On vector-valued twisted conjugation invariant functions on a group

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To Joseph Bernstein with deep admiration

Abstract. We study the space of vector-valued (twisted) conjugation invariant functions on a connected reductive group.

Contents

1. Introduction

We first describe the problems we are studying in this article. Let us fix an algebraically closed field k throughout this article. Let G be a connected simply-connected semisimple group over k, and let $k[G]$ denote its ring of regular functions. Consider the conjugation action of G on itself, and let $\boldsymbol{J} = k[G]^{c(G)}$ denote the space of conjugation invariant regular functions on G. More generally, for an algebraic representation V of G, let

$$\boldsymbol{J}(V) := (k[G] \otimes V)^{c(G)} = \{f : G \to V \mid f(gxg^{-1}) = g \cdot f(x), \quad x, g \in G\}$$

denote the $\boldsymbol{J}$-module of V-valued conjugation invariant (algebraic) functions on G, where G acts on $k[G] \otimes V$ diagonally. There are also similarly defined rings of invariants $\boldsymbol{J}_0$ and $\boldsymbol{J}_+$ and the corresponding modules $\boldsymbol{J}_0(V)$ and $\boldsymbol{J}_+(V)$, when G is replaced by its asymptotic cone As_G and the corresponding Vinberg monoid V_G. Our first result is as follows.

Theorem 1.0.1. *Assume that V admits a good filtration. Then the module $\boldsymbol{J}(V)$ (resp. $\boldsymbol{J}_+(V)$, resp. $\boldsymbol{J}_0(V)$) is finite free over $\boldsymbol{J}$ (resp. $\boldsymbol{J}_+$, resp. $\boldsymbol{J}_0$) of rank $\dim V(0)$, where $V(0)$ denotes the*

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zero weight space of V (with respect to a maximal torus of G). In addition, if $\operatorname{char} k = 0$*, there are natural bases of $\boldsymbol{J}(V)$, $\boldsymbol{J}_+(V)$, and $\boldsymbol{J}_0(V)$ determined by certain bases of V.*

We refer to § 4.4 for detailed discussions regarding to the last statement. This theorem resembles the classical result of Kostant ([**Ko63**]): Assume $\operatorname{char} k = 0$, and let $\mathfrak{g}$ be the Lie algebra of G. Then $(k[\mathfrak{g}] \otimes V)^G$ is a graded free $k[\mathfrak{g}]^G$-module with a basis given by harmonic polynomials. Recall that the study of $(k[\mathfrak{g}] \otimes V)^G$ leads to a natural filtration on the weight spaces (the Kostant–Brylinski filtration), which can be studied either via the geometry of the flag variety (as in [**Br89**]) or via the geometric Satake correspondence (as in [**Gi95**]). Likewise, our proof of the above theorem leads us to define a multi-filtration on the weight spaces (see § 3.3), which can also be studied either via the geometry of the flag variety (§ 3.4) or via the geometric Satake correspondence (§ 3.5). In particular, we obtain a multivariable analogue of the weight multiplicity $P_{\mu,\nu}(q)$ of a representation (see Remark 3.3.3). Note that Kostant's theorem has a very simple proof ([**BL96**]). It would be interesting to see whether there is a similar argument in the group case (at least when $\operatorname{char} k = 0$).

Note that finite freeness of $\boldsymbol{J}(V)$ over $\boldsymbol{J}$ was previously known to Richardson (see [**Ric79**]) when $\operatorname{char} k = 0$ and Donkin (see [**Do88b**] and the appendix) in positive characteristic, under the same assumption on V, but by a different method[1].

Now let V^* be the dual representation of V, which is also assumed to admit a good filtration. Then there is a natural $\boldsymbol{J}$-bilinear pairing

$$\boldsymbol{J}(V) \otimes \boldsymbol{J}(V^*) \to \boldsymbol{J} \tag{1.0.1}$$

induced by pairing V-valued functions with V^*-valued functions. One can show that this pairing is a certain deformation of the natural pairing between $V(0)$ and $V^*(0)$ (see Remark 6.1.3 (2)). Our main result (Theorem 6.1.2) calculates the determinant of this pairing as an element in $\boldsymbol{J}$ up to a unit (or more precisely as a divisor on $\operatorname{Spec} \boldsymbol{J}$.) We choose a maximal torus T of G. Let $\Phi(G,T)$ denote the root system of G. For a root α of G, let e^α denote the corresponding character function on T. For a weight λ of T, let $V(\lambda)$ denote the corresponding weight space. Using the classical Chevalley isomorphism, we identify $\boldsymbol{J} = k[G]^{c(G)}$ with the Weyl group invariant functions on T.

THEOREM 1.0.2. *Assume that* $\operatorname{char} k > 2$*. Then the determinant of the pairing* (1.0.1) *is*

$$c \prod_{\alpha \in \Phi(G,T)} (e^\alpha - 1)^{\zeta_\alpha},$$

where c is a non-zero constant and $\zeta_\alpha = \sum_{n \geq 1} \dim V(n\alpha)$.

As mentioned above, when $\operatorname{char} k = 0$, it is possible to construct bases of $\boldsymbol{J}(V)$ and $\boldsymbol{J}(V^*)$ so the pairing (1.0.1) can be represented by a matrix. We refer to § 6.4 for some examples where the matrices are calculated explicitly. Our main interest in these matrices (and their determinants) lies in the fact that under the Satake isomorphism they exactly correspond to the intersection matrices for certain cycles on the mod p fibers of some Shimura varieties; we refer to [**XZ17**] for more details.

In fact, in this article, we will consider a more general situation. Assume that τ is an automorphism of the algebraic group G. We consider the τ-twisted conjugation action of G on itself

$$c_\tau(h)(g) = hg\tau(h)^{-1}, \quad \text{for } g, h \in G. \tag{1.0.2}$$

This is equivalent to considering the usual conjugation action of G on the coset $G\tau$ inside the semidirect product $G \rtimes \langle \tau \rangle$. We can similarly define $\boldsymbol{J}(V) := (k[G] \otimes V)^{c_\tau(G)}$ as the space of V-valued functions f on G satisfying $f(gx\tau(g)^{-1}) = g \cdot f(x), \quad x, g \in G$, and in particular $\boldsymbol{J} := k[G]^{c_\tau(G)}$. We prove the above mentioned results for general τ (see Theorem 4.3.2 and Theorem 6.1.2 for

[1]For example, the method in *loc. cit.* requires the flatness of the Chevalley map $G \to G/\!\!/c(G)$ as an input, whereas our approach could deduce the flatness as a corollary (see Corollary 4.3.3).

the general statements). This generality is needed in [**XZ17**] for applications, but will cause some additional complications for this article. Readers who are happy with $\tau = \mathrm{id}$ can skip § 5 and parts of § 4 and § 6.

We expect that results in this article can be generalized to the case of symmetric pairs. In addition, by reformulating the argument purely algebraically, it is likely that they also extend to quantum groups.

We briefly describe the remaining parts of the note. In § 2, we discuss multi-filtered vector spaces and the corresponding Rees construction in some generality. Unlike the classical situation, the dimension of the associated graded vector space might be larger than the dimension of the original vector space. Whether the two dimensions are equal is a subtle question and is closely related to the flatness of certain Rees modules. We give some sufficient conditions for the equality, which might be of independent interest.

In § 3.1, we define the canonical filtration on a G-representations and study the associated Rees modules. We apply these discussions to the Vinberg monoid in § 3.2.

In § 5, we discuss twisted conjugacy classes of G for the action (1.0.2). In particular, we discuss the notion of τ-regular conjugacy class, and study the (twisted) Chevalley map and the Grothendieck–Springer resolution, generalizing some well-known results for $\tau = \mathrm{id}$.

In § 6.3, we study $\boldsymbol{J}(V \otimes V^*)$. It is naturally the space of endomorphisms of a vector bundle $\widetilde{V}$ on the quotient stack $[G/c_\tau(G)]$ for the τ-twisted conjugation action of G on itself, and therefore is a $\boldsymbol{J}$-algebra. This is non-commutative in general. But we will study a commutative subalgebra generated by a "tautological" element in $\gamma_{\mathrm{taut}} \in \boldsymbol{J}(V \otimes V^*)$.

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Notations and conventions. Throughout the note, let k denote an algebraically closed field. By a variety over k, we mean a separated, integral scheme of finite type over k. If X is an affine algebraic variety over k, let $k[X]$ denote the ring of regular functions on X. If $\mathfrak{X}$ is an algebraic stack of finite presentation over k, let $\mathrm{Coh}(\mathfrak{X})$ denote the category of coherent sheaves on $\mathfrak{X}$.

For an algebraic group H, let $\mathbb{B}H = [\operatorname{Spec} k/H]$ denote the classifying stack. We identify $\mathrm{Coh}(\mathbb{B}H)$ with the category $\mathrm{Rep}^f(H)$ of finite dimensional representations of H, and use them interchangeably. In particular, the trivial representation, sometimes denoted by $\mathbf{1}$ corresponds to the structure sheaf of $\mathbb{B}H$. If X is an H-space and V is an H-representation, let $V_{[X/H]} := X \times^H V$ be the locally free sheaf on $[X/H]$ via the usual associated construction. Alternatively, it is the pullback of V (as coherent sheaf on $\mathbb{B}H$) along the natural projection $[X/H] \to \mathbb{B}H$.

For a reductive group H acting on an affine variety X over k, we write $X/\!\!/H := \operatorname{Spec} k[X]^H$ for the GIT quotient. There is a natural morphism of stacks $[X/H] \to X/\!\!/H$.

The ind-completion $\mathrm{Rep}(H)$ of $\mathrm{Rep}^f(H)$ is the category of algebraic representations of H. If $H_1 \subset H$ is a closed subgroup, and V an H_1-representation, let

$$\operatorname{ind}_{H_1}^H V := (k[H] \otimes V)^{H_1} = \Gamma(H/H_1, V_{H/H_1})$$

be the induced H-representation. It is the *right* adjoint of the restriction functor $\operatorname{Res}_{H_1}^H$ from H-representations to H_1-representations. If no confusion will likely arise, for an H-representation V, we write V for $\operatorname{Res}_{H_1}^H V$ for simplicity.

For an algebraic group H, let $\mathrm{Dist}(H)$ denote the Hopf algebra of invariant distributions on H. Sometimes, $\mathrm{Dist}(H)$ is also called the hyperalgebra of H.

Let X be a set (or scheme) with an automorphism τ. We denote by X^τ the subset (subscheme) of fixed points. If X is an abelian group, let X_τ denote its group of coinvariants.

Throughout this note, $\mathbb{N}$ stands for the set of *nonnegative* integers.

2. Filtered vector spaces and Rees modules

2.1. Filtered vector spaces. We need to first discuss (multi)-filtered vector spaces and (multi)-graded modules. It seems best to start with the following setup. Let $(S, \leq)$ be a partially ordered set (or sometimes called a poset for simplicity). As usual, we write $s_1 < s_2$ if $s_1 \leq s_2$ but $s_1 \neq s_2$. Recall that S is called *directed* if for $s, s' \in S$, there exists $s'' \in S$ such that $s \leq s''$ and $s' \leq s''$. We say a partially ordered set *poly-directed* if $S = \sqcup_i S_i$ is a disjoint union of directed poset S_i, and if $s \in S_i$ and $s' \in S_j$ are incomparable if $i \neq j$.

Definition 2.1.1. Assume that S is poly-directed. We define an *S-filtered vector space* (or a vector space with an *S-filtration*) to be a vector space M, equipped with a collection of subspace $\{\mathrm{fil}_s M, s \in S\}$ such that $\mathrm{fil}_s M \subset \mathrm{fil}_{s'} M$ if $s \leq s'$, and that

$$M = \bigoplus_{S_i} \bigcup_{s \in S_i} \mathrm{fil}_s M. \tag{2.1.1}$$

Let $\mathrm{Vect}^{S-\mathrm{fil}}$ denote the category of S-filtered vector spaces, with morphisms given by k-linear maps preserving the filtrations. Similarly, we define an *S-graded vector space* to be a vector space M equipped with a direct sum decomposition $M = \oplus_{s \in S} M_s$ indexed by S. Let $\mathrm{Vect}^{S-\mathrm{gr}}$ denote the corresponding category. There is a natural functor

$$\mathrm{gr} : \mathrm{Vect}^{S-\mathrm{fil}} \to \mathrm{Vect}^{S-\mathrm{gr}}, \quad M \mapsto \mathrm{gr}_S M = \bigoplus_{s \in S} \mathrm{gr}_s M, \quad \mathrm{gr}_s M = \mathrm{fil}_s M \Big/ \sum_{s' < s} \mathrm{fil}_{s'} M.$$

In the above definition, if s is *minimal* in S, i.e. the set $\{s' \in S \mid s' < s\}$ is empty, then we set $\sum_{s' < s} \mathrm{fil}_{s'} M = 0$ so $\mathrm{gr}_s M = \mathrm{fil}_s M$. If S is clear from the context, we write $\mathrm{gr} M$ for $\mathrm{gr}_S M$ for simplicity.

We say a nonzero element $m \in \mathrm{gr} M$ is of *homogeneous* of degree s if $m \in \mathrm{gr}_s M$. For a homogeneous element m of degree s, a *lifting* of m is an element $\widetilde{m} \in \mathrm{fil}_s M$ whose projection to $\mathrm{gr}_s M$ is m.

Example 2.1.2. (1) If $S = \mathbb{N}$ equipped with the natural order, an S-filtered vector space is a vector space M equipped with an increasing filtration $\mathrm{fil}_0 M \subset \mathrm{fil}_1 M \subset \cdots$ in the usual sense such that $M = \cup_s \mathrm{fil}_s M$. In this case, for every non-zero element m, there is a unique s such that $m \in \mathrm{fil}_s M - \mathrm{fil}_{s-1} M$ and its projection to $\mathrm{gr}_s M$ is called the *symbol* of m, denoted by $\overline{m}$.

(2) Assume $S = \mathbb{N}^r$ with the partial order $(s_1, \ldots, s_r) \leq (s_1', \ldots, s_r')$ if $s_j \leq s_j'$ for every j. Let M be a vector space equipped with r independent increasing filtrations $\mathrm{fil}_0^j M \subset \mathrm{fil}_1^j M \subset \cdots$ such that $\bigcup_s \mathrm{fil}_s^j M = M$, for each $j = 1, \ldots, r$. Then one can define an S-filtration of M as follows: for $s = (s_1, \ldots, s_r)$, let $\mathrm{fil}_s M = \cap_j \mathrm{fil}_{s_j}^j M$.

The partially ordered sets we encounter in this article will satisfy the following descending chain condition

(DCC) Every descending chain $s_0 > s_1 > \cdots$ is finite.

It is clear that the condition (DCC) passes to subsets of a partially order subset.

The following lemma is usually referred to as the *graded Nakayama lemma*.

Lemma 2.1.3. *Let S be a poly-directed poset satisfying* (DCC). *Let M be an S-filtered vector space.*

(1) *If $\mathrm{gr} M = 0$, then $M = 0$.*

(2) *Let $\{m_i\}$ be a set of homogeneous elements that span* $\mathrm{gr}M$*, and let $\{\widetilde{m_i}\}$ be a set of liftings. Then $\{\widetilde{m_i}\}$ span M. In particular,* $\dim M \leq \dim \mathrm{gr}M$.

PROOF. (1) Assume $M \neq 0$. Then there should exist some $s_1 \in S$ such that $\mathrm{fil}_{s_1} M \neq 0$. Since $\mathrm{gr}M = 0$, there should be another $s_2 < s_1$ such $\mathrm{fil}_{s_2} M \neq 0$. In this way, one could produce a descending chain $s_1 > s_2 > \cdots$ of infinite length. Contradiction.

(2) We may assume that S is directed. Let $\widetilde{M} \subset M$ be the subspace spanned by $\{\widetilde{m_i}\}$, and let $M' = M/\widetilde{M}$, equipped with an S-filtration defined as $\mathrm{fil}_s M' = \mathrm{Im}(\mathrm{fil}_s M \to M')$. By construction, $\mathrm{gr}_S M' = 0$. Therefore $M' = 0$ by Part (1). □

REMARK 2.1.4. In the classical situation, i.e. Example 2.1.2 (1), it is always true that $\dim \mathrm{gr}M = \dim M$. One can also show that this is the case if S is as in Example 2.1.2 (2) with $r = 2$. However, the inequality $\dim \mathrm{gr}M \geq \dim M$ could be strict in general. For example, take S as in Example 2.1.2 (2) with $r = 3$ and let M be a 2-dimensional vector space equipped with three filtrations $0 = \mathrm{fil}_0^j M \subset \mathrm{fil}_1^j M \subset \mathrm{fil}_2^j M = M$, $j = 1, 2, 3$ such that $\mathrm{fil}_1^1 M, \mathrm{fil}_1^2 M, \mathrm{fil}_1^3 M$ are three different lines in M. Then if we define the S-filtration on M as in Example 2.1.2 (2), the graded spaces $\mathrm{gr}_{(1,2,2)}M, \mathrm{gr}_{(2,1,2)}M, \mathrm{gr}_{(2,2,1)}M$ are all nontrivial. It then follows that $\dim \mathrm{gr}M = 3 > 2 = \dim M$.

As we shall see soon, whether $\dim M = \dim \mathrm{gr}M$ is closely related to the flatness of certain Rees module. So it would be desirable to have some criteria for the equality. Here is one which is a direct consequence of the above lemma.

COROLLARY 2.1.5. *Let (S, M) be as in Lemma 2.1.3. If*

(*) *there is a basis $\{m_i\}$ of* $\mathrm{gr}M$ *consisting of homogeneous elements (such a basis is called a* homogeneous basis*), and for each i a lifting $\widetilde{m_i}$ of m_i, such that $\{\widetilde{m_i}\}$ form a basis of M,*

then $\dim M = \dim \mathrm{gr}M$.

Conversely, if $\dim M = \dim \mathrm{gr}M < \infty$*, then for every homogeneous basis of* $\mathrm{gr}M$*, any set of liftings of elements in this basis to elements in M form a basis of M.*

A particular situation where the above criterion is applicable is as follows.

LEMMA 2.1.6. *Consider an S-filtered module M as in Example 2.1.2 (2). Assume that there is a basis $\mathbb{B} = \{m_i\}$ of M such that for each j, the corresponding symbols $\{\overline{m_i}^j\}$ for the filtration $\{\mathrm{fil}^j\}$ form a basis of $\mathrm{gr}^j M = \oplus_d \mathrm{fil}_d^j M / \mathrm{fil}_{d-1}^j M$. Then* $\dim M = \dim \mathrm{gr}M$.

PROOF. For $m_i \in \mathbb{B}$, define its multidegree to be $\underline{d}(m_i) := (d_1(m_i), \ldots, d_r(m_i)) \in \mathbb{N}^r$ such that with respect to the jth filtration $\mathrm{fil}_\bullet^j M$, $m_i \in \mathrm{fil}_{d_j(m_i)}^j M - \mathrm{fil}_{d_j(m_i)-1}^j M$. Given $\underline{d} = (d_1, \ldots, d_r)$, let

$$\mathbb{B}_{\underline{d}} = \{m_i \in \mathbb{B} \mid d_j(m_i) = d_j\}, \quad \mathbb{B}_{\leq \underline{d}} = \sqcup_{\underline{d}' \leq \underline{d}} \mathbb{B}_{\underline{d}'}.$$

Then it is easy to show that $\mathbb{B}_{\leq \underline{d}}$ form a basis of $\mathrm{fil}_{\underline{d}} M$. Indeed, let $m \in \mathrm{fil}_{\underline{d}} M$, and write $m = \sum a_i m_i$ in terms of the basis $\mathbb{B}$. If not every element in $\{\underline{d}(m_i) \mid a_i \neq 0\}$ is $\leq \underline{d}$, we can find some $1 \leq j \leq r$ such that $d_j' := \max\{d_j(m_i) \mid a_i \neq 0\} > d_j$. Then projecting to $\mathrm{gr}_{d_j'}^j M$ gives $0 = \sum a_i \overline{m_i}^j$ for the sum over those m_i with $d_j(m_i) = d_j'$. This contradicts our assumption.

It follows that the projection of elements in $\mathbb{B}_{\underline{d}}$ in $\mathrm{gr}_{\underline{d}} M$ form a basis of $\mathrm{gr}_{\underline{d}} M$. The lemma follows from Corollary 2.1.5. □

We will also use of the following lemma.

LEMMA 2.1.7. *Let $(S, M, \{\mathrm{fil}_s M\}_{s \in S})$ be an S-filtered vector as above (in particular S is poly-directed), and assume that S satisfies* (DCC)*. Let $S' \subset S$ be a subset. Assume for every $s \in S$, the set $\{s' \in S' \mid s \leq s'\}$ is nonempty and has a unique least element, denoted by s^h. Then S' with the induced partial order is also poly-directed, and also satisfies (DCC). In addition,*

$\dim \mathrm{gr}_{S'} M \leq \dim \mathrm{gr}_S M$, *where* $\mathrm{gr}_{S'} M$ *denotes the associated graded of* M *with respect to the sub-filtration* $\{\mathrm{fil}_s M\}_{s \in S'}$. *In particular, if* $\dim M = \dim \mathrm{gr}_S M$, *then* $\dim M = \dim \mathrm{gr}_{S'} M$.

Proof. The statement for S' is clear. We prove the inequality. Then last equality follows from it by Lemma 2.1.3.

If $s \in S'$, the s-graded piece in $\mathrm{gr}_S M$ is denote by $\mathrm{gr}_s M$ while the s-graded piece in $\mathrm{gr}_{S'} M$ is denoted by $\mathrm{gr}'_s M$. For $t \in S'$, let $S_t = \{s \in S \mid s^h = t\} \subset S$. Note that this is a partially ordered set with the greatest element t and $S = \sqcup_{t \in S'} S_t$. We can define a filtration on $\mathrm{gr}'_t M$ labeled by S_t as $\mathrm{fil}_s \mathrm{gr}'_t M = \mathrm{Im}(\mathrm{fil}_s M \to \mathrm{gr}'_t M)$. Note that $\mathrm{fil}_s \mathrm{gr}'_t M \subset \mathrm{fil}_{s'} \mathrm{gr}'_t M$ if $s \leq s'$. Let

$$\mathrm{gr}_s \mathrm{gr}'_t M = \mathrm{fil}_s \mathrm{gr}'_t M \Big/ \sum_{s' < s,\, s' \in S_t} \mathrm{fil}_{s'} \mathrm{gr}'_t M.$$

Note that S_t also satisfies (DCC), so by Lemma 2.1.3, $\sum_{s \in S_t} \dim \mathrm{gr}_s \mathrm{gr}'_t M \geq \dim \mathrm{gr}'_t M$. On the other hand, for a fixed $s \in S_t$, if $s' < s$ and $s'^h \neq t$, then $s'^h < t$ and therefore the image of $\mathrm{fil}_{s'} M \to \mathrm{gr}'_t M$ is zero. It follows that $\mathrm{gr}_s \mathrm{gr}'_t M$ is a quotient of $\mathrm{gr}_s M = \mathrm{fil}_s M / \sum_{s' < s} \mathrm{fil}_{s'} M$. Therefore, it follows that for every $t \in S'$,

$$\sum_{s \in S_t} \dim \mathrm{gr}_s M \geq \sum_{s \in S_t} \dim \mathrm{gr}_s \mathrm{gr}'_t M \geq \dim \mathrm{gr}'_t M,$$

giving $\dim \mathrm{gr}_S M \geq \dim \mathrm{gr}_{S'} M$. □

2.1.8. *Monoid and partially ordered set.* The partially ordered sets we encounter in this paper will mostly arise in the following way. Let Γ be a commutative monoid, and let S be a non-empty set with a Γ-action (where multiplication and the action map will be written additively). Consider the following condition

(Can) The only invertible element in Γ is the unit element (such a monoid is called *sharp*), and the action of Γ on S is free, i.e. for $s \in S$ and $\gamma, \gamma' \in \Gamma$, the identity $s + \gamma = s + \gamma'$ implies $\gamma = \gamma'$.

This condition in particular implies that Γ is *integral*, i.e. the map from Γ to its group completion Γ^{gp} is injective. In addition, there is a well-defined partial order on S defined by $s \leq s'$ if $s' = s + \gamma$ for some $\gamma \in \Gamma$. Note that since $s_1 \leq s_2$ and $s_2 \leq s_1$ will imply $s_1 = s_2$, this is indeed a partial order.

Lemma 2.1.9. *The set* S *equipped with the above partial order is poly-directed.*

Proof. The only property of this partial order we need is the following: if $s \leq s_1$ and $s \leq s_2$, then there is s' such that $s_1 \leq s'$ and $s_2 \leq s'$. Indeed, by definition $s_1 = s + \gamma_1$ and $s = s + \gamma_2$. Then we can choose $s' = s + \gamma_1 + \gamma_2$. We show that any partially ordered set S satisfying this property is poly-directed.

Consider the equivalence relation on S generated by the partial order. It is enough to show that every equivalence class equipped with the induced partial order is directed. But if s, s' are in one equivalence class, then there exists a chain of elements $s = s_0, s_1, \ldots, s_r = s' \in S$ such that either $s_i \leq s_{i+1} \geq s_{i+2}$ or $s_i \geq s_{i+1} \leq s_{i+2}$. By replacing the second relation by $s_i \leq s'_{i+1} \geq s_{i+2}$ repeatedly, one find s'' such that $s \leq s'' \geq s'$. □

2.2. Rees construction. Let Γ be a commutative monoid acting on a set S. Let $R = k[\Gamma]$ be the monoid algebra. For $\gamma \in \Gamma$, the corresponding element in R is denoted by e^γ. Let $I_1 \subset R$ be the ideal generated by $e^\gamma - 1$ for all $\gamma \in \Gamma$, and let $I_0 \subset R$ be the ideal spanned by e^γ for all $0 \neq \gamma \in \Gamma$.

We define an *S-graded R-module* to be a k-vector space N with a direct sum decomposition $N = \oplus_{s \in S} N_s$, such that $e^\gamma N_s \subset N_{s+\gamma}$. They naturally form a category, with morphisms given

by R-module homomorphisms preserving the grading, denoted by $R\text{−Mod}^{S-\mathrm{gr}}$. There is a natural functor

$$R\text{−Mod}^{S-\mathrm{gr}} \to \mathrm{Vect}^{S-\mathrm{gr}}, \quad N \mapsto N/I_0N,$$

where $(N/I_0N)_s = N_s/\sum_{\gamma'+s'=s} e^{\gamma'}N_{s'}$. Given a homogeneous element $n \in N/I_0N$ of degree s, *a homogeneous lifting* of n to N is an element $\widetilde{n} \in N_s$ which projects to n.

If (Γ, S) satisfies (Can), there is also a natural functor

$$R\text{−Mod}^{S-\mathrm{gr}} \to \mathrm{Vect}^{S-\mathrm{fil}}, \quad N \mapsto N/I_1N,$$

where we define the S-filtration by $\mathrm{fil}_s(N/I_1N) = \mathrm{Im}(N_s \to N \to N/I_1N)$. The following lemma is another version of the graded Nakayama lemma.

LEMMA 2.2.1. *Let (Γ, S) be as above and assume that* (Can) *and* (DCC) *hold for (Γ, S). Let $R = k[\Gamma]$. Let N be an S-graded R-modules.*

1. *If $N/I_0N = 0$, then $N = 0$.*
2. *Let $\{n_i\}$ be a set of homogeneous elements that span N/I_0N, and let $\{\widetilde{n_i}\}$ be a set of homogeneous liftings. Then $\{\widetilde{n_i}\}$ generates N as an R-module.*
3. *Assume that in addition, the images of $\{\widetilde{n_i}\}$ in N/I_1N form a basis of N/I_1N. Then $N \cong \oplus_i R\widetilde{n_i}$. In particular, N is a free R-module.*

PROOF. The proof of Part (1) and (2) is the same as the proof of Lemma 2.1.3. For Part (3), let $\sum r_i\widetilde{n_i} = 0$ be a homogeneous linear relation in N, and write $r_i = a_ie^{\gamma_i}$. Then in N/I_1N, there is a linear relation $\sum a_i\widetilde{n_i} = 0$, which implies all $a_i = 0$. □

We continue to assume that (Γ, S) satisfies (Can). Then the functor $N \mapsto N/I_1N$ as above admits a right adjoint

$$\mathrm{Vect}^{S-\mathrm{fil}} \to R\text{−Mod}^{S-\mathrm{gr}}, \quad M \mapsto R_SM = \bigoplus_{s\in S} \mathrm{fil}_s M,$$

where we define the multiplication $e^\gamma : \mathrm{fil}_s M \to \mathrm{fil}_{s+\gamma} M$ (i.e. the R-module structure) to be the natural inclusion. The functor $M \to R_SM$ sometimes is called the *Rees construction* and R_SM is called the *Rees module*. It is convenient to introduce the formal symbols e^s, $s \in S$ and write elements in the degree s summand in R_SM as me^s for $m \in \mathrm{fil}_s M$. Then the R-module structure on R_SM is given by $e^\gamma \cdot me^s = me^{s+\gamma}$.

We need some criteria of flatness of R_SM over R. Note that if it is the case, then by (2.2.1) $\dim \mathrm{gr}M$ should be equal to $\dim M$ (if both are finite). It turns out that under some mild assumptions on M, this condition is also sufficient. First, note that

$$R_SM/I_0R_SM \cong \mathrm{gr}M, \quad R_SM/I_1R_SM \cong M, \tag{2.2.1}$$

where the second isomorphism makes use of (2.1.1).

LEMMA 2.2.2. *Let $(\Gamma, S, M, \{\mathrm{fil}_s M\}_{s\in S})$ be as above and assume that Conditions* (Can) *and* (DCC) *hold for (Γ, S). Let R_SM be the corresponding Rees module. Assume that Condition* (*) *in Corollary 2.1.5 holds. Then R_SM is free over R with basis $\{\widetilde{m_i}e^{s_i}\}$. In particular, if $\dim \mathrm{gr}M = \dim M < \infty$, then R_SM is free over R.*

PROOF. This follows from Lemma 2.2.1 and (2.2.1). □

2.2.3. *Rees algebra.* Let $(\Gamma, S, M, \{\mathrm{fil}_s M\}_{s\in S})$ be as before, where (Γ, S) satisfy condition (Can). If S itself is a commutative monoid such that the action of Γ on S is induced by a monoid homomorphism $f : \Gamma \to S$ (the action of S on itself is the natural translation), and if M is a (not necessarily commutative) k-algebra, then it makes sense to assume that the filtration $\{\mathrm{fil}_s M\}_{s\in S}$ satisfies the additional condition

(Alg) $k \subset \mathrm{fil}_0 M$ and $\mathrm{fil}_s M \cdot \mathrm{fil}_{s'} M \subset \mathrm{fil}_{s+s'} M$.

In this case, $R_S M$ is naturally a (not necessarily commutative) algebra over R, with the multiplication given by $me^s \cdot m'e^{s'} = mm'e^{s+s'}$, and the map $R \to R_S M$ is given by $e^\gamma \mapsto 1 \cdot e^{f(\gamma)}$. We call it the *Rees algebra* of M.

If in addition, M has a co-algebra structure such that each $\mathrm{fil}_s M$ is a sub-coalgebra, then $R_S M$ is also a coalgebra.

3. Filtration on representations

In this section, let G be a connected reductive group over k. We discuss the canonical filtration on G-representations and the associated Rees modules in § 3.1, and apply discussions to the Vinberg monoid in § 3.2. In § 3.3–3.5, we define and study a multi-filtration on the weight spaces of a representation of G. Just as the Kostant–Brylinski filtration plays an important role in the study of vector-valued invariant functions on $\mathfrak{g}$, this filtration is important for the study of vector-valued invariant functions on the group. We will make use of the following notations and conventions throughout this section.

We fix a maximal torus T of G, contained in a Borel subgroup B. Let U denote the unipotent radical of B. Let $\Phi = \Phi(G,T)$ denote the root system, and $\Delta \subset \Phi$ the set of simple roots with respect to B. For every $\alpha \in \Phi$, we fix an isomorphism $x_\alpha : \mathbb{G}_a \simeq U_\alpha$, where U_α is the root subgroup corresponding to α. The tuple $(G, B, T, \{x_\alpha\}_{\alpha\in\Delta})$ forms a pinning of G.

The Lie algebra $\mathfrak{u}_\alpha$ of U_α is spanned over k by $E_\alpha := \frac{d}{dy_\alpha}$ (here we regard $y_\alpha := x_\alpha^{-1}$ as a coordinate function on U_α). More generally, the algebra of invariant distributions $\mathrm{Dist}(U_\alpha)$ of U_α is spanned over k by $\{E_\alpha^{(n)},\ n \geq 0\}$, where $E_\alpha^{(n)}(y_\alpha^j) = \binom{j}{n} y_\alpha^{j-n}$. In particular, $E_\alpha^{(n)}(y_\alpha^j) = 0$ if $n > j$. Informally, we may think $E_\alpha^{(n)} = E_\alpha^n/n!$.

Let B^- be the opposite Borel with respect to (T,B) and U^- its unipotent radical. Let $N = N_G(T)$ be the normalizer of T in G, and let $W = N/T$ denote the (absolute) Weyl group. Let $w_0 \in W$ be the longest Weyl group element. Let $\mu \mapsto \mu^* := -w_0(\mu)$ be the involution on the character lattice $\mathbb{X}^\bullet(T)$, which preserves the set of dominant weights $\mathbb{X}^\bullet(T)^+$ (with respect to B). When we regard a weight $\nu \in \mathbb{X}^\bullet(T)$ as a regular function on T, we write it as e^ν.

Let Z_G denote the scheme-theoretic center of G. Let T_{ad} be the adjoint torus of G, i.e. the quotient of T by Z_G. Its character lattice $\mathbb{X}^\bullet(T_{\mathrm{ad}})$ is the subgroup of $\mathbb{X}^\bullet(T)$ generated by roots. We write $\mathbb{X}^\bullet(T_{\mathrm{ad}})_{\mathrm{pos}}$ for the monoid of nonnegative integer linear combinations of simple roots in Δ. We consider the partial order $\preceq$ on $\mathbb{X}^\bullet(T)$ induced by the action of $\mathbb{X}^\bullet(T_{\mathrm{ad}})_{\mathrm{pos}}$ in the sense of § 2.1.8, i.e. $\lambda_1 \preceq \lambda_2$ if and only if $\lambda_2 - \lambda_1$ is a nonnegative integral linear combination of simple roots of G.

For a root α, let G_α be the rank one subgroup of G generated by $T, U_\alpha, U_{-\alpha}$. Let $B_\alpha = TU_\alpha$ and $B_\alpha^- = TU_{-\alpha}$ be the pair of opposite Borel subgroups of G_α. We similarly have the partial order $\preceq_\alpha$ on $\mathbb{X}^\bullet(T)$ induced by the action of $\mathbb{Z}_{\geq 0}\alpha$. We say λ is *α-dominant* if $\langle \lambda, \alpha^\vee \rangle \geq 0$, where $\alpha^\vee$ is the coroot corresponding to α. Note that if $0 \preceq_\alpha \lambda$, then λ is α-dominant.

For a weight $\nu \in \mathbb{X}^\bullet(T)$, let k_ν denote the corresponding one-dimensional T-module. For a representation V of T and $\nu \in \mathbb{X}^\bullet(T)$, we write $V(\nu)$ for the ν-weight space, so $V(\nu) \cong \mathrm{Hom}_T(k_\nu, V) \otimes k_\nu$.

3.1. The canonical filtration on G-modules. We first review Weyl and Schur modules. Via inflation, the T-module k_ν can be regarded as a representation of B or B^-. Let

$$\mathtt{S}_\nu := \mathrm{ind}_{B^-}^G k_\nu \cong \mathrm{ind}_B^G k_{w_0(\nu)}$$

be the *Schur module* of highest weight ν, and let

$$\mathtt{W}_\nu := \mathtt{S}_{\nu^*}^*$$

denote the *Weyl module* of highest weight ν. More geometrically, we write

$$\mathcal{O}_{G/B}(\nu) = G \times^B k_\nu, \quad \mathcal{O}_{G/B^-}(\nu) = G \times^{B^-} k_\nu$$

to denote the line bundle on the flag variety. Then

$$\mathtt{S}_\nu = \Gamma(G/B^-, \mathcal{O}(\nu)) = \Gamma(G/B, \mathcal{O}(w_0(\nu))).$$

It is known that $\mathtt{S}_\nu = \mathtt{W}_\nu = 0$ unless ν is dominant.

We call a dominant weight $\omega \in \mathbb{X}^\bullet(T)^+$ *minuscule* if all weights in $\mathtt{S}_\omega$ form a single orbit under the action of the Weyl group W. Note that in this case, the multiplicity of each weight space is one-dimension and $\mathtt{S}_\omega \cong \mathtt{W}_\omega$. The set of minuscule weights is denoted by $\mathtt{Min} \subset \mathbb{X}^\bullet(T)^+$. Note that in our convention, the zero weight is minuscule.

