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# On Well-Defined Kinematic Metric Functions

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*Abstract*— **This paper presents both formal as well as practical well-definedness conditions for kinematic metric functions. To formulate these conditions, we introduce an intrinsic definition of a rigid body's configuration space. Based on this definition, the principle of objectivity is introduced to derive a formal condition for well-definedness of kinematic metric functions, as well as to gain physical insight into left, right and bi-invariances on the Lie group $SE(3)$. We then relate the abstract notion of objectivity to the more intuitive notion of frame-invariance, and show that frame-invariance can be used as a practical condition for determining objective functions. Examples demonstrate the utility of objectivity and frame-invariance.**

## 1 Introduction

*Rigid body kinematic metric functions* are real-valued functions of rigid body configurations, velocities, and applied wrenches. They can be used to assess certain *metric attributes*, such as "distance," "length," and "angle". While such functions are frequently needed in robotic task planning to assess the quality of a proposed solution, some commonly used metric functions are *not* well-defined, as their value depends on the choice of reference frames. This paper formally addresses the well-definedness issue for a general class of kinematic metric functions.

To motivate our study with a simple example, consider the candidate fixtures of a triangular object shown in Fig. 1. Note that when the bodies' compliance and surface curvatures are taken into account, both of these fixtures are actually stable [10], though not necessarily good fixtures. Suppose that the object is subjected to a drilling torque, $\tau$. It can be shown [10] that if the fixtures have frictionless contacts and are loaded in a certain specified manner, the object in each fixture is displaced by a small rotation, of equal magnitude $0 < \theta \ll 1$, about the axis centered at the common intersection of the contact normals. Let's assume that the two fixtures are to be compared based on the *size* of the object's displacement.

Small displacements can be approximated by rigid body velocity vectors, whose size can be measured by a *norm*, a common kinematic metric function that measures the *length* of the velocity vector. Let a planar rigid body velocity be represented as a vector, $\dot{q} = (v_x, v_y, \omega)$, where $(v_x, v_y)$ and $\omega$ describe the translation and rotation of a body-fixed reference frame.

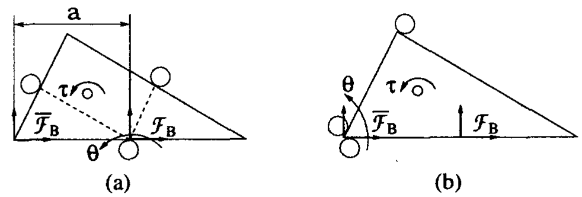


Fig. 1. Candidate Fixtures: (a) Fixture I, and (b) Fixture II.

The commonly used Euclidean norm of the velocity is: $\|\dot{q}\| = \sqrt{v_x^2 + v_y^2 + (\omega l)^2}$, where $l$ is a characteristic object length that is used to make the rotational and translational velocities comparable. Using this norm, we evaluate the object's displacements for reference frames $\mathcal{F}_B$ and $\bar{\mathcal{F}}_B$ (Fig. 1). With respect to $\mathcal{F}_B$, the fixtures' displacements are $\dot{q}_1 = (0, 0, \theta)$ and $\dot{q}_2 = (0, \theta a, \theta)$. Hence, $\|\dot{q}_1\| = \theta l$ and $\|\dot{q}_2\| = \theta\sqrt{l^2 + a^2}$, which indicates that the displacement is smaller in Fixture I. On the other hand, when measured in $\bar{\mathcal{F}}_B$, the displacements are $\dot{\bar{q}}_1 = (0, -\theta a, \theta)$ and $\dot{\bar{q}}_2 = (0, 0, \theta)$. Since $\|\dot{\bar{q}}_1\| = \theta\sqrt{l^2 + a^2}$ and $\|\dot{\bar{q}}_2\| = \theta$, the displacement in Fixture II is now smaller! Thus, the Euclidean velocity norm, which is *frame-dependent*, leads to inconsistent results, and is hence ill-defined. As this simple example shows, well-definedness is an key issue for kinematic metric functions.

### 1.1 Related Work and Our Contributions

To understand the content of this paper and our contributions, we now briefly review kinematic metric functions and the status of their study—Section 4 presents a more precise consideration. Let $M$ be a (smooth) $n$-dimensional manifold. Let $T_xM$ and $T_x^*M$ denote the tangent and cotangent spaces to $M$ at $x \in M$. Below, $M$ will be the configuration space of a rigid body, while tangent and cotangent vectors correspond to velocities and wrenches. A *metric function* on $M$ is a real-valued function of the form $\Theta(x_1, \ldots, x_m)$, or a map $\Phi$ that for each $x \in M$ assigns a real-valued function of the form

$$\Phi_x(\mathbf{v}_1, \ldots, \mathbf{v}_k, \boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_l), \tag{1}$$

where $\mathbf{v}_i \in T_xM$ and $\boldsymbol{\alpha}_j \in T_x^*M$. A *distance metric*, $\Theta(x_1, x_2)$, measures the "distance" between configurations, $x_1$ and $x_2$. A *norm* maps a rigid body velocity or wrench to a non-negative number representing its length. A *Riemannian metric function* is a symmetric and positive definite bilinear function of the form $\Phi_x(\mathbf{v}_1, \mathbf{v}_2)$ that can be used to measure the "length" of a tangent vector. Other types of metric functions may be useful for many applications (see Section 8).

Previous works have mainly focused on distance and Riemannian metrics. Well-definedness of such functions has been addressed by considering the effects of reference frame choices, as well as by applying Lie group theory. Frame-invariant distance metrics are developed in Refs. [5, 8, 13]. From the perspective of Lie group theory, the configuration space of a rigid body (Section 3) is represented by the set of rigid transformations of $\mathbb{R}^3$ (denoted by $SE(3)$), which is a Lie group [14]. Loncaric [11] showed that there exist no bi-invariant Riemannian metrics on $SE(3)$, and Park [15] showed that there are no differentiable bi-invariant distance metrics on $SE(3)$. Guided by these results, Park [15], and Tchon and Duleba [17] used Riemannian metrics to define distance metrics that are left or right invariant, while Larochelle and McCarthy [7], and Etzel and McCarthy [3] developed distance metrics that are approximately bi-invariant. Essentially focusing on Riemannian metrics, Duffy [2] showed that a commonly used notion of orthogonality depends on frame choices as well as the length scale used to compare translations and rotations. Li [9] showed that several manipulability measures, which are defined using the inner product structure discussed by Duffy [2], vary with frame changes.

