Nonsmooth controllability theory and an example

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Abstract

This paper extends recent results in local controllability analysis for Multiple Model Driftless Affine (MMDA) control systems. Such controllability results can be interpreted as non-smooth extensions of Chow's theorem, and use a set-valued Lie Bracket. In particular, we formulate controllability in terms of generalized differential quotients (GDQs). Additionally, we present an extensive example in order to illustrate how these results can provide insight into the control of some specific physical systems. Moreover, this paper indicates that a multiple model system consisting of individually controllable models is not necessarily controllable.

1 Introduction

This paper considers the issue of controllability for systems of the following form:

Definition 1.1. *A control system Σ is said to be a* multiple model driftless affine system (MMDA) *if it can be expressed in the form*

$$\Sigma: \quad \dot{q} = f_1(q)u_1 + f_2(q)u_2 + \cdots + f_n(q)u_n \quad (1.1)$$

where for any q and t, $f_i \in \{g_{\alpha_i} | \alpha_i \in I_i\}$, with I_i an index set, f_i measurable in (q,t), g_{α_i} analytic in (q,t) for all α_i, and the controls $u_i \in \mathbb{R}$ are piecewise constant and bounded for all i.

Definition 1.1 implies the control vector fields may change, or switch, amongst a finite collection of vector fields, each representing a model, P, in a set of models $\mathcal{P}$. Such systems are intimately related to multiple model systems such as studied in [6, 13, 14]. However, we should emphasize that the "switching" is *not* like the switching phenomena found in [2, 10, 4, 20], or as typically studied in the hybrid control systems literature (e.g., [15, 1]). In these studies, the switching is part of a control strategy to be implemented in the controller. Rather, it is switching induced by environmental factors, such as variations in the contact state between rigid bodies. Systems of this sort are actually quite common in engineering practice, and Section 3 develops an example in detail. As a first step in understanding such systems,

we would like a local controllability test that works in the presence of a priori unknown switching behavior. This, as in the smooth case, is a first step towards eventually having generic tools for stabilization. This paper presents an application of a "discontinuous" version of the Chow's theorem that extends the classical result in a natural way to the systems of Def 1.1.

This paper extends earlier work by the authors done in [13, 14]. Prior to this work, there was little controllability analysis of such systems. As far as the authors are aware, two groups have studied controllability of nonsmooth systems like these. In [13] we used a set-valued lie bracket to study the local controllability of MMDA systems that experience arbitrarily fast switching in neighborhood of the operating point. In [14] we used a set-valued lie bracket to consider local controllability of MMDA systems where the dynamics are determined by crossing a set of C^1 submanifolds of the configuration manifold. The current work extends the work done in [14] by taking away the need for a C^1 stratified system of manifolds forming the boundary of switching, and moreover uses a proof technique which has more potential in the future for developing stabilization techniques. While they did not study multi-model systems, [16] recently developed a nonsmooth version of Chow's theorem, using the same set-valued lie bracket, that applies to Lipschitz vector fields. The proofs found in this paper are adaptations of their proof methodology. These results are not appropriate for systems with switching, but nevertheless are important for some physical situations. An example of such a system is a carangiform fish swimming in a fluid with drag on the body (see [11]). The results obtained by [16] have strong analogues with our result, to the point that all three papers have essentially the same conditions for controllability, indicating that a set-valued lie bracket has wide potential application in the analysis of traditionally intractable systems. Moreover, the results presented in this paper represent the merging of their theory with our previous work.

In Section 2 we summarize these controllability results. Then, as an example of a physical system where these concepts are important, Section 3 analyzes a simple model of an overconstrained wheeled vehicle, which is inspired by novel high-mobility wheeled robots (e.g,

the Mars Sojourner) that operate in rough terrain.

2 Controllability Results

We use several aspects of the formalism of [5] for investigating the properties of ODEs with discontinuous right hand sides. Eq. (1.1) can be viewed as a differential inclusion, i.e., a system of the form $\dot{q} \in \mathbf{f}$, where $\mathbf{f}$ is a set valued multi-function which is almost everywhere (a.e.) single valued. For equations of the form $\dot{q} = f(q)$ with f discontinuous in q at a point q_*, one must generally allow f to take on the convex hull of limiting values $\lim_{q \to q_*} f$ at q_* in order to guarantee existence of solutions (see [5, Chapter 2] for details). To account for this issue, we define the following at each q:

$$\mathbf{f}_i(q) = co\{\lim_{q \to q_*} f_i(q)\} = co\{g_{\alpha_i}(q_*)|\alpha_i \in I_i(q)\}. \quad (2.1)$$

where $I_i(q)$ is an index on the set of limiting values of $f_i(q)$ at q_*, and $co\{\cdot\}$ denote the convex hull of a set. For notational convenience, let $s_i(\mathbf{f}_i)$ denote a *selection* of $\mathbf{f}_i(q)$—i.e., a choice of a particular vector from $\mathbf{f}_i(q)$. Let $S_{i_1,i_2,\dots,i_k}$ denote the set of all possible selections from $\mathbf{f}_{i_1}, \dots, \mathbf{f}_{i_k}$.