LEMMA 3.1.1. *Let $\mathbb{X}^\bullet(T)^+_{\mathrm{pos}} \subset \mathbb{X}^\bullet(T)$ be the submonoid generated by $\mathbb{X}^\bullet(T_{\mathrm{ad}})_{\mathrm{pos}}$ and $\mathbb{X}^\bullet(T)^+$. Then under the natural action of the monoid $\mathbb{X}^\bullet(T_{\mathrm{ad}})_{\mathrm{pos}}$,*

$$\mathbb{X}^\bullet(T)^+_{\mathrm{pos}} = \bigsqcup_{\omega \in \mathtt{Min}} \big(\omega + \mathbb{X}^\bullet(T_{\mathrm{ad}})_{\mathrm{pos}}\big).$$

In particular, the pair $(\Gamma, S) = (\mathbb{X}^\bullet(T_{\mathrm{ad}})_{\mathrm{pos}}, \mathbb{X}^\bullet(T)^+_{\mathrm{pos}})$ satisfies Conditions (DCC) *and* (Can) *in the previous section.*

PROOF. It is known that the set $\mathtt{Min}$ gives a collection of coset representatives of the quotient $\mathbb{X}^\bullet(T)/\mathbb{X}^\bullet(T_{\mathrm{ad}})$ so that every $\lambda \in \mathbb{X}^\bullet(T)$ can be uniquely written as $\lambda = \gamma_\lambda + \omega_\lambda$ with $\gamma_\lambda \in \mathbb{X}^\bullet(T_{\mathrm{ad}})$ and $\omega_\lambda \in \mathtt{Min}$. In addition, if $\lambda \in \mathbb{X}^\bullet(T)^+$, then $\gamma_\lambda \in \mathbb{X}^\bullet(T_{\mathrm{ad}})_{\mathrm{pos}}$. The lemma then clearly follows. □

For a representation V of G, we define an $\mathbb{X}^\bullet(T)^+_{\mathrm{pos}}$-filtration on V, called *the canonical filtration* of V,[2] as follows. For $\lambda \in \mathbb{X}^\bullet(T)$, we denote by $V_{\preceq\lambda}$ the *maximal* subrepresentation of G such that $V_{\preceq\lambda}(\nu) \neq 0$ implies $\nu \preceq \lambda$. Clearly, $V_{\preceq\lambda} \neq 0$ only if $\lambda \in \mathbb{X}^\bullet(T)^+_{\mathrm{pos}}$. Moreover, the functor $V \mapsto V_{\preceq\lambda}$ is left exact. Therefore, we obtained the quadruple

$$(\Gamma, S, M, \{\mathrm{fil}_s M\}_{s\in S}) = \big(\mathbb{X}^\bullet(T_{\mathrm{ad}})_{\mathrm{pos}}, \mathbb{X}^\bullet(T)^+_{\mathrm{pos}}, V, \{V_{\preceq\lambda}\}_{\lambda \in \mathbb{X}^\bullet(T)^+_{\mathrm{pos}}}\big).$$

Let $R = k[\mathbb{X}^\bullet(T_{\mathrm{ad}})_{\mathrm{pos}}]$, and let $R_{\mathbb{X}^\bullet(T)^+_{\mathrm{pos}}} V$ be the associated Rees module, which is an $\mathbb{X}^\bullet(T)^+_{\mathrm{pos}}$-graded R-module. Note that the functor

$$G\text{−Mod} \to R\text{−Mod}^{\mathbb{X}^\bullet(T)^+_{\mathrm{pos}}-\mathrm{gr}}, \quad V \mapsto R_{\mathbb{X}^\bullet(T)^+_{\mathrm{pos}}} V$$

is left exact.

There is an important class of G-modules, whose Rees module associated to the canonical filtration is R-flat. Recall that a *good filtration* of a representation V of G is a filtration of V by G-submodules (in the classical sense as in Example 2.1.2 (1)) whose associated graded are Schur modules. We recall some important properties of this class of representations.

THEOREM 3.1.2.
(1) *If V admits a good filtration, then its restriction to every Levi subgroup $M \subset G$ also admits a good filtration (as an M-module).*
(2) *The tensor product of two G-modules that admit a good filtration also admits a good filtration.*
(3) *The following are equivalent.*
   - *V admits a good filtration.*
   - *$\mathrm{Ext}^i(\mathtt{W}_{\nu^*}, V) = \mathrm{H}^i(G, \mathtt{S}_\nu \otimes V) = 0$ for every dominant weight ν and every $i > 0$.*
   - *$\mathrm{Ext}^1(\mathtt{W}_{\nu^*}, V) = \mathrm{H}^1(G, \mathtt{S}_\nu \otimes V) = 0$ for every dominant weight ν.*

[2] This is closely related, but not the same as the notion of canonical filtration in [**Ma90**, §3]. In particular, our definition is independent of any choice.

(4) *Regard $k[G]$ as a $G \times G$-bimodule via left and right translations. Then $k[G]$ admits a good filtration. For every dominant weight ν, $\mathtt{S}_\nu \otimes \mathtt{S}_{\nu^*}$ appears in the composition factors of any good filtration of $k[G]$ exactly once.*

Part (1) and (2) are due to Mathieu [**Ma90**] (and were already obtained earlier by Donkin [**Do85**] in most cases) and (3) is due to Donkin [**Do81**]. Part (4) is due to Donkin [**Do88a**] and independently Koppinen [**Ko84**]. It follows easily from Part (3) that the number of factors in the successive quotients of a good filtration of V that are isomorphic to $\mathtt{S}_\nu$ is equal to $\dim \mathrm{Hom}_G(\mathtt{W}_\nu, V)$, and therefore is independent of the choice of the good filtration. In particular, if V is finite dimensional

$$\dim V = \sum_\nu \dim \mathrm{Hom}_G(\mathtt{W}_\nu, V) \cdot \dim \mathtt{S}_\nu. \tag{3.1.1}$$

The main result of this subsection is as follows.

PROPOSITION 3.1.3. *Let V be a (finite dimensional) G-module that admit a good filtration. Then the Rees module $R_{\mathbb{X}^\bullet(T)^+_{\mathrm{pos}}}V$ associated to the canonical filtration is a (finite) flat R-module.*

PROOF. As every G-module V that admits a good filtration is the union of finite dimensional G-submodules V_i that admit a good filtration, and as $R_{\mathbb{X}^\bullet(T)^+_{\mathrm{pos}}}V$ is the union of $R_{\mathbb{X}^\bullet(T)^+_{\mathrm{pos}}}V_i$, we may assume that V is finite dimensional. Now by Lemma 2.2.2, it suffices to show that $\dim V = \dim \mathrm{gr}V$. But this follows from (3.1.1) and the following lemma. □

LEMMA 3.1.4. *Let V be a representation of G that admits a good filtration. Let λ be a weight.*

(1) *Both $V_{\preceq\lambda} \subset V$ and $V/V_{\preceq\lambda}$ admit good filtrations.*
(2) *Let $V_{\prec\lambda} = \sum_{\lambda' \preceq \lambda} V_{\preceq\lambda'} \subset V_{\preceq\lambda}$. Then $V_{\prec\lambda}$ admits a good filtration, and $V_{\preceq\lambda}/V_{\prec\lambda}$ is isomorphic to $\mathrm{Hom}_G(\mathtt{W}_\lambda, V) \otimes \mathtt{S}_\lambda$.*

PROOF. This is a special case of [**Do85**, 12.1.6]. We include a proof for completeness. (1) By Theorem 3.1.2 (3), it is enough to show that for every dominant ν, the map

$$\mathrm{Hom}_G(\mathtt{W}_\nu, V) \to \mathrm{Hom}_G(\mathtt{W}_\nu, V/V_{\preceq\lambda}) \tag{3.1.2}$$

is surjective. If $\nu \preceq \lambda$, then $\mathrm{Hom}_G(\mathtt{W}_\nu, V/V_{\preceq\lambda}) = \mathrm{Hom}_G(\mathtt{W}_\nu, (V/V_{\preceq\lambda})_{\preceq\lambda}) = 0$, since $(V/V_{\preceq\lambda})_{\preceq\lambda}$ is evidently zero. Assume $\nu \not\preceq \lambda$. By Frobenius reciprocity, the map (3.1.2) may be identified with the map $\mathrm{Hom}_B(k_\nu, V) \to \mathrm{Hom}_B(k_\nu, V/V_{\preceq\lambda})$. Let $L \subset V(\nu)$ be a line, such that B acts on its image in $V/V_{\preceq\lambda}$ via the character $B \to T \xrightarrow{\nu} \mathbb{G}_m$. Then B acts on L by the same way (as any weights $\succeq \nu$ does not appear in $V_{\preceq\lambda}$). Therefore (3.1.2) is an isomorphism in this case.

(2) The same argument as in the proof of Part (1) shows that

- both $V_{\prec\lambda}$ and $V_{\preceq\lambda}/V_{\prec\lambda}$ admit good filtrations; and that
- the natural map $\mathrm{Hom}_G(\mathtt{W}_\nu, V_{\preceq\lambda}) \to \mathrm{Hom}_G(\mathtt{W}_\nu, V_{\preceq\lambda}/V_{\prec\lambda})$ is zero unless $\nu = \lambda$, in which case, the map is an isomorphism.

Now the isomorphism $V_{\preceq\lambda}/V_{\prec\lambda} \cong \mathrm{Hom}_G(\mathtt{W}_\lambda, V) \otimes \mathtt{S}_\lambda$ follows by combining the isomorphisms

- $V_{\preceq\lambda}/V_{\prec\lambda} \cong (V_{\preceq\lambda}/V_{\prec\lambda})(\lambda) \otimes \mathtt{S}_\lambda$;
- $(V_{\preceq\lambda}/V_{\prec\lambda})(\lambda) \cong \mathrm{Hom}_G(\mathtt{W}_\lambda, V_{\preceq\lambda}/V_{\prec\lambda}) \cong \mathrm{Hom}_G(\mathtt{W}_\lambda, V_{\preceq\lambda}) \cong \mathrm{Hom}_G(\mathtt{W}_\lambda, V)$. □

We end this subsection with the following results.

LEMMA 3.1.5. *Let V be a G-module that admits a good filtration, and let $\lambda \in \mathbb{X}^\bullet(T)^+_{\mathrm{pos}}$. Then the following sequence is exact*

$$0 \to V_{\preceq\lambda-\alpha_1-\cdots-\alpha_r} \to \cdots \to \bigoplus_{i<j} V_{\preceq\lambda-\alpha_i-\alpha_j} \to \bigoplus_i V_{\preceq\lambda-\alpha_i} \to V_{\preceq\lambda} \to V_{\preceq\lambda}/V_{\prec\lambda} \to 0. \tag{3.1.3}$$

PROOF. Note that $(e^{\alpha_1}, \dots, e^{\alpha_r})$ form a regular sequence in R generating the ideal $I_0 \subset R$, and since $R_{\mathbb{X}^\bullet(T)^+_{\mathrm{pos}}} V$ is R-flat, they form a regular sequence in $R_{\mathbb{X}^\bullet(T)^+_{\mathrm{pos}}} V$. The lemma follows from taking the λ-graded piece of the corresponding Koszul complex. □

COROLLARY 3.1.6. *In the exact sequence* (3.1.3)*, both the kernel and the image of* $\bigoplus_i V_{\preceq \lambda - \alpha_i} \to V_{\preceq \lambda}$ *admit good filtrations.*

PROOF. By Theorem 3.1.2 (4) and Lemma 3.1.4, each term in (3.1.3) admits a good filtration. The corollary follows from the criteria for existence of good filtrations in Theorem 3.1.2 (3). □

3.2. Vinberg monoid via the canonical filtration. We apply the previous discussion to the $G \times G$-modules $k[G]$, with the $G \times G$-module structure given by left and right translation of G on itself. We regard a pair of weights (ν_1, ν_2) as a weight of $T \times T$. Then for $\nu \in \mathbb{X}^\bullet(T)$, by regarding $k[G]$ as a $G \times G$-representation, we can define

$$\mathrm{fil}_\nu k[G] := k[G]_{\preceq (\nu^*, \nu)}. \tag{3.2.1}$$

LEMMA 3.2.1. *The above* $\mathbb{X}^\bullet(T)^+_{\mathrm{pos}}$ *filtration* $\{\mathrm{fil}_\mu k[G]\}_{\mu \in \mathbb{X}^\bullet(T)^+_{\mathrm{pos}}}$ *of* $k[G]$ *satisfies the following properties.*

(1) $\mathrm{fil}_\nu k[G] \subset \mathrm{fil}_{\nu+\lambda} k[G]$ *if* $\lambda \in \mathbb{X}^\bullet(T_{\mathrm{ad}})_{\mathrm{pos}}$.
(2) $\mathrm{fil}_\nu k[G] \cdot \mathrm{fil}_{\nu'} k[G] \subset \mathrm{fil}_{\nu+\nu'} k[G]$;
(3) *Each* $\mathrm{fil}_\nu k[G]$ *is a sub-coalgebra of* $k[G]$.
(4) $\mathrm{gr}\, k[G] = \oplus_{\nu \in \mathbb{X}^\bullet(T)^+} \mathtt{S}_{\nu^*} \otimes \mathtt{S}_\nu$;

PROOF. Properties (1)–(3) are clear. Property (4) follows from Theorem 3.1.2 (4) and Proposition 3.1.3. □

Write $T^+_{\mathrm{ad}} = \mathrm{Spec}\, R$. This is a natural monoid (in fact, the affine space with coordinate function indexed by simple roots, and equipped with coordinate multiplication), containing the adjoint torus T_{ad} of G as the open subset of the group of invertible elements. In particular, R has a coalgebra structure. To avoid possible confusion of notations in later discussions, we write $\bar{e}^\alpha$ (instead of e^α) for the coordinate function on T^+_{ad} corresponding the simple root α. The comultiplication sends $\bar{e}^\alpha$ to $\bar{e}^\alpha \otimes \bar{e}^\alpha$.

According to the discussion of Rees algebra in § 2.2.3, $R_{\mathbb{X}^\bullet(T)^+_{\mathrm{pos}}} k[G]$ is an R-algebra, and the map $R \to R_{\mathbb{X}^\bullet(T)^+_{\mathrm{pos}}} k[G]$ is also a coalgebra homomorphism. Then

$$V_G := \mathrm{Spec}\, R_{\mathbb{X}^\bullet(T)^+_{\mathrm{pos}}} k[G]$$

is a monoid, which is usually called the *Vinberg monoid* (at least when G is semisimple and simply-connected, see Remark 3.2.3 below). In addition, it is equipped with a monoid homomorphism

$$\mathfrak{d} : V_G \to T^+_{\mathrm{ad}}, \tag{3.2.2}$$

usually called the *abelianization map*. We pick two distinguished representatives for the open and closed T_{ad}-orbit on T^+_{ad}: $\mathbf{1} = (1, \dots, 1), \mathbf{0} = (0, \dots, 0) \in \mathbb{A}^r \cong T^+_{\mathrm{ad}}$. Then by (2.2.1),

$$\mathfrak{d}^{-1}(\mathbf{1}) \cong \mathrm{Spec}(k[G]) = G, \quad \mathfrak{d}^{-1}(\mathbf{0}) \cong \mathrm{Spec}(\mathrm{gr}\, k[G]) =: \mathrm{As}_G\,.$$

The affine scheme As_G is usually called the *asymptotic cone* of G. We will make use of the following basic facts.

PROPOSITION 3.2.2. (1) *Let* $G \times^{Z_G} T$ *be the quotient of* $G \times T$ *by the action of* Z_G *given by* $z \cdot (g, t) = (zg, zt)$. *The affine monoid* V_G *contains the open affine scheme* $G \times^{Z_G} T$ *as the group of invertible elements, such that the abelianization map* (3.2.2) *extends the natural group homomorphism* $G \times^{Z_G} T \to T_{\mathrm{ad}}$, $(g, t) \mapsto (t \mod Z_G)$. *In particular, there is a natural* $G \times G \times T$*-action on* V_G*, where* $G \times G$ *acts on* V_G *by left and right translation and* T *acts on* V_G *by multiplication.*

(2) *The map $\mathfrak{d}$ is faithfully flat.*

PROOF. Part (1) is clear. Part (2) follows from Proposition 3.1.3. □

REMARK 3.2.3. The original construction of the Vinberg monoid as in [**Vi95**] (when $\operatorname{char} k = 0$) and in [**Rit01**] (in general) is different. However, it is easy to see that V_G is uniquely characterized by the properties in Proposition 3.2.2 and the fact that $\mathfrak{d}^{-1}(\mathbf{0}) = \mathrm{As}_G$. Therefore V_G is indeed what people usually call the Vinberg monoid.

Let V_T be the closure of $T \times^{Z_G} T \subset G \times^{Z_G} T$ in V_G. The image of the $\mathbb{X}^\bullet(T)^+_{\mathrm{pos}}$-filtration on $k[G]$ under the map $k[G] \to k[T]$ defines an $\mathbb{X}^\bullet(T)^+_{\mathrm{pos}}$-filtration on $k[T]$ given by

$$\mathrm{fil}_\nu k[T] = \bigoplus_{\lambda_{\mathrm{dom}} \preceq \nu} k \cdot e^\lambda, \tag{3.2.3}$$

where for a weight λ, λ_{dom} denotes the unique dominant element in the Weyl group orbit $W\lambda$ of λ. Then the embedding $T \times^{Z_G} T \to V_T$ is given by

$$k[V_T] = \bigoplus_{(\lambda,\nu),\ \nu-\lambda_{\mathrm{dom}} \in \mathbb{X}^\bullet(T_{\mathrm{ad}})_{\mathrm{pos}}} k(e_1^\lambda \otimes e_2^\nu) \subset \bigoplus_{(\lambda,\nu),\ \nu-\lambda \in \mathbb{X}^\bullet(T_{\mathrm{ad}})} k(e_1^\lambda \otimes e_2^\nu) = k[T \times^{Z_G} T],$$

where e_i^μ are the corresponding character functions on the ith factor of $T \times^{Z_G} T$, and the map $V_T \to T^+_{\mathrm{ad}}$ is given by $\bar{e}^\lambda \mapsto 1 \otimes e_2^\lambda$ for $\lambda \in \mathbb{X}^\bullet(T_{\mathrm{ad}})_{\mathrm{pos}}$.

3.3. The filtration on the weight spaces. To prepare our study of vector-valued conjugation invariant functions $\boldsymbol{J}_G(V)$, we need to introduce a different filtration on each weight space of a representation V of G. This is not directly related to the filtration we discussed in the previous subsections. Fix a simple root α of G. Let V be a representation of G and let ν be a weight of T. Define a filtration on $V(\nu)$ as follows

$$\mathrm{fil}_i^\alpha V(\nu) := V(\nu) \cap (\mathrm{Res}^G_{G_\alpha} V)_{\preceq_\alpha \nu + i\alpha}. \tag{3.3.1}$$

There are two equivalent descriptions of the filtration. First, we claim that

$$\mathrm{fil}_i^\alpha V(\nu) = \ker\Big(\bigoplus_{j\geq 1} E_\alpha^{(i+j)} : V(\nu) \to \bigoplus_{j\geq 1} V(\nu + (i+j)\alpha)\Big), \tag{3.3.2}$$

Indeed, let v be a vector in the right hand side of (3.3.2). Then $\mathrm{Dist}(G_\alpha)v$ is a G_α-module whose weights $\preceq_\alpha \nu + i\alpha$. Conversely, if $v \in V(\nu) \cap V_{\preceq_\alpha \nu + i\alpha}$, then clearly $E_\alpha^{(i+j)} v = 0$ for every $j \geq 1$. The claim follows.

In addition, recall that if V is a finite dimensional representation of G_α whose weights $\preceq_\alpha \lambda$, then all of its weights $\succeq_\alpha s_\alpha(\lambda)$. It follows that we can also define the filtration as

$$\mathrm{fil}_i^\alpha V(\nu) = \ker\Big(\bigoplus_{j\geq 1} F_\alpha^{(\langle \nu, \alpha^\vee\rangle + i + j)} : V(\nu) \to \bigoplus_{j\geq 1} V\big(\nu - (\langle \nu, \alpha^\vee\rangle + i + j)\alpha\big)\Big). \tag{3.3.3}$$

Next we define the multi-filtration we need. Let T_{sc} denote the preimage of T in the simply-connected cover G_{sc} of G, and $\Gamma = S = \mathbb{X}^\bullet(T_{\mathrm{sc}})^+$ be the monoid of dominant weights, which acts on itself by translations. We identify

$$\mathbb{X}^\bullet(T_{\mathrm{sc}})^+ \cong \mathbb{N}^\Delta,\ \mu \mapsto (\langle \mu, \alpha^\vee\rangle)_{\alpha\in\Delta}. \tag{3.3.4}$$

Under this identification, the partial order on $\mathbb{X}^\bullet(T_{\mathrm{sc}})^+$ induced by the translation action (as in § 2.1.8) is just the standard partial order on $\mathbb{N}^\Delta$, as in Example 2.1.2 (2). (Note that this is different from the restriction to $\mathbb{X}^\bullet(T_{\mathrm{sc}})^+$ of partial order $\preceq$ on $\mathbb{X}^\bullet(T_{\mathrm{sc}})$ induced by the action of $\mathbb{X}^\bullet(T_{\mathrm{ad}})_{\mathrm{pos}}$.) We define an $\mathbb{X}^\bullet(T_{\mathrm{sc}})^+$-filtration on $V(\nu)$ as in Example 2.1.2 (2), i.e.

$$\mathrm{fil}_\lambda V(\nu) = \bigcap_{\alpha\in\Delta} \mathrm{fil}^\alpha_{\langle \lambda, \alpha^\vee\rangle} V(\nu).$$

We obtain a quadruple $(\Gamma, S, M, \{\mathrm{fil}_s M\}_{s\in S}) = \big(\mathbb{X}^\bullet(T_{\mathrm{sc}})^+, \mathbb{X}^\bullet(T_{\mathrm{sc}})^+, V(\nu), \{\mathrm{fil}_\lambda V(\nu)\}_{\lambda\in\mathbb{X}^\bullet(T_{\mathrm{sc}})^+}\big)$. The main result of this section is

THEOREM 3.3.1. *Assume that V admits a good filtration, then* $\dim \mathrm{gr}V(\nu) = \dim V(\nu)$.

We prove this theorem here, assuming two ingredients that will be established in § 3.4–§ 3.5.

PROOF. We first reduce to the case when V is a Schur module. Indeed, suppose we have a short exact sequence of G-modules

$$0 \to V' \to V \to V'' \to 0.$$

We deduce easily from (3.3.2) or (3.3.3) an exact sequence $0 \to \mathrm{fil}_\lambda V'(\nu) \to \mathrm{fil}_\lambda V(\nu) \to \mathrm{fil}_\lambda V''(\nu)$. In addition, if V' admits a good filtration, by Theorem 3.1.2 and Proposition 3.4.1 below, this sequence is also exact on the right.

It follows that it is enough to prove the theorem for a Schur module V. In this case, by Lemma 2.1.6, it is a consequence of the following proposition. □

PROPOSITION 3.3.2. *Let V be a Schur module. Then there exists a basis $\{v_j\}$ of $V(\nu)$ such that for every simple root α, the corresponding symbols $\{\overline{v_j}\}$ for the filtration fil^α form a basis of the associated graded $\mathrm{gr}^\alpha V(\nu) = \oplus_i \mathrm{fil}^\alpha_i V(\nu)/\mathrm{fil}^\alpha_{i-1} V(\nu)$.*

There are several ways to construct such a basis. For example, the canonical basis constructed by Lusztig and Kashiwara, or the semi-canonical basis constructed by Lusztig satisfies the required properties in the proposition (see [**Lu00**, Theorem 3.1 and Corollary 3.9]). At the end of § 3.5, we will give an alternative construction of a basis with the needed properties using the MV basis from the geometric Satake correspondence.[3]

REMARK 3.3.3. For $\mu, \nu \in \mathbb{X}^\bullet(T)$ with μ dominant, we define

$$P_{\mu,\nu}(q) := \sum_{\lambda\in\mathbb{X}^\bullet(T_{\mathrm{sc}})^+} \dim \mathrm{gr}_\lambda \mathtt{S}_\mu(\nu) q^\lambda \in \mathbb{Z}[\mathbb{X}^\bullet(T_{\mathrm{sc}})^+]$$

as an element in the monoid algebra for $\mathbb{X}^\bullet(T_{\mathrm{sc}})^+$. (Here we use q^λ instead of e^λ to denote the element in $\mathbb{Z}[\mathbb{X}^\bullet(T_{\mathrm{sc}})^+]$ corresponding to λ.) If we identify $\mathbb{X}^\bullet(T_{\mathrm{sc}})^+$ with $\mathbb{N}^l$ as above, it becomes to a polynomial of l-variables

$$P_{\mu,\nu}(q_1, \ldots, q_l) = \sum \dim \mathrm{gr}_{(s_1,\ldots,s_l)} \mathtt{S}_\mu(\nu) q_1^{s_1} \cdots q_l^{s_l}.$$

Then Theorem 3.3.1 implies that $P_{\mu,\nu}(1) = \dim \mathtt{S}_\mu(\nu)$. This may be viewed as a multivariable analogue of the Lusztig–Kato polynomials (which give the q-analogue of the weight multiplicities [**Lu83**]).

3.4. The filtration via Borel–Weil. We will explain how to obtain the filtration defined in § 3.3 using the geometry of flag varieties.

Recall that by Frobenius reciprocity, there is a canonical morphism of B^--modules $\mathtt{S}_\mu \to k_\mu$, and a canonical morphism of B-modules $\mathtt{S}_\mu \to k_{w_0(\mu)}$.

Let μ, ν be two dominant weights. Let V be a representation of G. The map $\mathtt{S}_{\mu^*} \otimes \mathtt{S}_\nu \otimes V \to k_{-\mu} \otimes k_\nu \otimes V$ induces a natural map

$$\ell_{\mu,\nu} : (\mathtt{S}_{\mu^*} \otimes \mathtt{S}_\nu \otimes V)^G \to (k_{-\mu} \otimes k_\nu \otimes V)^T \cong V(\mu - \nu).$$

[3]This is the set of basis we use in [**XZ17**]. On the other hand, it is expected that the MV basis coincide with the semi-canonical basis. We thank Lusztig for pointing this out.

Note that the collection of the maps $\{\ell_{\mu,\nu}\}$ has the following property. For a dominant weight η, there is a canonical G-equivariant map in $\mathrm{Hom}_G(\mathtt{W}_\eta, \mathtt{S}_\eta)$, which gives a G-invariant element $\sigma_\eta \in \mathtt{S}_{\eta^*} \otimes \mathtt{S}_\eta$. Multiplying by σ_η induces

$$(\mathtt{S}_{\mu^*} \otimes \mathtt{S}_\nu \otimes V)^G \to (\mathtt{S}_{\mu^*+\eta^*} \otimes \mathtt{S}_{\nu+\eta} \otimes V)^G.$$

We denote this map still by σ_η. Then it is clear that

$$\ell_{\mu+\eta,\nu+\eta} \circ \sigma_\eta = \ell_{\mu,\nu}. \tag{3.4.1}$$

This subsection is devoted to proving the following.

Proposition 3.4.1. *The map $\ell_{\mu,\nu}$ above induces an isomorphism*

$$(\mathtt{S}_{\mu^*} \otimes \mathtt{S}_\nu \otimes V)^G \cong \mathrm{fil}_\nu V(\mu-\nu) \subset V(\mu-\nu).$$

Here by abuse of notations, the image of ν under the map $\mathbb{X}^\bullet(T) \to \mathbb{X}^\bullet(T_{\mathrm{sc}})$ is still denoted by ν.

We will later in § 3.4.4 deduce this proposition from the following natural exact sequence of B-modules

$$0 \to \mathtt{S}_\nu \to \mathrm{ind}_T^B k_\nu \to \bigoplus_\alpha \mathrm{ind}_{B_\alpha}^B \mathtt{M}^\alpha_{s_\alpha(\nu)-\alpha}. \tag{3.4.2}$$

Here, for a weight $\lambda \in \mathbb{X}^\bullet(T)$,

$$\mathtt{M}^\alpha_\lambda := \mathrm{Dist}(G_\alpha) \otimes_{\mathrm{Dist}(B_\alpha)} k_\lambda$$

is the *restricted Verma module* of G_α of highest weight λ. The sequence (3.4.2) is in fact the first two terms of the (restricted) dual BGG complex. Since this sequence in the above form (and in characteristic $p > 0$) might be not familiar to some readers, we give a self-contained construction.

3.4.2. *Case of* SL_2. First, we review some facts about representations of $G = \mathrm{SL}_2$. Let B (resp. B^-) be the subgroup of upper (resp. lower) triangular matrices in SL_2, and let $T = B \cap B^- \cong \mathbb{G}_m$ be the group of diagonal matrices. We identify $G/B^- = \mathbb{P}^1$ in the way such that

- $j : \mathbb{A}^1 \to \mathbb{P}^1$ corresponds to the open B-orbit and $i : \{\infty\} \to \mathbb{P}^1$ corresponds to the closed B-orbit,
- $0 \in \mathbb{A}^1$ is fixed by B^-.

For $n \in \mathbb{N}$, consider the exact sequence of sheaves

$$0 \to \mathcal{O}(n) \to j_* j^* \mathcal{O}(n) \to j_* j^* \mathcal{O}(n)/\mathcal{O}(n) \to 0 \tag{3.4.3}$$

on $\mathbb{P}^1$. Since the sequence is B-equivariant, we may also regard it as an exact sequence on $[B\backslash G/B^-]$ via descent.

We fix two nonzero sections t_0 and t_∞ of $\mathcal{O}(1)$ that vanish at 0 and ∞ respectively and view $x = t_0/t_\infty$ as a coordinate function on $\mathbb{A}^1$. Then

$$\Gamma(\mathbb{A}^1, \mathcal{O}(n)) = k[x] t_\infty^n \cong k[x] =: \mathtt{M}_n^\vee$$

is a $(\mathrm{Dist}(G), B)$-module with highest weight n, on which e acts as $\frac{d}{dx}$, h acts as $n - 2x\frac{d}{dx}$ and f acts as $x(n - x\frac{d}{dx})$. Note that as a $\mathrm{Dist}(G)$-module, it is isomorphic to the restricted dual Verma module, and as a B-module, it is isomorphic to the induced representation $\mathrm{ind}_T^B k_n$. The subspace

$$\Gamma(\mathbb{P}^1, \mathcal{O}(n)) \cong \{f(x) \in k[x] \mid \deg f \le n\} := \mathtt{S}_n$$

is the Schur module for SL_2 of highest weight n. The section of the quotient sheaf is

$$\Gamma(\mathbb{P}^1, j_* j^* \mathcal{O}(n)/\mathcal{O}(n)) \cong x^{n+1} k[x] =: \mathtt{M}_{-n-2},$$

which as a $\mathrm{Dist}(G)$-module is isomorphic to the restricted Verma module for SL_2 of highest weight $-n-2$.

For a B-module V, write $V = \oplus V(j)$ for the weight decomposition with respect to T, then

$$(V \otimes \mathtt{M}_n^\vee)^B = \Big\{ \sum_i (-1)^i e^{(i)} v \otimes x^i \mid v \in V(-n) \Big\} \cong V(-n).$$

It follows that the following diagram is commutative with horizontal sequences exact

$$\begin{array}{ccccccc} 0 & \longrightarrow & (V \otimes \mathtt{S}_n)^B & \longrightarrow & (V \otimes \mathtt{M}_n^\vee)^B & \longrightarrow & (V \otimes \mathtt{M}_{-n-2})^B \\ & & \| & & \downarrow \cong & & \downarrow \\ 0 & \longrightarrow & (V \otimes \mathtt{S}_n)^B & \longrightarrow & V(-n) & \xrightarrow{\oplus_{i\geq 1}(-1)^{n+i} e^{(n+i)}} & \bigoplus_{i\geq 1} V(n+2i), \end{array} \tag{3.4.4}$$

where the right vertical map is the inclusion $(V \otimes \mathtt{M}_{-n-2})^B \hookrightarrow (V \otimes \mathtt{M}_{-n-2})^T \cong \bigoplus_{i\geq 1} V(n+2i)$.

Note that the above discussions may equally apply to G_α, with n replaced by a weight ν of T such that $\langle \nu, \alpha^\vee \rangle \geq 0$. The corresponding $(\mathrm{Dist}(G_\alpha), B_\alpha)$-modules $\mathtt{M}_n^\vee$, $\mathtt{S}_n$, and $\mathtt{M}_{-n-2}$ will be denoted by $\mathtt{M}_\nu^{\alpha,\vee}$, $\mathtt{S}_\nu^\alpha$, and $\mathtt{M}_{s_\alpha(\nu)-\alpha}^\alpha$, respectively.

3.4.3. *General case.* Recall that the B-orbits on G/B^- are parameterized by the Weyl group W. For $w \in W$, let C_w denote the corresponding B-orbit through $\dot{w} \in G/B^-$, where $\dot{w}$ is any lifting of w to $N_G(T)$. In particular, C_e is open and isomorphic to B/T, and C_{s_α}'s are of codimension one, where s_α is the simple reflection corresponding to the simple root α. In addition, the natural map $B \times^{B_\alpha} (G_\alpha/B_\alpha^-) \to G/B^-$ is an open embedding, with the image $C_{\leq s_\alpha} = C_e \sqcup C_{s_\alpha}$. Let $C_\Delta = \cup_{\alpha\in\Delta} C_{\leq s_\alpha}$ be the open subset of G/B^- complement to the union of B-orbits of codimension at least two. The inclusion $j : C_e \to C_\Delta$ is open and the inclusion $i : \sqcup C_\alpha \to C_\Delta$ is closed. For simplicity, the restriction of $\mathcal{O}_{G/B^-}(\nu)$ to C_Δ is denoted by $\mathcal{O}(\nu)$. Consider the following exact sequence of B-equivariant quasi-coherent sheaves on C_Δ

$$0 \to \mathcal{O}(\nu) \to j_* j^* \mathcal{O}(\nu) \to j_* j^* \mathcal{O}(\nu)/\mathcal{O}(\nu) \to 0. \tag{3.4.5}$$

Note that $C_e \cong B/T$, and $\mathcal{O}(\nu)|_{C_e} \cong B \times^T k_\nu$. In addition, the restriction of the map $j_* j^* \mathcal{O}(\nu) \to j_* j^* \mathcal{O}(\nu)/\mathcal{O}(\nu)$ to $C_{\leq s_\alpha}$ is the pullback of (3.4.3) (with n replaced by ν) along the natural projection $C_{\leq s_\alpha} \cong B \times^{B_\alpha} (G_\alpha/B_\alpha^-) \to [B_\alpha \backslash G_\alpha / B_\alpha^-]$. It follows from these observations and the previous discussions about SL_2 that (3.4.2) is obtained by taking the global sections of (3.4.5).

3.4.4. *Proof of Proposition 3.4.1.* We may assume that V is finite dimensional. Using Frobenius reciprocity, we have

$$(\mathtt{S}_{\mu^*} \otimes \mathtt{S}_\nu \otimes V)^G = \mathrm{Hom}_G(\mathtt{W}_{\nu^*}, \mathtt{S}_{\mu^*} \otimes V) = \mathrm{Hom}_B(\mathtt{W}_{\nu^*}, k_{-\mu} \otimes V) = \mathrm{Hom}_B(k, \mathtt{S}_\nu \otimes k_{-\mu} \otimes V).$$

Then by (3.4.2), there is an exact sequence

$$0 \to \mathrm{Hom}_B(k, \mathtt{S}_\nu \otimes k_{-\mu} \otimes V) \to (k_\nu \otimes k_{-\mu} \otimes V)^T \to \bigoplus_\alpha (\mathtt{M}_{s_\alpha(\nu)-\alpha}^\alpha \otimes k_{-\mu} \otimes V)^{B_\alpha}.$$

It is easy to check that the resulting map $(\mathtt{S}_{\mu^*} \otimes \mathtt{S}_\nu \otimes V)^G = \mathrm{Hom}_B(k, \mathtt{S}_\nu \otimes k_{-\mu} \otimes V) \to (k_\nu \otimes k_{-\mu} \otimes V)^T$ is $\ell_{\mu,\nu}$. In addition, using (3.4.4) and (3.3.2), we see that the kernel of the map

$$V(\mu-\nu) = (k_\nu \otimes k_{-\mu} \otimes V)^T \to (\mathtt{M}_{s_\alpha(\nu)-\alpha}^\alpha \otimes k_{-\mu} \otimes V)^{B_\alpha}$$

is exactly $\mathrm{fil}^\alpha_{\langle \nu, \alpha^\vee \rangle} V(\mu-\nu)$. Proposition 3.4.1 now follows from the definition of the $\mathbb{X}^\bullet(T)^+$-filtration on $V(\mu-\nu)$.

3.5. The filtration via geometric Satake correspondence. We give a geometric construction of the above filtrations via the geometric Satake correspondence and in particular give a proof of Proposition 3.3.2.

We refer to [**Gi95, MV07**] for the geometric Satake correspondence (see also [**Zhu17, BR17**] for an exposition). Let $\hat{G}$ be the Langlands dual group of G over $\mathbb{C}$. Let $\mathrm{Gr}_{\hat{G}} = L\hat{G}/L^+\hat{G}$ denote

its affine Grassmannian over $\mathbb{C}$, equipped with the analytic topology. Here for an affine variety Z over $\mathbb{C}$, let LZ (resp. L^+Z) denote its loop (resp. jet) space as usual, so $LZ(\mathbb{C}) = Z(\mathbb{C}((t)))$ and $L^+Z(\mathbb{C}) = Z(\mathbb{C}[[t]])$. Let $\mathrm{P}_{L^+\hat{G}}(\mathrm{Gr}_{\hat{G}})$ be the category of $L^+\hat{G}$-equivariant perverse sheaves on $\mathrm{Gr}_{\hat{G}}$, with k-coefficients as in [**MV07**]. It is known from *loc. cit.* that this is an abelian tensor category. Recall that the geometric Satake correspondence is a natural equivalence of abelian tensor categories

$$\mathrm{Sat} : \mathrm{Rep}^f(G) \to \mathrm{P}_{L^+\hat{G}}(\mathrm{Gr}_{\hat{G}}),$$

such that its composition with hypercohomology functor $\mathrm{H}^*(\mathrm{Gr}_{\hat{G}}, -)$ is isomorphic to the forgetful functor from $\mathrm{Rep}^f(G)$ to the category of finite dimensional k-vector spaces. Let us also recall from *loc. cit.* the dictionary between some geometry and representation theory under this equivalence. For a weight $\nu \in \mathbb{X}^\bullet(T) = \mathbb{X}_\bullet(\hat{T})$, let t^ν denote the $\hat{T}$-fixed point of $\mathrm{Gr}_{\hat{G}}$ corresponding to ν as usual, i.e. it is the image of $t \in \mathbb{C}((t))^\times = L\mathbb{G}_m(\mathbb{C})$ under the map $L\mathbb{G}_m \xrightarrow{\nu} L\hat{T} \to L\hat{G} \to \mathrm{Gr}_{\hat{G}}$.