Note that all of these works focus on specific, rather than general, classes of metric functions. Many interesting types of metric functions have not been adequately addressed, and some ill-defined functions (such as Euclidean velocity and wrench norms) are still in common use. Kirkpatrick et al. [6], and Ferrari and Canny [4], who used ill-defined Euclidean norms to develop grasp quality measures, identified a need for well-definedness and frame-invariance. However, well-defined velocity and wrench norms, in particular those not expressible as a Riemannian metric, have not been formally investigated. In contrast to previous works that focus on specific types of functions, we analyze the well-definedness issue for a *general* class of kinematic metric functions that encompass all metric functions mapping rigid body configurations, velocities and wrenches to a real number. The generality of this class provides a much richer resource for developing practically useful metric functions.

Our approach is based on an intrinsic definition of a rigid body's configuration space. We then employ the principle of objectivity (or observer-indifference) [12] to derive a formal well-definedness condition for kinematic metric functions. This condition yields physical insight into the invariances on $SE(3)$. In particular, we show that for a kinematic metric function to be well-defined:

- left invariance is *necessary but not sufficient*,
- bi-invariance is *sufficient but not necessary*,
- right invariance is *not* implied by objectivity.

These precise results clear up some misconceptions in the literature. Finally, the notion of frame-invariance is clarified and provided as a practical condition for testing the well-definedness of kinematic metric functions. Hence, given a problem, one can often determine a metric function that captures the essential physics of the problem, and then use our test to evaluate its objectivity. If the function fails the test, it should either be discarded, or used carefully, with the knowledge that it will produce frame-dependent results. In the concluding section, we show that useful norms can be defined with or without inner products. A more lengthy discussion of this topic, including many of the proofs that are omitted from this paper, can be found in Ref. [10].

## 2 An Intrinsic C-Space Definition

The notion of the configuration space (c-space) of a rigid body is well-known and has been widely applied in robotics. A rigid body c-space has conventionally been defined as the set of all possible locations of a body-fixed frame relative to a stationary world frame. This approach essentially identifies the c-space of a rigid body with the Lie group $SE(3)$, and is often convenient for engineering applications. However, it is this approach, used without careful examination, that has hampered the understanding of some fundamental geometric properties of rigid body kinematic metric functions.

To truly understand how the choice of reference frame affects the behavior of a kinematic metric function, it is first necessary to establish an intrinsic definition of the c-space of a rigid body which is independent of the notion of reference frames. We can then purposely introduce reference frames (using the notion of representation functions described in the next section) and observe their influence on the well-definedness issue. Lin [10] presents a more comprehensive discussion of the concepts that are briefly reviewed in this section.

Our intrinsic c-space definition is based on the distinction between Euclidean and Cartesian spaces. Three-dimensional *Euclidean space*, denoted by $\mathbb{E}^3$, is a geometric model for the physical space and is *axiomatically* defined in terms of three systems of geometric objects: points, lines and planes [16]. It suffices for our purposes to recognize that $\mathbb{E}^3$ does *not* involve any coordinate frames, and consequently must be *distinguished* from $\mathbb{R}^3$. For example, a point in $\mathbb{E}^3$ is *not* a triple of real numbers, and a straight line in $\mathbb{E}^3$ is *not* a linear algebraic equation. Thus, $\mathbb{E}^3$ and $\mathbb{R}^3$ are *different* spaces.

While each point of $\mathbb{E}^3$ can be assigned a set of coordinates as discussed in Section 3, this representation of $\mathbb{E}^3$ by $\mathbb{R}^3$ is only achieved by embedding a coordinate frame in $\mathbb{E}^3$ and is hence unnatural.

It is also conceptually important to distinguish between rigid transformations of $\mathbb{E}^3$ and $\mathbb{R}^3$. Rigid transformations of $\mathbb{R}^3$ comprise the familiar set $SE(3)$, while rigid transformations of $\mathbb{E}^3$ (i.e., distance and orientation preserving maps of $\mathbb{E}^3$; [16]) make up a different set, which we will denote by $\widetilde{SE}(3)$. The spaces $\mathbb{E}^3$ and $\widetilde{SE}(3)$ can be used to define a rigid body and its configuration space as follows.

**Definition 1.** A set $\mathcal{B}$ consisting of at least four distinct elements[1] is said to be a **rigid body** if there is a nonempty set of mappings, denoted $\mathcal{C}$, with the following properties: (1) Each map $\chi \in \mathcal{C}$ is a bijection from $\mathcal{B}$ onto a closed subset of $\mathbb{E}^3$ such that $\chi(\mathcal{B})$ does not lie in a plane. (2) Given any $\chi_1, \chi_2 \in \mathcal{C}$, $\chi_2 \circ \chi_1^{-1}$ is a distance and orientation preserving map, i.e., a rigid transformation, from $\chi_1(\mathcal{B})$ onto $\chi_2(\mathcal{B})$. We call $\chi \in \mathcal{C}$ a **configuration** of $\mathcal{B}$, and $\mathcal{C}$ the intrinsic **configuration space** of $\mathcal{B}$. The elements of $\mathcal{B}$ are called the rigid body's *particles* or *points*.

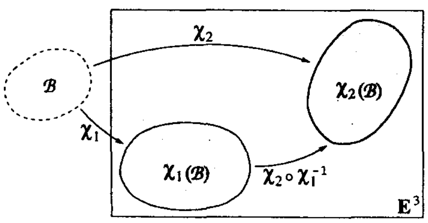


Fig. 2. Placement and displacement of a rigid body.

This definition is *intrinsic* since it involves *no* coordinate frames and each physical location of the body corresponds to a *unique* configuration. Intuitively, a configuration $\chi \in \mathcal{C}$ may be thought of as a *placement* of the rigid body at some location in $\mathbb{E}^3$, and given $\chi_1, \chi_2 \in \mathcal{C}$, $\chi_2 \circ \chi_1^{-1}$ may be regarded as a *(rigid) displacement* of the body, as shown in Fig. 2. Note that we can naturally identify $\chi_2 \circ \chi_1^{-1}$ with a rigid transformation of $\mathbb{E}^3$, i.e., $\chi_2 \circ \chi_1^{-1} \in \widetilde{SE}(3)$, since a rigid transformation defined on a subset of $\mathbb{E}^3$ can be uniquely extended to one defined on all of $\mathbb{E}^3$.