Remark: We should comment more on the relationship between differential inclusions (DIs) and differential equations with discontinuous right hand sides (DEDRHSs). Consider the case $\dot{q} = f(q,t)$ with $f(q,t) = \{f_i(q,t)|i \in I\}$ for some choice of i. When f changes from f_i to f_j, we typically must set $f = co\{f_i, f_j\}$ at that point in order to guarantee existence of solutions. Moreover, $\dot{q} = f(q,t)$ implicitly depends on the choice of switching function determining i at any given point - call it $\sigma(q,t)$. An elementary result from [5] is that the set of solutions to $\dot{q} = f(q,t)$ over all possible choices of σ coincides with the set of solutions to $\dot{q} \in \mathbf{f}(q,t)$ where $\mathbf{f} = co\{f_i|i \in I\}$. This, of course, is the relationship between DIs and DEDRHSs - for a given DEDRHS, we can choose a DI such that their solutions coincide.

Our result uses the notion of a set valued Lie bracket. This concept has its origin in two distinct areas. Previously in [13] we used a set-valued Lie bracket to consider the local controllability of MMDAs in the special case where switching occurs very rapidly. The use of a set-valued bracket was a natural consequence of the underlying assumptions in [13]. [16] uses a set-valued Lie bracket to prove the controllability of a driftless affine control system whose single-valued governing equation includes Lipschitz control vector fields. They showed that this choice of Lie bracket is a generalized differential quotient (GDQ) of the product of exponentials formulation of a lie bracket. Although these two applications seem different, the choice of Lie bracket is the same, and the resulting non-smooth versions of Chow's theorem are analogous. Hence, we use the following Lie bracket definition, adapted here to our situation:

Definition 2.1. *Let f_1 and f_2 be as in Def 1.1. I.e., $f_i \in \{g_{\alpha_i}|\alpha_i \in I_i(q)\}$. The Lie bracket of f_1 and f_2 is defined as*

$$[f_1, f_2](q) = co\{\lim_{j \to \infty}(Df_1(q_j) \cdot f_2(q_j) - Df_2(q_j) \cdot f_1(q_j))\}$$

for all sequences $\{q_j\}_{j \in \mathbb{N}}$ such that

1. *f_1 and f_2 are differentiable $\forall\, q_j$,*
2. *$\lim_{j \to \infty} q_j = q$,*
3. *the limit exists.*

Note that this lie bracket is a set valued object, which can be shown to be both compact and convex. It is moreover always well defined for MMDA systems which are almost everywhere analytic. In the case where $\mathbf{f} = co\{f_i|i \in I\}$ and $\mathbf{g} = co\{g_j|j \in J\}$ (with I and J index sets), it is straightforward to show that $[\mathbf{f}, \mathbf{g}] = co\{[f_i, g_j]|i \in I, j \in J\}$. Definition 2.1 is equivalent to a set-valued bracket defined previously in [12], where it was used to show local controllability for MMDA systems undergoing rapid switching. To analyze the controllability of MMDAs, we define:

Definition 2.2. *Let f_i be as in Def 1.1 and $\mathbf{f}_i$ as in Eq. (2.1). Define a distribution $\Delta_{s_1 s_2 \cdots s_n}(q)$ as*

$$\Delta_{s_1 s_2 \cdots s_n}(q) = span\{v|\ v = s_i(\mathbf{f}_i(q)),\ i = 1 \dots n\} \quad (2.2)$$

That is, $\Delta_{s_1 s_2 \cdots s_n}(q)$ is formed from a particular selection of vectors from each $\mathbf{f}_i(q)$. Define the distribution $\Delta(q)$ as:

$$\Delta(q) = \bigcap_{S_1,\dots,n} \Delta_{s_1 s_2 \cdots s_n}(q) \quad (2.3)$$

That is, $\Delta(q)$ is formed by intersecting the $\Delta_{s_1 s_2 \cdots s_n}(q)$ over all possible selections of $\mathbf{f}_1(q), \dots, \mathbf{f}_n(q)$. Next define

$$\Delta^1(q) = \bigcap_{S_{12,13,\dots}} (span\{v|\ v = s_{ij}([\mathbf{f}_i, \mathbf{f}_j])\}) \quad (2.4)$$

