = \sigma_z = -1350 \text{ psi} ,$$

$$\sigma_3 = \sigma_\theta = -2700 \text{ psi} .$$

From Eq. (210),

$$f = m\sigma_1 - \sigma_3 = (3)(-83) - (-2700) = 2451 \text{ psi} .$$

As shown by Eqs. (225) and (226), the uniaxial compressive stress-plastic strain curve is the effective stress-strain curve with both signs changed. Therefore

$$\epsilon_{eff}^P = \left(\frac{f}{B}\right)^2 ,$$

and consequently

$$\epsilon_{eff}^P = \left(\frac{2.451 \times 10^3}{1.34 \times 10^5}\right)^3 = 0.000334 .$$

Therefore the plastic strains in the concrete are

$$\epsilon_1^P = \epsilon_r^P = (3)(0.000334) = +0.00100 ,$$

$$\epsilon_2^P = \epsilon_z^P = 0 ,$$

$$\epsilon_3^P = \epsilon_\theta^P = -0.00033 .$$

Using Hooke's law, the elastic strains in the concrete are

$$\epsilon_r^E = +0.00007 ,$$

$$\epsilon_z^E = -0.00017 ,$$

$$\epsilon_\theta^E = -0.00042 .$$

The total strains are obtained by adding the elastic and the plastic strains according to Eq. (1),

$$\epsilon_r = +0.00107 ,$$

$$\epsilon_z = -0.00017 ,$$

$$\epsilon_\theta = -0.00075 .$$

Using Hooke's law again, the calculated circumferential stress in the steel liner for $E = 3 \times 10^7$ psi and $\nu = 0.30$ is

$$\sigma_\theta = \frac{E(\epsilon_\theta + \nu\epsilon_z)}{1 - \nu^2} = 26,400 \text{ psi} .$$

This corresponds to a pressure between the steel liner and the concrete of −73 psi (compression). Correcting the radial stress results in only a 1% change in the value of the yield function, so the total strains can be considered substantially correct. Thus the steel liner does not yield under the application of prestress alone.

LIMITATIONS AND EXTENSIONS OF THE THEORY

The majority of the work in plasticity completed to date that has a direct engineering application consists of analytical, experimental, or numerical solutions to particular types of problems. In most cases, the behavior of metals was involved, and the condition of zero plastic volume change was known or assumed to hold. However, as has been explained, the general flow rule in plasticity is not restricted to yield functions that dictate a zero plastic volume change. In fact, some research has already been done in the field of soil mechanics that utilizes the Mohr-Coulomb function, which does lead to a nonzero plastic volume change. However, this yield function does not always give satisfactory results when applied to real materials.

The assumption of fixed principal axes made in this paper is not actually a restriction on the theory but, rather, a simplifying assumption made for this paper. One restriction that does apply to the theory as discussed here is that the effects of unloading and reloading are not considered.[4] Since unloading and reloading may proceed according to a different effective stress-strain curve, the theory must be extended to cover these cases.

In order to handle the many unsolved problems of analyzing the inelastic strains in nonmetallic materials such as soil, concrete, and graphite, some new yield functions will have to be derived. These new yield functions may indicate the occurrence of plastic volume increases under tensile loading but plastic volume decreases under compressive loading. A possible means of determining such a yield function would be to determine the intersection curves of its surface with a set of planes in stress space. These intersection curves could then be fit by a surface mathematically. The individual intersection curves might be determined by a combination of triaxial and thin-walled tube tests.

Another very interesting extension of the theory of plasticity is into the realm of nonlinear creep and relaxation. Considerable work of this nature has already been done.[5,25-29] The theories of nonlinear creep and plasticity are very similar.[5,25,26] Again, in the case of nonlinear creep, a reexamination of the definition of the effective inelastic (creep) strain might prove useful.

SUMMARY

The basic equations of multiaxial plasticity have been derived with the use of only calculus and vector algebra. By clearly stating all assumptions and introducing the basic equations in the proper sequence, it has been demonstrated that all the familiar equations of multiaxial plasticity can be derived without making arbitrary assumptions, such as the algebraic form of the effective plastic strain increment.

By using the plastic work equation,[3]

$$dW_p = \sigma_{eff} \, d\epsilon^P_{eff} , \tag{10}$$

the flow rule has been derived without reference to the plastic work performed by the stress increments. Therefore, it appears that the classical concept of conservation of energy is sufficient, when combined with other accepted principles of mechanics, to establish a theory of plasticity.

The basic assumptions regarding plastic behavior used in this paper are listed below, in the order of their occurrence:

1. Plastic strains are time-independent nonlinear functions of the stresses and are permanent.

2. Elastic strains are time independent linear functions of the stresses and are recoverable.

3. Total strain equals elastic strain plus plastic strain combined linearly.

4. Elastic strains can be computed by Hooke's law.

5. In general, no unique relationship exists between the stresses and the total plastic strains.

6. A unique relationship exists between the stresses and the plastic strain increments.

7. The principal axes of stress and strain coincide and remain fixed. (This condition is not a general characteristic of plastic behavior. However, it is used in this paper to maintain algebraic simplicity.)

8. The initiation and progression of yielding are controlled by a yield function that is a function only of the principal stresses.

9. The yield function has the dimensions of a stress.

10. When yielding occurs, the increment in the yield function is either zero or positive.

11. A single effective stress-strain relation holds for all states of stress in the plastic range.

12. The effective stress is a function only of the principal stresses.

13. The effective stress has the dimensions of a stress.

14. The effective plastic strain is a function only of the principal plastic strains.

15. The effective plastic strain has the dimensions of a plastic strain.

16. The algebraic definition of the effective stress is always the same.

17. The numerical value of the effective plastic strain is always the same for a given state of stress, but its algebraic definition, in terms of the principal plastic strains, may depend on prior loading history. However, the algebraic definition of the effective plastic strain increment in terms of the principal plastic strain increments is always the same.

18. The ratios of the principal plastic strain increments are uniquely determined by the state of stress, independent of the ratios of the stress increments. (This assumption was shown to hold only at states of stress at which the yield function is continuous. At corners, the assumption does not hold because slip can occur independently in more than one direction.)

19. For every state of stress, there is some incremental loading path along which the plastic strain increment ratios remain constant.

20. The plastic work increment is the product of the effective stress and the effective plastic strain increment.

21. The effective plastic strain increment is a function only of the principal plastic strain increments.

22. The effective plastic strain increment has the dimensions of a plastic strain increment.

23. The plastic work performed by a set of stress increments is zero or positive. (This second-order differential plastic work quantity

was used in only one derivation of the flow rule. The assumption of its positive value was shown to be unnecessary by the proof that $dW_p = f\lambda$.)

24. The principal axes of stress and strain coincide with the axes of anisotropy, if any. (This too is a simplifying assumption, not a general condition. It is possible to consider the shear terms in basically the same manner as the normal stress terms.)

25. Slip causes no plastic strain in the direction of slip.

The fact that a total of 25 assumptions was used to construct a theory of plasticity helps to explain, at least in part, why assembling the various parts of the theory in the right order is difficult. However, without first putting the theory in proper order, it is difficult, if not impossible, to fully understand or utilize the techniques of plastic analysis. But without plastic analysis, there is no way to obtain a better understanding of some very important problems involving structural behavior and safety. For this reason, it is hoped that this report will contribute to a better understanding among practicing engineers of the basic principles of plasticity.

ACKNOWLEDGMENTS

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