## 3 Representation of C-Space

Our abstract c-space is not immediately useful for practical analysis. Commonly, abstract manifolds can be *represented* by other manifolds whose properties are more practically useful. In robotics, it is common to represent $\mathcal{C}$ by $SE(3)$. In this section we carefully examine the representation issue, first representing $\mathcal{C}$ by $\widetilde{SE}(3)$, and then representing $\widetilde{SE}(3)$ by $SE(3)$. Here $\widetilde{SE}(3)$ is the set of rigid transformations on $\mathbb{E}^3$ (in contrast, $SE(3)$ is such a set on $\mathbb{R}^3$). Our careful examination clarifies the effect of reference frame choice.

First, let us relate $\mathbb{E}^3$ to $\mathbb{R}^3$. Any point of $\mathbb{E}^3$ has a unique set of coordinates (a member of $\mathbb{R}^3$) with respect to a reference frame embedded in $\mathbb{E}^3$. Since the choice of embedded frame is arbitrary, we use the following convention.

**Notational Convention.** Choose a nominal embedded frame, denoted $\mathcal{F}_X$. An arbitrary embedded frame is denoted $\mathcal{F}_X^b$, where the superscript $b$ always means that $\mathcal{F}_X^b$ is displaced from $\mathcal{F}_X$ by $b \in SE(3)$.

Given an embedded frame, one can define a map $X^b\colon \mathbb{E}^3 \to \mathbb{R}^3$ such that the coordinates of each $p \in \mathbb{E}^3$ in $\mathcal{F}_X^b$ are given by $X^b(p) \in \mathbb{R}^3$. In particular, $X \triangleq X^e$ ($e$ is the identity element of $SE(3)$) corresponds to the frame $\mathcal{F}_X$.

We use the following procedure to represent $\widetilde{SE}(3)$ by $SE(3)$. Via the (invertible) map $X^b$, any $\widetilde{g} \in \widetilde{SE}(3)$ corresponds uniquely to $X^b \circ \widetilde{g} \circ (X^b)^{-1}\colon \mathbb{R}^3 \to \mathbb{R}^3$, which as can be shown is actually a rigid transformation on $\mathbb{R}^3$. Thus, this defines a map $F^b\colon \widetilde{SE}(3) \to SE(3)$ by $F^b(\widetilde{g}) = X^b \circ \widetilde{g} \circ (X^b)^{-1}$. Since $F^b$ can be shown to be invertible, it can be used to represent $\widetilde{SE}(3)$ by $SE(3)$. For convenience we use the notation $F \triangleq F^e$. The map $F^b$ has the following interpretation [10]. Imagine that the embedded frame $\mathcal{F}_X^b$ is "glued" to $\mathbb{E}^3$ during the action of $\widetilde{g} \in \widetilde{SE}(3)$ on $\mathbb{E}^3$. Then, $F^b(\widetilde{g})$ is precisely the displacement of $\mathcal{F}_X^b$ as $\mathbb{E}^3$ is mapped to $\widetilde{g}(\mathbb{E}^3)$.

We next represent $\mathcal{C}$ by $\widetilde{SE}(3)$: any configuration $\chi \in \mathcal{C}$ can be represented by a rigid displacement from a *reference configuration* to $\chi$ (Fig. 2). Analogous to embedded frames, the following notational convention is introduced to handle reference configuration choices.

**Notational Convention.** Choose a nominal reference configuration $\chi_0 \in \mathcal{C}$. Denote by $\chi_{\widetilde{a}}$ an arbitrary reference configuration, where the subscript $\widetilde{a}$ always indicates that $\chi_{\widetilde{a}}$ is determined by $\chi_{\widetilde{a}} \circ \chi_0^{-1} = \widetilde{a} \in \widetilde{SE}(3)$. That is, $\chi_{\widetilde{a}}$ is displaced from $\chi_0$ by $\widetilde{a}$.

Corresponding to an arbitrary reference configuration $\chi_{\widetilde{a}} \in \mathcal{C}$, we can define a map $J^{\widetilde{a}}\colon \mathcal{C} \to \widetilde{SE}(3)$ by $J^{\widetilde{a}}(\chi) = \chi \circ \chi_{\widetilde{a}}^{-1}$ for any $\chi \in \mathcal{C}$. That is, $J^{\widetilde{a}}(\chi)$ is the displacement of $\mathcal{B}$ from $\chi_{\widetilde{a}}$ to $\chi$. We write $J \triangleq J^{\widetilde{e}}$ ($\widetilde{e}$ is the identity element of $\widetilde{SE}(3)$), which corresponds to $\chi_0$. Clearly, $J^{\widetilde{a}}$ is invertible and can be used to represent $\mathcal{C}$ by $\widetilde{SE}(3)$.

Concatenating the representation of $\mathcal{C}$ by $\widetilde{SE}(3)$ and

[1] $\mathcal{B}$ is required to contain at least four distinct elements since four non-coplanar points in $\mathbb{E}^3$ are necessary to determine a rigid transformation of $\mathbb{E}^3$ [10].

that of $\widetilde{SE}(3)$ by $SE(3)$, we can represent $\mathcal{C}$ as follows.

**Definition 2.** The **c-space representation map** corresponding to a reference configuration $\chi_{\tilde{a}}$ and an embedded frame $\mathcal{F}_X^b$ is a map $\Gamma^{\tilde{a},b}: \mathcal{C} \to SE(3)$ defined by $\Gamma^{\tilde{a},b} = F^b \circ J^{\tilde{a}}$. In particular, $\Gamma \triangleq \Gamma^{\tilde{e},e} = F \circ J$ corresponds to the nominal reference configuration $\chi_0$ and embedded frame $\mathcal{F}_X$.

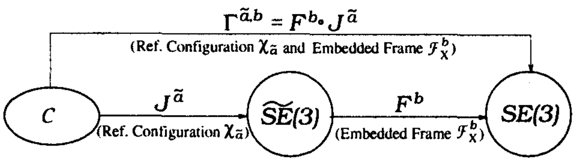


Fig. 3. C-space representation diagram.

The definition of c-space representation maps is illustrated in Fig. 3. In summary, since a c-space representation map is a one-to-one correspondence, we have established a relationship between $\mathcal{C}$ and $SE(3)$. However, as explicitly shown by its definition, *this representation depends on embedded frame and reference configuration choices.* Therefore, $\mathcal{C}$ and $SE(3)$ *cannot* be naturally identified and must be treated as *distinct* spaces for our developments.