(where s_{ij} and $S_{12,13,\dots}$ are defined similarly to before, with s_{ij} being selections of $[\mathbf{f}_i, \mathbf{f}_j]$ and $S_{12,13,\dots}$ being the set of all selections of $[\mathbf{f}_i, \mathbf{f}_j]$) and analogous higher order distributions formed from higher order set-valued lie brackets. Finally, define $\overline{\Delta}(q)$ as

$$\overline{\Delta}(q) = \Delta(q) \bigcup \Delta^1(q) \bigcup \cdots. \quad (2.5)$$

Now we cover some of the basic formalism found in [18] needed to prove our result. In particular, we need the notion of a *GDQ* (defined below) which can be thought of as a generalized differential. We will use an open mapping theorem for *GDQ*s from [17] to prove this extension of Chow's theorem. Let $\mathbb{N}$ denote the natural numbers. We denote a set-valued map by $\mathbf{h} = (S, T, Gr(\mathbf{h}))$, where S is the source set, T is the target set, and $Gr(\mathbf{h})$ is the set $\{(s,t)|\exists s \in S$ with $\mathbf{h}(s) = t\}$. Now, given a set-valued map $\mathbf{h}$, we say that a sequence $\{\mathbf{h}_j\}_{j \in \mathbb{N}}$ of set valued maps with compact graph inward

graph converges (denoted by $\mathbf{h}_j \xrightarrow{igr} \mathbf{h}$) to $\mathbf{h}$ (also a set valued map with compact graph) if for every open subset $\Omega \subseteq S \times T$ such that $Gr(\mathbf{h}) \subseteq \Omega$ there exists a $j_\Omega \in \mathbb{N}$ such that $Gr(\mathbf{h}_j) \subseteq \Omega$ whenever $j \geq j_\Omega$. Moreover, we define the *flow* of $\mathbf{h}, \Phi^\mathbf{h}$, to be the set-valued map from $\Omega \times \mathbb{R}$ to Ω whose value, for a given $(x_0, t) \in \Omega \times \mathbb{R}$, is the set of all $y \in \Omega$ such that there exists an integral curve $x(t) : [0, T] \to \Omega$ of $\mathbf{h}$ satisfying $x \in \mathbf{h}$ a.e. which is defined on some subinterval $[0, T] \subseteq \mathbb{R}$ such that $t \in [0, T]$, $x(0) = x_0$, and $x(t) = y$.

Definition 2.3. *Let X and Y be metric spaces. A regular set-valued map from X to Y is a set valued map $\mathbf{f}$ such that for every compact subset $K \subset X$ the restriction $\mathbf{f}|K$ is a set valued map with compact graph, and is a limit, in the sense of inward graph convergence, of a sequence of continuous single-valued maps from K to Y.*

Definition 2.4. *Let $\mathbf{f} : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ be a set-valued map, and let Λ be a nonempty compact subset of $\mathbb{R}^{n \times m}$. Let S be a subset of $\mathbb{R}^m$. We define Λ to be the* generalized differential quotient (GDQ) *of $\mathbf{f}$ at $(x, \mathbf{f}(x))$ in the direction of S (denoted by $\Lambda \in GDQ(\mathbf{f}, x, \mathbf{f}(x), S)$), if for every positive real number δ there exist U and $\mathbf{g}$ such that:*

1. *U is a compact neighborhood of $x \in \mathbb{R}^m$ and $U \cap S$ is compact.*

2. *$\mathbf{g}$ is a regular set-valued map from $U \cap S$ to the δ-neighborhood Λ^δ of Λ in $\mathbb{R}^{n \times m}$.*

3. *$\mathbf{g}(x) \cdot x \subseteq \mathbf{f}(x)$ for every $x \in U \cap S$.*

We should note that GDQ theory is a generalized differentiation theory in the sense of [3]. Moreover, the GDQ theory has a strong directional open mapping property which we use to prove controllability.

Now we will state and give the briefest possible sketch of a proof for our initial result, that $\mathbf{f}$ (as a set valued object) is a GDQ of $\Phi^\mathbf{f}$.

Theorem 2.1. *Assume Ω is an open subset of $\mathbb{R}^n$, f is a MMDA vector field on Ω as in Definition 1.1, and $q_* \in \Omega$. Then the set $L^\mathbf{f}$, defined by*

$$L^\mathbf{f}(q_*)(v, r) = v + rf(q_*) \text{ for } f \in \mathbf{f}, v \in \mathbb{R}^n, r \in \mathbb{R} \quad (2.6)$$

is a GDQ of the set-valued map $\Phi^\mathbf{f}$ at $(q_*, 0)$ in the direction of $\mathbb{R}^n$.