(1) For each $\lambda \in \mathbb{X}^\bullet(T)^+$, let $\mathring{\mathrm{Gr}}_{\hat{G},\lambda}$ denote the $L^+\hat{G}$-orbit through t^λ, and $\mathrm{Gr}_{\hat{G},\lambda}$ its closure. They are $\langle 2\check{\rho}, \lambda\rangle$-dimensional, where $2\check{\rho}$ is the sum of positive coroots of G. Let $i_\lambda : \mathrm{Gr}_{\hat{G},\lambda} \to \mathrm{Gr}_{\hat{G}}$ denote the closed embedding and $\mathring{i}_\lambda : \mathring{\mathrm{Gr}}_{\hat{G},\lambda} \to \mathrm{Gr}_{\hat{G}}$ the locally closed embedding. Let $k[\langle 2\check{\rho}, \lambda\rangle]$ denote the constant sheaf on $\mathring{\mathrm{Gr}}_{\hat{G},\lambda}$, shifted to degree $-\langle 2\check{\rho}, \lambda\rangle$. Then

$$\mathrm{Sat}(\mathtt{W}_\lambda) \simeq {}^p\mathrm{H}^0(\mathring{i}_\lambda)_! k[\langle 2\check{\rho}, \lambda\rangle], \quad \mathrm{Sat}(\mathtt{S}_\lambda) \simeq {}^p\mathrm{H}^0(\mathring{i}_\lambda)_* k[\langle 2\check{\rho}, \lambda\rangle].$$

(2) We fix a Borel subgroup $\hat{B}$ of $\hat{G}$, and a maximal torus $\hat{T} \subset \hat{B}$. Let $\hat{U} \subset \hat{B}$ denote the unipotent radical. Let S_ν be the $L\hat{U}$-orbit through t^ν. Then the functor

$$F_\nu := \mathrm{H}^{\langle 2\check{\rho}, \nu\rangle}_{S_\nu}(\mathrm{Gr}_{\hat{G}}, -) : D^b_c(\mathrm{Gr}_{\hat{G}}, k) \to \mathrm{Vect}_k \tag{3.5.1}$$

is exact when restricted to $\mathrm{P}_{L^+\hat{G}}(\mathrm{Gr}_{\hat{G}})$, which corresponds to the weight functor $V \mapsto V(\nu)$ under the geometric Satake correspondence.

(3) For a simple root α of G, let $\hat{P}_\alpha$ be the standard parabolic subgroup whose Levi quotient $\hat{G}_\alpha$ is the Langlands dual group of G_α. There is the following diagram

$$\begin{array}{ccc} & \mathrm{Gr}_{\hat{P}_\alpha} & \\ {\scriptstyle q_\alpha}\swarrow & & \searrow{\scriptstyle i_\alpha} \\ \mathrm{Gr}_{\hat{G}_\alpha} & & \mathrm{Gr}_{\hat{G}}. \end{array}$$

The morphism i_α is a locally closed embedding and hence we also regard S_ν as a subscheme of $\mathrm{Gr}_{\hat{P}_\alpha}$. The connected components of the affine Grassmannian $\mathrm{Gr}_{\hat{G}_\alpha}$ are parameterized by $\mathbb{X}^\bullet(T)/\mathbb{Z}\alpha$. For $\theta \in \mathbb{X}^\bullet(T)/\mathbb{Z}\alpha$, let $\mathrm{Gr}^\theta_{\hat{G}_\alpha}$ be the corresponding component, and $\mathrm{Gr}^\theta_{\hat{P}_\alpha} = q_\alpha^{-1}(\mathrm{Gr}^\theta_{\hat{G}_\alpha})$. The restrictions of i_α and q_α to $\mathrm{Gr}^\theta_{\hat{P}_\alpha}$ are denoted by i^θ_α and q^θ_α respectively. Recall that $\langle 2\check{\rho}, \alpha\rangle = \langle \check{\alpha}, \alpha\rangle$ and therefore $\langle 2\check{\rho} - \check{\alpha}, \theta\rangle$ makes sense, which we denote by l_θ. Then there is a perverse exact functor

$$\mathrm{CT}_\alpha := \bigoplus_\theta (q^\theta_\alpha)_* (i^\theta_\alpha)^! [l_\theta] : \mathrm{P}_{L^+\hat{G}}(\mathrm{Gr}_{\hat{G}}) \to \mathrm{P}_{L^+\hat{G}_\alpha}(\mathrm{Gr}_{\hat{G}_\alpha})$$

which corresponds under the geometric Satake correspondence to the restriction functor from G-representations to G_α-representations. For a weight ν, let $S^\alpha_\nu \subset \mathrm{Gr}_{\hat{G}_\alpha}$ be the $L\hat{U}_\alpha$-orbit through t^ν, and $F^\alpha_\nu = \mathrm{H}^{\langle \check{\alpha}, \nu\rangle}_{S^\alpha_\nu}(\mathrm{Gr}_{\hat{G}_\alpha}, -)$ the corresponding weight functor on $\mathrm{P}_{L^+\hat{G}_\alpha}(\mathrm{Gr}_{\hat{G}_\alpha})$. Since $q_\alpha^{-1}(S^\alpha_\nu) = S_\nu$, the proper base change theorem implies that there is a canonical isomorphism $F_\nu \cong F^\alpha_\nu \circ \mathrm{CT}_\alpha$, which corresponds to the natural identification $V(\nu) = (\mathrm{Res}^G_{G_\alpha} V)(\nu)$ under the geometric Satake correspondence.

Now we give a geometric construction of the filtration defined in § 3.3. First, we define a filtration on S_ν by open subsets

$$S_\nu^{>\nu+i\alpha} := q_\alpha^{-1}(S_\nu^\alpha \cap (\mathrm{Gr}_{\hat{G}_\alpha} - \mathrm{Gr}_{\hat{G}_\alpha,\nu+i\alpha})).$$

Proposition 3.5.1. *Let V be a finite dimensional representation of G. Then we have an exact sequence*

$$0 \to \mathrm{fil}_i^\alpha V(\nu) \to \mathrm{H}_{S_\nu}^{\langle 2\check{\rho},\nu\rangle}(\mathrm{Gr}_{\hat{G}}, \mathrm{Sat}(V)) \to \mathrm{H}_{S_\nu^{>\nu+i\alpha}}^{\langle 2\check{\rho},\nu\rangle}(\mathrm{Gr}_{\hat{G}}, \mathrm{Sat}(V)).$$

Remark 3.5.2. It will follow from the arguments in § 3.5.4 that if V is a Schur module (or more generally V admits a good filtration), the above sequence is also surjective at the right.

Proof. For an α-dominant weight ν of G, we denote the closed embedding $\mathrm{Gr}_{\hat{G}_\alpha,\nu} \to \mathrm{Gr}_{\hat{G}_\alpha}$ by i_ν^α, and the complementary open embedding by j_ν^α. Then for $\mathcal{A} \in \mathrm{P}_{L^+\hat{G}}(\mathrm{Gr}_{\hat{G}})$, there is the following distinguished triangle of sheaves on $\mathrm{Gr}_{\hat{G}_\alpha}$

$$(i^\alpha_{\nu+i\alpha})_*(i^\alpha_{\nu+i\alpha})^! \,\mathrm{CT}_\alpha(\mathcal{A}) \to \mathrm{CT}_\alpha(\mathcal{A}) \to (j^\alpha_{\nu+i\alpha})_*(j^\alpha_{\nu+i\alpha})^* \,\mathrm{CT}_\alpha(\mathcal{A}) \to .$$

Applying the functor F_ν^α as in (3.5.1), and noticing by the proper base change,

$$F_\nu^\alpha((j^\alpha_{\nu+i\alpha})_*(j^\alpha_{\nu+i\alpha})^* \,\mathrm{CT}_\alpha(\mathcal{A})) = \mathrm{H}_{S_\nu^{>\nu+i\alpha}}^{\langle 2\check{\rho},\nu\rangle}(\mathrm{Gr}_{\hat{G}}, \mathcal{A})),$$

we obtain

$$0 \to F_\nu^\alpha(i^\alpha_{\nu+i\alpha})_* {}^p\mathrm{H}^0(i^\alpha_{\nu+i\alpha})^! \,\mathrm{CT}_\alpha(\mathcal{A}) \to \mathrm{H}_{S_\nu}^{\langle 2\check{\rho},\nu\rangle}(\mathrm{Gr}_{\hat{G}}, \mathcal{A}) \to \mathrm{H}_{S_\nu^{>\nu+i\alpha}}^{\langle 2\check{\rho},\nu\rangle}(\mathrm{Gr}_{\hat{G}}, \mathcal{A}).$$

The injectivity at the left follows from the fact that $(j^\alpha_{\nu+i\alpha})_*(j^\alpha_{\nu+i\alpha})^* \,\mathrm{CT}_\alpha(\mathcal{A})$ lives in perverse cohomological degree ≥ 0 and the exactness of F_ν^α. Now let $\mathcal{A} = \mathrm{Sat}(V)$. To conclude the proof, we apply the following lemma (which follows from the description of the Weyl modules under the geometric Satake correspondence as in (1) above) and the definition of the filtration (3.3.1) to replace $F_\nu^\alpha(i^\alpha_{\nu+i\alpha})_* {}^p\mathrm{H}^0(i^\alpha_{\nu+i\alpha})^! \,\mathrm{CT}_\alpha(\mathrm{Sat}(V))$ by $\mathrm{fil}_i^\alpha V(\nu)$. □

Lemma 3.5.3. *Let λ be a dominant weight of G. Under the geometric Satake correspondence, the left exact functor $V \mapsto V_{\preceq\lambda}$ corresponds to $(i_\lambda)_* {}^p\mathrm{H}^0(i_\lambda)^! : \mathrm{P}_{L^+\hat{G}}(\mathrm{Gr}_{\hat{G}}) \to \mathrm{P}_{L^+\hat{G}}(\mathrm{Gr}_{\hat{G}})$.*

3.5.4. *Proof of Proposition 3.3.2.* We construct the needed basis explicitly via MV cycles. Let V be a Schur module so $\mathrm{Sat}(V) = {}^p\mathrm{H}^0(\mathring{i}_\mu)_* k[\langle 2\check{\rho},\mu\rangle]$ for some dominant weight μ of G. In this case, the natural map ${}^p\mathrm{H}^0(\mathring{i}_\mu)_* k[\langle 2\check{\rho},\mu\rangle] \to (\mathring{i}_\mu)_* k[\langle 2\check{\rho},\mu\rangle]$ induces the following commutative diagram

$$\begin{array}{ccc} \mathrm{H}_{S_\nu}^{\langle 2\check{\rho},\nu\rangle}(\mathrm{Gr}_{\hat{G}}, {}^p\mathrm{H}^0(\mathring{i}_\mu)_* k[\langle 2\check{\rho},\mu\rangle]) & \longrightarrow & \mathrm{H}_{S_\nu^{>\nu+i\alpha}}^{\langle 2\check{\rho},\nu\rangle}(\mathrm{Gr}_{\hat{G}}, {}^p\mathrm{H}^0(\mathring{i}_\mu)_* k[\langle 2\check{\rho},\mu\rangle]) \\ \downarrow & & \downarrow \\ \mathrm{H}_{S_\nu\cap\mathring{\mathrm{Gr}}_{\hat{G},\mu}}^{\langle 2\check{\rho},\nu+\mu\rangle}(\mathring{\mathrm{Gr}}_{\hat{G},\mu}, k) & \longrightarrow & \mathrm{H}_{S_\nu^{>\nu+i\alpha}\cap\mathring{\mathrm{Gr}}_{\hat{G},\mu}}^{\langle 2\check{\rho},\nu+\mu\rangle}(\mathring{\mathrm{Gr}}_{\hat{G},\mu}, k). \end{array} \tag{3.5.2}$$

The two vertical arrows are isomorphisms by (the proof of) [**MV07**, Proposition 3.10], and therefore we identify their sources and targets. The groups in the bottom row are canonically isomorphic to the top Borel–Moore homology of $S_\nu \cap \mathring{\mathrm{Gr}}_{\hat{G},\mu}$ and of $S_\nu^{>\nu+i\alpha} \cap \mathring{\mathrm{Gr}}_{\hat{G},\mu}$ respectively, and therefore have a basis given by the fundamental classes of their irreducible components. Such a fundamental class in the lower left corner maps to zero under the horizontal arrow if and only if the corresponding irreducible component does not intersect with $S_\nu^{>\nu+i\alpha}$.

Now we take the basis of $V(\nu) = F_\nu(\mathrm{Sat}(V))$ given by the aforementioned fundamental classes $\{v_j\}$ in $\mathrm{H}_{S_\nu\cap\mathring{\mathrm{Gr}}_{\hat{G},\mu}}^{\langle 2\check{\rho},\nu+\mu\rangle}(\mathring{\mathrm{Gr}}_{\hat{G},\mu}, k)$, i.e. the MV basis. The discussion above says that, for every $\alpha \in \Delta$ and every $i \in \mathbb{Z}_{\geq 0}$, those v_j's which lie in the kernel of the top horizontal arrow of (3.5.2), or equivalently in $\mathrm{fil}_i^\alpha V(\nu)$ by Proposition 3.5.1, in fact span $\mathrm{fil}_i^\alpha V(\nu)$. Therefore, their symbols $\overline{v_j}$

for the filtration fil^α form a basis of the associated graded $\mathrm{gr}^\alpha V(\nu)$. This concludes the proof of Proposition 3.3.2.

4. Vector-valued twisted conjugation invariant functions

In this section, we start to study the space of vector-valued (twisted) conjugation invariant functions on a group.

We will continue making use of the conventions and notations in § 3. In particular, we have a pinned reductive group $(G, B, T, \{x_\alpha\}_{\alpha\in\Delta})$ over k. Let σ be an automorphism of G preserving (B, T), and hence the root system $\Phi(G, T)$. We define the σ-action on the character group $\mathbb{X}^\bullet(T)$ by

$$\sigma(\alpha)(t) = \alpha(\sigma^{-1}(t)), \quad \text{for } \alpha \in \mathbb{X}^\bullet(T),\ t \in T.$$

so that $\sigma(U_\alpha) = U_{\sigma(\alpha)}$. For a G-representation V, let σV denote the representation

$$G \xrightarrow{\sigma^{-1}} G \to \mathrm{GL}(V).$$

We identify elements in σV with $\{\sigma v \,|\, v \in V\}$ and thus $g(\sigma v) = \sigma(\sigma^{-1}(g)v)$ for $g \in G$. This way, we have $\sigma \mathtt{S}_\mu = \mathtt{S}_{\sigma(\mu)}$ and $\sigma \mathtt{W}_\mu = \mathtt{W}_{\sigma(\mu)}$ for $\mu \in \mathbb{X}^\bullet(T)^+$.

4.1. Vector-valued twisted conjugation invariant functions on a group. Let H be a linear algebraic group over k equipped with an automorphism τ. We still denote the τ-twisted conjugation action of H on itself (as defined in (1.0.2)) by c_τ, i.e

$$c_\tau(h)(g) = hg\tau(h)^{-1}, \quad h, g \in H.$$

Let $\boldsymbol{J}_H = k[H]^{c_\tau(H)}$ denote the space of τ-twisted conjugation invariant functions on H, i.e. those $f \in k[H]$ satisfying $f(hg\tau(h)^{-1}) = f(g)$ for $h, g \in H$. More generally, for a representation V of H, we will denote by

$$\boldsymbol{J}_H(V) = (k[H] \otimes V)^{c_\tau(H)}$$

the space of vector-valued τ-twisted conjugation invariant functions on H. Equivalently, let $[H/c_\tau(H)]$ (or sometimes $[H\tau/H]$) denote the quotient stack of H by the τ-twisted conjugation action of H on itself, and let $\widetilde{V} := V_{[H/c_\tau(H)]}$ denote the corresponding vector bundle on $[H/c_\tau(H)]$ (see § 1 for the notations and conventions). Then

$$\boldsymbol{J}_H = \Gamma([H\tau/H], \mathcal{O}), \quad \boldsymbol{J}_H(V) = \Gamma([H\tau/H], \widetilde{V}).$$

Moreover, $\boldsymbol{J}_H(V)$ is naturally a $\boldsymbol{J}_H$-module.

Remark 4.1.1. Note that the spaces $\boldsymbol{J}_H(V)$ depend only on the image of τ in the outer automorphism group $\mathrm{Out}(H)$ of H. Indeed, if $\tau_2 = c(h) \circ \tau_1$, where $c(h) : H \to H,\ \ g \mapsto hgh^{-1}$ is the inner automorphism of H induced by some $h \in H$, then the map $H \to H,\ x \mapsto xh$ is an isomorphism intertwining the action c_{τ_1} and c_{τ_2}, and therefore induces isomorphisms between the spaces of vector-valued twisted conjugation invariant functions.

We discuss a few basic properties of these spaces. First, if $H = H_1 \times H_2$ and $V = V_1 \boxtimes V_2$ is the exterior tensor product of two representations of H_1 and H_2 respectively, then clearly $\boldsymbol{J}_H(V)$ is compatible with tensor product

$$\boldsymbol{J}_H(V) = \boldsymbol{J}_{H_1}(V_1) \otimes_k \boldsymbol{J}_{H_2}(V_2). \tag{4.1.1}$$

Next, if K is another linear algebraic group equipped with an automorphism τ and there is a τ-equivariant homomorphism $K \to H$, we have a natural homomorphism

$$\mathrm{Res}^\tau_V : \boldsymbol{J}_H(V) \to \boldsymbol{J}_K(V)$$

by restricting (a.k.a. pulling back) of the V-valued functions on H to K; this is compatible with the $\boldsymbol{J}_H$-module and $\boldsymbol{J}_K$-module structure through the natural restriction map $\mathrm{Res}^{\tau}_{\mathbf{1}} : \boldsymbol{J}_H \to \boldsymbol{J}_K$, and in particular induces the map of $\boldsymbol{J}_K$-modules

$$\mathrm{Res}^{\tau}_{V} \otimes 1 : \boldsymbol{J}_H(V) \otimes_{\boldsymbol{J}_H} \boldsymbol{J}_K \to \boldsymbol{J}_K(V).$$

Next we discuss the compatibility of $\boldsymbol{J}_H(V)$ with tensor induction. Let H_0 be a linear algebraic group equipped with an automorphism τ_0. Suppose we have an embedding $\langle \tau_0 \rangle \subset \langle \tau \rangle$ of cyclic groups of finite index so that $\tau_0 = \tau^d$. Then we can form the tensor induction $H = \mathrm{Ind}^{\langle \tau \rangle}_{\langle \tau_0 \rangle} H_0$, which is the group of τ_0-equivariant maps from $\langle \tau \rangle$ to H, where τ_0 acts on $\langle \tau \rangle$ by translation. In addition, τ acts on H since it acts on $\langle \tau \rangle$ by translation. Since $\{\mathrm{id}, \tau, \dots, \tau^{d-1}\}$ forms a set of representatives of $\langle \tau \rangle / \langle \tau_0 \rangle$, we may explicitly identify H with the product $\prod_{i=0}^{d-1} H_0$, on which τ acts by $(h_0, \dots, h_{d-1}) \mapsto (\tau_0(h_{d-1}), h_0, \dots, h_{d-2})$.

We have two natural maps from H_0 to H, given by embedding i_0 into the 0th factor of H and by the diagonal embedding Δ, i.e.

$$i_0(h) = (h, 1, \dots, 1), \quad \Delta(h) = (h, \dots, h).$$

It is straightforward to check that these two embeddings satisfy the following relation: for $g, h \in H_0$,

$$c_\tau(\Delta(h))(i_0(g)) = c_\tau(h, \dots, h)(g, 1, \dots, 1) = (hg\tau_0(h)^{-1}, hh^{-1}, \dots, hh^{-1}) = i_0\big(c_{\tau_0}(h)(g)\big),$$

i.e. i_0 intertwines the τ_0-twisted conjugation action of H_0 on the source, and the τ-twisted conjugation action of $\Delta(H_0)$ on the target:

$$\begin{array}{ccc} H_0 & \xrightarrow{\ i_0\ } & H \\ {\scriptstyle c_{\tau_0}}\circlearrowleft & & \circlearrowright{\scriptstyle c_\tau} \\ H_0 & \xrightarrow{\ \Delta\ } & H. \end{array}$$

From this, we naturally deduce a morphism of stacks

$$(i_0/\Delta) : [H_0/c_{\tau_0}(H_0)] \to [H/c_\tau(H)]. \tag{4.1.2}$$

If $K_0 \subset H_0$ is a τ_0-stable subgroup, then $K = \mathrm{Ind}^{\langle \tau \rangle}_{\langle \tau_0 \rangle} K_0$ is a τ-stable subgroup of H, and we obtain a commutative diagram

$$\begin{array}{ccc} [K_0/c_{\tau_0}(K_0)] & \longrightarrow & [K/c_\tau(K)] \\ \downarrow & & \downarrow \\ [H_0/c_{\tau_0}(H_0)] & \longrightarrow & [H/c_\tau(H)]. \end{array}$$

Lemma 4.1.2. (1) *The morphism (i_0/Δ) in* (4.1.2) *is an isomorphism of stacks.*

(2) *If V is an H-module, regarded as a representation of H_0 via the* diagonal *embedding Δ, then we may form $\boldsymbol{J}_{H_0}(V)$ and $\boldsymbol{J}_{K_0}(V)$ with respect to the τ_0-twisted conjugation action, and $\boldsymbol{J}_H(V)$ and $\boldsymbol{J}_K(V)$ with respect to the τ-twisted conjugation action. Then we have a natural commutative diagram,*

$$\begin{array}{ccc} \boldsymbol{J}_H(V) \otimes_{\boldsymbol{J}_H} \boldsymbol{J}_K & \xrightarrow{\mathrm{Res}^{\tau}_{V} \otimes 1} & \boldsymbol{J}_K(V) \\ {\scriptstyle (i_0/\Delta)^*}\downarrow & & \downarrow{\scriptstyle (i_0/\Delta)^*} \\ \boldsymbol{J}_{H_0}(V) \otimes_{\boldsymbol{J}_{H_0}} \boldsymbol{J}_{K_0} & \xrightarrow{\mathrm{Res}^{\tau_0}_{V} \otimes 1} & \boldsymbol{J}_{K_0}(V) \end{array}$$

where the two vertical morphisms are isomorphisms.

PROOF. (1) The group $\{1\} \times H_0^{d-1}$ is a coset representative of $H/\Delta(H_0)$. It is enough to show that the following map is an isomorphism

$$\begin{aligned} c_\tau(-, i_0(-)) : (\{1\} \times H_0^{d-1}) \times H_0 &\longrightarrow H = H_0^d \\ \big((h_1, \dots, h_{d-1}), g\big) &\longmapsto c_\tau(1, h_1, \dots, h_{d-1})(i_0(g)) \\ &= \big(g\tau_0(h_{d-1}^{-1}), h_1, h_2h_1^{-1}, \dots, h_{d-1}h_{d-2}^{-1}\big). \end{aligned}$$

But this is clear, and the inverse map is given by

$$(g_0, \dots, g_{d-1}) \mapsto \big(g_1, g_2g_1, \dots, g_{d-1}g_{d-2}\cdots g_1), g_0\tau_0(g_{d-1}g_{d-2}\cdots g_1)\big).$$

So (4.1.2) is an isomorphism.

(2) follows from (1) immediately. In an explicit form, the inverse of the pull back $(i_0/\Delta)^*$ is given by sending $\bar{f} \in \boldsymbol{J}_{H_0}(V)$ to a τ-twisted conjugation invariant function $f : H \to V$ defined by the following formula: for $g_0, \dots, g_{d-1} \in H_0$

$$f(g_0, \dots, g_{d-1}) = (g_1^{-1}g_2^{-1}\cdots g_{d-1}^{-1}, \dots, g_{d-2}^{-1}g_{d-1}^{-1}, g_{d-1}^{-1}, 1) \cdot \bar{f}(g_{d-1}g_{d-2}\cdots g_0). \qquad \square$$

Finally, let us discuss the compatibility of $\boldsymbol{J}_H(V)$ with central homomorphisms. Let $F \subset H$ denote a τ-stable central subgroup of H. We assume that F is of multiplicative type, i.e. if $\Lambda = \mathrm{Hom}(F, \mathbb{G}_m)$ denote its character group, then $F = \mathrm{Spec}\, k[\Lambda]$. The exact sequence

$$1 \to \Lambda^\tau \to \Lambda \xrightarrow{1-\tau} \Lambda \to \Lambda_\tau \to 1$$

induces

$$1 \to F^\tau \to F \xrightarrow{1-\tau} F \to F_\tau \to 1.$$

Let $H' = H/(1-\tau)F$ and $H'' = H/F$. Then the kernel of the map $H' \to H''$ is F_τ. Left multiplication by F induces an F-action on $k[H]$ via $(z \cdot f)(h) = f(zh)$. Then we have the decomposition $k[H] = \oplus_{\psi\in\Lambda} k[H]_\psi$ according to the weights of F, where $k[H]_{\psi=\mathbf{1}} = k[H'']$ and each $k[H']_\psi$ is an invertible $k[H'']$-module. In addition, $\oplus_{\psi\in\Lambda^\tau} k[H]_\psi = k[H']$, and for given $\chi \in (1-\tau)\Lambda$, $\oplus_{\tau(\psi)-\psi=\chi} k[H]_{\tau(\psi)}$ is an invertible $k[H']$-module.

Let V be an H-module, decomposed as $\oplus_\chi V_\chi$ according to the weights of F. Then

$$\boldsymbol{J}_H(V) = \bigoplus_{\psi\in\Lambda} \big(k[H]_{\tau(\psi)} \otimes V_{\tau(\psi)-\psi}\big)^{c_\tau(H'')} = \bigoplus_{\chi\in(\tau-1)\Lambda} \Big(\bigoplus_{\tau(\psi)-\psi=\chi} k[H]_{\tau(\psi)} \otimes V_\chi \Big)^{c_\tau(H'')}.$$

Each direct summand $\big(\oplus_{\tau(\psi)-\psi=\chi} k[H]_{\tau(\psi)} \otimes V_\chi\big)^{c_\tau(H'')}$ is acted on by F_τ. In particular,

$$\boldsymbol{J}_H = k[H']^{c_\tau(H'')}$$

is acted by F_τ, and the $\boldsymbol{J}_H$-module structure on $\boldsymbol{J}_H(V)$ is compatible with this action. Note that if V is a representation of H'', then

$$\boldsymbol{J}_H(V)^{F_\tau} = \boldsymbol{J}_{H''}(V). \tag{4.1.3}$$

In particular, $\boldsymbol{J}_H^{F_\tau} = \boldsymbol{J}_{H''}$.

LEMMA 4.1.3. *Let V be a representation of H''. Assume that the above F_τ-action realizes* $\mathrm{Spec}\, \boldsymbol{J}_H$ *as an F_τ-torsor (in fpqc topology) over* $\mathrm{Spec}\, \boldsymbol{J}_{H''}$*, then the natural map*

$$\mathrm{Res}_V^\tau \otimes 1 : \boldsymbol{J}_{H''}(V) \otimes_{\boldsymbol{J}_{H''}} \boldsymbol{J}_H \to \boldsymbol{J}_H(V)$$

is an isomorphism.

PROOF. Note that the action of F_τ on $\boldsymbol{J}_H(V)$ equips $\boldsymbol{J}_H(V)$ with a descent datum for the map $\boldsymbol{J}_{H''} \to \boldsymbol{J}_H$. The lemma now follows from (4.1.3) and faithfully flat descent. $\square$

4.2. Vector-valued twisted conjugation invariant functions on the Vinberg monoid. Now let $H = G$ be the reductive group as fixed at the beginning of the section. Let τ be an automorphism of G. Let V be a representation of G. Sometimes we write $\boldsymbol{J}(V) = \boldsymbol{J}_G(V)$ and $\boldsymbol{J} = \boldsymbol{J}_G$ for simplicity. We now define their their analogues $\boldsymbol{J}_+(V)$ (resp. $\boldsymbol{J}_0(V)$) of vector-valued twisted conjugation invariant functions on V_G (resp. As_G).

We consider the G-action on V_G via the twisted diagonal embedding $G \to G \times G,\ g \mapsto (g, \tau(g))$ so that its restriction to $\mathfrak{d}^{-1}(\mathbf{1}) = G$ is the τ-twisted conjugation action of G on itself. For this reason, we also denote this action of G on V_G and on As_G by c_τ. Let $[V_G/c_\tau(G)]$ and $[\mathrm{As}_G/c_\tau(G)]$ denote the stack quotients. The map $\mathfrak{d}$ induces

$$[\mathfrak{d}] : [V_G/c_\tau(G)] \to T^+_{\mathrm{ad}}, \quad [\mathfrak{d}]^{-1}(\mathbf{1}) = [G/c_\tau(G)], \quad [\mathfrak{d}]^{-1}(\mathbf{0}) = [\mathrm{As}_G/c_\tau(G)].$$

Write $\widetilde{V}_+ := V_{[V_G/c_\tau(G)]}$ (resp. $\widetilde{V}_0 = V_{[\mathrm{As}_G/c_\tau(G)]}$) for the vector bundle on $[V_G/c_\tau(G)]$ (resp. $[\mathrm{As}_G/c_\tau(G)]$) corresponding to V. Then clearly $\widetilde{V} := \widetilde{V}_+|_{[\mathfrak{d}]^{-1}(\mathbf{1})}$ and $\widetilde{V}_0 = \widetilde{V}_+|_{[\mathfrak{d}]^{-1}(\mathbf{0})}$. Now we can define

$$\boldsymbol{J}_+ = \Gamma([V_G/c_\tau(G)], \mathcal{O}), \quad \boldsymbol{J}_+(V) = \Gamma([V_G/c_\tau(G)], \widetilde{V}_+),$$

and

$$\boldsymbol{J}_0 = \Gamma([\mathrm{As}_G/c_\tau(G)], \mathcal{O}), \quad \boldsymbol{J}_0(V) = \Gamma([\mathrm{As}_G/c_\tau(G)], \widetilde{V}_0).$$

Then $\boldsymbol{J}_+(V)$ (resp. $\boldsymbol{J}_0(V)$) is a $\boldsymbol{J}_+$ (resp. $\boldsymbol{J}_0$)-module.

Remark 4.1.1 also applies to the study of the spaces $\boldsymbol{J}_+(V)$ and $\boldsymbol{J}_0(V)$. Recall that a choice of a pinning of G defines a section of the projection $\mathrm{Aut}(G) \to \mathrm{Out}(G)$. Therefore, to study $\boldsymbol{J}_*(V)$ for $* = +, 0, \emptyset$, without loss of generality we may and will assume that $\tau = \sigma$ is an automorphism preserving the pinning $(G, B, T, \{x_\alpha\}_{\alpha\in\Delta})$ we fix at the beginning of this section.

We explain the relations between $\boldsymbol{J}_+(V), \boldsymbol{J}_0(V)$ and $\boldsymbol{J}(V)$. Let $V_G/\!\!/c_\sigma(G) = \operatorname{Spec} \boldsymbol{J}_+$ be the GIT quotient. Then $[\mathfrak{d}]$ factors as

$$[V_G/c_\sigma(G)] \to V_G/\!\!/c_\sigma(G) \xrightarrow{\bar{\mathfrak{d}}} T^+_{\mathrm{ad}}.$$

LEMMA 4.2.1. (1) *For any representation V, $\boldsymbol{J}_+(V) \otimes_{\boldsymbol{J}_+} k[\bar{\mathfrak{d}}^{-1}(\mathbf{1})] \cong \boldsymbol{J}(V)$. In particular, $k[\bar{\mathfrak{d}}^{-1}(\mathbf{1})] \cong \boldsymbol{J}$.*

(2) *If V admits a good filtration, then $\boldsymbol{J}_+(V) \otimes_{\boldsymbol{J}_+} k[\bar{\mathfrak{d}}^{-1}(\mathbf{0})] \cong \boldsymbol{J}_0(V)$. In particular, $k[\bar{\mathfrak{d}}^{-1}(\mathbf{0})] \cong \boldsymbol{J}_0$.*

PROOF. Recall that there is an $\mathbb{X}^\bullet(T)^+_{\mathrm{pos}}$-filtration on $k[G]$ by (3.2.1). Then

$$\boldsymbol{J}_+(V) \otimes_{\boldsymbol{J}_+} k[\bar{\mathfrak{d}}^{-1}(\mathbf{1})] = \bigoplus_{\omega\in\mathsf{Min}} \varinjlim_{\mu\in\omega+\mathbb{X}^\bullet(T)^+_{\mathrm{pos}}} \left(\mathrm{fil}_\mu k[G] \otimes V\right)^{c_\sigma(G)}.$$

Since taking G-invariants commutes with taking direct limits, Part (1) follows.

To prove Part (2), first notice that for $\mu \in \mathbb{X}^\bullet(T)^+_{\mathrm{pos}}$, there is the following commutative diagram with all rows exact.

(4.2.1)

$$\begin{array}{ccccccc}
\bigoplus_i \left(\mathrm{fil}_{\mu-\alpha_i} k[G] \otimes V\right)^{c_\sigma(G)} & \longrightarrow & (\mathrm{fil}_\mu k[G] \otimes V)^{c_\sigma(G)} & \longrightarrow & \left(\boldsymbol{J}_+(V) \otimes_{\boldsymbol{J}_+} k[\bar{\mathfrak{d}}^{-1}(\mathbf{0})]\right)_\mu & \longrightarrow & 0 \\
\downarrow & & \| & & \downarrow & & \\
0 \longrightarrow \left(\sum_i \mathrm{fil}_{\mu-\alpha_i} k[G] \otimes V\right)^{c_\sigma(G)} & \longrightarrow & (\mathrm{fil}_\mu k[G] \otimes V)^{c_\sigma(G)} & \longrightarrow & (\mathrm{gr}_\mu k[G] \otimes V)^{c_\sigma(G)} & \longrightarrow & 0.
\end{array}$$

Here $(\boldsymbol{J}_+(V) \otimes_{\boldsymbol{J}_+} k[\bar{\mathfrak{d}}^{-1}(\mathbf{0})])_\mu$ denotes the μ-graded piece of $\boldsymbol{J}_+(V) \otimes_{\boldsymbol{J}_+} k[\bar{\mathfrak{d}}^{-1}(\mathbf{0})]$, and $\sum_i \mathrm{fil}_{\mu-\alpha_i} k[G]$ denotes the image of $\bigoplus_i \mathrm{fil}_{\mu-\alpha_i} k[G] \to \mathrm{fil}_\mu k[G]$. The second row is clearly left exact, and the right exactness follows from the fact that $\sum_i \mathrm{fil}_{\mu-\alpha_i} k[G]$ has a good filtration as a $G \times G$-module, by

Theorem 3.1.2, Corollary 3.1.6. By the same reasoning, the left vertical map is also surjective. It follows that the right vertical map is an isomorphism. Part (2) follows. □

REMARK 4.2.2. If V is the trivial representation, one can directly show the surjectivity of the map

$$(\mathrm{fil}_\mu k[G])^{c_\sigma(G)} \to (\mathrm{gr}_\mu k[G])^{c_\sigma(G)}$$

by constructing a splitting. Namely, note that

$$\dim(\mathrm{gr}_\mu k[G])^{c_\sigma(G)} = \begin{cases} 0 & \sigma(\mu) \neq \mu \\ 1 & \sigma(\mu) = \mu. \end{cases}$$

In the latter case, the composition

$$(\mathtt{W}_\mu^* \otimes \mathtt{W}_\mu)^{c_\sigma(G)} \to \mathrm{fil}_\mu k[G]^{c_\sigma(G)} \to (\mathtt{S}_{\mu^*} \otimes \mathtt{S}_\mu)^{c_\sigma(G)}$$

is an isomorphism, giving the splitting.

In addition, by applying Lemma 2.2.2 to the case $M = k[V_G]^{c_\sigma(G)}$, we conclude that $\bar{\mathfrak{d}} : V_G /\!\!/ c_\sigma(G) \to T^+_{\mathrm{ad}}$ is flat.

We finish this subsection with the following *twisted Chevalley isomorphism*. Recall that we denote by $W = N(T)/T$ the Weyl group. Let $W_0 = W^\sigma$ and let N_0 be the preimage of W_0 in N. It acts on T via the twisted conjugation c_σ and it also acts on V_T via c_σ.

PROPOSITION 4.2.3. *The restriction of a function on V_G to V_T induces an isomorphism*

$$\mathrm{Res}^\sigma_{+,\mathbf{1}} : k[V_G]^{c_\sigma(G)} \xrightarrow{\cong} k[V_T]^{c_\sigma(N_0)}.$$

In particular, restricting a σ-conjugation invariant function on G to T induces the twisted Chevalley isomorphism

$$\mathrm{Res}^\sigma_{\mathbf{1}} : \boldsymbol{J} = k[G]^{c_\sigma(G)} \xrightarrow{\cong} k[T]^{c_\sigma(N_0)}.$$

The last statement was also proved in [**Sp06**, Theorem 1], essentially by the same argument given below.

PROOF. Recall the $\mathbb{X}^\bullet(T)^+_{\mathrm{pos}}$-filtration on $k[T]$ defined by (3.2.3), which is the image of the $\mathbb{X}^\bullet(T)^+_{\mathrm{pos}}$-filtration on $k[G]$ defined by (3.2.1). Then

$$(\mathrm{fil}_\nu k[T])^{c_\sigma(N_0)} := \bigoplus_{\substack{\lambda \in \mathbb{X}^\bullet(T)^{+,\sigma} \\ \mu - \lambda \in \mathbb{X}^\bullet(T_{\mathrm{ad}})_{\mathrm{pos}}}} \Big(k \cdot \sum_{\nu \in W_0 \lambda} e^\nu \Big),$$

By Remark 4.2.2, it is straightforward to see that $\mathrm{gr} k[G]^{c_\sigma(G)} \cong \mathrm{gr} k[T]^{c_\sigma(N_0)}$. Therefore, $\mathrm{Res}^\sigma_{+,\mathbf{1}}$ is an isomorphism. □

4.3. Freeness. Here are the main results of this section. We define a number

$$r_V := \dim V|_{T^\sigma}(0).$$

ASSUMPTION 4.3.1. In this subsection, let G be a simply-connected semisimple group over k and V a G-representation that admits a good filtration.

THEOREM 4.3.2. *Keep Assumption 4.3.1. Then $\boldsymbol{J}_+(V)$ (resp. $\boldsymbol{J}_0(V)$, resp. $\boldsymbol{J}(V)$) is a free $\boldsymbol{J}_+$-module (resp. $\boldsymbol{J}_0$-module, resp. $\boldsymbol{J}$-module) of rank r_V.*

COROLLARY 4.3.3. *Keep Assumption 4.3.1. The morphisms*

$$\chi_+ : V_G \to V_G /\!\!/ c_\sigma(G), \quad \chi_0 : \mathrm{As}_G \to \mathrm{As}_G /\!\!/ c_\sigma(G), \quad \chi : G \to G /\!\!/ c_\sigma(G)$$

are faithfully flat.