## 4 Kinematic Metric Functions

Having formally defined the configuration space of a rigid body, the notion of kinematic metric functions can be precisely defined. First recall the following basic notions about smooth manifolds [1].

An $n$-dimensional *manifold* is a set $M$ that is locally similar to $\mathbb{R}^n$, i.e., for any $x \in M$ there is a one-to-one map $\phi$, called a coordinate map, from some neighborhood of $x$ onto an open subset of $\mathbb{R}^n$. We say $M$ is *smooth* if $\phi \circ \psi^{-1}$ is smooth, where $\phi$ and $\psi$ are any coordinate maps with overlapping domains. A *tangent vector* to $M$ at $x$ is the velocity of a smooth curve that lies in $M$ and passes through $x$, and the *tangent space* at $x$, denoted $T_xM$, is the set of all tangent vectors at $x$. A *covector* to $M$ at $x$ is a linear functional on $T_xM$, and the *cotangent space* at $x$, denoted $T_x^*M$, consists of all covectors at $x$. The *tangent bundle* is the union $TM = \cup_{x\in M} T_xM$, while the *cotangent bundle* is $T^*M = \cup_{x\in M} T_x^*M$.

As is well-known, $SE(3)$ is a smooth manifold. With the one-to-one relationship between $\mathcal{C}$ and $SE(3)$ established in Section 3, this fact implies that $\mathcal{C}$ is also a smooth manifold. To state the obvious, tangent vectors to $\mathcal{C}$ are precisely rigid body velocities, while covectors to $\mathcal{C}$ are wrenches.

**Definition 3.** A *metric function* $\Phi$ on a smooth manifold $M$ is a map (on $TM$ and $T^*M$) that for each $x \in M$ assigns a real-valued function $\Phi_x$ of the form[2]

$$\Phi_x(\mathbf{v}_1,\ldots,\mathbf{v}_k,\boldsymbol{\alpha}_1,\ldots,\boldsymbol{\alpha}_l) \triangleq \Phi(x,\mathbf{v}_1,\ldots,\mathbf{v}_k,\boldsymbol{\alpha}_1,\ldots,\boldsymbol{\alpha}_l),$$

where $x \in M$, $\mathbf{v}_i \in T_xM$ and $\boldsymbol{\alpha}_j \in T_x^*M$. If $M = \mathcal{C}$, then $\Phi$ is called a **kinematic metric function.**

Riemannian metrics on $\mathcal{C}$ take the form $\Phi_x(\mathbf{v}_1, \mathbf{v}_2)$, while velocity and wrench norms are expressed as $\Phi_x(\mathbf{v})$ and $\Phi_x(\boldsymbol{\alpha})$. It should however be noted that the notion of kinematic metric functions given by Definition 3 is quite general and can be used to assess a variety of metric attributes that may or may not be covered by norms and Riemannian metrics.

Section 5 discusses the well-definedness issue for general kinematic metric functions. Section 6 investigates how well-defined kinematic metric functions are represented by appropriately interrelated metric functions on $SE(3)$.

## 5 Objective Metric Functions

This section defines the notion of *objective* metric functions. Objectivity is a fundamental principle in mechanics [12], and is commonly used in continuum mechanics to require observer-indifference of constitutive relations. We introduce this notion in the abstract c-space, and then develop its implication in $SE(3)$.

Given a rigid transformation $\tilde{g} \in \widetilde{SE}(3)$, a configuration $\chi \in \mathcal{C}$ is mapped to $\kappa = \tilde{g} \circ \chi \in \mathcal{C}$. While $\chi$ and $\kappa$ can be regarded as two configurations that differ by a rigid displacement, it is more interesting to interpret them as *the same configuration of the rigid body as viewed by different observers,* whose locations differ by $\tilde{g}$. This interpretation allows us to introduce the following notion.

**Definition 4.** Let $\bar{c}_1: I \to \mathcal{C}$ and $\bar{c}_2: I \to \mathcal{C}$, where $I = (-\epsilon, \epsilon)$ with $\epsilon > 0$, be c-space curves. If there exists some $\tilde{g} \in \widetilde{SE}(3)$ such that $\bar{c}_2(t) = \tilde{g} \circ \bar{c}_1(t)$ for all $t \in I$, then $\bar{c}_2$ is said to be *equivalent* to $\bar{c}_1$ with respect to $\tilde{g}$.

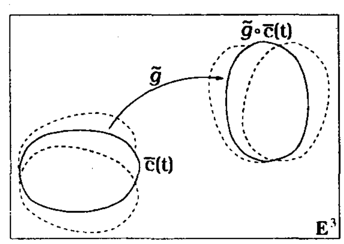


Fig. 4. Equivalent curves in c-space.

Two equivalent c-space curves can be brought into coincidence by a rigid displacement (Fig. 4). More

[2] Metric functions of the form $\Theta(x_1, \ldots, x_m)$ with $x_i \in M$ are not discussed here due to limited space, but the developments presented herein are equally applicable [10].

interestingly, in accordance with the above change-of-observer argument, such curves can be regarded as a *single* motion of the rigid body viewed by two observers whose locations differ by a rigid displacement. Thus, the notion of equivalent curves captures change-of-observer effects on rigid body motions. The equivalent curve notion can be used to define equivalent tangent vectors and covectors.

**Definition 5.** Given tangent vectors $\overline{\mathbf{v}}_1 \in T_{\chi_1}\mathcal{C}$ and $\overline{\mathbf{v}}_2 \in T_{\chi_2}\mathcal{C}$, we say that $\overline{\mathbf{v}}_2$ is *equivalent* to $\overline{\mathbf{v}}_1$ with respect to $\widetilde{g} \in \widetilde{SE}(3)$ if there are two curves $\overline{c}_1, \overline{c}_2 \colon (-\epsilon, \epsilon) \to \mathcal{C}$, where $\overline{c}_2$ is equivalent to $\overline{c}_1$ with respect to $\widetilde{g}$, such that $\overline{c}_1(0) = \chi_1$ and $\dot{\overline{c}}_1(0) = \overline{\mathbf{v}}_1$, and $\overline{c}_2(0) = \chi_2$ and $\dot{\overline{c}}_2(0) = \overline{\mathbf{v}}_2$. On the other hand, a covector $\overline{\boldsymbol{\alpha}}_2 \in T^*_{\chi_2}\mathcal{C}$ is said to be *equivalent* to another covector $\overline{\boldsymbol{\alpha}}_1 \in T^*_{\chi_1}\mathcal{C}$ with respect to $\widetilde{g} \in \widetilde{SE}(3)$, if $\chi_2 = \widetilde{g} \circ \chi_1$ and $\overline{\boldsymbol{\alpha}}_2(\overline{\mathbf{v}}_2) = \overline{\boldsymbol{\alpha}}_1(\overline{\mathbf{v}}_1)$ whenever $\overline{\mathbf{v}}_2 \in T_{\chi_2}\mathcal{C}$ is equivalent to $\overline{\mathbf{v}}_1 \in T_{\chi_1}\mathcal{C}$ with respect to $\widetilde{g} \in \widetilde{SE}(3)$.