Sketch of Proof: We need only satisfy the properties of Definition 2.4. We can choose a δ neighborhood of q_* to be U. We define the following:

$$\Lambda(q - q_*, t) = \{L^f(q_*)(q - q_*, t) + E^{n+1, n}_{(q - q_*, t), p - q - tf(q_*)}(q - q_*, t)\}$$

such that $p \in \Phi^f$, $f \in \mathbf{f}$ and where $E^{m,n}_{x,y}(u) = \frac{<x, u>}{\|x\|^2} y$ for $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, and $u \in \mathbb{R}^m$. We extend the

definition of Λ to $t = 0$ by defining $\Lambda(q, t) = \{L^f(q_*)\}$ (that is, the set of L^f with $f \in \mathbf{f}$ at q_*). The proof goes in three stages:

1. Show $\Lambda(q - q_*, t) \cdot (q - q_*) \subseteq \Phi^\mathbf{f}$ for a sufficiently small neighborhood of $q_* \in \mathbb{R}^n$.

2. Show Λ is a map from neighborhoods of q_* to Λ^δ.

3. Show Λ is regular.

1) Let $M \in \Lambda(q, t)$ and (q, t) be in a sufficiently small ϵ-ball of $(q_*, 0)$ (i.e. $(q, t) \in (q_* + B_{\epsilon_1}, B_{\epsilon_2})$ with B_ϵ denoting a ball of ϵ radius and ϵ_1 and ϵ_2 sufficiently small). Then $q_* + M(q - q_*, t) =$

$$q_* + L^f(q_*)(q - q_*, t) + E^{n+1, n}_{(q - q_*, t), p - x - tf(q_*)}(q - q_*, t)$$

$$= q_* + q - q_* + tf(q_*) + p - q - tf(q_*) = p$$

So $\Lambda(q_*, t) \cdot q \subseteq \Phi^\mathbf{f}$.

2) Now we show that $d(\Lambda, L^\mathbf{f}(q_*)) < \delta$, where $d(\cdot, \cdot)$ is the Haussdorff set distance function defined by:

$$d(A, B) = \max \left(\max_{b \in B} \min_{a \in A} d(a, b), \max_{a \in A} \min_{b \in B} d(a, b) \right).$$

To do so, we again let $M \in \Lambda(q, t)$. Then,

$$\sup \|M - L^\mathbf{f}(q_*)\| = \sup_{f \in \mathbf{f}} \frac{\|p - q - tf(q_*)\|}{(\|q - q_*\|^2 + t^2)^{\frac{1}{2}}}$$

This, in turn, we can show must be small for t and $\|q - q_*\|$ sufficiently small, showing that $d(\Lambda, L^\mathbf{f}(q_*)) < \delta$.
3) We know that MMDA systems are almost everywhere differentiable. This implies that we can construct regularizations of f at all points of discontinuity, where the regularization of the transition of f from f_i to f_j at q_* is defined by:

$$f^\xi_{ij} = \int_{\mathbb{R}^n} \phi(h) f(x + \xi) dh \quad (2.7)$$

where $\phi(h) \geq 0$, is C^∞, $\int_{\mathbb{R}^n} \phi(h) dh = 1$, $\phi(h) = 0$ for $\|h\| > 0$, and $\xi > 0$. This forms a finite set of regularizations. We then use these regularizations to form the smooth single valued maps which must converge to Λ as $\xi \to 0$. ∎

What we now show is that $[f, g]$ from Definition 2.1 is a GDQ of the map

$$\Xi^{\mathbf{f}, \mathbf{g}}(q, \varepsilon) = \begin{cases} \psi^{f, g}(q, \sqrt{\varepsilon}, \sqrt{\varepsilon}) & \text{if } \varepsilon \geq 0 \\ \psi^{f, g}(q, \sqrt{-\varepsilon}, \sqrt{-\varepsilon}) & \text{if } \varepsilon \leq 0 \end{cases} \quad (2.8)$$

for $f \in \mathbf{f}$ and $g \in \mathbf{g}$, and where
$\psi^{f, g}(x, t, s) = \Phi^{-sg} \circ \Phi^{-tf} \circ \Phi^{sg} \circ \Phi^{tf}(x)$
where $\circ$ is the composition operator.