We call these morphisms the *(twisted) Chevalley maps*. When $\sigma = \mathrm{id}$, this corollary is also proved by [**Bou15**] by a different method.

PROOF. Let Reg denote the regular representation of G, i.e. the representation of G on $k[G]$ induced by left multiplication, which as mentioned before admits a good filtration. Since for any affine variety $X = \operatorname{Spec} R$ with a G-action, $(R \otimes \mathrm{Reg})^G = R$, the statement follows from Theorem 4.3.2. □

REMARK 4.3.4. The Chevalley map $\chi : G \to G/\!\!/c_\sigma(G)$ is not flat in general, and therefore Theorem 4.3.2 cannot hold for arbitrary reductive group. We refer to [**Ric79**, Proposition 4.1] for a discussion of this point when $\sigma = \mathrm{id}$.

Nevertheless, there is a sufficient condition for the flatness of $\boldsymbol{J}(V)$ over $\boldsymbol{J}$ when G is a general connected reductive group. Let G_{sc} be the simply-connected cover of the derived subgroup of G, and let F be the kernel of the central isogeny $1 \to F \to G' := G_{\mathrm{sc}} \times Z_G \to G \to 1$. It is known that σ lifts to a unique automorphism of G_{sc} (e.g. see [**St68**, §9.16]). We decompose $V = \oplus_\psi V_\psi \otimes k_\psi$ according to the central character for the action of Z_G on V so that each V_ψ is a G_{sc}-module. Then by (4.1.1)

$$\boldsymbol{J}_{G'}(V) = \bigoplus_{\psi} \boldsymbol{J}_{G_{\mathrm{sc}}}(V_\psi) \otimes \boldsymbol{J}_{Z_G}(k_\psi) = \bigoplus_{\psi|_{Z_G^\sigma}=\mathbf{1}} \boldsymbol{J}_{G_{\mathrm{sc}}}(V_\psi) \otimes \boldsymbol{J}_{Z_G}(k_\psi),$$

which is free over $G'/\!\!/c_\sigma(G') \cong G_s/\!\!/c_\sigma(G_s) \times (Z_G)_\sigma$ of rank r_V by Theorem 4.3.2. It follows from Lemma 4.1.3 that if the action of F_σ on $G'/\!\!/c_\sigma(G')$ is free, then $\boldsymbol{J}_G(V)$ is finite projective of rank r_V over $G/\!\!/c_\sigma(G)$. In particular, if the map $F_\sigma \to (Z_G)_\sigma$ is injective, then $J_G(V)$ is finite projective over J_G. For example, this is the case if $G_{\mathrm{sc}} = G_{\mathrm{der}}$ and $\sigma = \mathrm{id}$.

The rest of this subsection is devoted to the proof of Theorem 4.3.2. We will first prove the statement for $\boldsymbol{J}_0(V)$, and then deduce from it the statement for $\boldsymbol{J}_+(V)$ and $\boldsymbol{J}(V)$. Note that by Lemma 3.2.1 (4),

$$\boldsymbol{J}_0(V) = \Gamma([\mathrm{As}_G/c_\sigma(G)], \widetilde{V}_0) = \bigoplus_{\nu \in \mathbb{X}^\bullet(T)^+} (\mathtt{S}_{\sigma(\nu^*)} \otimes \mathtt{S}_\nu \otimes V)^G,$$

Let us fix $\xi \in \mathbb{X}^\bullet(T)$. Clearly, the part of Theorem 4.3.2 for $\boldsymbol{J}_0(V)$ will follow from the following refinement.

PROPOSITION 4.3.5. *Assume that V admits a good filtration. Then*

$$\boldsymbol{J}_0(V)_\xi := \bigoplus_{\sigma(\nu)-\nu=\xi} (\mathtt{S}_{\sigma(\nu^*)} \otimes \mathtt{S}_\nu \otimes V)^G$$

is a finite free $\boldsymbol{J}_0$-module of rank $= \dim V(\xi)$. Moreover, we may choose a basis $\{e_i\}$ of $\boldsymbol{J}_0(V)_\xi$ as a $\boldsymbol{J}_0$-module such that each $e_i \in (\mathrm{gr}_{\nu_i} k[G] \otimes V)^{c_\sigma(G)}$ for some $\nu_i \in \mathbb{X}^\bullet(T)^+$ satisfying $\sigma(\nu_i) - \nu_i = \xi$.

PROOF. Set $\mathbb{X}^\bullet(T)^{+,\sigma} = \{\nu \in \mathbb{X}^\bullet(T)^+ \mid \sigma(\nu) = \nu\}$ and $\mathbb{X}^\bullet(T)^+_\xi = \{\nu \in \mathbb{X}^\bullet(T)^+ \mid \sigma(\nu) - \nu = \xi\}$. Then $\mathbb{X}^\bullet(T)^{+,\sigma}$ naturally acts on $\mathbb{X}^\bullet(T)^+_\xi$. We shall apply the discussions from § 2 to the following tuple

$$(\Gamma, S, M, \{\mathrm{fil}_s M\}_{s\in S}) = \big(\mathbb{X}^\bullet(T)^{+,\sigma}, \mathbb{X}^\bullet(T)^+_\xi, V(\xi), \{\mathrm{fil}_\nu V(\xi)\}_{\nu \in \mathbb{X}^\bullet(T)^+_\xi}\big).$$

The following lemma implies that $(\mathbb{X}^\bullet(T)^{+,\sigma}, \mathbb{X}^\bullet(T)^+_\xi)$ satisfies (Can) and (DCC), and Lemma 2.1.7 is applicable (with $S' \subset S$ being $\mathbb{X}^\bullet(T)^+_\xi \subset \mathbb{X}^\bullet(T)^+ = \mathbb{X}^\bullet(T_{\mathrm{sc}})^+$).

LEMMA 4.3.6. *We equip $\mathbb{X}^\bullet(T)^+$ with a partial order by identifying $\mathbb{X}^\bullet(T)^+ \cong \mathbb{N}^\Delta$ as in* (3.3.4)*, i.e. $\lambda_1 \geq \lambda_2$ if and only if $\langle \lambda_1, \check{\alpha}\rangle \geq \langle \lambda_2, \check{\alpha}\rangle$ for every simple coroot $\check{\alpha}$. Then for every $\nu_0 \in \mathbb{X}^\bullet(T)^+$, the set*

$$\{\nu \in \mathbb{X}^\bullet(T)^+ \mid \sigma(\nu) - \nu = \xi, \nu \geq \nu_0\}$$

has a unique minimal element, denoted by ν_0^h. In addition, in each σ-orbit of simple coroots, there is at least one $\check{\alpha}$ such that $\langle \nu_0^h, \check{\alpha}\rangle = \langle \nu_0, \check{\alpha}\rangle$.

Specializing this discussion to $\nu_0 = 0$, we deduce that $\mathbb{X}^\bullet(T)^+_\xi = \nu_0^h + \mathbb{X}^\bullet(T)^{+,\sigma}$.

PROOF. The last sentence of the lemma is a direct corollary of the existence and the properties of ν_0^h. We focus on constructing the needed element ν_0^h. Since G is a simply-connected semisimple group, we may take the set of fundamental weights $\{\omega_\alpha\}_{\alpha\in\Delta}$ such that $\langle \omega_\alpha, \check{\beta}\rangle = \delta_{\alpha\beta}$ for any pair $\alpha, \beta \in \Delta$. Then every weight can be written as $\xi = \sum\langle \xi, \check{\alpha}\rangle\omega_\alpha$.

If we write $\nu = \sum_\alpha \nu_\alpha\omega_\alpha$ with $\nu_\alpha \in \mathbb{Z}$, the equality $\sigma(\nu) - \nu = \xi$ is equivalent to the system of equations

$$\nu_\alpha - \nu_{\sigma(\alpha)} = \langle \xi, \sigma(\check{\alpha})\rangle, \quad \alpha \in \Delta,$$

with variables $\{\nu_\alpha\} \in \mathbb{Z}^\Delta$, and the condition $\nu \geq \nu_0$ is equivalent to that

$$\nu_\alpha \geq \langle \nu_0, \check{\alpha}\rangle \quad \text{for any} \quad \alpha \in \Delta. \tag{4.3.1}$$

For every σ-orbit $\mathcal{O} \subset \Delta$, let $\omega_\mathcal{O} = \sum_{\alpha\in\mathcal{O}} \omega_\alpha$. Then after modifying a solution ν by a multiple of $\omega_\mathcal{O}$ if necessary, we can always find ν satisfying the additional inequalities (4.3.1) and such that in each σ-orbit $\mathcal{O} \subset \Delta$, there is at least one $\alpha \in \mathcal{O}$ such that the equality in (4.3.1) holds. Then this is the desired ν_0^h. □

We now return to the proof of Proposition 4.3.5. By Proposition 3.4.1 and (3.4.1), we have an isomorphism

$$\boldsymbol{J}_0(V)_\xi = \bigoplus_{\nu\in\mathbb{X}^\bullet(T)^+_\xi} (\mathtt{S}_{\sigma(\nu^*)} \otimes \mathtt{S}_\nu \otimes V)^G \overset{\oplus_\nu \ell_{\sigma(\nu),\nu}}{\cong} \bigoplus_{\sigma(\nu)-\nu=\xi} \mathrm{fil}_\nu V(\xi) = R_{\mathbb{X}^\bullet(T)^+_\xi} V(\xi),$$

as modules over $\boldsymbol{J}_0 \cong k[\mathbb{X}^\bullet(T)^{+,\sigma}]$.

Therefore, by Lemma 2.2.2, the lemma will follow if we can show that

$$\dim \mathrm{gr}_{\mathbb{X}^\bullet(T)^+_\xi} V(\xi) = \dim V(\xi). \tag{4.3.2}$$

But by Lemma 4.3.6, we can apply Lemma 2.1.7 to deduce this equality from Theorem 3.3.1. □

By Proposition 4.3.5,we may choose a basis $\{e_i\}$ of $\boldsymbol{J}_0(V)_\xi$ as a $\boldsymbol{J}_0$-module such that each $e_i \in (\mathrm{gr}_{\nu_i} k[G] \otimes V)^{c_\sigma(G)}$ for some $\nu_i \in \mathbb{X}^\bullet(T)^+$. By the exactness of the bottom row of (4.2.1), we can lift each element in $\{e_i\}$ to $\boldsymbol{J}_+(V)$ so that $\widetilde{e_i} \in (\mathrm{fil}_{\nu_i} k[G] \otimes V)^{c_\sigma(G)}$. By Lemma 2.2.1 (2), the natural map

$$\bigoplus_i \boldsymbol{J}_+\widetilde{e_i} \to \boldsymbol{J}_+(V)$$

is surjective. In particular, $\boldsymbol{J}_+(V)$ is finitely generated over $\boldsymbol{J}_+$. By Lemma 4.2.1 (1), $\boldsymbol{J}(V)$ is also finitely generated over $\boldsymbol{J}$.

Now Theorem 4.3.2 is reduced to show that for every point $x \in V_G/\!\!/c_\sigma(G)$, the fiber of the module $\boldsymbol{J}_+(V)$ over x has dimension $\geq r_V$. By the semi-continuity, it is a consequence of the following.

LEMMA 4.3.7. *Over the generic point of $V_G/\!\!/c_\sigma(G)$, the rank of $\boldsymbol{J}_+(V)$ is r_V.*

PROOF. Since the map $V_G \to T^+_{\mathrm{ad}}$ is T-equivariant, and this T-action commutes with the $G\times G$-action, it is enough to prove a similar statement for $\boldsymbol{J}_+(V)|_{\mathfrak{d}^{-1}(\mathbf{1})} = \boldsymbol{J}(V)$.

If V is a representation of G, the restriction from G to T gives a natural map

$$\mathrm{Res}^\sigma_V : \boldsymbol{J}(V) \to (k[T] \otimes V)^{c_\sigma(N_0)}, \tag{4.3.3}$$

compatible with the isomorphism $\boldsymbol{J} \cong k[T]^{c_\sigma(N_0)}$ from Proposition 4.2.3. We call Res^σ_V the *twisted Chevalley restriction homomorphism*. But unlike the case when V is the trivial representation,

the map (4.3.3) in general fails to be an isomorphism and the failure is studied in details in [**Ba11, KNV11**] (when $\sigma = \mathrm{id}$). In the next section, we will also study in details (a variant of) this map. Currently, we just notice the following. By Remark 5.2.12 and Remark 5.2.15 below, there is an affine open N_0-stable subset

$$T^{\sigma\text{-reg}} = \left\{ t \in T \,\middle|\, I_t := \{g \in G \mid gt\sigma(g)^{-1} = t\} = T^\sigma \right\} \subset T,$$

an open subset $G^{\sigma\text{-rs}}$ of G, and an isomorphism $G \times^{N_0} T^{\sigma\text{-reg}} \xrightarrow{\cong} G^{\sigma\text{-rs}}$. It follows that

LEMMA 4.3.8. *The map* (4.3.3)*, regarded as a map of coherent sheaves on* $G/\!\!/c_\sigma(G) \cong T/\!\!/c_\sigma(N_0)$*, restricts to an isomorphism over a (non-empty) open subset of* $T^{\sigma\text{-reg}}/\!\!/c_\sigma(N_0)$*.*

Note that the fiber of $(k[T] \otimes V)^{c_\sigma(N_0)}$ over any point of $T^{\sigma\text{-reg}}/\!\!/c_\sigma(N_0)$ is isomorphic to V^{T^σ}, whose dimension is $\dim V^{T^\sigma} = r_V$. Therefore, the generic fiber of $\boldsymbol{J}(V)$ over $\operatorname{Spec} \boldsymbol{J}$ is of dimension r_V. Again, since $\mathfrak{d} : V_G \to T^+_{\mathrm{ad}}$ is T-equivariant, Lemma 4.3.7 follows. □

4.4. Construction of the basis. In this subsection, we assume that $\operatorname{char} k = 0$, and G is semisimple and simply-connected. Then every representation V of G admits a good filtration. The proof of Theorem 4.3.2 in fact gives a method to construct a basis of $\boldsymbol{J}(V)$, from certain basis of V.

For every weight $\xi \in (\sigma - 1)\mathbb{X}^\bullet(T)$, let $\mathbb{B}(\xi)$ be a basis of $V(\xi)$ satisfying the conditions in Proposition 3.3.2; for example, we may choose the MV basis as in § 3.5.

For each $\mathbf{b} \in \mathbb{B}(\xi)$ and each simple root α, let $\varepsilon_\alpha(\mathbf{b}) \geq 0$ be the integer such that $\mathbf{b} \in \mathrm{fil}^\alpha_{\varepsilon_\alpha(\mathbf{b})} V(\xi) - \mathrm{fil}^\alpha_{\varepsilon_\alpha(\mathbf{b})-1} V(\xi)$, i.e.

$$E_\alpha^{\varepsilon_\alpha(\mathbf{b})}(\mathbf{b}) \neq 0, \quad E_\alpha^{\varepsilon_\alpha(\mathbf{b})+1}(\mathbf{b}) = 0.$$

By Lemma 4.3.6, there is a unique minimal $\nu_{\mathbf{b}} \in \mathbb{X}^\bullet(T)^+$ such that $\sigma(\nu_{\mathbf{b}}) - \nu_{\mathbf{b}} = \xi$ and $\langle \nu_{\mathbf{b}}, \check{\alpha} \rangle \geq \varepsilon_\alpha(\mathbf{b})$ for every $\alpha \in \Delta$. Then by Proposition 3.4.1, we have an isomorphism

$$\ell_{\sigma(\nu_{\mathbf{b}}),\nu_{\mathbf{b}}} : (\mathtt{S}_{\sigma(\nu_{\mathbf{b}}^*)} \otimes \mathtt{S}_{\nu_{\mathbf{b}}} \otimes V)^G \to \mathrm{fil}_{\nu_{\mathbf{b}}} V(\sigma(\nu_{\mathbf{b}}) - \nu_{\mathbf{b}}).$$

It follows that $\mathbf{b}$ comes from a unique element in $(\mathtt{S}_{\sigma(\nu_{\mathbf{b}}^*)} \otimes \mathtt{S}_{\nu_{\mathbf{b}}} \otimes V)^G$, denoted by $f_{\mathbf{b},0}$.

REMARK 4.4.1. Recall that $\boldsymbol{J}_0 = k[\mathbb{X}^\bullet(T)^{+,\sigma}]$. For $\omega \in \mathbb{X}^\bullet(T)^{+,\sigma}$, let q^ω the corresponding element in $\boldsymbol{J}_0$. Let $x_1 \in \operatorname{Spec} \boldsymbol{J}_0$ be the point defined by $q^\omega(x_1) = 1$ for any ω. Then by the proof of Proposition 4.3.5, the fiber of $\boldsymbol{J}_0(V)$ at x_1 is canonically isomorphic to $\oplus_{\xi \in (\sigma-1)\mathbb{X}^\bullet(T)} V(\xi)$, and the restriction of $f_{\mathbf{b},0}$ to x_1 is just $\mathbf{b}$.

Since in characteristic zero, Schur and Weyl modules are isomorphic, there is a canonical G-equivariant map

$$\mathtt{S}_{\nu_{\mathbf{b}}} \otimes \mathtt{S}_{\sigma(\nu_{\mathbf{b}}^*)} \cong \mathtt{W}_{\nu_{\mathbf{b}}} \otimes \mathtt{S}_{\sigma(\nu_{\mathbf{b}}^*)} \to k[G\sigma] \tag{4.4.1}$$

given by taking (a twisted version of) matrix coefficients. Thus $f_{\mathbf{b},0}$ defines an element $f_{\mathbf{b}} \in (k[G\sigma] \otimes V)^G = \boldsymbol{J}(V)$. Explicitly, we write $f_{\mathbf{b},0}$ as $\sum_{i,j} \sigma e_i^* \otimes e_j \otimes v_{ij}$, where $\{e_i\}$ is a basis of $\mathtt{S}_{\nu_{\mathbf{b}}}$ and $\{e_i^*\}$ the dual basis, and $v_{ij} \in V$. Then

$$f_{\mathbf{b}}(g\sigma) = \sum_{i,j} \langle \sigma e_i^*, g\sigma e_j \rangle \cdot v_{ij}, \tag{4.4.2}$$

where $\langle \cdot, \cdot \rangle$ is the natural pairing between $\mathtt{S}_{\sigma(\nu_{\mathbf{b}})}$ and $\mathtt{S}_{\sigma(\nu_{\mathbf{b}}^*)}$. In fact, (4.4.2) is valid as long as $\mathtt{S}_{\nu_{\mathbf{b}}} \cong \mathtt{W}_{\nu_{\mathbf{b}}}$ (even in positive characteristic).

PROPOSITION 4.4.2. *The collection* $\{f_{\mathbf{b}} \mid \mathbf{b} \in \sqcup_{\xi \in (\sigma-1)\mathbb{X}^\bullet(T)} \mathbb{B}(\xi)\}$ *forms a basis of* $\boldsymbol{J}(V)$ *as a* $\boldsymbol{J}$*-module.*

PROOF. This follows from the proof of the freeness of $\boldsymbol{J}(V)$ over $\boldsymbol{J}$. Indeed, since $\boldsymbol{J}_+(V) = \sum_\nu (\mathrm{fil}_\nu k[G] \otimes V)^{c_\sigma(G)}$ and the matrix coefficient map (4.4.1) lands in $(\mathrm{fil}_{\nu_{\mathbf{b}}} k[G] \otimes V)^{c_\sigma(G)}$, we may regard $f_{\mathbf{b}}$ as a homogeneous element in $\boldsymbol{J}_+(V)$, denoted as $f_{\mathbf{b},+}$. By definition, Lemma 2.2.2 and (4.3.2) from the proof of Proposition 4.3.5, the image of $\{f_{\mathbf{b},+}\}$ under the restriction $\boldsymbol{J}_+(V) \otimes_{\boldsymbol{J}_+} \boldsymbol{J}_0 \cong \boldsymbol{J}_0(V)$ forms a basis. The proof of Theorem 4.3.2 shows that $\{f_{\mathbf{b},+}\}$ forms a basis of $\boldsymbol{J}_+(V)$ over $\boldsymbol{J}_+$. Therefore, $\{f_{\mathbf{b}}\}$ forms a basis of $\boldsymbol{J}(V) = \boldsymbol{J}_+(V) \otimes_{\boldsymbol{J}_+} \boldsymbol{J}$ over $\boldsymbol{J}$. □

5. Chevalley groups with an automorphism

In this section, we establish a few results about Chevalley groups equipped with a pinned automorphism σ. Our conventions and notations are as in § 4. In particular, we have a pinned reductive group $(G, B, T, \{x_\alpha\}_{\alpha\in\Delta})$ over k. We further assume that σ is a finite order automorphism of G preserving the pinning, i.e. $\sigma \circ x_\alpha = x_{\sigma(\alpha)}$ for $\alpha \in \Delta$. Put $A = T/(\sigma - 1)T$. Let $W_0 = W^\sigma$, which acts on A, and let N_0 be the preimage of W_0 in N, which acts on T by twisted conjugation c_σ. Let $(\mathfrak{g}, \mathfrak{b}, \mathfrak{t}, \mathfrak{u})$ denote the Lie algebra of (G, B, T, U). Write G_{sc} for the simply-connected cover of G, and T_{sc} the maximal torus of G_{sc} which is the preimage of T.

5.1. Root datum with an automorphism. We start with some discussion of a version of folding of root systems. We also refer to [**Sp06**] for some related discussions.

For each σ-orbit $\mathcal{O} \subset \Phi(G, T)$, we write

$$\alpha_{\mathcal{O}} := \sum_{\gamma\in\mathcal{O}} \gamma,$$

which belongs to $\mathbb{X}^\bullet(A) \cong \mathbb{X}^\bullet(T)^\sigma \subset \mathbb{X}^\bullet(T)$. If we pick $\gamma \in \mathcal{O}$, then $\alpha_{\mathcal{O}} = \gamma + \sigma\gamma + \cdots + \sigma^{|\mathcal{O}|-1}\gamma$, where $|\mathcal{O}|$ denotes the cardinality of $\mathcal{O}$. Note that $\alpha_{\mathcal{O}}$ may be different from the image of γ under the usual norm map $\mathbb{X}^\bullet(T) \to \mathbb{X}^\bullet(A)$.

LEMMA 5.1.1. *The collection of $\alpha_{\mathcal{O}}$ for all σ-orbits $\mathcal{O} \subset \Phi(G, T)$, regarded as a subset of $\mathbb{X}^\bullet(A)$, has a structure of a root datum. Let G_σ denote the corresponding reductive group over k containing A as a maximal torus,*[4] *and let $\Phi(G_\sigma, A)$ denote the corresponding root system. Then the map $\mathcal{O} \mapsto \alpha_{\mathcal{O}}$ establishes a bijection between the set of σ-orbits in Δ and a set of simple roots in $\Phi(G_\sigma, A)$. With this choice of simple roots of $\Phi(G_\sigma, A)$, its subset of positive roots are $\Phi(G_\sigma, A)^+ = \{\alpha_{\mathcal{O}} \mid \mathcal{O} \subset \Phi(G, T)^+\}$.*

PROOF. Since σ lifts to a unique automorphism of the simply-connected cover of the derived group of G (e.g. see [**St68**, §9.16]), we may assume that G is semisimple and simply-connected. Then $(G, T) = \prod(G_i, T_i)$, where G_i is simple and simply-connected, and the action of σ permutes the simple factors. To prove the lemma, clearly we may assume that there are r simple factors, all of which are isomorphic to (G_0, T_0) and are cyclically permuted by σ, and that σ^r is an automorphism of $(\mathbb{X}^\bullet(T_0), \Phi(G_0, T_0))$ of order d. Then if $d = 1$, $G_\sigma = G_0$. If $d > 1$, then $\Phi(G, T)$ is of type A_n, D_n, or E_6, and one can check case by case that the reductive group G_σ is determined by the following table ($n \geq 1$).

G_0	SL_{2n+1}	SL_{2n+2}	$\mathrm{Spin}_{2n+2}, d = 2$	E_6	$\mathrm{Spin}_8, d = 3$
G_σ	SO_{2n+1}	Spin_{2n+3}	Sp_{2n}	F_4	G_2

The detailed calculations for the case $G_0 = \mathrm{SL}_{2n+1}, \mathrm{SL}_{2n+2}$ and $\mathrm{Spin}_{2n+2}, d = 2$ can be found in Example 5.1.5, Example 5.1.6 and Example 6.4.4 below. □

[4]We choose the notation G_σ for the group because it has a maximal torus $A = T_\sigma$. This does not suggest that it relates to the σ-coinvariants of G, whatever it means.

REMARK 5.1.2. Assume that G is semisimple and simply-connected. Note that the group of invariants G^σ of G under the σ-action is a connected semisimple group by [**St68**, Theorem 8.2] containing T^σ as a maximal torus (note however that the group of invariants is denoted by G_σ in *loc. cit.*). The root system $\Phi(G^\sigma, T^\sigma)$ is isogenous to $\Phi(G_\sigma, T_\sigma)$ but in general is not isomorphic to it. For example, if $G = \mathrm{SL}_{2n}$ $(n > 1)$ with σ nontrivial, then $G^\sigma = \mathrm{Sp}_{2n}$, whereas $G_\sigma = \mathrm{Spin}_{2n+1}$.

REMARK 5.1.3. Let $\hat{G}$ denote the (adjoint semisimple) Deligne–Lusztig dual group with dual torus $\hat{T}$ over a finite field κ (so that the absolute root datum of $(\hat{G}, \hat{T})$ is dual to the root datum $\Phi(G,T) \subset \mathbb{X}^\bullet(T)$ and the Weil descent datum defining $(\hat{G}, \hat{T})$ over κ induces the σ-action on $\Phi(G,T)$). Then the maximal split torus $\hat{A}$ of $\hat{G}$ is dual to A. For example, if $G = \mathrm{SL}_n$ with the non-trivial involution σ as in Example 5.1.5, Example 5.1.6 below, then $\hat{G}$ is isomorphic to the projectivel unitary group PU_n over κ. Then the dual root datum of $\Phi(G_\sigma, A)$ is equal to the sub-root system of the relative root system $\Phi_{\mathrm{rel}}(\hat{G}, \hat{A}) \subset \mathbb{X}^\bullet(\hat{A})$ consisting of those $\alpha^\vee \in \Phi_{\mathrm{rel}}(\hat{G}, \hat{A})$ such that $2\alpha^\vee \notin \Phi_{\mathrm{rel}}(\hat{G}, \hat{A})$.

It is easy to see (e.g. by a case-by-case inspection) that every root in $\Phi(G_\sigma, A)$ comes from one or two σ-orbits in $\Phi(G,T)$. In the latter case, the cardinality of one orbit is twice of the cardinality of the other.

DEFINITION 5.1.4. A root in $\Phi(G_\sigma, A)$ is called of *type* A if there is a unique σ-orbit $\mathcal{O} \subset \Phi(G,T)$ such that this root is $\alpha_{\mathcal{O}}$. In this case, the corresponding orbit $\mathcal{O}$ is also called of *type* A. Note that $\langle \alpha, \check{\beta} \rangle = 0$ for all pairs of distinct roots $\alpha, \beta \in \mathcal{O}$. If $\alpha_{\mathcal{O}}$ is a simple root in $\Phi(G_\sigma, A)$, then $\mathcal{O}$ is a σ-orbit of simple roots of $\Phi(G,T)$ and all vertices in the sub Dynkin diagram corresponding to $\mathcal{O}$ are isolated.

A root in $\Phi(G_\sigma, A)$ is called of *type* BC if there are two σ-orbits $\mathcal{O}^-$ and $\mathcal{O}^+$ such that this root is $\alpha_{\mathcal{O}^-} = \alpha_{\mathcal{O}^+}$, and that $|\mathcal{O}^-| = 2|\mathcal{O}^+|$. The orbit $\mathcal{O}^-$ (resp. $\mathcal{O}^+$) is called of *type* BC^- (resp. *type* BC^+). In this case, $\langle \sigma^{|\mathcal{O}^+|}\alpha, \check{\alpha} \rangle = 1$ for every $\alpha \in \mathcal{O}^-$, and $\beta := \alpha + \sigma^{|\mathcal{O}^+|}\alpha \in \mathcal{O}^+$. If $\alpha_{\mathcal{O}^-}$ is a simple root in $\Phi(G_\sigma, A)$, then $\mathcal{O}^-$ is a σ-orbit of simple roots in $\Phi(G,T)$, and the sub Dynkin diagram corresponding to $\mathcal{O}^-$ is a product of $|\mathcal{O}^+|$ copies of the root system A_2.

EXAMPLE 5.1.5. Let G be SL_{2r+1} $(r \geq 1)$ with row and column indices in $\{-r, \dots, r\}$, the pinning (B, T, e) given by the group of standard upper triangular matrices, the subgroup of diagonal matrices, and $e = \sum_{i=-r}^{r-1} E_{i,i+1}$. Then the unique non-trivial pinned automorphism σ is given by

$$\sigma(X) = J\, {}^tX^{-1} J \quad \text{for } X \in \mathrm{SL}_{2r+1}, \tag{5.1.1}$$

where J is the anti-diagonal matrix, with entries $J_{i,-i} = (-1)^{r+i}$. Then

$$\Phi(G,T) = \{\, \varepsilon_i - \varepsilon_j \mid -r \leq i, j \leq r,\ i \neq j\},$$

where ε_i is the character of T given by evaluating at the (i,i)-entry. Since σ acts on $\Phi(G,T)$ by $\sigma(\varepsilon_i) = -\varepsilon_{-i}$, the σ-orbits on $\Phi(G,T)$ are

$$\mathcal{O}_{i,j} = \{\varepsilon_{-i} - \varepsilon_{-j},\, \varepsilon_j - \varepsilon_i\}, \quad \mathcal{O}_i^- = \{\varepsilon_{-i} - \varepsilon_0,\, \varepsilon_0 - \varepsilon_i\}, \quad \mathcal{O}_i^+ = \{\varepsilon_{-i} - \varepsilon_i\} \tag{5.1.2}$$

for $i \in \{\pm 1, \dots, \pm r\}$ and $j \in \{\pm 1, \dots, \pm(|i| - 1)\}$. They are of type A, BC^-, and BC^+ respectively.

As $A = T/(\sigma - 1)T$, its character group is

$$\mathbb{X}^\bullet(A) = \mathbb{X}^\bullet(T)^\sigma = \bigoplus_{i=1}^{r} \mathbb{Z}(\varepsilon_{-i} - \varepsilon_i).$$

The simple roots of $\Phi(G_\sigma, A)$ are $\varepsilon_{-i-1} - \varepsilon_{-i} + \varepsilon_i - \varepsilon_{i+1}$ for $i = 1, \dots, r-1$ and $\varepsilon_{-1} - \varepsilon_1$. The former ones are of type A (being equal to $\alpha_{\mathcal{O}_{i+1,i}}$) and the latter one is of type BC (being equal to $\alpha_{\mathcal{O}_1^-} = \alpha_{\mathcal{O}_1^+}$).

EXAMPLE 5.1.6. Let G be SL_{2r} ($r \geq 2$) with row and column indices in $\{-r, \dots, -1, 1, \dots, r\}$, the pinning (B, T, e) given by the group of standard upper triangular matrices, the subgroup of diagonal matrices, and $e = \sum_{i=-r}^{-2}(E_{i,i+1} + E_{-i-1,-i}) + E_{-1,1}$. Then the unique non-trivial pinned automorphism σ is given by

$$\sigma(X) = J\,{}^tX^{-1}J \quad \text{for } X \in \mathrm{SL}_{2r},$$

where J is the anti-diagonal matrix, with entries $J_{i,-i} = -J_{-i,i} = (-1)^{r+i}$ for $i = 1, \dots, r$. Then

$$\Phi(G,T) = \{\, \varepsilon_i - \varepsilon_j \mid -r \leq i, j \leq r,\ i \neq j,\ ij \neq 0\},$$

where ε_i is the character of T given by evaluating at the (i,i)-entry. Since σ acts on $\Phi(G,T)$ by $\sigma(\varepsilon_i) = -\varepsilon_{-i}$, the σ-orbits on $\Phi(G,T)$ are

$$\mathcal{O}_{i,j} = \{\varepsilon_{-i} - \varepsilon_{-j},\ \varepsilon_j - \varepsilon_i\}, \quad \mathcal{O}_i = \{\varepsilon_{-i} - \varepsilon_i\}$$

for $i \in \{\pm 1, \dots, \pm r\}$ and $j \in \{\pm 1, \dots, \pm(|i| - 1)\}$. They are all of type A.

As $A = T/(\sigma - 1)T$, its character group is

$$\mathbb{X}^\bullet(A) = \mathbb{X}^\bullet(T)^\sigma = \Big(\bigoplus_{i=1}^{r} \mathbb{Z}(\varepsilon_{-i} - \varepsilon_i)\Big) + \mathbb{Z}(\varepsilon_{-r} + \cdots + \varepsilon_{-1}).$$

The simple roots of $\Phi(G_\sigma, A)$ are $\varepsilon_{-i-1} - \varepsilon_{-i} + \varepsilon_i - \varepsilon_{i+1}$ for $i = 1, \dots, r-1$ and $\varepsilon_{-1} - \varepsilon_1$.

LEMMA 5.1.7. *The action of the Weyl group $N_{G_\sigma}(A)/A$ on A is identified with the natural action of W_0 on A.*

PROOF. For a root $\alpha \in \Phi(G,T)$, let s_α denote the corresponding reflection acting on $\mathbb{X}^\bullet(T)$. Let $\alpha_\mathcal{O}$ be a simple root of $\Phi(G_\sigma, A)$ and $s_{\alpha_\mathcal{O}}$ the corresponding simple reflection. One checks easily that $s_{\alpha_\mathcal{O}} = \prod_{\gamma \in \mathcal{O}} s_\alpha \in W_0$ if $\alpha_\mathcal{O}$ is of type A, and $s_{\alpha_\mathcal{O}} = \prod_{\alpha \in \mathcal{O}^+} s_\alpha \in W_0$ if $\alpha_\mathcal{O}$ is of type BC. Therefore, $N_{G_\sigma}(A)/A \subset W_0$ as automorphisms of A. It remains to notice that they are isomorphic as abstract Coxeter groups, as can be checked easily from the table in Lemma 5.1.1. □

We discuss subgroups of G associated to σ-orbits of roots, which specialize to "root SL_2" when $\sigma = \mathrm{id}$. First, we have the following lemma.

LEMMA 5.1.8. *Let $\alpha \in \Phi(G,T)$. If the σ-orbit $\mathcal{O}$ containing α is of type A or BC^-, then $\sigma^{|\mathcal{O}|} \circ x_\alpha = x_\alpha$. If the σ-orbit $\mathcal{O}$ containing α is of type BC^+, then $\sigma^{|\mathcal{O}|} \circ x_\alpha = -x_\alpha$.*

PROOF. First, if α is a simple root, then $\sigma^{|\mathcal{O}|} \circ x_\alpha = x_\alpha$ since σ acts by pinned automorphisms. In general, every σ-orbit $\mathcal{O}$ of type A or of type BC^- is conjugated to a σ-orbit of simple roots by an element $w \in W_0 = W^\sigma$. We can choose a lifting of w to a σ-invariant element $\dot{w}$ in N (see [**St68**, (5), p. 55]). For a root $\alpha \in \mathcal{O}$, write $\mathrm{Ad}_{\dot{w}}(E_\alpha) = cE_{w(\alpha)}$ for some invertible constant c. Then $\mathrm{ord}(|\mathcal{O}|) = \mathrm{ord}(|w(\mathcal{O})|)$ and

$$\mathrm{Ad}_{\dot{w}}(E_\alpha) = cE_{w(\alpha)} = \sigma^{|\mathcal{O}|}(cE_{w(\alpha)}) = \sigma^{|\mathcal{O}|}(\mathrm{Ad}_{\dot{w}}(E_\alpha)) = \mathrm{Ad}_{\dot{w}}(\sigma^{|\mathcal{O}|}(E_\alpha)).$$

Therefore, $\sigma^{|\mathcal{O}|} \circ x_\alpha = x_\alpha$.

Next, assume that α belongs to a σ-orbit $\mathcal{O}^+$ of type BC^+, then $\alpha = \beta + \sigma^{|\mathcal{O}^+|}(\beta)$ for some root β in a σ-orbit $\mathcal{O}^-$ of type BC^-, and $E_\alpha = c[E_\beta, E_{\sigma^{|\mathcal{O}^+|}(\beta)}]$ for some invertible constant c. Note that $\sigma^{|\mathcal{O}^+|}$ will send E_β to $c' E_{\sigma^{|\mathcal{O}^+|}(\beta)}$ and $E_{\sigma^{|\mathcal{O}^+|}(\beta)}$ to $c'^{-1}E_\beta$ for some invertible c'. Hence $\sigma^{|\mathcal{O}^+|}(E_\alpha) = -E_\alpha$, and therefore $\sigma^{|\mathcal{O}^+|} \circ x_\alpha = -x_\alpha$. □

Now, for a σ-orbit $\mathcal{O} \subset \Phi(G,T)^+$ of positive roots, let $G_\mathcal{O}$ be the subgroup of G generated by T and root subgroups $U_{\pm\alpha}$ for $\alpha \in \mathcal{O}$. Clearly, this is a σ-stable reductive subgroup of G. Let $U_\mathcal{O}$ (resp. $U_{\mathcal{O},-}$) be the subgroup of U generated by U_α for $\alpha \in \mathcal{O}$ (resp. $-\alpha \in \mathcal{O}$), and let $B_\mathcal{O} = U_\mathcal{O} T$. If $\mathcal{O}$ is a σ-orbit of simple roots, we also let $U^\mathcal{O}$ denote the subgroup of U generated by U_β, for

those positive roots β that are not in the sub-root system spanned by $\mathcal{O}$. Then $B = U^{\mathcal{O}} B_{\mathcal{O}}$ is a semi-direct product decomposition, and $P_{\mathcal{O}} = U_{\mathcal{O}} G_{\mathcal{O}}$ is a σ-stable standard parabolic subgroup of G.

Note that $(G_{\mathcal{O}}, B_{\mathcal{O}}, T, \{x_\alpha\}_{\alpha \in \mathcal{O}})$ is a pinning of $G_{\mathcal{O}}$. When G is semisimple and simply-connected, there are essentially two cases for its derived subgroup $G_{\mathcal{O},\mathrm{der}}$ (which is always simply-connected). The following lemma also follows from Lemma 5.1.8 (and in fact is equivalent to Lemma 5.1.8).