Fig. 5 illustrates the physical intuition for equivalent tangent vectors and covectors. Two equivalent tangent vectors are instantaneous motions that differ only by a rigid displacement, and can be interpreted as a *single* instantaneous motion of the rigid body as viewed by different observers. For example, the equivalent tangent vectors in Fig. 5(a) can be regarded as an instantaneous planar rigid body motions whose instantaneous center of rotation is at $p$ for one observer, and at $\widetilde{g}(p)$ for a different observer. On the other hand, as shown in Fig. 5(b), two equivalent covectors, say $\overline{\boldsymbol{\alpha}}_1 \in T^*_{\chi_1}\mathcal{C}$ and $\overline{\boldsymbol{\alpha}}_2 \in T^*_{\chi_2}\mathcal{C}$, can be interpreted as a *single* physical wrench that acts on the object and is observed by different observers. The equivalent covectors do the same work on equivalent tangent vectors.

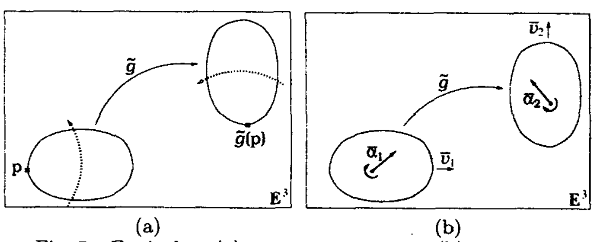

Fig. 5. Equivalent (a) tangent vectors and (b) covectors.

The well-definedness of kinematic metric functions can be based on the notions of equivalent tangent vectors and covectors. Clearly, for a metric function on $\mathcal{C}$ to be well-defined, the metric measurements it represents must be consistent with respect to different observers. Thus, the function must yield the *same* value at equivalent tangent vectors and covectors. This requirement is formalized by the following definition.

**Definition 6.** Consider a kinematic metric function $\overline{\Phi}$ as defined in Definition 3. This function is said to be **objective** if $\overline{\Phi}_\chi(\overline{\mathbf{v}}_1, \ldots, \overline{\mathbf{v}}_k, \overline{\boldsymbol{\alpha}}_1, \ldots, \overline{\boldsymbol{\alpha}}_l) = \overline{\Phi}_\kappa(\overline{\mathbf{u}}_1, \ldots, \overline{\mathbf{u}}_k, \overline{\boldsymbol{\beta}}_1, \ldots, \overline{\boldsymbol{\beta}}_l)$ whenever $\overline{\mathbf{v}}_i \in T_\chi\mathcal{C}$ and $\overline{\mathbf{u}}_i \in T_\kappa\mathcal{C}$ are equivalent for each $1 \leqslant i \leqslant k$, and $\overline{\boldsymbol{\alpha}}_j \in T^*_\chi\mathcal{C}$ and $\overline{\boldsymbol{\beta}}_j \in T^*_\kappa\mathcal{C}$ are equivalent for each $1 \leqslant j \leqslant l$.

The notion of objectivity provides a fundamental characterization of well-defined kinematic metric functions. This notion is also quite intuitive, since it represents the physically motivated requirement of observer-indifference. The next sections show that the objectivity notion can be used to gain insight into invariances in $SE(3)$, and affords a convenient representation in terms of frame-invariance.

## 6 Left, Right and Bi-Invariances

This section studies the relationship between objectivity in $\mathcal{C}$ and left, right and bi-invariances in $SE(3)$, and thereby gives such invariances an intuitive physical interpretation. To enable these discussions, it is convenient to use the notion of pull-backs and push-forwards of metric functions [1]. In the following definition, $T\mathsf{g}$ and $T^*\mathsf{g}$ are tangent and cotangent maps[3], respectively.

**Definition 7.** Let $M$ and $N$ be manifolds, $\mathsf{f} \colon M \to N$ and $\mathsf{g} \colon N \to M$ diffeomorphisms, and $\Phi$ a metric function on $M$ given in Definition 3. The *pull-back* of $\Phi$ by $\mathsf{g}$, denoted $\mathsf{g}^*\Phi$, is a metric function on $N$ defined by

$$\begin{aligned}&\mathbf{g}^*\Phi_y(\mathbf{u}_1, \ldots, \mathbf{u}_k, \boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_l)\\ &= \Phi_{\mathbf{g}(y)}(T\mathbf{g}(\mathbf{u}_1), \ldots, T\mathbf{g}(\mathbf{u}_k), T^*\mathbf{g}^{-1}(\boldsymbol{\beta}_1), \ldots, T^*\mathbf{g}^{-1}(\boldsymbol{\beta}_l))\end{aligned}$$

where $y \in N$, $\mathbf{u}_i \in T_yN$, and $\boldsymbol{\beta}_j \in T^*_yN$. The *push-forward* of $\Phi$ by $\mathsf{f}$ is defined by $\mathsf{f}_*\Phi = (\mathsf{f}^{-1})^*\Phi$.

Given a kinematic metric function $\overline{\Phi}$ (on $\mathcal{C}$), the push-forward can be used to induce a metric function on $SE(3)$ given by $\Phi^{\widetilde{a},b} = (\Gamma^{\widetilde{a},b})_*\overline{\Phi}$ where $\Gamma^{\widetilde{a},b}$ is the c-space representation map corresponding to reference configuration $\chi_{\widetilde{a}}$ and embedded frame $\mathcal{F}^b_\chi$. Note that we write $\Phi \triangleq \Phi^{\widetilde{e},e} = \Gamma_*\overline{\Phi}$. Thus, we can investigate $\Phi \triangleq \Phi^{\widetilde{e},e}$ in terms of its left, right, or bi-invariance property. These invariances are reviewed as follows with the aid of Definition 7 (setting $M = N = SE(3)$, and $\mathsf{g}$ to a left or right translation).