Theorem 2.2. *Assume Ω is an open subset of $\mathbb{R}^n$, $\mathbf{f}$ and $\mathbf{g}$ are MMDA vector fields on Ω as in Definition 1.1, and $q_* \in \Omega$. Then the set $\{L^w(q_*)|w \in [\mathbf{f}, \mathbf{g}]\}$ is a GDQ of the map $\Xi^{\mathbf{f}, \mathbf{g}}$ at $(q_*, 0)$ in the direction of $\mathbb{R}^n$.*

Very Brief Sketch of Proof: The proof of this statement is significantly more involved than the proof of

Theorem 2.1, and we therefore only give it in the broadest details. As in the previous proof, the only case of interest to us is q_* at a point of discontinuity of σ. At such a point we know that $\dot{x} \in \mathbf{f}u_1 + \mathbf{g}u_2$. where $\mathbf{f} = co\{f_i\}$ and $\mathbf{g} = co\{g_j\}$. As in the previous proof we choose regularizations of f and g, f_{ij} and g_{kl}, which form, for every value of ξ, a finite set of C^∞ maps with which one can take classical lie brackets. We then show that this set of lie brackets inward graph converge to $[f, g]$ as defined in Definition 2.1. In particular, we show that

$$\sup_{M \in L^w(q_*)} \|\Xi^{f,g}(q, \varepsilon) - \Xi^{f,g}(q_*, 0) - M(q - q_*, \varepsilon)\|$$
$$= \mathcal{O}(\varepsilon + \|q - q_*\|)$$

This then satisfies the sufficient condition given in [18] for a set to be a GDQ of a map. ∎

Having shown both of those are $GDQs$ of their respective maps, we can use the considerable machinery developed in [17] to prove controllability using an open mapping theorem developed therein. This leads us to the following sufficient condition for MMDA systems to be controllable.

Theorem 2.3. *Let Σ be a control system as in Definition 1.1, and q_* be a point such that $\overline{\Delta}(q_*) = \mathbb{R}^n$. Then Σ is small time locally controllable at q_*.*

The proof is nearly the same as that found in [16]. It relies on constructing a map composing flows of all the MMDA fields and their associated brackets. Then one uses the theory in [17] to show that this map is open, thus showing controllability.

3 The Example

Here we consider the relatively simple example of a six wheeled vehicle such as that found in Fig. 1 and apply the result of Section 2. This *overconstrained wheeled vehicle* is a simplified model of the six-wheeled rocker-bogey mobility system of the Sojourner vehicle that landed on Mars in 1997. This chassis geometry will also be the basis for near-term Mars rover missions.

This system is "overconstrained" because not all the non-holonomic kinematic wheel constraints can be simultaneously satisfied. Consequently, the vehicle's motion can not be determined directly from kinematic constraints (i.e., it's governing equations of motion can not be put in the form $\dot{q} = \sum_i g_i(q)u_i$ with the g_i smooth).

Except when the vehicle moves straight ahead, at least one of the non-powered wheels must be slipping at all times. Hence, classical nonholonomic control theories do not apply to this vehicle. In [12] we proposed a *power dissipation method* for determining the governing equations of such overconstrained systems when they are moving slowly.

Definition 3.1. *The Dissipation or Friction Functional for an n-contact state is defined to be*

$$\mathcal{D} = \sum_{i=1}^{n} \alpha_i \mid \omega^i(q)\dot{q} \mid \qquad (3.1)$$

where q represents the configuration, $\alpha_i = \mu_i F_i$, with μ_i and F_i being the Coulomb friction coefficient and normal force at the ith wheel/ground contact, which are assumed known. $\omega^i(q)\dot{q}$ represents the velocity of the ith wheel's point of contact with the ground.

Since one or more of the contact points must always be in a slipping state due to the overconstrained geometry, the power dissipation approach states that the vehicle's motion at any instant is the one that minimizes $\mathcal{D}$, the power lost to slip. In [12] we showed that the minimum of power dissipation function yields governing equations that are MMDA systems (Definition 1.1). The power dissipation function measures the object's total energy dissipation due to contact slippage. It essentially restricts attention to those Lagrangian states which can be reduced to kinematic systems. See [8] for some details of such a reduction in the smooth case.

Section 3.1 discusses the full input space for a six-wheeled, fully actuated system. The complexity of such a system leads us to do an input state reduction to two inputs which we discuss in Section 3.2. In Section 3.3, as an example of how poor choices can be made in such a reduction, we attempt to gain more actuation by allowing both front wheels to be independently steerable and then show that such a system does not satisfy the sufficient conditions for controllability.

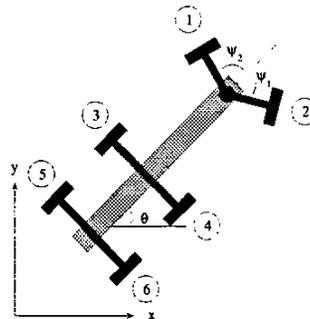### 3.1 Six wheels and two steerable wheels