LEMMA 5.1.9. *Assume that G is semisimple and simply-connected.*

(1) *If $\mathcal{O}$ is of type A, then $G_{\mathcal{O},\mathrm{der}} \cong \prod_{i=1}^{|\mathcal{O}|} \mathrm{SL}_2$ where the σ acts by permuting factors and preserves the pinning (up to possibly rescaling the x_α's).*

(2) *If $\mathcal{O}$ is of type BC^-, then $G_{\mathcal{O},\mathrm{der}} \cong \prod_{i=1}^{|\mathcal{O}|/2} \mathrm{SL}_3$ where the σ acts by permuting factors and $\sigma^{|\mathcal{O}|/2}$ acts on each factor SL_3 as in Example 5.1.5, which also preserves the pinning (up to possibly rescaling the x_α's).*

(3) *If $\mathcal{O}$ is of type BC^+, then $G_{\mathcal{O},\mathrm{der}} \cong \prod_{i=1}^{|\mathcal{O}|} \mathrm{SL}_2$, where the σ acts by*

$$(g_1, \dots, g_{|\mathcal{O}|}) \longmapsto \left(\mathrm{Ad}_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)}(g_{|\mathcal{O}|}), g_1, \dots, g_{|\mathcal{O}|-1}\right). \tag{5.1.3}$$

Let $\mathbf{T} = B/U$ denote the abstract Cartan. Note that σ acts on $\mathbf{T}$, and $T \to B \to \mathbf{T}$ is a σ-equivariant isomorphism. Let $\mathbf{A} = \mathbf{T}/(1-\sigma)\mathbf{T}$, which is canonically isomorphic to A. By transport of structures, W_0 acts on $\mathbf{A}$, and for every root $\alpha_{\mathcal{O}} \in \Phi(G_\sigma, A)$, $e^{\alpha_{\mathcal{O}}}$ can be regarded as a regular function on $\mathbf{A}$. Let

$$q_B : B \to \mathbf{T} \to \mathbf{A} \tag{5.1.4}$$

denote the quotient map.

DEFINITION 5.1.10. Let $\mathcal{O} \subset \Phi(G, A)$ be a σ-orbit. We define a divisor

$$\mathbf{A}_{\mathcal{O}} = \begin{cases} \{t \in \mathbf{A} \mid e^{\alpha_{\mathcal{O}}}(t) = 1\} & \mathcal{O} \text{ is of type } \mathsf{A} \text{ or } \mathsf{BC}^-, \\ \{t \in \mathbf{A} \mid e^{\alpha_{\mathcal{O}}}(t) = -1\} & \mathcal{O} \text{ is of type } \mathsf{BC}^+. \end{cases}$$

Note that $\mathbf{A}_{\mathcal{O}} = \mathbf{A}_{-\mathcal{O}}$. Let

$$\mathring{\mathbf{A}} = \mathbf{A} - \cup_{\mathcal{O}} \mathbf{A}_{\mathcal{O}},$$

where the union is taken over all σ-orbits $\mathcal{O} \subset \Phi(G, T)$. This is a W_0-invariant open subset of $\mathbf{A}$. For a σ-orbit $\mathcal{O} \subset \Phi(G, T)$, let

$$\mathbf{A}^{[\mathcal{O}]} = \mathbf{A} - \bigcup_{\mathcal{O}' \neq \mathcal{O}} \mathbf{A}_{\mathcal{O}'}.$$

In particular, it is an open neighborhood of the generic points of $\mathbf{A}_{\mathcal{O}}$ in $\mathbf{A}$.

Let $\mathring{B}$ and $\mathring{T}$ be the preimage of $\mathring{\mathbf{A}}$ under the natural projections $q_B : B \to \mathbf{A}$ and $T \cong \mathbf{T} \to \mathbf{A}$. SImilarly, let $B^{[\mathcal{O}]}$ and $B^{[\mathcal{O}]}_{\mathcal{O}'}$ be the preimage of $\mathbf{A}^{[\mathcal{O}]}$ under the projections $B \to \mathbf{A}$ and $B_{\mathcal{O}'} \to \mathbf{A}$. Finally, via the isomorphism $A \cong \mathbf{A}$, we have similarly defined spaces $A_{\mathcal{O}}$, $\mathring{A}$ and $A^{[\mathcal{O}]}$.

REMARK 5.1.11. Note that $A_{\mathcal{O}}$ may not be irreducible nor reduced in general. For example, if $G_\sigma = \mathrm{Sp}_{2n}$, then every long root in $\Phi(G_\sigma, A)$ is twice of a weight of A and therefore the corresponding divisor $A_{\mathcal{O}}$ consists of two connected components if $\mathrm{char}\, k > 2$, and is non-reduced if $\mathrm{char}\, k = 2$. On the other hand, if $\mathrm{char}\, k > 2$, then $A_{\mathcal{O}}$ is always reduced.

LEMMA 5.1.12. *Assume that G is semsimple and simply-connected. Then $\mathbf{A}/\!\!/W_0$ is isomorphic to an affine space, and the natural morphism $\mathbf{A} \to \mathbf{A}/\!\!/W_0$ is a finite and flat W_0-cover, with branch loci $\cup_{\mathcal{O}} \mathbf{A}_{\mathcal{O}}$. In particular, the restriction $\mathring{\mathbf{A}} \to \mathring{\mathbf{A}}/\!\!/W_0$ is a finite étale Galois cover.*

PROOF. We may identify the action of W_0 on $\mathbf{A}$ with the action of $W_0 \cong N_{G_\sigma}(A)/A$ on A by Lemma 5.1.7. Note that by table in the proof of Lemma 5.1.1, G_σ is isomorphic to a product of simply-connected groups and odd orthogonal groups. Therefore, $k[\mathbf{A}]^{W_0}$ is a polynomial algebra (see [**St75**]), and $\mathbf{A} \to \mathbf{A}/\!\!/W_0$ is finite flat. The branch locus of the covering $\mathbf{A} \to \mathbf{A}/\!\!/W_0$ is the union of divisors $\{t \in \mathbf{A} \mid \check{\alpha}_\mathcal{O}(e^{\alpha_\mathcal{O}}(t)) = 1, \alpha_\mathcal{O} \in \Phi(G_\sigma, A)\}$. Here $\check{\alpha}_\mathcal{O}$ denotes the coroot of $\alpha_\mathcal{O}$, regarded as a homomorphism $\check{\alpha}_\mathcal{O} : \mathbb{G}_m \to A \cong \mathbf{A}$. Its construction is as follows. Let $\gamma \in \mathcal{O}$ if $\alpha_\mathcal{O}$ is of type A or $\gamma \in \mathcal{O}^+$ if $\alpha_\mathcal{O}$ is of type BC. Let $\check{\gamma}$ be the corresponding coroot, regarded as a homomorphism $\mathbb{G}_m \to T$. Then $\check{\alpha}_\mathcal{O}$ is the composition of this homomorphism with the projection $T \to A \cong \mathbf{A}$. Again, by checking the table in the proof of Lemma 5.1.1, one sees that $\ker \check{\alpha}_\mathcal{O}$ is trivial if $\alpha_\mathcal{O}$ is of type A and is $\{\pm 1\}$ if $\alpha_\mathcal{O}$ is of type BC. The lemma follows. □

5.2. Twisted conjugacy classes. In this subsection, we study σ-regular elements of G, generalizing some of the well-known results of Steinberg [**St65**] for $\sigma = \mathrm{id}$. Some results with restriction of the characteristic of k were also obtained by Mohrdieck [**Mo03**] before.

Let G be a reductive group over k. Recall the σ-twisted conjugation (or σ-conjugation for brevity)

$$c_\sigma : G \times G\sigma \to G\sigma, \quad (h, g\sigma) = hg\sigma(h)^{-1}\sigma =: c_\sigma(h)(g).$$

Since σ preserves (B, T), it acts on the set $\Delta \subset \Phi(G, T)$ of simple roots.

Let I be the *centralizer group scheme* for the action of G on itself by c_σ, i.e. it is the group scheme over G defined by the Cartesian diagram

$$\begin{array}{ccc} I & \longrightarrow & G \times G\sigma \\ \downarrow & & \downarrow{\scriptstyle c_\sigma \times \mathrm{pr}_2} \\ G & \longrightarrow & G \times G\sigma, \end{array}$$

where pr_2 denotes the projection to the second factor and $G \to G \times G\sigma$ denotes the "diagonal" embedding $g \mapsto (g, g\sigma)$. Its fiber over $g \in G$ is denoted by I_g.

DEFINITION 5.2.1. Let $G^{\sigma\text{-reg}}$ denote the *σ-regular locus* of G. Namely,

$$G^{\sigma\text{-reg}} = \{g \in G \mid \dim I_g = \dim T^\sigma =: r\} \subset G.$$

Let $B^{\sigma\text{-reg}} := G^{\sigma\text{-reg}} \cap B$, $T^{\sigma\text{-reg}} = G^{\sigma\text{-reg}} \cap T$ and $U^{\sigma\text{-reg}} = G^{\sigma\text{-reg}} \cap U$.

REMARK 5.2.2. (1) It will be clear from the following discussion that $\dim I_g \geq r$ for any $g \in G$. Therefore, $G^{\sigma\text{-reg}}$ is an open subset of G by semi-continuity.

(2) When $\mathrm{char}\, k = 0$, $\dim I_g = \mathrm{Lie}\, I_g = \ker(\mathrm{id} - \mathrm{Ad}_g\sigma : \mathfrak{g} \to \mathfrak{g})$. Complication arises in the positive characteristic case; for example, the center $\mathfrak{z}$ of $\mathfrak{g}$, i.e.

$$\mathfrak{z} := \bigcap_{\alpha \in \Phi(G,T)} \mathrm{Ker}(d\alpha : \mathfrak{t} \to k),$$

may not be trivial even if G is semisimple.

(3) For every automorphism τ of the algebraic group G, one can define the open subset of τ-regular elements $G^{\tau\text{-reg}} \subset G$ as those $g \in G$ such that I_g achieves the minimal dimension. But since every τ differs from some pinned automorphism σ by an inner automorphism, the study of $G^{\tau\text{-reg}}$ reduces to the study of $G^{\sigma\text{-reg}}$, by virtual of Remark 4.1.1.

We first study $U^{\sigma\text{-reg}}$. Let $V = [U, U] \cdot (\sigma - 1)(\prod_{\alpha \in \Delta} U_\alpha) \subset U$, and let

$$q_U : U \to W = U/V$$

denote the quotient map. The natural conjugation of T^σ on U induces an action of T^σ on W, and there is a unique T^σ-open orbit $\mathring{W} \subset W$. Explicitly, if we choose an element $\alpha \in \mathcal{O}$ for each σ-orbit

of simple roots so that there is an isomorphism

$$\prod_{\text{chosen } \alpha} x_\alpha : \mathbb{G}_a^{|\Delta^\sigma|} \to U \to W$$

giving a basis of W (as a vector space over k), then $v \in \mathring{W}$ if and only if all the coefficients of v with respect to this basis are non-zero. Let $\mathring{U} = q_U^{-1}(\mathring{W}) \subset U$.

We say that an element $w \in W$ is a *σ-twisted Coxeter element* in W, if one can choose an element $\alpha \in \mathcal{O}$ for each σ-orbit $\mathcal{O}$ of simple roots and write w as the product of simple reflections corresponding to these chosen α's (in some order). Let $\dot{w}$ be a representative of w in N.

LEMMA 5.2.3. *Assume that G is simply-connected. Let w be a σ-twisted Coxeter element in W. Then for every $b \in B$,*

$$\dim \ker(\mathrm{id} - \mathrm{Ad}_{b\dot{w}}\sigma : \mathfrak{g} \to \mathfrak{g}) \leq \dim \mathfrak{z}^\sigma + r.$$

PROOF. Given the properties of σ-twisted Coxeter elements presented in [**Sp74**, Theorem 7.6], the proof of [**St65**, Lemma 4.3] applies literally to our situation. □

Now we come to the first key result.

PROPOSITION 5.2.4. *For each σ-orbit of simple roots $\mathcal{O} \subset \Phi(G,T)$, choose one $\alpha \in \mathcal{O}$ and define*

$$x := \prod_\alpha x_\alpha^{-1}(1) \in U,$$

where the product is taken over the chosen simple roots in a fixed order. Then

1. $(\mathrm{id} - \mathrm{Ad}_x\sigma)^{-1}(\mathfrak{u}) = \mathfrak{t}^\sigma + \mathfrak{u}$.
2. *Let $g \in G$. If $gx\sigma(g)^{-1} \in U$, then $g \in B$. In addition, $I_x \subset B$.*
3. *$x \in U^{\sigma\text{-reg}}$; in particular, $U^{\sigma\text{-reg}}$ is nonempty.*
4. *$\mathring{U} = U^{\sigma\text{-reg}}$ is a single orbit under the σ-conjugation action of $T^\sigma \cdot U$ on U. In particular, $I_u \subset B$ and $(\mathrm{id} - \mathrm{Ad}_u\sigma)^{-1}(\mathfrak{u}) = \mathfrak{t}^\sigma + \mathfrak{u}$ for every $u \in U^{\sigma\text{-reg}}$.*

PROOF. Write $\mathfrak{g}_x$ for the Lie algebras of I_x, and similarly $\mathfrak{b}_x$ and $\mathfrak{u}_x$ for the Lie algebra of $I_x \cap B$ and $I_x \cap U$. We have the following lemma.

LEMMA 5.2.5. *We have $\dim \mathfrak{u}_x = |\Delta^\sigma|$ and $\mathfrak{g}_x = \mathfrak{b}_x \subset \mathfrak{t}^\sigma + \mathfrak{u}$.*

PROOF. The inclusion $\mathfrak{b}_x \subset \mathfrak{t}^\sigma + \mathfrak{u}$ is clear. We prove the two equalities in the statement. We first assume that G is semisimple and simply-connected. Then $r = \dim T^\sigma = |\Delta^\sigma|$. Let w_0 be the longest element in the Weyl group and $\dot{w}_0 \in N_0$ a representative of w_0. Then $\dot{w}_0 x \dot{w}_0 \in B\dot{w}B$ for some σ-twisted Coxeter element w. Then as in [**St65**, Theorem 4.6],

$$\dim \mathfrak{z}^\sigma + \dim \mathfrak{u}_x \leq \dim \mathfrak{b}_x \leq \dim \mathfrak{g}_x \leq \dim \mathfrak{z}^\sigma + r.$$

Therefore $\dim \mathfrak{u}_x \leq r$ and $\dim(I_x \cap U) \leq r$. On the other hand, U acts on fibers of q_U via σ-conjugation. Therefore, $\dim(I_x \cap U) \geq \dim W = r$. Putting them together gives $\dim \mathfrak{u}_x = \dim(I_x \cap U) = r$ and $\mathfrak{b}_x = \mathfrak{g}_x$. Now for a general reductive group G, let G_{sc} be the simply-connected cover of its derived group. Then σ lifts to a unique automorphism of G_{sc} (e.g. see [**St68**, §9.16]). Since the central isogeny $G_{\mathrm{sc}} \to G$ induces an isomorphism on unipotent subgroups, we have $\dim(I_x \cap U) = |\Delta^\sigma|$. In addition, since the kernels and the cokernels of the two maps $\mathfrak{b}_{\mathrm{sc}} \to \mathfrak{b}$ and $\mathfrak{g}_{\mathrm{sc}} \to \mathfrak{g}$ are equal, and since $(\mathfrak{b}_{\mathrm{sc}})_x = (\mathfrak{g}_{\mathrm{sc}})_x$, $\mathfrak{b}_x = \mathfrak{g}_x$ also holds for G. □

Now we prove the proposition. (1) First, note that $(\mathrm{id} - \mathrm{Ad}_x\sigma)(\mathfrak{u}) \subset \mathfrak{v}$, where $\mathfrak{v}$ is the Lie algebra of V. But since $\dim \mathfrak{u}_x = \dim \mathfrak{u} - \dim \mathfrak{v}$, we have $(\mathrm{id} - \mathrm{Ad}_x\sigma)(\mathfrak{u}) = \mathfrak{v}$. Next, if G is of adjoint type, then the composition of maps

$$\mathfrak{t}^\sigma \xrightarrow{\mathrm{id}-\mathrm{Ad}_x\sigma} \mathfrak{u} \to \mathfrak{u}/\mathfrak{v}$$

is an isomorphism. It follows that in this case $(\mathrm{id}-\mathrm{Ad}_x\sigma)(\mathfrak{t}^\sigma+\mathfrak{u})=\mathfrak{u}$. Since $\mathfrak{g}_x=\mathfrak{b}_x\subset\mathfrak{t}^\sigma+\mathfrak{u}$, we see that $(\mathrm{id}-\mathrm{Ad}_x\sigma)^{-1}(\mathfrak{u})=\mathfrak{t}^\sigma+\mathfrak{u}$. In general, let G_{ad} be the adjoint quotient of G. Since the central isogeny induces an isomorphism $\mathfrak{u}\cong\mathfrak{u}_{\mathrm{ad}}$, it is easy to see that $(\mathrm{id}-\mathrm{Ad}_x\sigma)^{-1}(\mathfrak{u})\subset\mathfrak{b}$. In addition, if $Y\in\mathfrak{b}$ such that $Y-\mathrm{Ad}_x\sigma(Y)\in\mathfrak{u}$, then $(Y \mod \mathfrak{u})\in\mathfrak{t}^\sigma$. Therefore, $(\mathrm{id}-\mathrm{Ad}_x\sigma)^{-1}(\mathfrak{u})=\mathfrak{t}^\sigma+\mathfrak{u}$ holds in general.

(2) Write $g=u_1nu_2$ under the Bruhat decomposition for some $n\in N,u_1,u_2\in U$. Then $nu_2x\sigma(u_2)^{-1}=u_1^{-1}u\sigma(u_1)\sigma(n)$ for some $u\in U$. It follows that $n=\sigma(n)$ by the uniqueness of the Bruhat decomposition. Then $nu_2x\sigma(u_2)^{-1}n^{-1}=u_1^{-1}u\sigma(u_1)\in U$. Since $q_U(u_2x\sigma(u_2)^{-1})=q_U(x)\in\mathring{W}$, we must have $n\in T$ and therefore $g\in B$. It follows that $I_x^{\mathrm{red}}\subset B$. Together with $\mathfrak{g}_x=\mathfrak{b}_x$, we deduce that $I_x\subset B$.

(3) Let $b\in I_x\subset B$. We write $b=tu$ according to the decomposition $B=TU$. Then $tux=x\sigma(t)\sigma(u)$. By projecting along $B\to\mathbf{T}$, we see that $\sigma(t)=t$, and thus $t^{-1}xtx^{-1}=ux\sigma(u)^{-1}x^{-1}\in V$. This shows that $t\in(\cap_{\alpha\in\Delta}\ker\alpha)^\sigma$, which has dimension $\dim T^\sigma-|\Delta^\sigma|$, and that $u\in I_x\cap U$. Therefore,

$$\dim I_x=(\dim T^\sigma-|\Delta^\sigma|)+\dim(I_x\cap U)=\dim T^\sigma,$$

i.e. x is σ-regular.

(4) Note that $q_U^{-1}(x)\subset U$ is a single orbit under the σ-conjugation action of U on itself. Indeed, since U is unipotent, the orbit through x is a closed subset of $q_U^{-1}(x)$. On the other hand, since $\dim(I_x\cap U)=\dim W$, the dimension of this orbit is equal to the dimension of $q_U^{-1}(x)$. Therefore, $q_U^{-1}(x)$ is an orbit. Now, let $u\in\mathring{U}$. After taking a conjugation by elements in T^σ, we may assume that $q_U(x)=q_U(u)$, and therefore x and u are σ-conjugate by an element in U. This shows that $\mathring{U}\subset U^{\sigma\text{-reg}}$, and $\mathring{U}$ is a single $T^\sigma\cdot U$-orbit. On the other hand, let $u\in U^{\sigma\text{-reg}}$. Since $\dim I_u=\dim T^\sigma$, the $T^\sigma\cdot U$-orbit through u is $\dim U$-dimensional, and therefore must meet $\mathring{U}$. This implies that $\mathring{U}=U^{\sigma\text{-reg}}$. The last statement follows from Part (1) and (2). □

Lemma 5.2.6. *An element $u\in U$ is σ-regular if and only if the set $\mathcal{B}_u:=\{gB\in G/B\mid g^{-1}u\sigma(g)\in B\}$ is finite.*

Proof. Proposition 5.2.4 implies that if $u\in U^{\sigma\text{-reg}}$, then $\mathcal{B}_u$ consists of only one element. Now, let $u\in U-U^{\sigma\text{-reg}}$. Then after a σ-conjugation by an element in U, we may assume that there is a σ-orbit $\mathcal{O}$ of simple roots, such that $u\in U^{\mathcal{O}}$, where $U^{\mathcal{O}}$ is the subgroup of U introduced before Lemma 5.1.9. Then $\mathcal{B}_u$ contains a positive dimensional subvariety $(G_{\mathcal{O}}/B_{\mathcal{O}})^\sigma$. The lemma is proved. □

Next, we study $T^{\sigma\text{-reg}}$, and then $B^{\sigma\text{-reg}}$. Recall the map q_B as in (5.1.4) and Defintion 5.1.10. For $t\in T(k)$, let $\Phi(G,T)_t\subset\Phi(G,T)$ be the smallest sub-root system containing those σ-orbits $\mathcal{O}$ such that $q_B(t\sigma)\in\mathbf{A}_{\mathcal{O}}(k)$. We allow $\Phi(G,T)_t=\emptyset$ if $t\in\mathring{T}$. Let $G_t\subset G$ denote the corresponding reductive subgroup, and $B_t=G_t\cap B$ a Borel subgroup of G_t. Its unipotent radical U_t is the subgroup generated by $\{U_\alpha\}$, for those positive roots $\alpha\in\Phi(G,T)_t$. We write $B_t=U_tT$. Let $\mathfrak{g}_t$ denote the Lie algebra of G_t.

Lemma 5.2.7. *The sub-root system $\Phi(G,T)_t$ is exactly the union of σ-orbits of roots in $\Phi(G,T)$ of the following three types:*

(1) *$\mathcal{O}$, if $\mathcal{O}$ is of type A and $e^{\alpha_{\mathcal{O}}}(t)=1$;*
(2) *$\mathcal{O}^-\cup\mathcal{O}^+$, if $\mathcal{O}^\pm$ is of type $\mathsf{BC}^\pm$ and $e^{\alpha_{\mathcal{O}}}(t)=1$;*
(3) *$\mathcal{O}$, if $\mathcal{O}$ is of type BC^+ and $e^{\alpha_{\mathcal{O}}}(t)=-1$.*

Proof. Indeed, the smallest sub-root system containing those σ-orbits $\mathcal{O}$ such that $q_B(t\sigma)\in\mathbf{A}_{\mathcal{O}}(k)$ would necessarily contain the these roots. Therefore, it is enough to show that they indeed form a sub-root system of $\Phi(G,T)$, or equivalently, if $\mathcal{O}_1,\mathcal{O}_2$ are two σ-orbits from the above types,

and $\alpha_i \in \mathcal{O}_i$, then the σ-orbit $\mathcal{O}$ containing $s_{\alpha_i}(\alpha_j)$ would also be one of the above types. To check this, write $\beta = s_{\alpha_i}(\alpha_j)$, and $\beta_{\mathcal{O}}$ the sum over the σ-orbit of β. We may assume that $\beta \neq \pm\alpha_j$ so in particular α_i and α_j are in the same irreducible factor of the root system $\Phi(G,T)$. Assume that $\tau = \sigma^r$ fixes this irreducible factor Φ. If (Φ,τ) is as in Example 5.1.5, one checks directly that $\beta_{\mathcal{O}}$ also belongs to one of the above types. If (Φ,τ) is not as in Example 5.1.5, then it is readily to see that $\beta_{\mathcal{O}}$ is an integral linear combination of $\alpha_{\mathcal{O}_1}$ and $\alpha_{\mathcal{O}_2}$ and all $\alpha_{\mathcal{O}_1}$, $\alpha_{\mathcal{O}_2}$ and $\beta_{\mathcal{O}}$ are of type (1). The lemma is proven. □

LEMMA 5.2.8. *Let $\Delta_t \subset \Phi(G,T)_t$ be the set of simple roots (with respect to (B_t,T)). Then after possible rescaling x_a's, the automorphism $G_t \to G_t, g \mapsto t\sigma(g)t^{-1}$ preserves the pinning $(G_t, B_t, T, \{x_\alpha\}_{\alpha\in\Delta_t})$.*

PROOF. Note that if $\mathcal{O} \subset \Delta_t$ is a σ-orbit, then $e^{\alpha_{\mathcal{O}}}(t) = 1$ if $\mathcal{O}$ is of type A and BC^-, and $e^{\alpha_{\mathcal{O}}}(t) = -1$ if $\mathcal{O}$ is of type BC^+. Then the lemma follows from Lemma 5.1.9. □

LEMMA 5.2.9. *Let $b = ut \in U_tT$. Then $\mathrm{id} - \mathrm{Ad}_b\sigma : \mathfrak{g}/\mathfrak{g}_t \to \mathfrak{g}/\mathfrak{g}_t$ is an isomorphism. In particular,*

$$\ker(\mathrm{id} - \mathrm{Ad}_b\sigma : \mathfrak{g} \to \mathfrak{g}) = \ker(\mathrm{id} - \mathrm{Ad}_b\sigma : \mathfrak{g}_t \to \mathfrak{g}_t), \tag{5.2.1}$$

and $\dim I_b = \dim(I_b \cap G_t)$.

PROOF. For $i \in \mathbb{Z}$, let Φ_i denote the set of positive roots α of height i, i.e. $\langle\alpha,\check{\rho}\rangle = i$, where $\check{\rho}$ is the half of the sum of positive coroots. We choose the following basis of $\mathfrak{g}$ (in the given order): first a basis in $\mathfrak{g}_t$; then the standard root basis $\{E_\alpha\}$ associated to the roots in $\ldots, \Phi_i, \Phi_{i-1}, \ldots$ but not in $\Phi(G,T)_t$, with σ-orbits grouped together; and finally the standard root basis $\{E_\alpha\}$ associated to the corresponding negative roots in $\ldots, \Phi_{i-1}, \Phi_i, \ldots$ but not in $\Phi(G,T)_t$, with σ-orbits grouped together. With respect to this choice of basis, the linear operator $\mathrm{id} - \mathrm{Ad}_b\sigma$ is represented by a block upper triangular matrix M, where the first block corresponds to $\mathfrak{g}_t$, and other blocks correspond to σ-orbits of roots not in $\Phi(G,T)_t$.

Note that a diagonal block that corresponds to a σ-orbit $\mathcal{O}'$ not in $\Phi(G,T)_t$ is invertible due to: (i) its determinant is $\pm(e^{\alpha'_{\mathcal{O}}}(t) - 1)$ if $\mathcal{O}'$ is of type A or BC^- and $\pm(e^{\alpha'_{\mathcal{O}}}(t) + 1)$ if $\mathcal{O}'$ is of type BC^+, which follows from Lemma 5.1.8 and an easy computation; and (ii) $q_B(t) \notin \mathbf{A}_{\mathcal{O}'}$. The first claim of the lemma follows. Then clearly (5.2.1) holds, which in turn implies that $\mathrm{Lie}(I_b^{\mathrm{red}}) \subset \mathrm{Lie} I_b \subset \mathfrak{g}_t$. Therefore the neutral connected component of I_b^{red} is a closed subgroup of G_t. The lemma is proved. □

A very similar argument yields the following "σ-twisted" Jordan decomposition.

LEMMA 5.2.10. *Let $b = ut \in UT$. Then b is σ-conjugated by an element in U to an element $u't$ with $u' \in U_t$.*

PROOF. Let $u_1 = u$. By induction on i, one can show that we can σ-conjugate b by an element in U to u_it, where u_i is in the subgroup of U generated by U_t and U_α for $\alpha \in \Phi_i \cup \Phi_{i+1} \cup \cdots$. This is because if $t \notin \mathbf{A}_{\mathcal{O}'}$, then $\mathrm{id} - \mathrm{Ad}_t\sigma$ is invertible on the space $\oplus_{\alpha\in\mathcal{O}'} kE_\alpha$. □

LEMMA 5.2.11. *We have equalities $T^{\sigma\text{-reg}} = \mathring{T}$.*

PROOF. Note that for $t \in \mathring{T}$, $G_t = T$, and $I_t \cap T = T^\sigma$. It follows from Lemma 5.2.9 that $t \in T^{\sigma\text{-reg}}$.

Next we show that $T^{\sigma\text{-reg}} \subset \mathring{T}$, i.e. if $q_B(t\sigma) \in \mathbf{A}_{\mathcal{O}}(k)$ for some σ-orbit $\mathcal{O} \subset \Phi(G,T)$, then I_t contains a unipotent subgroup and therefore $\dim I_t > \dim T^\sigma$ (as clearly $T^\sigma \subset I_t$). If $\mathcal{O}$ is of type A or BC^+, pick $\alpha \in \mathcal{O}$ and write $\alpha_i = \sigma^i(\alpha)$ for $i = 1, \ldots, |\mathcal{O}|$. These α_i's are pairwise orthogonal

and we may assume that $\sigma \circ x_{\alpha_i} = x_{\alpha_{i+1}}$ for $i = 1, \dots, |\mathcal{O}| - 1$ and thus $\sigma \circ x_{|\mathcal{O}|}$ is equal to x_{α_1} if $\mathcal{O}$ is of type A and is equal to $-x_{\alpha_1}$ if $\mathcal{O}$ is of type BC^+. One checks that, for $c \in k$,

$$\prod_{i=1}^{|\mathcal{O}|} x_{\alpha_i}\big(e^{\alpha_2+\cdots+\alpha_i}(t)c\big)$$

form a unipotent subgroup of I_t.

It remains to consider the case $\mathcal{O}$ is of type BC^- and $\operatorname{char} k \neq 2$. Let $d = |\mathcal{O}|/2$. Pick $\alpha \in \mathcal{O}$ and write $\alpha_i = \sigma^i(\alpha)$ for $i = 1, \dots, 2d$ and $\beta_i = \alpha_i + \alpha_{i+d}$ for $i = 1, \dots, d$. We may assume that

- $\sigma \circ x_{\alpha_{2d}} = x_1$, $\sigma \circ x_{\alpha_i} = x_{\alpha_{i+1}}$ for $i = 1, \dots, 2d-1$, and
- $x_{\alpha_i}(u)x_{\alpha_{d+i}}(v) = x_{\alpha_{d+i}}(v)x_{\alpha_i}(u)x_{\beta_i}(uv)$ for $i = 1, \dots, d$.

This in particular implies that $\sigma \circ x_{\beta_i} = x_{\beta_{i+1}}$ for $i = 1, \dots, d-1$, and $\sigma \circ x_{\beta_d} = -x_{\beta_1}$. Now one can check that for every $c \in k$,

$$\prod_{i=1}^{d} \Big(x_{\alpha_i}\big(e^{\alpha_2+\cdots+\alpha_i}(t)c\big) \cdot x_{\alpha_{d+i}}\big(e^{\alpha_2+\cdots+\alpha_{d+i}}(t)c\big) \cdot x_{\beta_i}\big(-\tfrac{1}{2}e^{\beta_2+\cdots+\beta_i+\alpha_2+\cdots+\alpha_{d+1}}(t)c^2\big)\Big) \in I_t,$$

giving the desired unipotent subgroup. □

Remark 5.2.12. We say an element $t \in T$ strongly σ-regular if $I_t = T^\sigma$. We claim that they form a non-empty affine open subset of $T^{\sigma\text{-reg}}$, sometimes denoted by $T^{\mathrm{s}.\sigma\text{-reg}}$. Indeed, let $t \in \mathring{T}$, and let $g \in I_t(k)$. Using the Bruhat decomposition of $g = u_1 n u_2$ with $u_1, u_2 \in U$ and $n \in N$, it is easy to see that $g = n \in N^\sigma$, and $g \in T^\sigma$ if and only if no non-trivial elements in W_0 fixes the image of t under the projection $T \to A$. But the last condition clearly defines a non-empty affine open subset of T, verifying the claim. Note that if G is semsimple and simply-connected, then by Lemma 5.1.12, $T^{\mathrm{s}.\sigma\text{-reg}} = T^{\sigma\text{-reg}} = \mathring{T}$.

To continue, we need the following lemma. We identify the tangent space T_gG of G at g with $\mathfrak{g}$ via the right translation R_g by g. Let $(h, g\sigma) \in G \times G\sigma$ and $g' = c_\sigma(h)(g)$. A direct calculation shows the following.

Lemma 5.2.13. *The differential of c_σ at $(h, g\sigma)$ is*

$$dc_\sigma : T_hG \oplus T_{g\sigma}G\sigma \to T_{g'}G\sigma, \quad (X, Y) \mapsto X - \mathrm{Ad}_{g'}(\sigma(X)) + \mathrm{Ad}_h(Y).$$

Recall that we denote by N_0 the preimage of $W_0 = W^\sigma \subset W$ in N. It acts on T via the twisted conjugation c_σ preserving $T^{\sigma\text{-reg}}$. Consider the map

$$G \times^{N_0} T^{\sigma\text{-reg}} \to G. \tag{5.2.2}$$

Lemma 5.2.14. *The map* (5.2.2) *is an open embedding.*

Let $G^{\sigma\text{-rs}}$ denote the image of this map, called the *σ-regular semisimple locus* of G. Note that $G^{\sigma\text{-rs}} \cap T = T^{\sigma\text{-reg}}$ by Lemma 5.2.11.

Proof. It follows from Lemma 5.2.13 that the map is étale. It remains to prove that it is also injective. Assume that $gt\sigma(g)^{-1} = t'$ with $t, t' \in T^{\sigma\text{-reg}} = \mathring{T}$. Using the Bruhat decomposition $g = u_1 n u_2$ as in Remark 5.2.12, one deduces that $u_1 = u_2 = 1$ and $t\sigma(n) = nt'$. The injectivity follows. □

Remark 5.2.15. Note that N_0 also preserves $T^{\mathrm{s}.\sigma\text{-reg}}$, and the map $G \times^{N_0} T^{\mathrm{s}.\sigma\text{-reg}} \to G$ is open. The image is denoted by $G^{\mathrm{s}.\sigma\text{-rs}}$, called the *strongly σ-regular semisimple locus* of G. Then $I|_{G^{\mathrm{s}.\sigma\text{-rs}}}$ is smooth and fiberwise conjugate to T^σ in G. By Lemma 5.1.12, $G^{\mathrm{s}.\sigma\text{-rs}} = G^{\sigma\text{-rs}}$ if G is semisimple and simply-connected.

REMARK 5.2.16. Note that σ is of finite order (since it preserves a pinning). If $\mathrm{char}\, k$ does not divide the order of σ, then $G^{\sigma\text{-rs}}\sigma$ is contained in the set of semisimple elements of the non-connected algebraic group $G \rtimes \langle\sigma\rangle$.

LEMMA 5.2.17. (1) *The restriction of c_σ to $U \times \mathring{T}\sigma \to \mathring{B}\sigma$ is an isomorphism.*

(2) *For a σ-orbit $\mathcal{O}$ of simple roots, let $U^{\mathcal{O}}$ be the subgroup of U introduced before Lemma 5.1.9. Then the restriction of c_σ to $U^{\mathcal{O}} \times B_{\mathcal{O}}^{[\mathcal{O}]}\sigma \to B^{[\mathcal{O}]}\sigma$ is an isomorphism.*

PROOF. We only prove (2) since the proof of (1) is similar (and simpler). Note that $U^{\mathcal{O}}$ is a normal subgroup in B and $B = U^{\mathcal{O}}B_{\mathcal{O}}$ as a semidirect product (of algebraic groups).

First, we claim that for any $b\sigma \in B_{\mathcal{O}}^{[\mathcal{O}]}\sigma$, the group

$$\{u \in U^{\mathcal{O}} \mid b\sigma(u)b^{-1} = u\}$$

is trivial. Indeed, let Φ_i be as in the proof of Lemma 5.2.9, and for $i \geq 1$, let U_i denote the group generated by the root groups U_α with $\alpha \in \Phi_i \cup \Phi_{i+1} \cup \cdots$. Then we obtain a filtration $U = U_1 \supset U_2 \supset \cdots$ by normal subgroups. Let $U_i^{\mathcal{O}} = U^{\mathcal{O}} \cap U_i$. Then $U_i^{\mathcal{O}}/U_{i+1}^{\mathcal{O}}$ is abelian, isomorphic to its Lie algebra. As argued in Lemma 5.2.9, $\{u \in U_i^{\mathcal{O}}/U_{i+1}^{\mathcal{O}} \mid b\sigma(u)b^{-1} = u\}$ is trivial because $q_B(b)$ does not lie in $\mathbf{A}_{\mathcal{O}'}$ for any σ-orbits $\mathcal{O}'$ that appear in $U^{\mathcal{O}}$. The claim follows by induction.

It follows that the map in Part (2) is a monomorphism, and that the map $\mathrm{Lie}\, U^{\mathcal{O}} \to \mathrm{Lie}\, U^{\mathcal{O}}$ given by $X \mapsto X - \mathrm{Ad}_b(\sigma(X))$ is an isomorphism. On the other hand, for any $b_1 \in U^{\mathcal{O}}$, and $Y \in \mathrm{Lie} B_{\mathcal{O}}$,

$$Y = \mathrm{Ad}_{b_1} Y \mod \mathrm{Lie}\, U^{\mathcal{O}}.$$

It follows from Lemma 5.2.13 that $U^{\mathcal{O}} \times B_{\mathcal{O}}^{[\mathcal{O}]}\sigma \to B^{[\mathcal{O}]}\sigma$ is étale and therefore is an open embedding. Note that the following diagram is commutative

$$\begin{array}{ccc} U^{\mathcal{O}} \times B_{\mathcal{O}}^{[\mathcal{O}]}\sigma & \xrightarrow{\quad c_\sigma \quad} & B^{[\mathcal{O}]}\sigma \\ & {\scriptstyle \mathrm{pr}_2} \searrow \quad \swarrow & \\ & B_{\mathcal{O}}^{[\mathcal{O}]}\sigma, & \end{array}$$

where $B^{[\mathcal{O}]}\sigma \to B_{\mathcal{O}}^{[\mathcal{O}]}\sigma$ is induced by the projection $B = U^{\mathcal{O}}B_{\mathcal{O}} \to B_{\mathcal{O}}$. In addition, for every point $b\sigma \in B_{\mathcal{O}}^{[\mathcal{O}]}\sigma$, the $U^{\mathcal{O}}$-orbit through this point is closed in $B^{[\mathcal{O}]}\sigma$ since $U^{\mathcal{O}}$ is unipotent. It follows that the $B_{\mathcal{O}}^{[\mathcal{O}]}\sigma$-morphism $c_\sigma : U^{\mathcal{O}} \times B_{\mathcal{O}}^{[\mathcal{O}]}\sigma \to B^{[\mathcal{O}]}\sigma$ is fiberwise open and closed. Therefore $U^{\mathcal{O}} \times B_{\mathcal{O}}^{[\mathcal{O}]}\sigma \to B^{[\mathcal{O}]}\sigma$ is open and surjective. Part (2) follows. □

We also need the following companion result. If $\mathcal{O}^+$ is a σ-orbit of type BC^+ such that $\alpha_{\mathcal{O}^+}$ is a simple root of $\Phi(G_\sigma, A)$, let $\mathcal{O}^-$ denote the corresponding σ-orbit of simple roots of type BC^- and we fix an order of roots in $\mathcal{O}^-$.