Given any $g \in SE(3)$, define $L_g \colon SE(3) \to SE(3)$, called a *left translation*, and $R_g \colon SE(3) \to SE(3)$, a *right translation*, by $L_g(h) = gh$ and $R_g(h) = hg$ for

[3]In manifold theory [1], the *tangent map of* $\mathsf{f} \colon M \to N$ *at* $x \in M$ ($M$ and $N$ are manifolds) is a map $T_x\mathsf{f} \colon T_xM \to T_{\mathsf{f}(x)}N$ given by $T_x\mathsf{f}(\dot{c}(0)) = (\mathsf{f} \circ c)'(0)$ for any tangent vector $\dot{c}(0) \in T_xM$. The *tangent map of* $\mathsf{f}$ is a map $T\mathsf{f} \colon TM \to TN$ that is given by $T_x\mathsf{f}$ when restricted to $T_xM$ for each $x \in M$. On the other hand, if $\mathsf{f}$ is a diffeomorphism (a *diffeomorphism* is a smooth invertible map that has a smooth inverse), its *cotangent* is a map $T^*\mathsf{f} \colon T^*N \to T^*M$ given by $T^*\mathsf{f}(\beta) = \beta \circ T\mathsf{f}$ for any $\beta \in T^*N$.

$h \in SE(3)$, respectively. The pull-backs of $\Phi$ by such translations, i.e., $L_g^*\Phi$ and $R_g^*\Phi$, are metric functions on $SE(3)$, which are *generally different from* $\Phi$. We say that $\Phi$ is *left invariant* if $\Phi = L_g^*\Phi$, and *right invariant* if $\Phi = R_g^*\Phi$, for all $g \in SE(3)$. If $\Psi$ is both left and right invariant, it is said to be *bi-invariant.*

Now suppose that $\overline{\Phi}$ is objective. In Ref. [10] we prove that the invariance properties of the induced function $\Phi^{\widetilde{a},b}$ are characterized as follows.

**Proposition 1.** *A metric function $\overline{\Phi}$ on $C$ is objective if and only if $\Phi^{\widetilde{a},b} = (\Gamma^{\widetilde{a},b})_*\overline{\Phi}$, a metric function on $SE(3)$, is* ***left invariant****. In addition, $\Phi^{\widetilde{a},b}$ and $\Phi$ are related by $\Phi^{\widetilde{a},b} = R^*_{b^{-1}F(\widetilde{a})}\Phi$.*

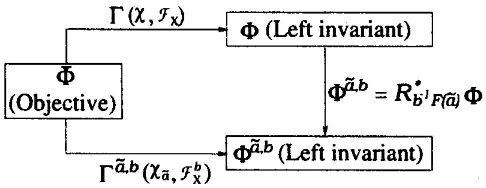


Fig. 6. Objectivity and invariances in $SE(3)$.

As illustrated in Fig. 6, this proposition yields the following interpretations of left, right and bi-invariances (similar interpretations have been given to Lagrangian density functions by Marsden and Hughes [12] in the context of Hamiltonian mechanics). If $\overline{\Phi}$ is objective, then $\Phi^{\widetilde{a},b}$ is left invariant but need **not** be right invariant, and in general **differs** for different choices of reference configuration and embedded frame (i.e., $\Phi^{\widetilde{a},b}$ is a different function for each choice of $\widetilde{a} \in \widetilde{SE}(3)$ and $b \in SE(3)$). In other words, *left invariance is necessary but not sufficient for objectivity in $C$,* since an objective function corresponds to a *family* of generally distinct left invariant functions, which can be indexed by the choices of embedded frame and reference configuration. When $\Phi^{\widetilde{a},b}$ happens to be right invariant for any $\widetilde{a}$ and $b$, then it is a *unique* bi-invariant function independent of $\widetilde{a}$ and $b$. Hence, *bi-invariance is sufficient, but* **not** *necessary for objectivity.* We can conclude that left invariance is justified by the physical requirement of observer-indifference, while viewing right invariance as an equal alternative to left invariance (e.g., [15]) is incorrect. Right invariant functions that are not left invariant do not lead to well-defined metric functions and should therefore be avoided.

These results also help us to more carefully interpret the works of others. For example, it has been shown that there does not exist a bi-invariant Riemannian metric on $SE(3)$. This fact has been interpreted by some to mean that there is no invariant way to measure the length of a velocity vector or a wrench. However, (see Section 8) objective, or frame-invariant, Riemannian metrics *do* exist. Furthermore, other functions can be used to measure velocity or wrench length in an invariant way. Prior works have also shown a preference for left invariant functions as a way to implement frame-invariance. However, our work shows that left-invariance is not a guarantee of frame-invariance.

## 7 Frame-Invariant Metric Functions

This section considers the notion of *frame-invariance*, which can be used as a practical condition to characterize well-defined metric functions in $SE(3)$. We use a conventional approach, which represents $\mathcal{B}$'s configuration as the displacement of a body frame relative to a world frame. Recall that a world frame is a stationary frame in $\mathbb{E}^3$, and that a body frame is fixed to $\mathcal{B}$ so that its particles have the same coordinates in this frame for all configurations of $\mathcal{B}$. The following convention is used.

**Notational Convention.** Denote a nominal world frame by $\mathcal{F}_W$ and a nominal body frame by $\mathcal{F}_B$. Arbitrary world and body frames are denoted by $\mathcal{F}_W^b$ and $\mathcal{F}_B^a$, respectively, where the superscripts indicate that $\mathcal{F}_W^b$ is displaced from $\mathcal{F}_W$ by $b \in SE(3)$, and that $\mathcal{F}_B^a$ is displaced from $\mathcal{F}_B$ by $a \in SE(3)$.

To describe objective functions using world and body frames, we must interpret locations of a body frame relative to a world frame in terms of the formal approach given in Section 3, where $C$ was represented by $SE(3)$ using a reference configuration and an embedded frame. As shown in Fig. 7, choose the nominal embedded frame $\mathcal{F}_X$ to be coincident with $\mathcal{F}_W$, and choose the nominal reference configuration $\chi_0$ such that $\mathcal{F}_B$ and $\mathcal{F}_W$ coincide when $\mathcal{B}$'s configuration is $\chi_0$. It follows from Section 3 that corresponding to a configuration $\chi \in C$, the location of $\mathcal{F}_B$ relative to $\mathcal{F}_W$ is $\Gamma(\chi)$.