Figure 1: 6 driven wheels, 2 steerable wheels

First we consider a full kinematic model of the Rocky 7 Sojourner robot. It has two steerable front wheels (inputs u_1, u_2) and all 6 of its wheels are driven (inputs $u_3, \ldots, u_8$). We will assume it is on flat ground with only the coefficient of friction altering the contact state. We idealize the steering of the wheels as a rotation about a vertical axis. Additionally, let the distance from that axis to the middle axle be l, and the distance from the middle axle to the back axle be r. In

this model, we naively have a total of 12 nonholonomic constraints on the system, with each wheel contributing a "no side way slip" constraint and a "no rolling" constraint. Clearly, not all of these constraints can be simultaneously satisfied except in non-generic cases, 1) the two rear axles are parallel, and therefore can only accommodate forward motion without slipping, and 2) there is no a priori reason to believe that the inputs u_i will produce mutually compatible velocities. Thus, the system is overconstrained. Applying the PDM to this system, we get

$$\binom{12}{3} = \frac{12!}{9!3!} = 220$$

kinematic states. That is, there are 220 different combinations of slipping motions. This is clearly unacceptable both from a complexity and practical standpoint.

To make progress on the analysis front, we reduce the number of inputs by introducing "matching" constraints. Observe that for any choice of u_1 and u_3 one can choose the other u_i inputs to be kinematically compatible with the motion produced by u_1 and u_3. Therefore, we reduce the dimension of the input space by requiring the following to hold:

$$u_6 = Ad_{g_{63}}^{[2]} u_3$$
$$u_7 = Ad_{g_{73}}^{[2]} u_3$$
$$u_8 = Ad_{g_{83}}^{[2]} u_3$$
$$u_5 = Ad_{g_{53}}^{[2]} u_3$$
$$u_4 = Ad_{g_{43}}^{[2]} u_3$$
$$u_2 = Ad_{g_{12}}^{[2]} u_1$$

where $Ad_{g_{ij}}^{[k]}$ is the kth component of the Adjoint operator of the rigid body transformation going from frame i (associated with the point where the input u_i acts on the system) to frame j. Practically speaking, these constraints enforce the two front wheels to steer together, and the remaining wheels to have a compatible motion. Therefore, u_1 and u_3 completely determine all the other control inputs. One can think of this system as being driven by two "virtual" inputs, v_1 and v_2 so as to get an underactuated vehicle. This effectively determines the model in Section 3.2. This technique is essentially an ad-hoc reduction in the shape space instead of the traditional group space reduction.

3.2 Reduction 1

Here we take the reduction as posed in the previous section. In this example, the front wheel is driven and steerable, while the middle and back wheels are passive. Additionally, the front wheel is always assumed to be in contact with the ground.

Using the power dissipation approach, one can show that the minimum of $\mathcal{D}$ must occur when either the middle or back wheel slips. If the vehicle configuration is
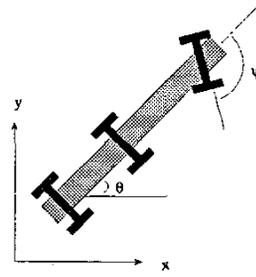

Figure 2: 1 driven and steered wheel, 2 passive wheels
$q = [x, y, \theta]^T$ and the controls u_1 and u_2 are associated with the drive and steering velocities respectively, the vehicle's governing equations of motion are:

$$\dot{q} = g_{\sigma_1}(q)u_1 + g_3(q)u_2 \qquad \sigma_1 : (q, t) \to \{a, b\}$$

$$g_{1a} = \begin{bmatrix} \cos(\psi)\cos(\theta) & \cos(\psi)\sin(\theta) & \frac{1}{l}\sin(\psi) & 0 \end{bmatrix}^T$$

$$g_{1b} = \begin{bmatrix} \cos(\theta)\cos(\psi) - \frac{r\sin(\theta)\sin(\psi)}{l+r} \\ \cos(\psi)\sin(\theta) + \frac{r\cos(\theta)\sin(\psi)}{l+r} \\ \frac{1}{l+r}\sin(\psi) \\ 0 \end{bmatrix}$$

$$g_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$$

The function which determines the switching boundaries is:

$$\Psi(q) = \left(\frac{F_1\mu_1}{F_2\mu_2}\right)^2 \left(\frac{l-r}{r}\right)^2 - 1.$$

where F_i are the normal forces above the middle axis and back axis, and the μ_i are the coefficients of friction at the two rear wheel contacts. When $\Psi(q) > 0$, $\sigma_1 = a$; when $\Psi(q) < 0$, $\sigma_1 = b$. Therefore switching is determined by $q \in Q$ such that $\Psi(q) = 0$. Intuitively, variations in tire-ground friction and vehicle weight distribution can cause alternations in the choice of the slipping wheel.