LEMMA 5.2.18. *Keep the notations as above. Then the restriction*

$$c_\sigma : \Big(U^{\mathcal{O}^-} \times \prod_{\alpha \in \mathcal{O}^-} U_\alpha\Big) \times B_{\mathcal{O}^+}^{[\mathcal{O}^+]}\sigma \to B^{[\mathcal{O}^+]}\sigma$$

is an isomorphism, where the product $\prod_{\alpha\in\mathcal{O}^-} U_\alpha$ is taken with respect to a chosen order.

PROOF. The same proof of Lemma 5.2.17 (2) for the σ-orbit $\mathcal{O}^-$ shows that the restriction of c_σ to $U^{\mathcal{O}^-} \times B_{\mathcal{O}^-}^{[\mathcal{O}^+]}\sigma \to B^{[\mathcal{O}^+]}\sigma$ is also an isomorphism. Therefore, it reduces to prove that

$$c_\sigma : \prod_{\alpha \in \mathcal{O}^-} U_\alpha \times B_{\mathcal{O}^+}^{[\mathcal{O}^+]}\sigma \to B_{\mathcal{O}^-}^{[\mathcal{O}^+]}\sigma$$

is an isomorphism. This follows from a simple explicit computation. Namely, we set $d = |\mathcal{O}^+|$, fix a root $\alpha \in \mathcal{O}^-$, and write $\alpha_i = \sigma^i(\alpha)$ for $1 \le i \le 2d$ and $\beta_i = \alpha_i + \sigma^d(\alpha_i)$. Then for $b = t\prod x_{\beta_i}(b_i)$ with $e^{\alpha_{\mathcal{O}}}(t) \neq 1$, and $u = \prod x_{\alpha_i}(a_i)$,

$$ub\sigma(u)^{-1} = t\prod x_{\alpha_i}(e^{-\alpha_i}(t)a_i - a_{i-1})\prod x_{\beta_i}(b_i'),$$

where $b_i' - b_i$ is a polynomial in $(a_1, \dots, a_{2d})$ of degree two (with coefficients involving $e^{\alpha_i}(t)$'s). Since $e^{\alpha_{\mathcal{O}}}(t) - 1$ is invertible, for every $(c_1, \dots, c_{2d})$, there is a unique $(a_1, \dots, a_{2d})$ such that $c_i = e^{\alpha_i}(t)a_i - a_{i-1}$, the map in the lemma is an isomorphism. □

PROPOSITION 5.2.19. (1) *The map $q_B : B^{\sigma\text{-reg}} \to \mathbf{A}$ is surjective.*
(2) *Each fiber of $q_B : B^{\sigma\text{-reg}} \to \mathbf{A}$ is a single B-orbit for the σ-conjugation action.*
(3) *For every $b \in B^{\sigma\text{-reg}}$, $(\mathrm{id} - \mathrm{Ad}_b\sigma)^{-1}(\mathfrak{u}) = \mathfrak{t}^\sigma + \mathfrak{u}$.*
(4) *Assume that G is semisimple and simply-connected. Fix $b \in B^{\sigma\text{-reg}}$. Let $g \in G$ such that $gb\sigma(g)^{-1} \in B$ and $q_B(gb\sigma(g)^{-1}) = q_B(b)$. Then $g \in B$. In addition, $I_b \subset B$.*

PROOF. (1) For every $t \in T$, consider the automorphism of G_t given in Lemma 5.2.8, denoted by $t\sigma$. Applying Proposition 5.2.4 to $(G_t, t\sigma)$ gives an element $u \in U_t$ which is $t\sigma$-regular in G_t. Let $b = ut \in U_tT$. Then $I_b \cap G_t = \{g \in G_t \mid gut\sigma(g)^{-1}t^{-1} = g\}$ is the $t\sigma$-twisted centralizer of u in G_t. Therefore, $b = ut$ is also σ-regular in G_t by Lemma 5.2.9.

(2) If $b \in B^{\sigma\text{-reg}}$, then the dimension of the B-orbit through b under the σ-conjugation action of B on itself is equal to $\dim q_B^{-1}(q_B(b))$. This shows that any two σ-regular B-orbits in a fiber of q_B must meet since fibers of q_B are irreducible. Therefore, there is exact one B-orbit in each fiber of $q_B : B^{\sigma\text{-reg}} \to \mathbf{A}$.

(3) We may assume that $b = ut$, with $u \in U_t$ being $t\sigma$-regular in G_t. Then the claims follows from Lemma 5.2.9, and Proposition 5.2.4.

(4) We may assume that $b = ut$, with $u \in U_t$ being $t\sigma$-regular in G_t. Write an element $g \in I_b(k)$ as $g = u_1nu_2$ with $u_1, u_2 \in U$ and $n \in N$. Then $u_1nu_2b = b\sigma(u_1)\sigma(n)\sigma(u_2)$. It follows that $\sigma(n) = n$, so

$$nu_2b\sigma(u_2)^{-1}n^{-1} = u_1^{-1}b\sigma(u_1).$$

Taking projection to $\mathbf{A}$, we see that $w(q_B(t)) = q_B(t)$, where $w = n \mod T$. Since G is simply-connected, by Lemma 5.1.12 and Lemma 5.2.7, w must be in the Weyl group of G_t. Then by applying the same argument as in the proof of Proposition 5.2.4 to G_t, we deduce that $w = 1$. In particular, $I_b^{\mathrm{red}} \subset B$. On the other hand,

$$\ker(\mathrm{id} - \mathrm{Ad}_b\sigma : \mathfrak{g} \to \mathfrak{g}) = \ker(\mathrm{id} - \mathrm{Ad}_b\sigma : \mathfrak{g}_t \to \mathfrak{g}_t) = \ker(\mathrm{id} - \mathrm{Ad}_b\sigma : \mathfrak{b} \cap \mathfrak{g}_t \to \mathfrak{b} \cap \mathfrak{g}_t).$$

where the first equality follows from Lemma 5.2.9, and the second follows from Lemma 5.2.5. Putting these together, we see that $I_b \subset B$.

More generally let $g \in G$ such that $gb\sigma(g)^{-1} \in B$ and $q_B(gb\sigma(g)^{-1}) = q_B(b)$, there is some $b' \in B$ such that $gb\sigma(g)^{-1} = b'b\sigma(b')^{-1}$, by Part (2). Therefore $g^{-1}b' \in I_b \subset B$ and $g \in B$. □

COROLLARY 5.2.20. *Let $K = \ker q_B = (\sigma - 1)T \cdot U$, and let $\mathfrak{k}$ be its Lie algebra. Then for every $b \in B^{\sigma\text{-reg}}$, $(\mathrm{id} - \mathrm{Ad}_b\sigma)^{-1}(\mathfrak{k}) = \mathfrak{b}$.*

PROOF. Let $X \in \mathfrak{g}$ such that $X - \mathrm{Ad}_b\sigma(X) = (1-\sigma)H + \mathfrak{u}$, with $H \in \mathfrak{t}$. Then $(\mathrm{id} - \mathrm{Ad}_b\sigma)(X - H) \in \mathfrak{u}$. Therefore $X - H \in \mathfrak{t}^\sigma + \mathfrak{u}$ and $X \in \mathfrak{b}$. □

COROLLARY 5.2.21. *The codimension of $B - B^{\sigma\text{-reg}}$ is at least two.*

PROOF. This follows from Lemma 5.2.17 (1) and Proposition 5.2.19. □

COROLLARY 5.2.22. *An element $b \in B$ is σ-regular if and only if the set $\mathcal{B}_b = \{g \in G/B \mid g^{-1}b\sigma(b) \in B\}$ is finite.*

PROOF. This follows from Lemma 5.2.17 by applying Lemma 5.2.6 to $u \in U_t$. □

5.3. Twisted Grothendieck–Springer resolution. In this subsection, we assume that G is semisimple and simply-connected.

Note that $k[T]^{c_\sigma(N_0)} = k[A]^{W_0}$. Then the twisted Chevalley isomorphism (Proposition 4.2.3) implies that $G/\!\!/c_\sigma(G) \cong \mathbf{A}/\!\!/W_0 := \operatorname{Spec} k[\mathbf{A}]^{W_0}$. So we write the (twisted) Chevalley map as

$$\chi : G \to G/\!\!/c_\sigma(G) = \mathbf{A}/\!\!/W_0,$$

which we recall is faithfully flat by Corollary 4.3.3.

As in the untwisted case, there is the following commutative diagram

$$\begin{array}{ccc} \widetilde{G} & \xrightarrow{q} & \mathbf{A} \\ \downarrow & & \downarrow \\ G & \xrightarrow{\chi} & \mathbf{A}/\!\!/W_0, \end{array}$$

where the left vertical map

$$\widetilde{G} := \{(gB, x) \in G/B \times G \mid x \in gB\sigma(g)^{-1}\} \longrightarrow G \qquad (gB, x) \mapsto x,$$

is what we call the *(σ-twisted) Grothendieck–Springer resolution*. The map q is induced by

$$\widetilde{G} \cong G \times^B (B\sigma) \xrightarrow{q_B} G \times^B \mathbf{A} \to \mathbf{A}.$$

Together, these two maps induce a proper map $\pi : \widetilde{G} \to G \times_{\mathbf{A}/\!\!/W_0} \mathbf{A}$.

Let $\widetilde{G}^{\sigma\text{-reg}}$ be the preimage of $G^{\sigma\text{-reg}}$. We have the following proposition.

PROPOSITION 5.3.1. *The induced map $\widetilde{G}^{\sigma\text{-reg}} \to G^{\sigma\text{-reg}} \times_{\mathbf{A}/\!\!/W_0} \mathbf{A}$ is an isomorphism.*

PROOF. We start with the following special case.

LEMMA 5.3.2. *Let $\widetilde{G}^{\sigma\text{-rs}}$ be the preimage of $G^{\sigma\text{-rs}}$. Then the induced map*

$$\pi|_{\widetilde{G}^{\sigma\text{-rs}}} : \widetilde{G}^{\sigma\text{-rs}} \to G^{\sigma\text{-rs}} \times_{\mathring{\mathbf{A}}/\!\!/W_0} \mathring{\mathbf{A}} \tag{5.3.1}$$

is an isomorphism.

PROOF. By (5.2.2) and Lemma 5.2.11, we have $G^{\sigma\text{-rs}} \cong G \times^{N_0} \mathring{T}$. So we can write

$$\widetilde{G}^{\sigma\text{-rs}} = \left\{(gB, g't\sigma(g')^{-1}) \in G/B \times (G \times^{N_0} \mathring{T}) \,\middle|\, g't\sigma(g')^{-1} \in gB\sigma(g)^{-1}\right\}.$$

The last condition is equivalent to $g^{-1}g't\sigma(g^{-1}g')^{-1} \in B$. By Proposition 5.2.19 (4), $g^{-1}g' \in B \times^T N_0$. From this, we deduce that $\widetilde{G}^{\sigma\text{-rs}} \cong G \times^T \mathring{T}$ and therefore (5.3.1) is an isomorphism. □

We return to the proof of Proposition 5.3.1. Since $\mathbf{A} \to \mathbf{A}/\!\!/W_0$ is finite and flat by Lemma 5.1.12, the fiber product $G^{\sigma\text{-rs}} \times_{\mathring{\mathbf{A}}/\!\!/W_0} \mathring{\mathbf{A}}$ is open and dense in $G \times_{\mathbf{A}/\!\!/W_0} \mathbf{A}$. But π is proper as G/B is, so π is surjective. In particular, $G \times_{\mathbf{A}/\!\!/W_0} \mathbf{A}$ is irreducible. Moreover, since $\mathbf{A}$ and G are smooth and $G \to \mathbf{A}/\!\!/W_0$ is flat, $G \times_{\mathbf{A}/\!\!/W_0} \mathbf{A}$ is a complete intersection (in particular Cohen–Macaulay), and it follows from the above lemma that $G \to \mathbf{A}/\!\!/W_0$ is generically smooth. Therefore, $G \times_{\mathbf{A}/\!\!/W_0} \mathbf{A}$ is also reduced. So $G \times_{\mathbf{A}/\!\!/W_0} \mathbf{A}$ is a closed subvariety of $G \times \mathbf{A}$.

Since $\widetilde{G} \to G$ is surjective, every element in G is σ-conjugate to an element in B. In particular, $\widetilde{G}^{\sigma\text{-reg}} = G \times^B B^{\sigma\text{-reg}}$. By Proposition 5.2.19 (4), the map $G \times^B B^{\sigma\text{-reg}} \to G \times \mathbf{A}$ is injective on points. By Lemma 5.2.13 and Corollary 5.2.20, the tangent map for $G \times^B B^{\sigma\text{-reg}} \to G \times \mathbf{A}$ is injective at every point, and therefore the morphism is unramified. It follows that $\widetilde{G}^{\sigma\text{-reg}} \to G^{\sigma\text{-reg}} \times \mathbf{A}$ is a closed embedding, with image (as topological space) $G^{\sigma\text{-reg}} \times_{\mathbf{A}/\!\!/W_0} \mathbf{A}$. But since $G^{\sigma\text{-reg}} \times_{\mathbf{A}/\!\!/W_0} \mathbf{A}$ is also reduced, the map in the proposition is indeed an isomorphism. □

Here are some standard corollaries.

COROLLARY 5.3.3. *The morphism $\chi : G^{\sigma\text{-reg}} \to \mathbf{A}/\!\!/W_0$ is smooth.*

PROOF. This follows from the smoothness of $\widetilde{G}^{\sigma\text{-reg}} \to \mathbf{A}$ and the flatness of $\mathbf{A} \to \mathbf{A}/\!\!/W_0$. □

COROLLARY 5.3.4. *The map π induces an isomorphism $k[G \times_{\mathbf{A}/\!\!/W_0} \mathbf{A}] \cong \Gamma(\widetilde{G}, \mathcal{O})$.*

PROOF. It is enough to show that $G \times_{\mathbf{A}/\!\!/W_0} \mathbf{A}$ is an integral normal scheme. Then it follows that π induces an isomorphism between rings of global regular functions, since it is proper birational.

We already seen that $G \times_{\mathbf{A}/\!\!/W_0} \mathbf{A}$ is integral, and a complete intersection. By Proposition 5.3.1, $G^{\sigma\text{-reg}} \times_{\mathbf{A}/\!\!/W_0} \mathbf{A}$ is smooth. By Corollary 5.2.21, the complement of $G^{\sigma\text{-reg}} \times_{\mathbf{A}/\!\!/W_0} \mathbf{A}$ in $G \times_{\mathbf{A}/\!\!/W_0} \mathbf{A}$ has codimension at least two. It follows from this lemma that $G \times_{\mathbf{A}/\!\!/W_0} \mathbf{A}$ is normal. □

COROLLARY 5.3.5. (1) *For every $a \in \mathbf{A}/\!\!/W_0$, $\chi^{-1}(a) \cap G^{\sigma\text{-reg}}$, is a single G-orbit, and the codimension of the complement of this G-orbit in $\chi^{-1}(a)$ at least two.*

(2) *Each fiber of χ is a complete intersection and normal variety.*

PROOF. (1) Pick $\tilde{a} \in \mathbf{A}(k)$ that lifts $a \in \mathbf{A}/\!\!/W_0$. By Proposition 5.3.1, the fiber $\chi^{-1}(a) \cap G^{\sigma\text{-reg}}$ is isomorphic to the fiber of $q : \widetilde{G}^{\sigma\text{-reg}} \cong G \times^B B^{\sigma\text{-reg}} \to \mathbf{A}$ at $\tilde{a}$, which is clearly a single G-orbit by Proposition 5.2.19 (2). On the other hand, by Corollary 5.2.22, the fibers of $\widetilde{G} \to G$ over $G - G^{\sigma\text{-reg}}$ have positive dimension. Since $q_B^{-1}(\tilde{a}) - (q_B^{-1}(\tilde{a}) \cap B^{\sigma\text{-reg}})$ is a proper closed subset of $q_B^{-1}(\tilde{a})$, $q^{-1}(a) - (q^{-1}(a) \cap G^{\sigma\text{-reg}})$ has codimension at least two in $q^{-1}(a)$.

(2) Since $\chi : G \to \mathbf{A}/\!\!/W_0$ is flat, each of its fiber is a complete intersection and hence Cohen–Macaulay. By (1), each fiber of χ contains a (smooth) G-orbit whose complement has codimension at least two. So the fiber is also normal. □

6. The determinant of the pairing $\boldsymbol{J}(V) \otimes \boldsymbol{J}(V^*) \to \boldsymbol{J}$

ASSUMPTION 6.0.1. In this section, assume that $\operatorname{char} k > 2$. Let G be a simply-connected semisimple group over k. Let V be a finite dimensional representation of G and V^* the dual representation. Assume that both V and V^* admit good filtrations.

We keep conventions and notations as in § 5. Then $\widetilde{V^*}$ is the dual of $\widetilde{V}$ as a vector bundle on $[G\sigma/G]$. The perfect pairing $\widetilde{V} \otimes \widetilde{V^*} \to \mathcal{O}_{[G\sigma/G]}$ induces a $\boldsymbol{J}$-bilinear pairing $\boldsymbol{J}(V) \otimes \boldsymbol{J}(V^*) \to \boldsymbol{J}$ of global sections, which however is not perfect in general. Our main result (Theorem 6.1.2) calculates the determinant of this pairing of two finite free $\boldsymbol{J}$-modules. A main intermediate step is to study the failure of the surjectivity of the twisted Chevalley restriction homomorphism (4.3.3).

6.1. Main results.

DEFINITION 6.1.1. For each σ-orbit $\mathcal{O} \subset \Phi(G, T)$, we choose $\alpha \in \mathcal{O}$ and view it as a character of T^σ by restriction, which is clearly independent of the choice of α. Define the number

$$\zeta_{\mathcal{O}} = \zeta_{\mathcal{O}}(V) := \sum_{n \geq 1} \dim V|_{T^\sigma}(n\alpha).$$

The main theorem of this section is the following.

THEOREM 6.1.2. *Keep Assumption 6.0.1. The determinant of the natural $\boldsymbol{J}$-bilinear pairing*[5]

$$\langle \cdot, \cdot \rangle_V : \boldsymbol{J}(V) \otimes \boldsymbol{J}(V^*) \to \boldsymbol{J} \tag{6.1.1}$$

is of the form

$$\big(\textit{some unit in } k\big) \cdot \prod_{\mathcal{O} \textit{ type } \mathsf{A} \textit{ or } \mathsf{BC}^-} (e^{\alpha_{\mathcal{O}}} - 1)^{\zeta_{\mathcal{O}}} \cdot \prod_{\mathcal{O} \textit{ type } \mathsf{BC}^+} (e^{\alpha_{\mathcal{O}}} + 1)^{\zeta_{\mathcal{O}}}. \tag{6.1.2}$$

[5]Taking the determinant makes sense because Theorem 4.3.2 shows that $\boldsymbol{J}(V)$ and $\boldsymbol{J}(V^*)$ are both free $\boldsymbol{J}$-modules of the same rank.

REMARK 6.1.3. (1) Note that $\zeta_{\mathcal{O}}$ depends only on the W_0-orbits of $\mathcal{O}$, and therefore the product in (6.1.2) does belong to $\boldsymbol{J}$.

(2) When σ is trivial, the rank of $\boldsymbol{J}(V)$ as a $\boldsymbol{J}$-module is $r_V = \dim V(0)$ by Theorem 4.3.2. In this case, then $\zeta_{\mathcal{O}} = \zeta_\alpha$ can be alternatively computed via the multi-filtration on the weight spaces introduced in § 3.3 as

$$\zeta_\alpha = \sum_{\nu \in \mathbb{X}^\bullet(T)^+} \langle \nu, \check{\alpha} \rangle \cdot \dim \mathrm{gr}_\nu V(0).$$

(3) Similarly we have the pairing $\boldsymbol{J}_+(V) \otimes \boldsymbol{J}_+(V^*) \to \boldsymbol{J}_+$. Its restriction to $\operatorname{Spec} \boldsymbol{J}$ is the pairing in the theorem, and restriction to the point $x_1 \in \operatorname{Spec} \boldsymbol{J}_0$ as defined in Remark 4.4.1 is the natural pairing between $(V|_{T^\sigma})(0)$ and $(V|_{T^\sigma})^*(0)$.

(4) When $\operatorname{char} k = 0$, we have explained in § 4.4 a construction of a basis of $\boldsymbol{J}(V)$ (resp. $\boldsymbol{J}(V^*)$) from a certain basis of V (resp. V^*) (e.g. the MV basis used in § 3.5). Then the $\langle \cdot, \cdot \rangle_V$ is represented by a square matrix. It seems to be an interesting (although probably difficult) question to calculate the entries of this matrix explicitly. See § 6.4 for some calculations and further discussions. We also refer to [**XZ17**] for the arithmetic and geometric meaning of this square matrix.

In what follows, we shall relate the pairing (6.1.1) to the twisted Chevalley restriction map (4.3.3). As we will show in Lemma 6.1.5 that Theorem 6.1.2 follows from Proposition 6.1.4 below. More precisely, we will not study the Chevalley restriction homomorphism (4.3.3) itself, but rather the induced $\boldsymbol{J}_T$-module homomorphism

$$\mathrm{Res}_V^\sigma \otimes 1 : \boldsymbol{J}_G(V) \otimes_{\boldsymbol{J}_G} \boldsymbol{J}_T \to \boldsymbol{J}_T(V), \tag{6.1.3}$$

where explicitly, we may write the target as

$$\boldsymbol{J}_T(V) = (k[T] \otimes V)^{c_\sigma(T)} \cong \bigoplus_{\xi \in (\sigma-1)\mathbb{X}^\bullet(T)} k[A] e^{\nu_\xi} \otimes V(\xi) \tag{6.1.4}$$

with $\nu_\xi \in \mathbb{X}^\bullet(T)$ some weight such that $\sigma(\nu_\xi) - \nu_\xi = \xi$. In particular, if $\sigma = \mathrm{id}$, $\boldsymbol{J}_T(V) \cong k[T] \otimes V(0)$.

We view $\mathrm{Res}_V^\sigma \otimes 1$ as a morphism of coherent sheaves over A. By Lemma 4.3.8 (which relies on Lemma 5.2.11), it is an isomorphism over $\mathring{A}$ (which is defined in Definition 5.1.10).

Now, let η be a generic point of the divisor $\bigcup_{\mathcal{O}} A_{\mathcal{O}}$ (which is reduced since we assumed that $\operatorname{char} k > 2$; see Remark 5.1.11). Then the complete local ring of A at η is isomorphic to $k(\eta)[[\varpi]]$, where $\varpi = e^{\alpha_{\mathcal{O}}} - 1$ or $e^{\alpha_{\mathcal{O}}} + 1$ for some σ-orbit $\mathcal{O} \subset \Phi(G, T)$. Note that $\boldsymbol{J}_G(V)$ is always a torsion free $\boldsymbol{J}_G$-module (even if V does not admit a good filtration). Therefore $\boldsymbol{J}_G(V) \otimes_{\boldsymbol{J}_G} k(\eta)[[\varpi]]$ is always free, and (6.1.3), base changed to $k(\eta)[[\varpi]]$, is a map Res_η of finite free $k(\eta)[[\varpi]]$-modules which becomes an isomorphism when further base changed to $k(\eta)((\varpi))$. (Such map is called a *modification* of vector bundles on $\operatorname{Spec} k(\eta)[[\varpi]]$ (in the sense as in [**XZ17**, § 3.1.3]). The top exterior power of this map Res_η is an element in $k(\eta)[[\varpi]] - \{0\}$, well-defined up to multiplying an element in $k[[\varpi]]^\times$, and therefore gives a well-defined element in

$$(k(\eta)[[\varpi]] - \{0\})/k(\eta)[[\varpi]]^\times \cong \mathbb{Z}_{\geq 0}.$$

We call this number *the length of the modification*.

Here is the main result regarding the map (6.1.3).

PROPOSITION 6.1.4. *For every G-module V with good filtration, the length of the modification* (6.1.3) *at every generic point of $A_{\mathcal{O}}$ is exactly $\zeta_{\mathcal{O}}$.*

This proposition will be proved in § 6.2. We note the following first.

LEMMA 6.1.5. *Proposition 6.1.4 implies Theorem 6.1.2.*

PROOF. Consider the following commutative diagram

$$\begin{array}{ccccc}
\boldsymbol{J}(V) & \times & \boldsymbol{J}(V^*) & \xrightarrow{\langle\cdot,\cdot\rangle_V} & \boldsymbol{J} \\
\downarrow & & \downarrow & & \downarrow \\
\boldsymbol{J}(V)\otimes_{\boldsymbol{J}}\boldsymbol{J}_T & \times & \boldsymbol{J}(V^*)\otimes_{\boldsymbol{J}}\boldsymbol{J}_T & \xrightarrow{\langle\cdot,\cdot\rangle_V} & \boldsymbol{J}_T \\
\downarrow{\scriptstyle \mathrm{Res}^\sigma_V\otimes 1} & & \downarrow{\scriptstyle \mathrm{Res}^\sigma_{V^*}\otimes 1} & & \| \\
\boldsymbol{J}_T(V) & \times & \boldsymbol{J}_T(V^*) & \xrightarrow{\langle\cdot,\cdot\rangle_V} & \boldsymbol{J}_T.
\end{array}$$

By Theorem 4.3.2, $\boldsymbol{J}(V)$ and $\boldsymbol{J}(V^*)$ are free $\boldsymbol{J}$-modules. So second row is simply a base change of the first row, and hence the matrices for the top two pairings are the same (when choosing compatible bases). Now the bottom row is a *perfect* $\boldsymbol{J}_T$-bilinear pairing (as can be easily seen from the explicit expression of $\boldsymbol{J}_T(V)$ in (6.1.4)). So the determinant of the middle row is the product of the determinant of the map $\mathrm{Res}^\sigma_V\otimes 1$ and the determinant of the map $\mathrm{Res}^\sigma_{V^*}\otimes 1$ (up to a unit). Thus, Proposition 6.1.4 would imply that the determinant of (6.1.1), as a divisor on A, is given by

$$\sum_{\mathcal{O}\subset\Phi(G,T)} (\zeta_{\mathcal{O}}(V)+\zeta_{\mathcal{O}}(V^*))\cdot A_{\mathcal{O}}.$$

By Lemma 6.1.6 below, this gives the same expression as the formula (6.1.2) up to a unit in $k[A]$. But note that the product in (6.1.1) is W_0-invariant, so is the determinant of (6.1.1). It follows that the ambiguous unit in $k[A]$ belongs to $(k[A]^{W_0})^\times$. As mentioned in the proof of Lemma 5.1.12 (see also [**Sp06**, Corollary 2]), $k[A]^{W_0}$ is a polynomial algebra and hence its units are just $k^\times$. Therefore, the determinant of (6.1.1) is given by (6.1.2) up to a unit in k. □

LEMMA 6.1.6. *For a representation V of G and a σ-orbit $\mathcal{O}\subset\Phi(G,T)$, we have $\zeta_{\mathcal{O}}(V)=\zeta_{\mathcal{O}}(V^*)$.*

PROOF. This follows from the following sequence of equalities

$$\dim V^*|_{T^\sigma}(n\alpha)=\dim V|_{T^\sigma}(-n\alpha)=\dim V|_{T^\sigma}(n\alpha),$$

where the first equality follows from the duality and the last equality follows from the fact that α and $-\alpha$ lie in the same W-orbit. □

6.2. Proof of Proposition 6.1.4. We will first reduce Proposition 6.1.4 to the cases of SL_2 and SL_3.

LEMMA 6.2.1. *Let V be a representation of G. The map $\boldsymbol{J}_G(V)\otimes_{\boldsymbol{J}_G}\boldsymbol{J}_T\to\boldsymbol{J}_B(V)$ is an isomorphism.*

PROOF. Recall that by Corollary 5.3.4, the pushforward of the structure sheaf along $\pi:[B\sigma/B]\to[G\sigma/G]\times_{\mathbf{A}/\!/W_0}\mathbf{A}$ is the structure sheaf. Then it follows from the projection formula that

$$\boldsymbol{J}_G(V)\otimes_{\boldsymbol{J}_G}\boldsymbol{J}_T=\Gamma([G\sigma/G]\otimes_{\mathbf{A}/\!/W_0}\mathbf{A},\widetilde{V}\boxtimes\mathcal{O})=\Gamma([B\sigma/B],\pi^*\widetilde{V})=\boldsymbol{J}_B(V)$$

is an isomorphism. □

Using the W_0-action on $\Phi(G_\sigma,A)$, it suffices to prove Proposition 6.1.4 for a σ-orbit $\mathcal{O}$ that is either a σ-orbit of *simple* roots (of type A or BC^-), or a σ-orbit of roots of type BC^+ that are sum of two roots from a σ-orbit $\mathcal{O}^-$ of simple roots. In either case, let $G_{\mathcal{O}}$ be the subgroup of G generated by T and U_α with $\alpha\in\mathcal{O}\cup-\mathcal{O}$; and $B_{\mathcal{O}}$ the Borel subgroup generated by T and U_α with $\alpha\in\mathcal{O}$.

LEMMA 6.2.2. *The map $\boldsymbol{J}_B(V)\to\boldsymbol{J}_{B_{\mathcal{O}}}(V)$ is an isomorphism over $\mathbf{A}^{[\mathcal{O}]}$.*

PROOF. It follows from Lemma 5.2.17 and Lemma 5.2.18 that $[B^{[\mathcal{O}]}\sigma/B] \cong [B_{\mathcal{O}}^{[\mathcal{O}]}\sigma/B_{\mathcal{O}}]$. So the map in the lemma induces an isomorphism when restricted to $\mathbf{A}^{[\mathcal{O}]}$. □

LEMMA 6.2.3. *To prove Proposition 6.1.4, it is enough to prove it for two cases:*

- $G = \mathrm{SL}_2$ *with* $\sigma = \mathrm{id}$*;*
- $G = \mathrm{SL}_3$ *with* σ *given as in Example 5.1.5 and* $\mathcal{O}$ *being a* σ*-orbit consisting of two roots.*

PROOF. By Lemma 6.2.1 and Lemma 6.2.2, we reduce Proposition 6.1.4 to prove that, for the σ-orbit $\mathcal{O}$ considered above and over every generic point η of $\mathbf{A}_{\mathcal{O}}$, the map $\boldsymbol{J}_{B_{\mathcal{O}}}(V) \to \boldsymbol{J}_T(V)$ is a modification of length $\zeta_{\mathcal{O}}$. (Note that $V|_{G_{\mathcal{O}}}$ also admits a good filtration by Theorem 3.1.2 (1).) We consider the central isogeny

$$G'_{\mathcal{O}} := G_{\mathcal{O},\mathrm{der}} \times Z_{G_{\mathcal{O}}} \to G_{\mathcal{O}}.$$

with kernel $F = G_{\mathcal{O},\mathrm{der}} \cap Z_{G_{\mathcal{O}}}$. Let T' denote the preimage of T and $A' = T'_\sigma$. Note that the right exact sequence $F_\sigma \to T'_\sigma \to T_\sigma \to 1$ is also exact on the left, so the induced map $A' \to A$ is an F_σ-torsor. Note that F_σ is indeed étale (given the classification of $G_{\mathcal{O},\mathrm{der}}$ by Lemma 5.1.9) since we assume $\mathrm{char}\, k \neq 2$. By Lemma 4.1.3 (applied to $B'_{\mathcal{O}} = B_{\mathcal{O},\mathrm{der}} \times Z_{G_{\mathcal{O}}}$ and $T_{\mathcal{O},\mathrm{der}} \times Z_{G_{\mathcal{O}}}$) and the fact that modifications commute with étale base change, the length of modification of $\boldsymbol{J}_{B_{\mathcal{O}}}(V) \to \boldsymbol{J}_T(V)$ at every generic point η of $\mathbf{A}_{\mathcal{O}}$ is the same as the length of modification of $\boldsymbol{J}_{B'_{\mathcal{O}}}(V) \to \boldsymbol{J}_{T'}(V)$ (or equivalently $\boldsymbol{J}_{G'_{\mathcal{O}}}(V) \otimes_{\boldsymbol{J}_{G'_{\mathcal{O}}}} \boldsymbol{J}_T \to \boldsymbol{J}_T(V)$ by Lemma 6.2.1) at any preimage of η under the map $\pi : \mathbf{A}' \to \mathbf{A}$.

We decompose $V = \oplus_\psi V_\psi \otimes k_\psi$ according to the central character for the action of $Z_{G_{\mathcal{O}}}$ on V so that each V_ψ is a $G_{\mathcal{O},\mathrm{der}}$-module. Then by (4.1.1)

$$\boldsymbol{J}_{G'_{\mathcal{O}}}(V) = \bigoplus_\psi \boldsymbol{J}_{G_{\mathcal{O},\mathrm{der}}}(V_\psi) \otimes \boldsymbol{J}_{Z_{G_{\mathcal{O}}}}(k_\psi) = \bigoplus_{\psi|_{Z^\sigma_{G_{\mathcal{O}}}} = \mathbf{1}} \boldsymbol{J}_{G_{\mathcal{O},\mathrm{der}}}(V_\psi) \otimes \boldsymbol{J}_{Z_{G_{\mathcal{O}}}}(k_\psi).$$

We write $A_{\mathrm{der}} = (T_{\mathcal{O},\mathrm{der}})_\sigma$, where $T_{\mathcal{O},\mathrm{der}} = T \cap G_{\mathcal{O},\mathrm{der}}$. Then $A' = A_{\mathrm{der}} \times (Z_{G_{\mathcal{O}}})_\sigma$, and one checks that $\pi : \mathbf{A}_{\mathrm{der},\mathcal{O}} \times (Z_{G_{\mathcal{O}}})_\sigma \to \mathbf{A}_{\mathcal{O}}$ is surjective. Clearly, the modification

$$\boldsymbol{J}_{G'_{\mathcal{O}}}(V) = \bigoplus_\psi \boldsymbol{J}_{G_{\mathcal{O},\mathrm{der}}}(V_\psi) \otimes \boldsymbol{J}_{Z_{G_{\mathcal{O}}}}(k_\psi) \to \boldsymbol{J}_{T'}(V) = \bigoplus_\psi \boldsymbol{J}_{T_{\mathcal{O},\mathrm{der}}}(V_\psi) \otimes \boldsymbol{J}_{Z_{G_{\mathcal{O}}}}(k_\psi)$$

only happens on the first factor. Since the second factor is of rank one over $\boldsymbol{J}_{Z_{G_{\mathcal{O}}}}$, and since

$$\dim(V|_{T^\sigma})(n\alpha) = \sum_{\psi|_{Z^\sigma_{G_{\mathcal{O}}}} = \mathbf{1}} (V_\psi|_{T^\sigma_{\mathcal{O},\mathrm{der}}})(n\alpha),$$

we see that to prove Proposition 6.1.4, it is enough to assume that $G = G_{\mathcal{O},\mathrm{der}}$.

By Lemma 5.1.9, there are three cases for $G_{\mathcal{O},\mathrm{der}}$. By Lemma 5.2.18, and Lemma 4.1.2, we reduce the proof of Proposition 6.1.4 to the case $G = \mathrm{SL}_2$ with $\sigma = \mathrm{id}$, and $G = \mathrm{SL}_3$ with σ given as in Example 5.1.5 and $\mathcal{O}$ being a σ-orbit consisting of two roots. This completes the proof of the lemma. □

Finally, we treat the above mentioned two cases.

LEMMA 6.2.4. *Proposition 6.1.4 holds for* $G = \mathrm{SL}_2$*.*

PROOF. It is enough to assume that V is the Schur module $\mathtt{S}_n$ (of dimension $n+1$), in which case we make explicit computations. We set

$$B = \left\{ \left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right) \;\middle|\; a \in k^\times, b \in k \right\}, \quad T = \left\{ \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right) \;\middle|\; a \in k^\times \right\}, \quad \text{and} \quad U = \left\{ \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \;\middle|\; b \in k \right\}.$$

By Lemma 6.2.1, we need to show that

(1) $J_T(\mathtt{S}_n)$ is zero when n is odd, and

(2) the length of the modification of $J_B(\mathtt{S}_n) \to J_T(\mathtt{S}_n)$ at the generic point of T is exactly $n/2$ when n is even.