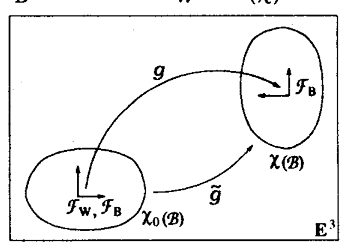


Fig. 7. A configuration specified as $\mathcal{F}_B$'s location relative to $\mathcal{F}_W$.

When the frames $\mathcal{F}_W^b$ and $\mathcal{F}_B^a$ are used, we need to choose an embedded frame $\mathcal{F}_X^c$ and reference configuration $\chi_{\widetilde{h}}$ such that $\Gamma^{\widetilde{h},c}(\chi)$ is the location of $\mathcal{F}_B^a$ relative to $\mathcal{F}_W^b$, where $\Gamma^{\widetilde{h},c}$ is the c-space representation map corresponding to $\mathcal{F}_X^c$ and $\chi_{\widetilde{h}}$. Assuming, without loss of generality, that either of the world or body frames remains unchanged, it can be verified [10] that $\mathcal{F}_X^c$ and $\chi_{\widetilde{h}}$ should be chosen as follows.

**Lemma 2.** *For $\Gamma^{\tilde{h},c}(\chi)$ to be $\mathcal{F}_B^a$'s location relative to $\mathcal{F}_W^b$, one should choose $\tilde{h}$ and $c$ as follows. When frames $\mathcal{F}_W^b$ and $\mathcal{F}_B$ are used (no body frame change), $\tilde{h} = F^{-1}(b)$ and $c = b$; when frames $\mathcal{F}_W$ and $\mathcal{F}_B^a$ are used (no world frame change), $\tilde{h} = F^{-1}(a^{-1})$ and $c = e$.*

While the embedded frame is stationary and always coincides with the given world frame, embedded and world frames play *different* roles. In particular, *a change of world frame usually involves changes in* **both** *embedded frame and reference configuration.*

According to Lemma 2, a configuration $\chi \in \mathcal{C}$, tangent vector $\overline{\mathbf{v}} \in T_\chi\mathcal{C}$ or covector $\overline{\boldsymbol{\alpha}} \in T_\chi^*\mathcal{C}$ is represented by a *different* element in $SE(3)$, $TSE(3)$ or $T^*SE(3)$ with respect to different choices of world and body frames. The relationship between these representations, as can be derived from Lemma 2 [10], is presented below using *body coordinates* [14] of tangent vectors and covectors on $SE(3)$, which are first briefly reviewed.

For $g \in SE(3)$ and $\mathbf{v} \in T_gSE(3)$ written in matrix notation, one has $g^{-1}\mathbf{v} = \left(\begin{smallmatrix}\widehat{\omega} & v\\ 0 & 0\end{smallmatrix}\right)$, where $v, \omega \in \mathbb{R}^3$ and $\widehat{\omega}$ is a skew symmetric matrix such that $\widehat{\omega}x = \omega \times x$ for all $x \in \mathbb{R}^3$. The body coordinates of $\mathbf{v}$, called a *body velocity*, are given by $\dot{q} = \left(\begin{smallmatrix}v\\ \omega\end{smallmatrix}\right)$, while the body coordinates of $\boldsymbol{\alpha}$, called a *body wrench*, are $\mathbf{w} = \left(\begin{smallmatrix}f\\ \tau\end{smallmatrix}\right) \in \mathbb{R}^3 \times \mathbb{R}^3$ such that $\mathbf{w}^T\dot{q} = \boldsymbol{\alpha}(\mathbf{v})$ for any $\mathbf{v} \in T_gSE(3)$ with body coordinates $\dot{q}$. It can be shown that, if $g$ is the location of a body frame relative to a world frame, then $v$ and $\omega$ give the translation and rotation of the body frame (with coordinates in the body frame), while $f$ and $\tau$ are a force and a torque with respect to the body frame.

**Lemma 3.** *Let $\chi \in \mathcal{C}$, $\overline{\mathbf{v}} \in T_\chi\mathcal{C}$ and covector $\overline{\boldsymbol{\alpha}} \in T_\chi^*\mathcal{C}$ be a configuration, a tangent vector and a covector. Suppose that they are respectively represented by a relative frame location $g \in SE(3)$, a body velocity $\dot{q} \in \mathbb{R}^6$, and a body wrench $\mathbf{w} \in \mathbb{R}^6$ with respect to the frames $\mathcal{F}_W$ and $\mathcal{F}_B$. Then with respect to the frames $\mathcal{F}_W^b$ and $\mathcal{F}_B^a$ they are respectively represented by $g^{\{b,a\}} = b^{-1}ga$, $\dot{q}^{\{b,a\}} \triangleq \dot{q}^a = Ad_a^{-1}\dot{q}$, and $\mathbf{w}^{\{b,a\}} \triangleq \mathbf{w}^a = Ad_a^T\mathbf{w}$, where $Ad_a = \left(\begin{smallmatrix}R & \widehat{d}R\\ 0 & R\end{smallmatrix}\right)$.*

A kinematic metric function $\overline{\Phi}$ on $\mathcal{C}$ induces a metric function $\Phi^{\{b,a\}} = (\Gamma^{\tilde{h},c})_*\overline{\Phi}$, where $\tilde{h}$ and $c$ are given by Lemma 2. Now suppose that $\overline{\Phi}$ is objective. Then, $\Phi^{\{b,a\}}$ is necessarily left invariant according to Proposition 1. In this case, it can be shown [10] that the function $\Phi^{\{b,a\}}$ can be represented as a world-frame-independent function of body coordinates, denoted $\phi^a$ and given by

$$\begin{aligned}&\Phi_g^{\{b,a\}}(\mathbf{v}_1, \ldots, \mathbf{v}_k, \boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_l)\\ &= \phi^a(\dot{q}_1^a, \ldots, \dot{q}_k^a, \mathbf{w}_1^a, \ldots, \mathbf{w}_l^a),\end{aligned}$$

where $g \in SE(3)$ (location of $\mathcal{F}_B^a$ relative to $\mathcal{F}_W^b$), and $\mathbf{v}_i \in T_gSE(3)$ and $\boldsymbol{\alpha}_j \in T_g^*SE(3)$, which have body coordinates $\dot{q}_i^a$ and $\mathbf{w}_j^a$, respectively.