Controllability is determined by the rank of the distribution:

$$(g_3, g_{\sigma_1}, [g_3, g_{\sigma_1}], [[g_3, g_{\sigma_1}], g_{\sigma_1}]) \qquad (3.2)$$

Computing accordingly, we get $[g_{\sigma_1}, g_3] =$

$$co\left\{ \begin{bmatrix} \cos(\theta)\sin(\psi) \\ \sin(\theta)\sin(\psi) \\ \frac{-\cos(\psi)}{l} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{r\cos(\psi)\sin(\theta)}{l+r} + \cos(\theta)\sin(\psi) \\ \frac{-r\cos(\theta)\cos(\psi)}{l+r} + \sin(\theta)\sin(\psi) \\ -\left(\frac{\cos(\psi)}{l+r}\right) \\ 0 \end{bmatrix} \right\}$$

and

$$[g_{\sigma_1}, [g_{\sigma_1}, g_3]] = co\left\{ \begin{bmatrix} \frac{-1}{l}\sin(\theta) \\ \frac{1}{l}\cos(\theta) \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{-1}{l+r}\sin(\theta) \\ \frac{1}{l+r}\cos(\theta) \\ 0 \\ 0 \end{bmatrix} \right\}$$

Now, substituting in the algebraic equivalent for $co\{\cdot, \cdot\}$ we can evaluate the determinant of Equation 3.2:

$[g_3, co\{g_{1a}, g_{1b}\}, [g_3, co\{g_{1a}, g_{1b}\}], [[g_3, co\{g_{1a}, g_{1b}\}], co\{g_{1a}, g_{1b}\}]]$
$= [g_3, co\{g_{1a}, g_{1b}\}, co\{[g_3, g_{1a}], [g_3, g_{1b}]\},$
$\qquad co\{[g_{1a}, [g_{1a}, g_3]], [g_{1b}, [g_{1b}, g_3]]\}]$
$= [g_3, \delta_1 g_{1a} + (1 - \delta_1)g_{1b}, \delta_2[g_3, g_{1a}] + (1 - \delta_2)[g_3, g_{1b}],$
$\qquad \delta_3[g_{1a}, [g_{1a}, g_3]] + (1 - \delta_3)[g_{1b}, [g_{1b}, g_3]]]$

where $\delta_i \in [0,1]$ for $i = 1,2,3$. The determinant is

$$\frac{(l + r\delta_3)(2l + r(\delta_1 + \delta_2) + r(\delta_2 - \delta_1)\cos(2\psi))}{2l^2(l+r)^2}$$

which equals 0 only if $\delta_3 < 0$ which is not an admissible δ_3. Hence, the vehicle is always STLC, as expected. Physically, this result implies that the vehicle remains locally controllable even as the status of slipping wheel alters unexpectedly.

3.3 Reduction 2

It is clearly necessary that all of the individual models in an MMDA system be controllable in order for the condition in Theorem 2.3 to be satisfied. Now we ask, is it sufficient that all of the individual models in an MMDA system be controllable? If it is, then we need only concern ourselves with the analysis of the individual smooth models. The answer to this question, perhaps surprisingly, is no. This example illustrates why.

We now consider the case where the front two wheels are independently steered. This choice, though seemingly reasonable, will cause the resulting system to fail our controllability test. To illustrate this concept, we consider a vehicle with the same two front steerable wheels as the Rocky 7, but with only one back wheel for simplicity (Fig. 3). For simplicity we assume that
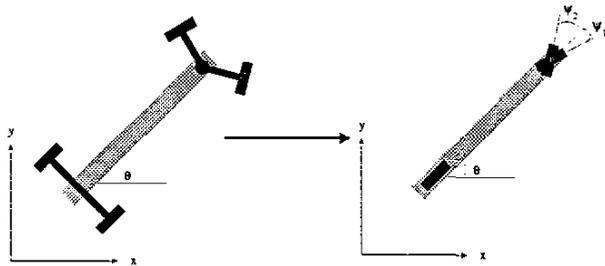

Figure 3: 2 driven and steered wheels, 1 passive wheel

the two front wheels are collapsed down along their respective axes so that they both are connected at the same point (see Figure 3). (This simplification does not change the outcome of the analysis significantly.) We will see that this vehicle is *only* controllable when $\psi_1 = \psi_2$ (when it is indistinguishable from a kinematic car). An analysis of this system using the PDM shows that there are two distinct kinematic states for this vehicle (i.e., two different permutations of wheel slip). The governing equations of both slipping cases look like kinematic cars, i.e.,

$$\dot{q} = g_{\sigma_1}(q)u_1 + g_3(q)u_2 \qquad \sigma_1 : (q,t) \to \{a,b\}$$

where

$$g_{1a} = \begin{bmatrix} \cos\psi_1\cos\theta \\ \cos\psi_1\sin\theta \\ \sin\psi_1 \\ 0 \end{bmatrix}, g_{1b} = \begin{bmatrix} \cos\psi_2\cos\theta \\ \cos\psi_2\sin\theta \\ \sin\psi_2 \\ 0 \end{bmatrix}, g_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