(1) is obvious by looking at the action of $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathrm{SL}_2$. For (2) (and hence n is even), we explicitly identify $\mathtt{S}_n \cong k[z]^{\deg \le n}$ and $\mathcal{O}_B \cong k[x^{\pm 1}, y]$, where B acts on $\mathtt{S}_n$ by $\left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right)(h)(z) = a^n h\big(a^{-2}z - a^{-1}b\big)$ and the conjugation action of B on $\mathcal{O}_B$ is given by

$$\left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right)(h)(x,y) = h(x, a^{-2}y + a^{-1}b(x - x^{-1})).$$

From this, we see that $z(x - x^{-1}) + y$ is invariant under the U-action and explicitly

$$(\mathcal{O}_B \otimes \mathtt{S}_n)^U \cong \bigoplus_{i=0}^{n-1} k[x^{\pm 1}] \cdot \big(z(x - x^{-1}) + y\big)^i.$$

But an element $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right) \in T$ acts on $(z(x - x^{-1}) + y)^i$ by multiplication by a^{n-2i}. It follows that

$$(\mathcal{O}_B \otimes \mathtt{S}_n)^B \cong k[x^{\pm 1}] \cdot (z(x - x^{-1}) + y)^{n/2}.$$

Restricting to T, or equivalently setting $y = 0$, we see that the map $\boldsymbol{J}_B(V) \to \boldsymbol{J}_T(V)$ can be identified with the inclusion

$$k[x^{\pm 1}] \cdot (z(x - x^{-1}))^{n/2} \to k[x^{\pm 1}] \cdot z^{n/2}.$$

The length of the modification of the above map at each point of the divisor $x - x^{-1}$ is $n/2$. This completes the proof of the lemma. □

Lemma 6.2.5. *Proposition 6.1.4 holds for $G = \mathrm{SL}_3$ with the automorphism σ given in Example 5.1.5, at the generic points of $A_{\mathcal{O}}$ for the σ-orbit $\mathcal{O}$ consisting of two roots.*

Proof. Let $\alpha, \sigma\alpha, \beta = \alpha + \sigma\alpha$ denote the positive roots and let $\mathcal{O} = \{\alpha, \sigma\alpha\}$ so that $\alpha_{\mathcal{O}} = \beta$.

We consider the principal SL_2 of SL_3. Its image is exactly $H = \mathrm{SO}_3 = (\mathrm{SL}_3)^\sigma$, where SO_3 is the orthogonal group defined by $\left(\begin{smallmatrix} & & 1 \\ & -1 & \\ 1 & & \end{smallmatrix}\right)$, regarded as a symmetric bilinear form. The map $T^\sigma \to T_\sigma = A$ is the square mapping on $\mathbb{G}_m$. One checks immediately that when $\operatorname{char} k \neq 2$, $\boldsymbol{J}_T \to \boldsymbol{J}_{T^\sigma}$ is finite étale of degree two and the induced map

$$\boldsymbol{J}_T(V) \otimes_{\boldsymbol{J}_T} \boldsymbol{J}_{T^\sigma} \to \boldsymbol{J}_{T^\sigma}(V)$$

is an isomorphism.

The embedding $H \to G$ induces $\operatorname{Spec} \boldsymbol{J}_H = H/\!\!/c(H) \to \operatorname{Spec} \boldsymbol{J}_G = G/\!\!/c_\sigma(G)$, where c denotes the usual conjugation of H on H. Here $\boldsymbol{J}_H = k[y]$ (resp. $\boldsymbol{J}_G = k[x]$) is the space of conjugation invariant functions on SO_3 (resp. twisted conjugation invariant functions on SL_3), with the variable y (resp. x) representing the character of the standard representation of SO_3 (resp. the adjoint representation of SL_3). Note that $x = (y-1)^2$ when viewed as functions on T^σ. So $k[x, x^{-1}] \to k[y, (y-1)^{-1}]$ is finite étale of degree two.

The point $x = 4$ is the image of the divisor $A_{\mathcal{O}}$ under the isomorphism $A/\!\!/W_0 \cong \operatorname{Spec} \boldsymbol{J}_G$. Its preimages under the map $\operatorname{Spec} \boldsymbol{J}_H \to \operatorname{Spec} \boldsymbol{J}_G$ are points represented by $y = -1$ and $y = 3$, where $y = 3$ corresponds to the Weyl group orbit of the zero of $e^\beta - 1$ on T^σ. Let $u = x - 4$, and $v = y - 3$. Then the map $k[x] \to k[y]$ induces an isomorphism $k[[u]] \cong k[[v]]$. We claim that the natural map

$$\boldsymbol{J}_G(V) \otimes_{\boldsymbol{J}_G} k[[u]] \to \boldsymbol{J}_H(V) \otimes_{\boldsymbol{J}_H} k[[v]] \tag{6.2.1}$$

is an isomorphism.

By [**St65**, Theorem 6.11 c)] (or a direct computation), the preimage of $y = 3$ under the quotient map $\mathrm{SO}_3^{\mathrm{reg}} \to \operatorname{Spec} \boldsymbol{J}_H = \operatorname{Spec} k[y]$ is a single H-orbit of $g(0) = \left(\begin{smallmatrix} 1 & 1 & \frac{1}{2} \\ & 1 & 1 \\ & & 1 \end{smallmatrix}\right)$. One can easily check that the image of $g(0)$ in SL_3, denoted by $h(0)$, is also σ-regular. Using the smoothness of the

Chevalley map $\mathrm{SO}_3^{\mathrm{reg}} \to \operatorname{Spec} k[y]$, we may lift $g(0)$ to $g : \operatorname{Spec} k[[v]] \to \mathrm{SO}_3^{\mathrm{reg}}$. Since $\mathrm{SL}_3^{\sigma\text{-reg}}$ is open and naturally $\operatorname{Spec} k[[u]] \cong \operatorname{Spec} k[[v]]$, we get a composition of maps

$$h : \operatorname{Spec} k[[u]] \cong \operatorname{Spec} k[[v]] \xrightarrow{g} \mathrm{SO}_3 \to \mathrm{SL}_3$$

whose image lands in $\mathrm{SL}_3^{\sigma\text{-reg}}$. Now, viewing g and h as $k[[u]]$-valued points on $\mathrm{SO}_3^{\mathrm{reg}}$ and $\mathrm{SL}_3^{\sigma\text{-reg}}$ respectively, the centralizer $Z_H(g)$ of g in SO_3 is naturally contained in the twisted centralizer I_h of h in SL_3. We claim that $Z_H(g) = I_h$. Indeed, this is true over the generic point of $\operatorname{Spec} k[[u]] \cong \operatorname{Spec} k[[v]]$ as both spaces are connected by Lemma 5.2.11. Over the special point, one can compute explicitly that

$$Z_H(g(0)) = I_{h(0)} = \left\{ \begin{pmatrix} 1 & x & \frac{1}{2}x^2 \\ & 1 & x \\ & & 1 \end{pmatrix} \ ; \ x \in k \right\}.$$

Thus $Z_H(g) = I_h$. Since a V-valued function is determined by its restriction to the regular locus, by Corollary 5.3.5 (1), we deduce that

$$\boldsymbol{J}_H(V) \otimes_{\boldsymbol{J}_H} k[[u]] = (V \otimes k[[u]])^{Z_H(g)} = (V \otimes k[[v]])^{I_h} = \boldsymbol{J}_G(V) \otimes_{\boldsymbol{J}_G} k[[v]].$$

This shows that (6.2.1) is an isomorphism.

Now we immediately reduce the lemma to the SL_2-case (Lemma 6.2.4) by the same argument as in the last two paragraphs of the proof of Lemma 6.2.3 (based on Lemma 5.2.18), provided that we can show that $V|_{\mathrm{SO}_3}$ admits a good filtration. It suffices to check this for Schur modules $\mathtt{S}_{a\omega_1+b\omega_2}$ with $a, b \geq 0$, where ω_1 and ω_2 are the fundamental weights of SL_3. We check by induction on $a+b$. This is true for ω_1 and ω_2 as they are standard representations of SO_3, and therefore true for $\mathtt{S}_{\omega_1}^{\otimes a} \otimes \mathtt{S}_{\omega_2}^{\otimes b}$ by Theorem 3.1.2 (2). By the Frobenius reciprocity, there is a unique zero (up to scalar) map $\mathtt{S}_{\omega_1}^{\otimes a} \otimes \mathtt{S}_{\omega_2}^{\otimes b} \to \mathtt{S}_{a\omega_1+b\omega_2}$. One easily identify the kernel V' as $(\mathtt{S}_{\omega_1}^{\otimes a} \otimes \mathtt{S}_{\omega_2}^{\otimes b})_{\prec a\omega_1+b\omega_2}$ in Lemma 3.1.4 (2), and therefore admits a good filtration (whose graded pieces are Schur modules with highest weights strictly lower than $a\omega_1+b\omega_2$). So the restriction of V' to SO_3 admits a filtration by inductive hypothesis. Moreover, by Lemma 3.1.4 (2), $\mathtt{S}_{\omega_1}^{\otimes a} \otimes \mathtt{S}_{\omega_2}^{\otimes b} \to \mathtt{S}_{a\omega_1+b\omega_2}$ is surjective. By Theorem 3.1.2 (3), we conclude that $\mathtt{S}_{a\omega_1+b\omega_2}$ admits a good filtration when restricted to SO_3. □

Remark 6.2.6. As in the proof of the lemma, we have an analogous map

$$\boldsymbol{J}_G(V) \otimes_{\boldsymbol{J}_G} k[[x-4]] \to \boldsymbol{J}_H(V) \otimes_{\boldsymbol{J}_H} k[[y+1]].$$

But it is not an isomorphism in general. The reason is that $y = -1$ lies on $\mathring{T}^\sigma /\!\!/ W_0$, so every regular element in $\mathrm{SO}_3^{\mathrm{reg}}$ that maps to the point $y = -1$ in $\operatorname{Spec} k[y]$ is conjugate to the *semisimple* matrix $\left(\begin{smallmatrix} -1 & 0 & 0 \\ & 1 & 0 \\ & & -1 \end{smallmatrix}\right)$; yet this matrix when viewed inside SL_3 is *not* σ-regular.

6.3. The ring $\operatorname{End}(\widetilde{V})$. Motivated by some applications in [**XZ17**], in this subsection we study the endomorphism ring $\operatorname{End}(\widetilde{V})$ of the vector bundle $\widetilde{V}$ on $[G\sigma/G]$. Here are some basic facts.

(1) The ring $\operatorname{End}(\widetilde{V})$ is a $\boldsymbol{J}$-algebra. In particular, if W is a representation of $G \rtimes \langle\sigma\rangle$, its character χ_W, restricted to $G\sigma$, defines a regular function on $[G\sigma/G]$ and therefore an element in $\operatorname{End}(\widetilde{V})$, still denoted by χ_W.

(2) The Chevalley restriction map is the pullback map $\operatorname{End}(\widetilde{V}) \to \operatorname{End}(\widetilde{V}|_{[T\sigma/T]})$ along $[T\sigma/T] \to [G\sigma/G]$. Note that

$$\operatorname{End}(\widetilde{V}|_{[T\sigma/T]}) \cong (k[T] \otimes \operatorname{End}_{T^\sigma} V)^{T/T^\sigma}.$$

So (6.1.3) gives an injective map

$$\operatorname{Res}^\sigma_{V \otimes V^*} \otimes 1 : \operatorname{End}(\widetilde{V}) \otimes_{\boldsymbol{J}_G} \boldsymbol{J}_T \to (k[T] \otimes \operatorname{End}_{T^\sigma} V)^{T/T^\sigma},$$

which is an isomorphism generically over $\boldsymbol{J}_T$. It follows that $\mathrm{End}(\widetilde{V})$ is commutative if and only if V is multiplicity free as a T^σ-module.

(3) Assume that V is the restriction of a representation $\rho : G \rtimes \langle\sigma\rangle \to \mathrm{GL}(V)$ to G. There is an element $\gamma_{\mathrm{taut}} \in \mathrm{End}(\widetilde{V})$, given by a tautological automorphism of $\widetilde{V}$. Namely, consider the trivial vector bundle $\mathcal{O}_{G\sigma} \otimes V$ over $G\sigma$. There is the tautological automorphism of this bundle whose restriction to the fiber over $g\sigma \in G\sigma$ is given by the automorphism $\rho(g\sigma)$ of V. This automorphism commutes with the G-equivariant structure on $\mathcal{O}_{G\sigma} \otimes V$, and therefore descends to the desired γ_{taut}. Note that

$$(\mathrm{Res}^\sigma_{V\otimes V^*} \otimes 1)(\gamma_{\mathrm{taut}}) = \sum_\mu e^{\sigma(\mu)} \otimes \sigma_\mu,$$

where the sum is taken over all weights in V and $\sigma_\mu : V(\mu) \to V(\sigma(\mu))$ is the natural isomorphism.

Proposition 6.3.1 (Cayley–Hamilton). *Write $d = \dim V$. Define a degree d polynomial $f(x) \in \boldsymbol{J}[x]$ as*

$$f(x) = \sum (-1)^i \chi_{\wedge^{d-i}V} x^i.$$

Then there is an injective map of $\boldsymbol{J}$-algebras defined by

$$\boldsymbol{J}[x]/(f(x)) \to \mathrm{End}(\widetilde{V}), \quad x \mapsto \gamma_{\mathrm{taut}}. \tag{6.3.1}$$

In particular,

$$\sum (-1)^i \chi_{\wedge^{d-i}V} \gamma^i_{\mathrm{taut}} = 0 \tag{6.3.2}$$

as elements in $\mathrm{End}(\widetilde{V})$.

Proof. For $G = \mathrm{GL}_n$, $\sigma = \mathrm{id}$ and $V = \mathrm{std}$ being the standard representation, the usual Cayley–Hamilton theorem gives (6.3.2). In general, the representation defines a homomorphism $G \rtimes \langle\sigma\rangle \to \mathrm{GL}(V)$, which induces $[G\sigma/G] \to [\mathrm{GL}(V)/\mathrm{GL}(V)]$. The pullback of the bundle $\widetilde{\mathrm{std}}$ on $[\mathrm{GL}(V)/\mathrm{GL}(V)]$ along this map is exactly $\widetilde{V}$, the pullback of the tautological automorphism γ_{taut} of $\widetilde{\mathrm{std}}$ is the tautological automorphism of $\widetilde{V}$, and the pullback of the regular function $\chi_{\wedge^i \mathrm{std}}$ is just $\chi_{\wedge^i V}$. Therefore, (6.3.2) holds in general. Or equivalently, (6.3.1) is well-defined. It remains to show that it is injective.

If $G = \mathrm{GL}_n, \sigma = \mathrm{id}$ and $V = \mathrm{std}$, by Lemma 6.3.4 below, (6.3.1) is an isomorphism. On the other hand, for a representation $G \rtimes \langle\sigma\rangle \to \mathrm{GL}(V)$, the map

$$\mathrm{End}(\widetilde{\mathrm{std}}) \otimes_{\boldsymbol{J}_{\mathrm{GL}(V)}} \boldsymbol{J}_G \to \mathrm{End}_{\mathrm{Coh}([G\sigma/G])}(\widetilde{V})$$

is injective. Therefore, the injectivity of (6.3.1) in the general case follows. □

Remark 6.3.2. Note that the proof of the above proposition holds for *any* algebraic group G.

Remark 6.3.3. We also briefly explain how to study $\mathrm{End}(\widetilde{V})$ if V is not a representation of $G \rtimes \langle\sigma\rangle$, but only a representation of a subgroup $G \rtimes \langle\tau\rangle$, where $\tau = \sigma^f$. In this case, we have the map

$$[G\sigma/G] \to [G\tau/G], \quad g\sigma \mapsto (g\sigma)^f = g\sigma(g)\sigma^2(g)\cdots\sigma^{f-1}(g)\tau.$$

Then we can pullback everything from $[G\tau/G]$. Note that for a representation W of $G \rtimes \langle\tau\rangle$, $W \otimes \sigma(W) \otimes \cdots \otimes \sigma^{f-1}(W)$ is a natural representation of $G \rtimes \langle\sigma\rangle$, usually called the tensor induction representation. Let us denote it by $\otimes^\sigma_\tau W$. Then the pullback of χ_W to $[G\sigma/G]$ is $\chi_{\otimes^\sigma_\tau W}$. So we have

$$\sum (-1)^i \chi_{\otimes^\sigma_\tau(\wedge^{d-i}V)} \gamma^i_{\mathrm{taut}} = 0.$$

As mentioned above, for a general V which is not multiplicity free as a T^σ-module, the ring $\mathrm{End}(\widetilde{V})$ is non-commutative. So $\mathrm{End}(\widetilde{V})$ is not generated by γ_{taut} as a $\boldsymbol{J}$-algebra. Even if $\mathrm{End}(\widetilde{V})$ is commutative, (6.3.1) will only be an isomorphism away from a divisor on $\mathrm{Spec}\,\boldsymbol{J}_G$. However, we can say more in the following special case.

LEMMA 6.3.4. *The map* (6.3.1) *is an isomorphism if*

- $G = \mathrm{GL}_n$ *or* SL_n *equipped with the trivial* σ*-action, and* V *is the standard representation or its dual.*
- $G = \mathrm{Sp}_{2n}$ *and* V *is the standard representation;*

PROOF. It is enough to show that $\boldsymbol{J}_G[x]/(f(x)) \otimes_{\boldsymbol{J}_G} \boldsymbol{J}_T = \boldsymbol{J}_T[x]/(f(x)) \to \mathrm{End}(\widetilde{V}) \otimes_{\boldsymbol{J}_G} \boldsymbol{J}_T$ is an isomorphism. For this, we calculate the composition

$$\boldsymbol{J}_G[x]/(f(x)) \otimes_{\boldsymbol{J}_G} \boldsymbol{J}_T = \boldsymbol{J}_T[x]/(f(x)) \to \mathrm{End}(\widetilde{V}) \otimes_{\boldsymbol{J}_G} \boldsymbol{J}_T \to \boldsymbol{J}_T \otimes \mathrm{End}_T(V). \tag{6.3.3}$$

Under this map (6.3.3), the image of γ_{taut} is $\sum e^{\lambda_j} \otimes \mathrm{id}_{V(\lambda_j)}$, where the sum is taken over all weights $\{\lambda_j\}$ of V, and $\mathrm{id}_{V(\lambda_j)}$ is the identity map of $V(\lambda_j)$. Then the image of γ_{taut}^i is $\sum e^{i\lambda_j} \otimes \mathrm{id}_{V(\lambda_j)}$. Since V is minuscule, $\{\mathrm{id}_{V(\lambda_j)}\}$ form a basis of $\boldsymbol{J}_T \otimes \mathrm{End}_T(V)$ as a $\boldsymbol{J}_T$-module. By Vandermonde, the determinant of the composed map (6.3.3) is

$$\prod_{j<j'} (e^{\lambda_j} - e^{\lambda_{j'}}) = e^{\mu} \prod_{j<j'} (1 - e^{\lambda_j - \lambda_{j'}}),$$

from some $\mu \in \mathbb{X}^\bullet(T)$. For the cases in the lemma, the difference $\lambda_j - \lambda_{j'}$ is a root α, which is equal to the determinant (up to a unit in $\boldsymbol{J}_T$) of the second map in (6.3.3) by Proposition 6.1.4. (Note that Proposition 6.1.4 also holds for GL_n.) Therefore, the first map in (6.3.3) is an isomorphism. □

6.4. Some examples. In this subsection, we present some examples of the calculation of the determinant of (6.1.1).

EXAMPLE 6.4.1. Consider the case where $G = \prod_{i=1}^d \mathrm{SL}_2$ and σ permutes all d factors, and let $V = V_{a_1} \boxtimes \cdots \boxtimes V_{a_d}$ be the exterior tensor product representation, where V_a is the ath symmetric power representation of SL_2 (i.e. the representation of SL_2 on the space of homogeneous polynomials in (x, y) of degree a). By Lemma 4.1.2, this is equivalent to the case where $G = \mathrm{SL}_2$, $\sigma = \mathrm{id}$, and $V = V_{a_\bullet} := V_{a_1} \otimes \cdots \otimes V_{a_d}$ (so that $r_V = V_{a_\bullet}(0) \neq 0$). Now assume that $a_1 + \cdots + a_d$ is even. In this case, Theorem 6.1.2 says that the determinant of the matrix (6.1.1) is equal to

$$c \cdot (e^\alpha - 1)^\zeta (e^{-\alpha} - 1)^\zeta$$

for some $c \in k^\times$, where

$$\zeta = \sum_{n \geq 1} \dim V_{a_\bullet}(n) = \frac{\dim V_{a_\bullet} - \dim V_{a_\bullet}(0)}{2}.$$

In fact, the matrix for the pairing (6.1.1) (for some appropriate basis) can be obtained via a combinatorial description in terms of periodic meanders, which carries a quantum deformation. In the special case when $G = \mathrm{SL}_2$, d is even, and $a_1 = \cdots = a_d = 1$, this matrix (6.1.1) is the Gram matrix for the periodic meanders (see [**TX14**]).

EXAMPLE 6.4.2. Consider the case when $G = \mathrm{SL}_{2r+1}$ with the non-trivial pinned σ-action as explained in Example 5.1.5. We use freely the notation therein. In particular, $\mathbb{X}^\bullet(T)$ is generated by the characters $\varepsilon_{-r}, \ldots, \varepsilon_r$ with the relation that $\sum_{i=-r}^r \varepsilon_i = 0$, and σ acts by $\sigma(\varepsilon_i) = -\varepsilon_{-i}$ for each i. The Bruhat partial order on $\mathbb{X}^\bullet(T)$ is generated by $\varepsilon_i \succeq \varepsilon_{i+1}$ for $i = -r, \ldots, r-1$. For

$A = T/(\sigma-1)T$, $\mathbb{X}^\bullet(A) = \mathbb{X}^\bullet(T)^\sigma = \bigoplus_{i=1}^r \mathbb{Z}\cdot(\varepsilon_{-i}-\varepsilon_i)$. The absolute Weyl group $W \cong S_{2r+1}$ permutes $\varepsilon_{-r},\dots,\varepsilon_r$, and its σ-invariant elements are

$$W_0 = W^\sigma = \langle (i,-i), (ij)(-i,-j) \mid i,j = 1,\dots,r\rangle = (\mathbb{Z}/2\mathbb{Z})^r \rtimes S_r.$$

Recall the σ-orbits of roots in (5.1.2); in particular, $\alpha_{\mathcal{O}_i^-} = \alpha_{\mathcal{O}_i^+} = \varepsilon_{-i}-\varepsilon_i$ for $i = 1,\dots,r$. Write $\mathfrak{S}_i$ for the ith elementary symmetric power in $e^{\alpha_{\mathcal{O},1}}+e^{-\alpha_{\mathcal{O},1}},\dots,e^{\alpha_{\mathcal{O},r}}+e^{-\alpha_{\mathcal{O},r}}$, then

$$\boldsymbol{J} := k[G\sigma]^G \cong k[\mathbb{X}^\bullet(A)]^{W_0} \cong k[\mathfrak{S}_1,\dots,\mathfrak{S}_r] \subset k[\mathbb{X}^\bullet(T)] = k[e^{\pm\varepsilon_{-r}},\dots,e^{\pm\varepsilon_{-1}},e^{\pm\varepsilon_1},\dots,e^{\pm\varepsilon_r}].$$

When $r=1$, $\boldsymbol{J} = k[e^{\alpha_{\mathcal{O},1}}+e^{-\alpha_{\mathcal{O},1}}]$.

We consider the standard representation $V = \mathtt{S}_{\varepsilon_{-r}} = \mathrm{std}$ of $G = \mathrm{SL}_{2r+1}$. Observe that the only nonzero weight space of V with weights in $(\sigma-1)\mathbb{X}^\bullet(T)$ is $V(\varepsilon_0)$. We write

$$\varepsilon_0 = \sigma(\nu)-\nu, \quad \text{for} \quad \nu = \varepsilon_{-r}+\cdots+\varepsilon_{-1}.$$

The representation $\mathtt{S}_\nu \cong \mathtt{S}_{\sigma(\nu)^*} \cong \wedge^r\mathrm{std}$ is the rth wedge product of the standard representation. Then it is easy to see that

$$(\mathtt{S}_{\sigma(\nu^*)}\otimes\mathtt{S}_\nu\otimes V)^G = (\wedge^r\mathrm{std}\otimes\wedge^r\mathrm{std}\otimes\mathrm{std})^G \cong (\wedge^{2r+1}\mathrm{std})^G \cong k,$$

Let $\mathbf{b}$ denote the element on the left hand side that corresponds to $1\in k$, or more precisely $e_{-r}\wedge\cdots\wedge e_r$. Then the recipe in (4.4.2) defines a function $f_{\mathbf{b}} \in \boldsymbol{J}(V)$ (note that in our case $\mathtt{W}_\nu \cong \mathtt{S}_\nu$). We will compute its restriction to $T\sigma$.

Let $e_{-r},\dots,e_r$ denote the standard basis of the representation std. Then $\wedge^r\mathrm{std}$ has a basis $(e_I)_{I\subset\{-r,\dots,r\},|I|=r}$, where for $I=\{i_1,\dots,i_r\}$ with elements ordered increasingly, we write $e_I = e_{i_1}\wedge\cdots\wedge e_{i_r}$. The natural map $\sigma : \mathtt{S}_\nu = \wedge^r\mathrm{std} \to \mathtt{S}_{\sigma(\nu)} \cong (\wedge^r\mathrm{std})^*$ is given by sending e_i to $(-1)^i e^*_{-i}$.[6] So $\sigma(e_I) = \mathrm{sgn}(I)\cdot e^*_{-I}$, where $-I = \{-i_r,\dots,-i_1\}$ and $\mathrm{sgn}(I) = (-1)^{i_1+\cdots+i_r+r(r-1)/2}$. Using this notation, we write explicitly

$$\mathbf{b} = \sum_{|I|=r,|I'|=r} e_I\otimes e_{I'}\otimes v_{I,I'}, \quad f_{\mathbf{b}}(t\sigma) = \sum_{|I|=r,|I'|=r}\langle e_I, t\sigma e_{I'}\rangle\cdot v_{I,I'},$$

where $v_{I,I'}\in V=\mathrm{std}$, $t\in T$, and the sums run through all subsets I,I' of $\{-r,\dots,r\}$ of cardinality r. In the expression for $f_{\mathbf{b}}(t\sigma)$, the pairing $\langle e_I, t\sigma e_{I'}\rangle$ is nonzero precisely when $I' = -I$. In the expression for $\mathbf{b}$, the vector $v_{I,I'}$ is nonzero precisely when I and I' are disjoint. So we need only to discuss the case when there is $\underline{s} = \{s_1,\dots,s_r\}\in\{\pm1\}^r$ such that $I = I_{\underline{s}} = \{s_1\cdot 1,\dots,s_r\cdot r\}$ and $I' = -I$. In this case, $\mathrm{sgn}(-I_{\underline{s}}) = (-1)^{r(r+1)/2+r(r-1)/2} = (-1)^r$ is independent of $\underline{s}$. Moreover, we may take $v_{I,I'} = (-1)^r\mathrm{sgn}(\underline{s})\cdot e_0$, where the sign $\mathrm{sgn}(\underline{s}) = (-1)^u$ with u equal to the number of -1's in $\underline{s}$. Thus,

$$\mathrm{Res}^\sigma_V(f_{\mathbf{b}}) = \sum_{\underline{s}\in\{\pm1\}^r}\mathrm{sgn}(\underline{s})e^{-\varepsilon_{-s_1}-\varepsilon_{-2s_2}-\cdots-\varepsilon_{-rs_r}}(t)\cdot e_0 = \prod_{i=1}^r\left(e^{-\varepsilon_{-i}}-e^{-\varepsilon_i}\right)\cdot e_0 \in \boldsymbol{J}_T(V).$$

In the dual picture, we consider an element

$$\mathbf{b}^\vee \in (V_{\sigma(\nu)}\otimes V_{\nu^*}\otimes V^*)^G = \left((\wedge^r\mathrm{std}^*)\otimes(\wedge^r\mathrm{std}^*)\otimes\mathrm{std}^*\right)^G \cong (\wedge^{2r+1}\mathrm{std}^*)^G \cong k.$$

Explicitly, we write

$$\mathbf{b}^\vee = \sum_{|I|=r,|I'|=r} e_I\otimes e^*_{I'}\otimes v^\vee_{I,I'}, \quad f_{\mathbf{b}^\vee}(t\sigma) = \sum_{|I|=r,|I'|=r}\langle e_I, t\sigma e^*_{I'}\rangle\cdot e^\vee_{I,I'}.$$

[6] This is in fact $(-1)^r$ times the natural map given by (5.1.1).

Similar to above, we need only to discuss the case when $I = I_{\underline{s}}$ for some $\underline{s} \in \{\pm 1\}^r$ and $I' = -I$. In this case, we may choose $v_{I,I'} = (-1)^r \mathrm{sgn}(\underline{s}) \cdot e_0^*$. Thus,

$$\mathrm{Res}^{\sigma}_{V^*}(f_{\mathbf{b}^\vee}) = \sum_{\underline{s}\in\{\pm1\}^r} \mathrm{sgn}(\underline{s}) e^{\varepsilon_{-s_1}+\varepsilon_{-2s_2}+\cdots+\varepsilon_{-rs_r}}(t) \cdot e_0^* = \prod_{i=1}^{r} \left(e^{\varepsilon_{-i}} - e^{\varepsilon_i}\right) \cdot e_0^* \in \boldsymbol{J}_T(V^*).$$

Now we compute $\langle f_{\mathbf{b}}, f_{\mathbf{b}^\vee}\rangle$ under the pairing $\boldsymbol{J}(V) \otimes_{\boldsymbol{J}} \boldsymbol{J}(V^*) \to \boldsymbol{J} \cong \boldsymbol{J}_T^{W_0}$:

$$\langle f_{\mathbf{b}}, f_{\mathbf{b}^\vee}\rangle = \prod_{i=1}^{r} \left(e^{-\varepsilon_{-i}} - e^{-\varepsilon_i}\right) \cdot \prod_{i=1}^{r} \left(e^{\varepsilon_{-i}} - e^{\varepsilon_i}\right) = \prod_{i=1}^{r} \left((e^{\alpha_{\mathcal{O}_i^-}} - 1)(e^{-\alpha_{\mathcal{O}_i^-}} - 1)\right).$$

This agrees with Theorem 6.1.2 by noting that $\zeta_{\mathcal{O}_i^-}(V) = 1$ and $\zeta_{\mathcal{O}_i^+}(V) = \zeta_{\mathcal{O}_{i,j}}(V) = 0$, for the σ-orbits in (5.1.2).

Example 6.4.3. Consider SL_4 with row and column indices in $\{-2,-1,1,2\}$, the pinning (B,T,e) given by the subgroup of standard upper triangular matrices, the subgroup of diagonal matrices, and $e = E_{-2,-1} + E_{-1,1} + E_{1,2}$. The unique non-trivial pinned automorphism σ is given by $\sigma(X) = J\,{}^tX^{-1}J^{-1}$ for $X \in \mathrm{SL}_4$, with $J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$.

Let ε_i for $i = \pm 1, \pm 2$ be the character of T given by evaluating at the (i,i)-entry; so that $\mathbb{X}^\bullet(T)$ is generated by these ε_i with the relation $\varepsilon_{-2} + \varepsilon_{-1} + \varepsilon_1 + \varepsilon_2 = 0$. Under the σ-action $\sigma(\varepsilon_i) = -\varepsilon_i$, the following are the σ-orbits of $\Phi(G,T)$:

$$\mathcal{O}_i = \{\varepsilon_{-i} - \varepsilon_i\}, \quad \mathcal{O}_{\pm2,\pm1} = \{\varepsilon_{\mp2} - \varepsilon_{\mp1}, \varepsilon_{\pm1} - \varepsilon_{\pm2}\}, \quad \mathcal{O}_{\mp2,\pm1} = \{\varepsilon_{\pm2} - \varepsilon_{\mp1}, \varepsilon_{\pm1} - \varepsilon_{\mp2}\}$$

with $i \in \{\pm1, \pm2\}$. They are all of type A. For $A = T/(\sigma-1)T$,

$$\mathbb{X}^\bullet(A) = \mathbb{X}^\bullet(T)^\sigma = \mathbb{Z}(\varepsilon_{-2} - \varepsilon_2) \oplus \mathbb{Z}(\varepsilon_{-1} + \varepsilon_{-2}).$$

(Note that $(\varepsilon_{-2} - \varepsilon_2) + (\varepsilon_{-1} - \varepsilon_1) = 2(\varepsilon_{-2} + \varepsilon_{-1})$.) The root system $\Phi(G_\sigma, A)$ consists of short roots $\alpha_{\mathcal{O}_i}$ and long roots $\alpha_{\mathcal{O}_{i,j}}$; the group $G_\sigma \cong \mathrm{Spin}_5$. The absolute Weyl group of $\Phi(G,T)$ is $W \cong S_4$ given by permuting the ε_i's, and its σ-invariant elements are $W_0 = W^\sigma = \langle (-1,1), (-2,2), (1,2)(-1,-2)\rangle = (\mathbb{Z}/2\mathbb{Z})^2 \rtimes S_2$. Write

$$\mathfrak{S}_1 = e^{\varepsilon_{-1}-\varepsilon_1} + e^{\varepsilon_1-\varepsilon_{-1}} + e^{\varepsilon_{-2}-\varepsilon_2} + e^{\varepsilon_2-\varepsilon_{-2}}, \quad \mathfrak{S}_2 = e^{\varepsilon_{-2}+\varepsilon_{-1}} + e^{-\varepsilon_{-2}-\varepsilon_{-1}} + e^{\varepsilon_1+\varepsilon_2} + e^{-\varepsilon_1-\varepsilon_2}.$$

Then

$$\begin{aligned} \boldsymbol{J} := k[G\sigma]^G \cong k[\mathbb{X}^\bullet(A)]^{W_0} &\cong k[\mathfrak{S}_1, \mathfrak{S}_2] \\ &\subset k[\mathbb{X}^\bullet(T)] = k[e^{\pm\varepsilon_{-2}}, e^{\pm\varepsilon_{-1}}, e^{\pm\varepsilon_1}, e^{\pm\varepsilon_2}]/(e^{\varepsilon_{-2}+\varepsilon_{-1}+\varepsilon_1+\varepsilon_2} - 1). \end{aligned}$$

Consider the minuscule representation $V = \mathtt{S}_{\varepsilon_{-2}+\varepsilon_{-1}}$ of SL_4; it is isomorphic to $\wedge^2\mathrm{std}$ for the standard representation std of SL_4. There are two nonzero weight spaces with weights in $(\sigma-1)\mathbb{X}^\bullet(T)$: $V(\lambda_1)$ and $V(\lambda_2)$ with $\lambda_i = \varepsilon_{-i} + \varepsilon_i$. They are both of one-dimensional. So $r_V = 2$. The minimal dominant weights ν_i (in the sense as in Lemma 4.3.6) satisfying $\sigma(\nu_i) - \nu_i = \lambda_i$ are

$$\nu_1 = \varepsilon_{-2}, \quad \nu_2 = -\varepsilon_2.$$

We remark here that for SL_n with $n \geq 5$ and the non-trivial automorphism σ, and for $V = \wedge^2\mathrm{std}$, we have $r_V = \lfloor \frac{n}{2} \rfloor$ and some (or rather most) of the minimal dominant weights ν above are no longer minuscule. So the computation will be much more involved.

Back to our case, $\mathtt{S}_{\nu_1} \cong \mathtt{S}_{\sigma(\nu_1)^*} \cong \mathrm{std}$ and $\mathtt{S}_{\nu_2} \cong \mathtt{S}_{\sigma(\nu_2)^*} \cong \mathrm{std}^*$. In what follows, we will describe nonzero elements

$$\mathbf{b}_1 \in (\mathtt{S}_{\sigma(\nu_1)^*} \otimes \mathtt{S}_{\nu_1} \otimes V)^G = (\mathrm{std} \otimes \mathrm{std} \otimes \wedge^2\mathrm{std})^G, \quad \mathbf{b}_2 \in (\mathtt{S}_{\sigma(\nu_2)^*} \otimes \mathtt{S}_{\nu_2} \otimes V)^G = (\mathrm{std}^* \otimes \mathrm{std}^* \otimes \wedge^2\mathrm{std})^G,$$

use the recipe in (4.4.2) to defines a function $f_{\mathbf{b}_1}, f_{\mathbf{b}_2} \in \boldsymbol{J}(V)$ (note that in our case $\mathtt{W}_{\nu_i} \cong \mathtt{S}_{\nu_i}$), and describe the restriction of $f_{\mathbf{b}_i}$ to $T\sigma$.