Thus, objective kinematic metric functions can be expressed as functions of body coordinates. Then, it is practically interesting to seek a condition that a given function of body coordinates, say $\phi^a$, should satisfy in order to represent an objective kinematic metric function. The desired condition can be obtained from Proposition 1, and utilizes the following notion of frame invariance. For convenience we write $\phi \triangleq \phi^e$, which corresponds to the nominal frames $\mathcal{F}_W$ and $\mathcal{F}_B$.

**Definition 8.** A real-valued function $\phi^a$ of body velocities and wrenches is said to be **frame-invariant** if

$$\phi^a(\dot{q}_1^a, \ldots, \dot{q}_k^a, \mathbf{w}_1^a, \ldots, \mathbf{w}_l^a) = \phi(\dot{q}_1, \ldots, \dot{q}_k, \mathbf{w}_1, \ldots, \mathbf{w}_l),$$

where $\dot{q}_i^a = Ad_a^{-1}\dot{q}_i$ and $\mathbf{w}_j^a = Ad_a^T\mathbf{w}_j$.

**Proposition 4.** *A real-valued function $\phi^a$ of body coordinates determines an objective metric function on $\mathcal{C}$ if and only if it is frame-invariant.*

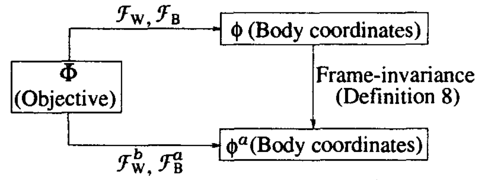


Fig. 8. Objectivity and frame-invariance.

Therefore, as illustrated in Fig. 8, the notion of frame-invariance provides a simple and practical test for objective kinematic metric functions. This notion should be distinguished from that of bi-invariance. Since bi-invariance is only sufficient for objectivity, it is a *stronger* condition than frame-invariance. As can be shown from Proposition 1 [10], a real-valued function $\phi^a$ of body coordinates determines a bi-invariant function if it is frame-invariant *and* satisfies the additional condition $\phi^a = \phi$ for any frame choices.

Finally, note that the use of body coordinates is important; it would be *incorrect* to use the alternative tangent vector and covector representation by "spatial coordinates" [14]. This is discussed in [10].

## 8 Examples

The first of the following two examples clarifies the difference between frame-invariance and bi-invariance, while the second one proposes an interesting velocity norm that is not induced from a Riemannian inner product.

**Example 1.** Consider a quadratic form in body velocities: $\phi^a(\dot{q}_1^a, \dot{q}_2^a) = (\dot{q}_1^a)^T M^a \dot{q}_2^a$, where $M^a$ is a $6 \times 6$ positive definite matrix. Suppose that $\phi^a$ is frame-invariant.

Then Definition 8 implies that

$$M^a = (Ad_a)^T M Ad_a, \tag{2}$$

where $M \triangleq M^e$. On the other hand, suppose that $\phi^a$ is a bi-invariant Riemannian metric. Then, as discussed in Section 7, $\phi^a = \phi$ and hence $M^a = M$ for all frame choices. Therefore, $M = (Ad_a)^T M Ad_a$ for all $a \in SE(3)$. Since no positive definite matrix $M$ exists to satisfy this condition [9], there exists *no* bi-invariant Riemannian metrics on $SE(3)$. However, there exist frame-invariant Riemannian metrics on $SE(3)$, such as the kinetic energy metric [10], since Eq. (2) is a weaker requirement than bi-invariance.

**Example 2.** We now present an interesting velocity norm. Let $\mathcal{B}$ also denote the region in $\mathbb{R}^3$ (with respect to a body frame $\mathcal{F}_B$) occupied by a rigid body $\mathcal{B}$. Given any body velocity $\dot{q} = (v, \omega)$ associated with $\mathcal{F}_B$, define the following real-valued function:

$$\phi(\dot{q}) = \max_{r \in \mathcal{B}} |\omega \times r + v|$$

This function is clearly frame-invariant, and, as shown below, is a norm. Letting $u_r(\dot{q}) = \omega \times r + v$, we see that $\phi$ is positive definite and homogeneous. In addition, for body velocities $\dot{q}_1$ and $\dot{q}_2$, we have $|u_r(\dot{q}_1) + u_r(\dot{q}_2)| \leqslant |u_r(\dot{q}_1)| + |u_r(\dot{q}_2)|$. Thus,

$$\begin{aligned}\phi(\dot{q}_1 + \dot{q}_2) &= \max_{r \in \mathcal{B}} |u_r(\dot{q}_1 + \dot{q}_2)| \\ &\leqslant \max_{r \in \mathcal{B}} |u_r(\dot{q}_1)| + \max_{r \in \mathcal{B}} |u_r(\dot{q}_2)|\end{aligned}$$

Hence, $\phi(\dot{q}_1 + \dot{q}_2) \leqslant \phi(\dot{q}_1) + \phi(\dot{q}_2)$. We have hence shown that $\phi$ satisfies the defining properties of a norm. Physically $\phi(\dot{q})$ corresponds to the *maximal velocity* of $\mathcal{B}$'s particles when $\mathcal{B}$ moves with speed $\dot{q}$. This norm is of practical interest—for example it can be used to define the maximal deflection of a compliantly fixtured object [10]. Also note that this norm is *not* inducible from a Riemannian metric. An appreciation of the fact that norms are a more basic concept than inner products gives us a richer choice of kinematic metric functions in practical applications.

Note that the frame-invariant norm $\phi$ can be used to consistently compare the fixtures in Fig. 1. Specifically, $\phi(\dot{q}_1) = \theta a$ and $\phi(\dot{q}_2) = 2\theta a$ regardless of frame choices, and the consistent conclusion can be reached that Fixture I allows the smaller displacement.

## 9 Conclusions

This paper considered formal and practical well-definedness conditions for kinematic metric functions, which are often needed for robotic task planning and analysis. Based on an intrinsic rigid body c-space definition, we applied the notion of objectivity (observer indifference) to kinematic metric functions. This approach yielded new insight into invariance properties on the Lie group $SE(3)$. The notion of frame-invariance has been clarified and developed into a practical test for well-defined kinematic metric functions.

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