As before in Equation (3.2), controllability is determined by the rank of:

$$(g_3, g_{\sigma_1}, [g_3, g_{\sigma_1}], [[g_3, g_{\sigma_1}], g_{\sigma_1}])$$

Computing, we get:

$$[g_3, g_{\sigma_1}] = co \left\{ \begin{bmatrix} \cos\psi_1\sin\theta \\ \cos\psi_1\cos\theta \\ -\cos\psi_1 \\ 0 \end{bmatrix}, \begin{bmatrix} \cos\psi_2\sin\theta \\ \cos\psi_2\cos\theta \\ -\cos\psi_2 \\ 0 \end{bmatrix} \right\}$$

and

$$[[g_3, g_{\sigma_1}], g_{\sigma_1}] = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \\ 0 \end{bmatrix}$$

Evaluating the determinant as we did before, we now compute if there exists a selection of the differential inclusion such that this is *not* full rank, i.e., we need to find out if there exists δ_1, δ_2 such that

$$\det[g_3, \delta_1 g_{1a} + (1-\delta_1)g_{1b}, \delta_2[g_3, g_{1a}] + (1-\delta_2)[g_3, g_{1b}],$$
$$\delta_3[g_{1a}, [g_{1a}, g_3]] + (1-\delta_3)[g_{1b}, [g_{1b}, g_3]]] = 0$$

Computing, we find that the above determinant equals

$$\delta_1 + \delta_2 - 2\delta_1\delta_2 - 1 + (\delta_1 + \delta_2 - 2\delta_1\delta_2)\cos(\psi_1 - \psi_2)$$

Now, if

$$\chi(\delta_1, \delta_2) \overset{def}{=} \frac{1}{\delta_1 + \delta_2 - 2\delta_1\delta_2} - 1$$

then the controllability condition depends on whether there exists δ_i such that $\chi(\delta_1, \delta_2) \in [-1, 1]$. Clearly there do exist such δ_i, just choose $\delta_1 = 0$ and $\delta_2 = 1$. Then $\chi = 0$. However, one may ask if making $|\psi_1 - \psi_2|$ sufficiently small helps. Doing so changes the requirement to $\chi \in [-\varepsilon, \varepsilon]$, which again has a solution for the same choice of δ_i. Therefore, this vehicle design is arbitrarily sensitive to switching with respect to local controllability. That is, although on the one hand it is desirable to get extra actuation by controlling ψ_1 and ψ_2 independently, on the other hand it is desirable to have a model we can show is controllable.

Now, what if the selections of $[g_{\sigma_1}, g_3]$ and $[[g_{\sigma_1}, g_3], g_3]$ are coupled in some way? Consider the situation where, if we again parameterize the selections of $[g_{\sigma_1}, g_3]$ by $\delta_1 \in [0,1]$ and the selections of $[[g_{\sigma_1}, g_3], g_3]$ by $\delta_2 \in [0,1]$, what if $\delta_2 = h(\delta_1)$. For example, take h to be the identity. Then

$$\chi(\delta_1, \delta_2) = \frac{1}{2\delta_1 - 2\delta_1^2} - 1$$

which only has a solution in $[-1, 1]$ for $\delta_1 = \frac{1}{2}$. Therefore, if we allow a slightly weaker notion of controllability by only requiring that the system be controllable almost always, this system would then be controllable. The key point here is that this system does not satisfy the requirements for controllability. This has two implications. First, it is a weakness in the theory, because we also do not know that the system is *not* locally controllable. Second, it indicates that we should avoid algorithm designs that rely on a paramterization like that found in Figure 3 until we have developed analytical tools that make such a system tractable.

4 Conclusions

This paper extends recent controllability results for MMDA systems that can be viewed as extensions of Chow's classical controllability theorem. This paper additionally studied an important MMDA example where the individual plants are driftless affine. Moreover, the variations of the model studied in Section 3.3 illustrate that controllability of the individual models that make up an MMDA system is *not* sufficient to guarantee controllability of the overall MMDA system. Because of the potential importance of the vehicles discussed in Section 3 in future planetary exploration missions, future work will investigate algorithms for stabilizing the multi-model systems of Definition 1.1. In particular, we will use the formalism of *GDQs* to study stability. That said, showing stability of such systems is complicated by a number of factors. Stability is difficult even for underactuated mechanical system with smooth dynamics. Stability theorems have often used significant insights into geometry (see [7, 9, 19]). For systems like those discussed in this paper, such geometric techniques must be extended to nonsmooth systems, which will inevitably be a formidable task.

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