Write $\{e_{-2}, e_{-1}, e_1, e_2\}$ for the standard basis of std and $\{e^*_{-2}, e^*_{-1}, e^*_1, e^*_2\}$ the dual basis; and the isomorphism $\sigma : \mathrm{std} \cong \mathrm{std}^*$ is given by

$$e_{-2} \mapsto e^*_2, \quad e_{-1} \mapsto -e^*_1, \quad e_1 \mapsto e^*_{-1}, \quad e_2 \mapsto -e^*_{-2}.$$

With this notation, we write explicitly,

$$\mathbf{b}_1 = \sum_{i,j \in \{\pm 1, \pm 2\}} e_i \otimes e_j \otimes v_{i,j}, \quad f_{\mathbf{b}_1}(t\sigma) = \sum_{i,j \in \{\pm 1, \pm 2\}} \langle e_i, t\sigma e_j \rangle \cdot v_{i,j},$$

where $v_{i,j} \in V = \wedge^2 \mathrm{std}$ and $t \in T$. In the expression for $f_{\mathbf{b}_1}(t\sigma)$, the pairing $\langle e_i, t\sigma e_j \rangle$ is nonzero precisely when $i = -j$. In the expression for $\mathbf{b}_1$, the vector $v_{i,j}$ is nonzero precisely when $i \neq j$. So we need only to discuss the case when $i = -j \in \{\pm 1, \pm 2\}$. In this case, we may choose $v_{-2,2} = -v_{2,-2} = -e_{-1} \wedge e_1$ and $v_{-1,1} = -v_{1,-1} = -e_{-2} \wedge e_2$. Thus,

$$\mathrm{Res}^{\sigma}_V(f_{\mathbf{b}_1}) = (e^{-\varepsilon_{-2}} + e^{-\varepsilon_2}) e_{-1} \wedge e_1 - (e^{-\varepsilon_{-1}} + e^{-\varepsilon_1}) e_{-2} \wedge e_2 \in \boldsymbol{J}_T(V).$$

In a similar way, we may compute (up to choosing a sign for $\mathbf{b}_2$)

$$\mathrm{Res}^{\sigma}_V(f_{\mathbf{b}_2}) = (e^{\varepsilon_{-1}} + e^{\varepsilon_1}) e_{-1} \wedge e_1 - (e^{\varepsilon_{-2}} + e^{\varepsilon_2}) e_{-2} \wedge e_2 \in \boldsymbol{J}_T(V).$$

On the dual side, we choose elements

$$\mathbf{b}^\vee_1 \in (\mathtt{S}_{\sigma(\nu_1)} \otimes \mathtt{S}_{\nu_1^*} \otimes V^*)^G = (\mathrm{std}^* \otimes \mathrm{std}^* \otimes \wedge^2 \mathrm{std}^*)^G, \quad \mathbf{b}^\vee_2 \in (\mathtt{S}_{\sigma(\nu_2)} \otimes \mathtt{S}_{\nu_2^*} \otimes V^*)^G = (\mathrm{std} \otimes \mathrm{std} \otimes \wedge^2 \mathrm{std}^*)^G,$$

and a computation similar to above gives the explicit formulas (up to choosing a sign for $\mathbf{b}^\vee_i$)

$$\mathrm{Res}^{\sigma}_{V^*}(f_{\mathbf{b}^\vee_1}) = (e^{\varepsilon_{-2}} + e^{\varepsilon_2}) e^*_{-1} \wedge e^*_1 - (e^{\varepsilon_{-1}} + e^{\varepsilon_1}) e^*_{-2} \wedge e^*_2 \in \boldsymbol{J}_T(V^*)$$
$$\mathrm{Res}^{\sigma}_{V^*}(f_{\mathbf{b}^\vee_2}) = (e^{-\varepsilon_{-1}} + e^{-\varepsilon_1}) e^*_{-1} \wedge e^*_1 - (e^{-\varepsilon_{-2}} + e^{-\varepsilon_2}) e^*_{-2} \wedge e^*_2 \in \boldsymbol{J}_T(V^*).$$

From this, we deduce that under the natural $\boldsymbol{J}$-linear pairing $\boldsymbol{J}(V) \times \boldsymbol{J}(V^*) \to \boldsymbol{J}$, we have

$$\langle f_{\mathbf{b}_1}, f_{\mathbf{b}^\vee_1} \rangle = \langle f_{\mathbf{b}_2}, f_{\mathbf{b}^\vee_2} \rangle = (e^{\varepsilon_{-2}} + e^{\varepsilon_2})(e^{-\varepsilon_{-2}} + e^{-\varepsilon_2}) + (e^{\varepsilon_{-1}} + e^{\varepsilon_1})(e^{-\varepsilon_{-1}} + e^{-\varepsilon_1}) = \mathfrak{S}_1 + 4,$$
$$\langle f_{\mathbf{b}_1}, f_{\mathbf{b}^\vee_2} \rangle = \langle f_{\mathbf{b}_2}, f_{\mathbf{b}^\vee_1} \rangle = 2(e^{-\varepsilon_{-1}} + e^{-\varepsilon_1})(e^{-\varepsilon_{-2}} + e^{-\varepsilon_2}) = 2(e^{\varepsilon_{-1}} + e^{\varepsilon_1})(e^{\varepsilon_{-2}} + e^{\varepsilon_2}) = 2\mathfrak{S}_2.$$

(Note that $e^{\varepsilon_{-2} + \varepsilon_{-1} + \varepsilon_1 + \varepsilon_2} = 1$.) We can compute the determinant of the pairing as

$$\begin{aligned}
\det \begin{pmatrix} \langle f_{\mathbf{b}_1}, f_{\mathbf{b}^\vee_1} \rangle & \langle f_{\mathbf{b}_1}, f_{\mathbf{b}^\vee_2} \rangle \\ \langle f_{\mathbf{b}_2}, f_{\mathbf{b}^\vee_1} \rangle & \langle f_{\mathbf{b}_2}, f_{\mathbf{b}^\vee_2} \rangle \end{pmatrix} &= \big((e^{\varepsilon_{-2}} + e^{\varepsilon_2})(e^{-\varepsilon_{-2}} + e^{-\varepsilon_2}) + (e^{\varepsilon_{-1}} + e^{\varepsilon_1})(e^{-\varepsilon_{-1}} + e^{-\varepsilon_1})\big)^2 \\
&\quad - 4(e^{\varepsilon_{-1}} + e^{\varepsilon_1})(e^{\varepsilon_{-2}} + e^{\varepsilon_2})(e^{\varepsilon_{-1}} + e^{\varepsilon_1})(e^{\varepsilon_{-2}} + e^{\varepsilon_2}) \\
&= \big((e^{\varepsilon_{-2}} + e^{\varepsilon_2})(e^{-\varepsilon_{-2}} + e^{-\varepsilon_2}) - (e^{\varepsilon_{-1}} + e^{\varepsilon_1})(e^{-\varepsilon_{-1}} + e^{-\varepsilon_1})\big)^2 \\
&= \big((e^{\varepsilon_{-2} - \varepsilon_2} - e^{\varepsilon_{-1} - \varepsilon_1})(1 - e^{\varepsilon_2 - \varepsilon_{-2} + \varepsilon_1 - \varepsilon_{-1}})\big)^2 \\
&= (e^{\alpha_{\mathcal{O}_{-2,-1}}} - 1)(e^{\alpha_{\mathcal{O}_{-2,1}}} - 1)(e^{\alpha_{\mathcal{O}_{2,-1}}} - 1)(e^{\alpha_{\mathcal{O}_{2,1}}} - 1).
\end{aligned}$$

One can compare this with the computation that $\zeta_{\mathcal{O}_i}(V) = 0$ and $\zeta_{\mathcal{O}_{\pm 2, \pm 1}}(V) = \zeta_{\mathcal{O}_{\pm 2, \mp 1}}(V) = 1$.

Example 6.4.4. Assume that $\mathrm{char}\, k = 0$. Consider the case of $G = \mathrm{Spin}_{2r}$. More precisely, consider $Q := k^{\oplus 2r}$ with basis $e_{-r}, \ldots, e_{-1}, e_1, \ldots, e_r$ and a symmetric quadratic form $\langle e_i, e_{-j} \rangle = \delta_{ij}$. Let

$$Cl = Cl(Q) = Cl^{\mathrm{even}} \oplus Cl^{\mathrm{odd}} = \Big(\bigoplus_{n \geq 0} Q^{\otimes n} \Big) \Big/ \big(w \otimes w - \tfrac{1}{2} \langle w, w \rangle;\, w \in Q \big)$$

denote the associated ($\mathbb{Z}/2\mathbb{Z}$-graded) Clifford algebra. There is a natural (anti-commutative) involution $\bullet^t$ on Cl generated by sending $w \in Q$ to $-w$. The pin group Pin_{2r} is formed by all elements

$x \in Cl$ such that $x \cdot x^t = 1$, and $xQx^t \subset Q$. The spin group $\mathrm{Spin}_{2r} := \mathrm{Pin}_{2r} \cap Cl^{\mathrm{even}}$ is the neutral connected component of Pin_{2r}. The maximal torus T of Spin_{2r} is given by the image of

$$\begin{aligned} \Gamma : \mathbb{G}_m^r &\longrightarrow \mathrm{Spin}_{2r} \\ (z_1, \dots, z_r) &\longmapsto \textstyle\prod_{i=1}^r \left(z_i e_i e_{-i} + z_i^{-1} e_i e_{-i}\right), \end{aligned}$$

where the kernel is $\{(z_1, \dots, z_r) \in \{\pm 1\}^r \mid z_1 \cdots z_r = 1\}$. Let $\frac{\varepsilon_1}{2}, \dots, \frac{\varepsilon_r}{2}$ denote the usual basis of the character group of $\mathbb{G}_m^r$; then

$$\mathbb{X}^\bullet(T) = \mathbb{Z} \cdot \tfrac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_r) \oplus \bigoplus_{i=1}^{r-1} \mathbb{Z} \cdot \varepsilon_i.$$

A weight $\sum_{i=1}^r a_i \varepsilon_i$ is dominant if $a_1 \geq \cdots \geq a_{r-1} \geq |a_r|$.

We equip with Spin_{2r} with the outer automorphism σ, induced by the conjugation by $e_r + e_{-r}$. A simple computation shows that

$$(e_r + e_{-r})(z_i e_i e_{-i} + z_i^{-1} e_i e_{-i})(e_r + e_{-r}) = \begin{cases} z_i e_i e_{-i} + z_i^{-1} e_i e_{-i} & \text{if } i \neq r \\ z_r^{-1} e_r e_{-r} + z_r e_r e_{-r} & \text{if } i = r \end{cases}$$

So σ fixes $\varepsilon_1, \dots, \varepsilon_{r-1}$ and maps ε_r to $-\varepsilon_r$. So for $A = T/(\sigma - 1)T$, its character group $\mathbb{X}^\bullet(A) = \mathbb{X}^\bullet(T)^\sigma$ is the free abelian group with basis $\varepsilon_1, \dots, \varepsilon_{r-1}$.

The absolute Weyl group $W \cong H_r \rtimes S_r \subset \{\pm 1\}^r \times S_r$, where $H_r \subset \{\pm 1\}^r$ is the subgroup consisting of even number of -1's, S_r permutes $\varepsilon_1, \dots, \varepsilon_r$ and $(h_1, \dots, h_r) \in H_r$ sends ε_i to $h_i \varepsilon_i$ for $i = 1, \dots, r$. The σ-invariant elements of W are $W_0 := W^\sigma = \{\pm 1\}^{r-1} \rtimes S_{r-1}$, where S_{r-1} permutes $\varepsilon_1, \dots, \varepsilon_{r-1}$ and $(h_1, \dots, h_r) \in \{\pm 1\}^{r-1}$ sends ε_i to $h_i \varepsilon_i$ for each $i = 1, \dots, r-1$. It follows that the invariants of $\mathbb{C}[\mathbb{X}^\bullet(A)]$ under W_0 are

$$\boldsymbol{J} := \mathbb{C}[\mathbb{X}^\bullet(A)]^{W_0} = \mathbb{C}[\mathfrak{S}_1, \dots, \mathfrak{S}_{r-1}],$$

where $\mathfrak{S}_i$ for $i = 1, \dots, r-1$ is the ith elementary symmetric polynomial in $e^{\varepsilon_1} + e^{-\varepsilon_1}, \dots, e^{\varepsilon_{r-1}} + e^{-\varepsilon_{r-1}}$.

The root system $\Phi(G, T)$ consists of $\pm\varepsilon_i \pm \varepsilon_j$ and $\pm\varepsilon_i \mp \varepsilon_j$ for $i \in \{1, \dots, r\}$ and $j \in \{1, \dots, i-1\}$. The σ-orbits of these roots are

$$\mathcal{O}_{i,j}^{\pm} = \{\pm\varepsilon_i \pm \varepsilon_j\}, \quad \mathcal{O}_{j,i}^{\pm} = \{\pm\varepsilon_i \mp \varepsilon_j\}, \quad \mathcal{O}_i^{\pm} = \{\pm\varepsilon_i + \varepsilon_r, \pm\varepsilon_i - \varepsilon_r\},$$

with $i \in \{1, \dots, r\}, j \in \{1, \dots, i-1\}$; they are all of type A. Thus the root system $\Phi(G_\sigma, A)$ consists of short roots $\alpha_{\mathcal{O}_{i,j}^{\pm}} = \pm\varepsilon_i \pm \varepsilon_j$ and long roots $\alpha_{\mathcal{O}_i^{\pm}} = \pm 2\varepsilon_i$.

We shall consider $V = \mathtt{S}_{\varepsilon_1} \cong Q$, the vector representation of $G = \mathrm{Spin}_{2r}$. Its nonzero weight spaces with weights in $(\sigma - 1)\mathbb{X}^\bullet(T)$ are $V(-\varepsilon_r)$ and $V(\varepsilon_r)$. We write

$$\mp\varepsilon_r = \sigma(\nu_\pm) - \nu_\pm \quad \text{for} \quad \nu_\pm = \tfrac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{r-1} \pm \varepsilon_r).$$

(Be careful with the signs here.) So $\sigma(\nu_\pm) = \nu_\mp$. We shall see that

$$(\mathtt{S}_{\sigma(\nu_\pm^*)} \otimes \mathtt{S}_{\nu_\pm} \otimes V)^G \cong k. \tag{6.4.1}$$

Indeed, since V is minuscule, $\mathtt{S}_{\nu_\pm} \otimes V$ is a direct sum of $\mathtt{S}_{\nu_\pm + \tau}$ with all $\tau \in \{\pm\varepsilon_1, \dots, \pm\varepsilon_r\}$ such that $\nu_\pm + \tau$ is dominant. In particular, $\nu_\mp$ is among those weights $\nu_\pm + \tau$. Dually, we have

$$(\mathtt{S}_{\sigma(\nu_\pm)} \otimes \mathtt{S}_{\nu_\pm^*} \otimes V^*)^G \cong k. \tag{6.4.2}$$

LEMMA 6.4.5. *There exist bases* $\mathbf{b}_\pm$ *of* (6.4.1) *and bases* $\mathbf{b}_\pm^\vee$ *of* (6.4.2) *such that if we use* $f_{\mathbf{b}_\pm} \in \boldsymbol{J}(V)$ *and* $f_{\mathbf{b}_\pm^\vee} \in \boldsymbol{J}(V^*)$ *to denote the associated class functions via the recipe* (4.4.2)*, then*

$$\mathfrak{M} := \begin{pmatrix} \langle f_{\mathbf{b}_+}, f_{\mathbf{b}_+^\vee}\rangle & \langle f_{\mathbf{b}_+}, f_{\mathbf{b}_-^\vee}\rangle \\ \langle f_{\mathbf{b}_-}, f_{\mathbf{b}_+^\vee}\rangle & \langle f_{\mathbf{b}_-}, f_{\mathbf{b}_-^\vee}\rangle \end{pmatrix} = \begin{pmatrix} \sum_{i\geq 0 \text{ even}} 2^{r-1-i}\mathfrak{S}_i & \sum_{i>0 \text{ odd}} 2^{r-1-i}\mathfrak{S}_i \\ \sum_{i>0 \text{ odd}} 2^{r-1-i}\mathfrak{S}_i & \sum_{i\geq 0 \text{ even}} 2^{r-1-i}\mathfrak{S}_i \end{pmatrix}. \tag{6.4.3}$$

The determinant of $\mathfrak{M}$ *is*

$$\prod_{i=1}^{r-1}(1-e^{2\varepsilon_i})(1-e^{2\varepsilon_{-i}}).$$

This agrees with our Theorem 6.1.2 because, for $i \neq j \in \{1,\dots,r-1\}$, one can check easily that $\zeta_{\mathcal{O}_i^\pm}(V) = 1$, $\zeta_{\mathcal{O}_{i,j}^\pm}(V) = 0$.

PROOF. Consider the (isotropic) subspace $U = \bigoplus_{i=1}^r ke_i$ of Q and its ($\mathbb{Z}/2\mathbb{Z}$-graded) wedge product space $\wedge^* U = (\wedge^* U)^{\text{even}} \oplus (\wedge^* U)^{\text{odd}}$. A basis of the latter is given by $e_I = e_{i_1} \wedge \cdots \wedge e_{i_m}$ for $I = \{i_1,\dots,i_m\}$ in which elements are ordered so that $i_1 > \cdots > i_m$. (Our unusual ordering is to avoid the appearance of unnecessary signs later.) The pin group Pin_{2r} acts on $\wedge^* U$ as, for $1 \leq i \leq m$,

$$\begin{aligned} e_i \bullet (u_1 \wedge \cdots \wedge u_m) &= e_i \wedge u_1 \wedge \cdots \wedge u_m, \\ e_{-i} \bullet (u_1 \wedge \cdots \wedge u_m) &= \textstyle\sum_{i=1}^m (-1)^{i-1}\langle e_{-i}, u_i\rangle \cdot u_1 \wedge \cdots \wedge \widehat{u_i} \wedge \cdots \wedge u_m. \end{aligned} \tag{6.4.4}$$

Restricting to Spin_{2r}, both $(\wedge^* U)^{\text{even}}$ and $(\wedge^* U)^{\text{odd}}$ are irreducible representations; when r is even, their highest weights are $\nu_+ = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_r)$ and $\nu_- = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{r-1} - \varepsilon_r)$, respectively; when r is odd, the highest weights are exchanged. In what follows, we assume that r is even. The other case can be treated similarly. We henceforth identify $\mathtt{S}_{\nu_+} \cong (\wedge^* U)^{\text{even}}$ and $\mathtt{S}_{\nu_-} \cong (\wedge^* U)^{\text{odd}}$. More generally, the weight of e_I (with $I = \{i_1,\dots,i_m\}$) is

$$\varepsilon_{i_1} + \cdots + \varepsilon_{i_m} - \nu_+.$$

Moreover, the action (6.4.4) also defines natural morphisms

$$V \otimes (\wedge^* U)^{\text{even}} \to (\wedge^* U)^{\text{odd}}, \quad V \otimes (\wedge^* U)^{\text{odd}} \to (\wedge^* U)^{\text{even}}.$$

The isomorphism $\sigma : \mathtt{S}_{\nu_+} \to \mathtt{S}_{\sigma(\nu_+)} = \mathtt{S}_{\nu_-}$ may be identified with the natural map

$$(\wedge^* U)^{\text{even}} \to (\wedge^* U)^{\text{odd}}, \quad x \mapsto (e_r + e_{-r}) \bullet x,$$

intertwining the natural action of Spin_{2r} on the source and the σ-twisted action of Spin_{2r} on the target. Explicitly, this map sends e_I to $e_{I\triangle\{r\}}$, where $I\triangle\{r\}$ is the symmetric difference.

With this discussion, we may express a basis element of

$$(\mathtt{S}_{\sigma(\nu_+)^*} \otimes \mathtt{S}_{\nu_+} \otimes V)^G = \big((\wedge^* U)^{\text{odd},*} \otimes (\wedge^* U)^{\text{even}} \otimes V\big)^G$$

and its class function as

$$\mathbf{b}_+ = \sum_{|I| \text{ odd}} \sum_{|I'| \text{ even}} e_I^* \otimes e_{I'} \otimes v_{I,I'}, \quad f_{\mathbf{b}_+}(t\sigma) = \sum_{|I| \text{ odd}} \sum_{|I'| \text{ even}} \langle e_I^* \otimes t\sigma e_{I'}\rangle \cdot v_{I,I'}$$

for $v_{I,I'} \in V$ and $t \in T$, where the sum is taken over all subsets $I, J \subset \{1,\dots,r\}$ whose sizes are of the specified parity. In the expression for $f_{\mathbf{b}}(t\sigma)$, the pairing $\langle e_I^*, \sigma e_{I'}\rangle$ is nonzero precisely when

$I' = I \triangle \{r\}$; in this case, we may rewrite $\{I, I'\}$ as $\{J, J \cup \{r\}\}$ for some $J \subset \{1, \dots, r-1\}$, and we may take the vector $v_{I,I'}$ to be e_r if $r \in I$ and to be e_{-r} if $r \notin I$. Therefore, we have

$$\begin{aligned} f_{\mathbf{b}_+}(t\sigma) &= \sum_{|J| \text{ even}} \langle e^*_{J\cup\{r\}}, e^{\varepsilon_J + \varepsilon_r - \nu_+}(t)\sigma e_J\rangle \cdot e_r + \sum_{|J| \text{ odd}} \langle e^*_J, e^{\varepsilon_J - \nu_+}(t) e_{J\cup\{r\}}\rangle \cdot e_{-r} \\ &= \sum_{|J| \text{ even}} e^{\varepsilon_J + \varepsilon_r - \nu_+}(t) \cdot e_r + \sum_{|J| \text{ odd}} e^{\varepsilon_J - \nu_+}(t) \cdot e_{-r}, \end{aligned}$$

where the sums are taken over subsets $J \subset \{1, \dots, r-1\}$ whose sizes are of the given parity, and for $J = \{i_1, \dots, i_m\}$, $\varepsilon_J = \varepsilon_{i_1} + \cdots + \varepsilon_{i_m}$. To simplify notation, we put

$$C_+ = e^{-\nu_+} \cdot \sum_{|J| \text{ even}} e^{\varepsilon_J}, \quad C_- = e^{-\nu_+} \cdot \sum_{|J| \text{ odd}} e^{\varepsilon_J}$$

so that $\mathrm{Res}^\sigma_V(f_{\mathbf{b}_+}) = C_+ e^{\varepsilon_r} \cdot e_r + C_- \cdot e_{-r} \in \boldsymbol{J}_T(V)$.

Similarly, we may choose a basis element

$$\mathbf{b}_- \in (\mathtt{S}_{\sigma(\nu_-)^*} \otimes \mathtt{S}_{\nu_-} \otimes V)^G = \big((\wedge^* U)^{\mathrm{even},*} \otimes (\wedge^* U)^{\mathrm{odd}} \otimes V\big)^G,$$

with class function

$$\mathrm{Res}^\sigma_V(f_{\mathbf{b}_-}) = C_- e^{\varepsilon_r} \cdot e_r + C_+ \cdot e_{-r} \in \boldsymbol{J}_T(V).$$

On the dual side, we choose elements

$$\begin{aligned} \mathbf{b}^\vee_+ &\in (\mathtt{S}_{\sigma(\nu_+)} \otimes \mathtt{S}_{\nu^*_+} \otimes V^*)^G = \big((\wedge^* U)^{\mathrm{odd}} \otimes (\wedge^* U)^{\mathrm{even},*} \otimes V^*\big)^G, \\ \mathbf{b}^\vee_- &\in (\mathtt{S}_{\sigma(\nu_-)} \otimes \mathtt{S}_{\nu^*_-} \otimes V)^G = \big((\wedge^* U)^{\mathrm{even}} \otimes (\wedge^* U)^{\mathrm{odd},*} \otimes V\big)^G. \end{aligned}$$

If we write $D_+ = e^{\nu_+} \cdot \sum_{|J| \text{ even}} e^{-\varepsilon_J}$, $D_- = e^{\nu_+} \cdot \sum_{|J| \text{ odd}} e^{-\varepsilon_J}$, the class functions $f_{\mathbf{b}^\vee_+}, f_{\mathbf{b}^\vee_-} \in \boldsymbol{J}(V^*)$ have restrictions

$$\mathrm{Res}^\sigma_{V^*}(f_{\mathbf{b}^\vee_+}) = D_+ e^{-\varepsilon_r} \cdot e^*_r + D_- \cdot e^*_{-r}, \quad \mathrm{Res}^\sigma_{V^*}(f_{\mathbf{b}^\vee_-}) = D_- e^{-\varepsilon_r} \cdot e^*_r + D_+ \cdot e^*_{-r} \quad \in \boldsymbol{J}_T(V^*).$$

Combining all above, we deduce that

$$\mathfrak{M} = \begin{pmatrix} \langle f_{\mathbf{b}_+}, f_{\mathbf{b}^\vee_+}\rangle & \langle f_{\mathbf{b}_+}, f_{\mathbf{b}^\vee_-}\rangle \\ \langle f_{\mathbf{b}_-}, f_{\mathbf{b}^\vee_+}\rangle & \langle f_{\mathbf{b}_-}, f_{\mathbf{b}^\vee_-}\rangle \end{pmatrix} = \begin{pmatrix} C_+D_+ + C_-D_- & C_+D_- + C_-D_+ \\ C_+D_- + C_-D_+ & C_+D_+ + C_-D_- \end{pmatrix}.$$

Now the formula (6.4.3) follows from the following identity

$$C_+D_+ + C_-D_- = \sum_{i \geq 0 \text{ even}} 2^{r-1-i}\mathfrak{S}_i, \quad C_+D_- + C_-D_+ = \sum_{i \geq 0 \text{ odd}} 2^{r-1-i}\mathfrak{S}_i.$$

It is easier to compute the determinant of $\mathfrak{M}$ in the following way:

$$\det \mathfrak{M} = (C_+D_+ + C_-D_-)^2 - (C_+D_- + C_-D_+)^2 = (C_+ + C_-)(D_+ + D_-)(C_+ - C_-)(D_+ - D_-).$$

On the other hand, it is easy to see that

$$C_+ \pm C_- = e^{-\nu_+} \prod_{i=1}^{r-1}(1 \pm e^{\varepsilon_i}), \quad D_+ \pm D_- = e^{\nu_+} \prod_{i=1}^{r-1}(1 \pm e^{-\varepsilon_i}).$$

So

$$\det \mathfrak{M} = \prod_{i=1}^{m-1} \big((1 - e^{2\varepsilon_i})(1 - e^{-2\varepsilon_i})\big). \qquad \square$$

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Appendix A. A remark on freeness

Stephen Donkin[7]

Let G be a semisimple, simply connected algebraic group over an algebraically closed field k. Then conjugation gives rise to a G-module structure on the coordinate algebra $k[G]$ and the algebra of class functions $C(G)$ is the invariant algebra. Let V be a finite dimensional rational G-module which admits a good filtration. In Theorem 1.0.1 it is shown that the space of invariants $(k[G]\otimes V)^G$ is free as a $C(G)$-module.

We here obtain this as a special case of Theorem A.2.2 below. However, we would like to stress that this does not cover the freeness of $\boldsymbol{J}_0(V)$ and $\boldsymbol{J}_+(V)$ established in Theorem 1.0.1, nor does it cover the cases in which the action on the coordinate algebra is twisted via an automorphism, considered by the authors.

In [**3**] we considered the situation in which G acts rationally on a finitely generated k-algebra A, see [**3**], 1.5 Theorem and deduced freeness over the algebra of invariants, in some cases, in particular in the case $A = k[G]$ with the conjugation action. The argument we give below is essentially a re-run of the argument from [**3**], but where we now consider invariants of a finitely generated A-module and rational G-module. Our argument relies on flatness and we should mention that in the cases considered in § 2–§ 4 a more constructive approach is taken and flatness is obtained as a corollary of freeness.

A.1. Some General Recollections. By a G-algebra we mean a commutative k-algebra on which G acts rationally as k-algebra automorphisms. We will denote the action of $g \in G$-algebra on $a \in A$ by $g \cdot a$. Let A be a G-algebra. By an (A, G)-module we mean a k-space M which is an A-module and a rational G-module in such a way that $g(am) = (g \cdot a)gm$, for all $g \in G$, $a \in A$, $m \in M$. The space of invariants of a G-module M will be denoted M^G and C denotes A^G.

We here modify the arguments of [**3**], pp139–140, to give a generalization to (A, G)-modules of the main result of [**3**]. We begin by describing some properties of (A, G)-modules. These are no doubt well known but we include them for the convenience of the reader.

Recall that, from the Mumford Conjecture (proved by Haboush in [**5**]) if A is finitely generated then so is C. Moreover, if $\pi : A \to B$ is a surjection of G-algebras then for $b \in B^G$ we have $b^n = \pi(a)$ for some $a \in A$ and positive integer n (see [**7**], p54). In particular B^G is integral over $\pi(A^G)$ and so if A is finitely generated then B^G is a finitely generated A^G-module.

Now suppose that A is finitely generated and that M is a finitely generated (A, G)-module. We claim that M^G is finitely generated as a $C = A^G$-module. If this holds we say that M has the finite generation property. Note that a submodule of an (A, G)-module with the finite generation property has the finite generation property. Note also that an extension of (A, G)-modules with this property also has this property: Suppose $0 \to X \to Y \to Z \to 0$ is a short exact sequence of (A, G)-modules and X, Z have the property. Then we have a short exact sequence of C-modules

$$0 \to X^G \to Y^G \to (X + Y^G)/X \to 0.$$

Now $X + Y^G/X$ embeds in $(Y/X)^G$ and so is finitely generated as a C-module. Thus X^G and $(X + Y^G)/X$ are finitely generated C-module so that Y^G is finitely generated too.

We consider the case in which M is generated by a non-zero invariant $m_0 \in M^G$. Thus $M = Am_0$ and M is isomorphic to A/I, where I is the annihilator of m_0. We have the natural map $\pi : A \to B = A/I$ and by the above B^G is integral over $B_0 = \pi(A^G)$. Hence B^G is finitely generated as a C-module and M has the finite generation property.

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Now consider the general case. Let M be a finitely generated (A, G)-module and let X be a submodule maximal subject to the condition that it has the finite generation property. If $(M/X)^G = 0$ then $M^G = X^G$ so that M has the required property. Otherwise we choose $m_0 \in M\backslash X$ with $(m_0 + X) \in (M/X)^G$. Put $Y = X + Am_0$. Then Y/X is generated by an invariant so has the property. Hence X and Y/X have the finite generation property and therefore so has Y. But X is strictly contained in Y so we have a contradiction.

Thus we have the following.

(1) *If A is a finitely generated G-algebra and M is a finitely generated (A, G)-module then M^G is a finitely generated A^G-module.*

We need to improve (1) so that we can take invariants for the action of a subgroup of G. Now suppose that H is a Grosshans subgroup of G, i.e., a closed subgroup such that the the algebra

$$k[H\backslash G] = \{f \in k[G] | f(hg) = f(g) \text{ for all } h \in H, g \in G\}$$

is finitely generated. (In fact we only in which need the case H is a maximal unipotent subgroup of G.). Suppose that A is a finitely generated G-algebra and M is a finitely generated (A, G)-module. We shall need that M^H is finitely generated as an A^H-module. By Frobenius reciprocity and the tensor identity one has $M^H = (M \otimes k[H\backslash G])^G$ as k-spaces and $(A \otimes k[H\backslash G])^G = A^H$ so the argument should be to regard $(M \otimes k[H\backslash G])$ as an $A \otimes k[H\backslash G]$-module and take invariants.

To see that this really works we write down explicitly the maps involved in identifying H-invariants and G-invariants. This is a slight extension of the context of [**8**], Theorem 4.

Let $\sigma : k[G] \to k[G]$ be the antipode, so $\sigma(f)(x) = f(x^{-1})$, for $f \in k[G]$, $x \in G$. Let A be a G-algebra and let M be an (A, G)-module. We choose a k-basis $(m_i)_{i\in I}$ of M. Let $f_{ij} \in k[G]$ be the corresponding coefficient functions so that

$$gm_i = \sum_{j\in I} f_{ji}(g)m_j$$

for $g \in G$, $i \in I$.

It is easy to check that there are inverse k-linear isomorphisms $\phi_M : (M \otimes k[H\backslash G])^G \to M^H$ and $\psi_M : M^H \to (M \otimes k[H\backslash G])^G$ satisfying

$$\phi_M(\sum_{i\in I} m_i \otimes b_i) = \sum_{i\in I} b_i(1)m_i$$

for $\sum_{i\in I} m_i \otimes b_i \in (M \otimes k[H\backslash G])^G$ and

$$\psi_M(\sum_{\in I} \lambda_i m_i) = \sum_{i,j\in I} \lambda_i m_j \otimes \sigma(f_{ji})$$

for $\sum_{i\in I} \lambda_i m_i \in M^H$.

For $a \in A^H$ and $m \in M^H$ one checks that $\psi_M(am) = \psi_A(a)\psi_M(m)$. We regard $M \otimes k[H\backslash G]$ as an $(A \otimes k[H\backslash G], G)$-module in the natural way. By (1), $(M \otimes k[H\backslash G])^G$ is a finitely generated $(A \otimes k[H\backslash G])^G$-module, i.e.,

$$(M \otimes k[H\backslash G])^G = \sum_{i=1}^{n} (A \otimes k[H\backslash G])^G y_i$$

for some $y_1, \ldots, y_n \in (M \otimes k[H\backslash G])^G$. Let $x_i \in M^H$ be such that $\psi_M(x_i) = y_i$ for $1 \le i \le n$ and put $D = \sum_{i=1}^{n} A^H x_i \le M^H$. Then

$$\psi_M(D) = \sum_{i=1}^{n} \psi_A(A^H)\psi_M(x_i) = \sum_{i=1}^{n} (A \otimes k[H\backslash G])^G y_i = (M \otimes k[H\backslash G])^G$$

and so $D = M^H$ and M^H is finitely generated.

To summarise, we have the following.

(2) *If A is a finitely generated G-algebra, H is a Grosshans subgroup of G and M is a finitely generated (A, G)-module then M^H is a finitely generated A^H-module.*

A.2. Freeness. We are now in a position to extend the main result of [**3**] to the context of (A, G)-modules. We adopt the notation of [**6**]. In particular we have the maximal torus T and Weyl group W. We attach a root system to G, with respect to T, and let B denote the negative Borel subgroup. We have the character group $X(T)$ with natural partial order $\leq$ and set of dominant weight $X^+(T)$. For $\lambda \in X^+(T)$ we write $L(\lambda)$ for the irreducible rational G-module with highest weight λ, write k_λ for the one dimensional B-module on which T acts via λ and write $\nabla(\lambda)$ for the induced module $\mathrm{ind}_B^G k_\lambda$. A subset π of $X^+(T)$ is said to be saturated if whenever $\lambda \in \pi$ and $\mu \in X^+(T)$ with $\mu \leq \lambda$ then $\mu \in \pi$. For π a saturated subset of $X^+(T)$ and V a rational G-module the set of submodules of V which have all composition factors belonging to $\{L(\lambda) | \lambda \in \pi\}$ has a unique maximal element, which we denote $O_\pi(V)$.

Theorem 1.5 in [**3**], which we now extend to the context of (A, G)-modules, is obtained via three Propositions. The first, Proposition 1.2. and the third, Proposition 1.4, require no modification.

The modified Proposition 1.3 and its proof require only minimal changes, given the remarks on finite generation above, but we give it again for the sake of completeness, in the form of the following lemma.

LEMMA A.2.1. *Let A be a finitely generated G-algebra and let $C = A^G$. Suppose that M is a finitely generated (A, G)-module and that M has a good filtration. Then, for every finite saturated subset π of $X^+(T)$, the C-module $O_\pi(M)$ is a finitely generated.*

PROOF. Let λ be a maximal element of π and let $\pi' = \pi \backslash \{\lambda\}$. Then $O_\pi(M)/O_{\pi'}(M) \cong C \otimes O_\pi(M)^\lambda$, by [**3**], 1.2 Proposition. By induction on the size of π it therefore suffices to prove that $O_\pi(M)^\lambda$ is a finitely generated C-module. Moreover, multiplication by a coset representative of w_0 (the longest element of the Weyl group W) induces an isomorphism $O_\pi(M)^\lambda \to O_\pi(M)^{w_0\lambda}$, so it suffices to show that $O_\pi(M)^{w_0\lambda}$ is finitely generated.

Let $M_0 = M^U$ and $A_0 = A^U$. Since $w_0\lambda$ is a lowest weight of $O_\pi(M)$, we have $O_\pi(M)^{w_0\lambda} \leq M_0^{w_0\lambda}$. On the other hand $\bar{M}^{w_0\lambda} = 0$, by [**2**], (12.1.6) and (1.5.2), where where $\bar{M} = (M/O_\pi(M))^U$, so that $O_\pi(M)^{w_0\lambda} = M_0^{w_0\lambda}$. Furthermore, M_0 is a T-module and $A_0^T = A^B = A^G$, by [**1**], (2.1) Theorem. Moreover, U is a Grosshans subgroup, e.g., by [**4**], so that A_0 is finitely generated and, by (2), M_0 is a finitely generated A_0-module. Hence we may (and do) replace M by M^U and G by T. So it suffices to prove that if A is finitely generated T-algebra and M is a finitely generated (A, T)-module then for any $\mu \in X(T)$ the weight space M^μ is a finitely generated A^T-module. We choose a finite dimensional T-invariant subspace of M which generates M, as a (A, T)-module. Then we have a surjective (A, T)-module map $A \otimes V \to M$ inducing a surjection on invariants (by complete reducibility of T-modules). Hence we may assume $M = A \otimes V$ and, by complete reducibility of T-modules again, that V is one dimensional, with weight τ say. Then $(A \otimes V)^\mu$ is isomorphic to $A^{\mu-\tau}$. Hence it suffices to prove that A^λ is a finitely generated A^T-module, for $\lambda \in X(T)$. So let $\chi = -\lambda$. Then $A \otimes k[\chi]$ is a finitely generated T-algebra and so $(A \otimes k[\chi])^T$ is finitely generated by $a_i \otimes \chi^{d_i}$ say, for $1 \leq i \leq n$, $d_i \geq 0$. Then A^λ is generated as a A^T-module by $\{a_i : 1 \leq i \leq n \text{ and } d_i = 1\}$. □

Now the proofs of [**3**] 1.5 Theorem and its Corollary go through in the context of (A, G)-modules with Lemma A.2.1 replacing [**3**], Proposition 1.3, and we obtain the following.

THEOREM A.2.2. *Suppose that A is a finitely generated G-algebra whose algebra of invariants $C = A^G$ is a free polynomial algebra. Suppose that M is a finitely generated (A, G)-module such that M is flat as an A-module and has a good filtration (as a G-module).*

Let π be any finite saturated subset π of $X^+(T)$, let λ be a maximal element and put $\pi' = \pi \backslash \{\lambda\}$. Then, as a (C, G)-module, $O_\pi(M)/O_{\pi'}(M)$ is isomorphic to $E \otimes C$, where E is isomorphic to a direct sum of finitely many copies of $\nabla(\lambda)$.

Given a C-module N and a k-space of G-module E we write $|E| \otimes N$ for the vector space $E \otimes N$ viewed as a C-module with action $c(e \otimes n) = e \otimes cn$, for $c \in C$, $e \in E$, $n \in N$.

For a finite dimensional G-module E admitting a good filtration and $\lambda \in X^+(T)$, we write $(E : \nabla(\lambda))$ for the multiplicity of $\nabla(\lambda)$ as a section in a good filtration of E.

COROLLARY A.2.3. *Under the hypotheses of the Theorem, M has an ascending (C, G)-module filtration $0 = M_0, M_1, \ldots$, where $M_i/M_{i-1} \cong |E_i| \otimes C$, E_i is a finite direct sum of copies of $\nabla(\lambda_i)$ $(i \geq 1)$ and $\lambda_1, \lambda_2, \ldots$ is a labelling of the elements of $X^+(T)$ such that $i < j$ whenever $\lambda_i < \lambda_j$. For a given labelling the multiplicity $(E_i : \nabla(\lambda_i))$ is independent of the choice of such a filtration.*

In particular M is a free C-module.

We specialize further to the situation of this paper, Theorem 1.0.1.

COROLLARY A.2.4. *Regard $k[G]$ as a G-module via the conjugating action and let V be a finite dimensional rational G-module which admits a good filtration. Then $(k[G] \otimes V)^G$ is free over $C(G)$.*

Note that $C(G)$ is a free polynomial algebra by [**10**], 6.1 Theorem. Moreover, $k[G] \otimes V$ is free, and hence flat over $k[G]$ and $k[G]$ is flat over $C(G)$, by [**9**], Proposition 2.3, so that $k[G] \otimes V$ is flat over $C(G)$. We take $A = k[G]$, considered as a G-algebra via the conjugating representation and $M = k[G] \otimes V$ and $\pi = \{0\}$. Then M has a good filtration, by [**6**], II, 4.20 Proposition and 4.21 Proposition so we may apply Corollary A.2.3 .

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