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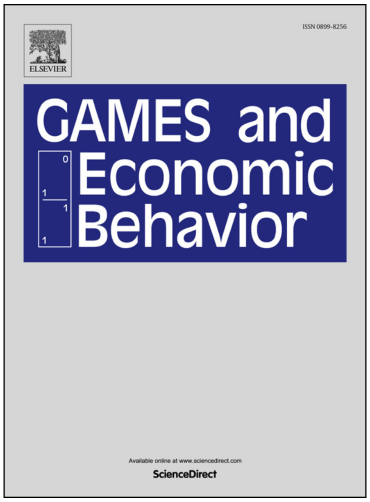

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THE DUALITY GAP FOR TWO-TEAM ZERO-SUM GAMES

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Abstract. We consider multiplayer games in which the players fall in two teams of size k, with payoffs equal within, and of opposite sign across, the two teams. In the classical case of $k = 1$, such zero-sum games possess a unique value, independent of order of play. However, this fails for all $k > 1$; we can measure this failure by a duality gap, which quantifies the benefit of being the team to commit last to its strategy. We show that the gap equals $2(1 - 2^{1-k})$ for $m = 2$ and $2(1 - m^{-(1-o(1))k})$ for $m > 2$, with m being the size of the action space of each player. Extensions hold also for different-size teams and players with various-size action spaces.

We further study the effect of exchanging order of commitment among individual players (not only among the entire teams).

The class of two-team zero-sum games is motivated from the weak selection model of evolution, and from considering teams such as firms in which independent players (ideally) have shared utility.

JEL code: C72 Noncooperative Games

1. Introduction

Games between *teams* of players are ubiquitous; in the economy this occurs most prominently in competition between firms. Another case of significance is that in which a team is a biological species and the players on the team are the genes of the species. What makes a set of players a team, in our idealization, is that in any outcome the players in the set receive identical payoffs. This is consistent with existing terminology in economics [8, 9].[1]

Competition among firms or species need not be zero-sum; however, the zero-sum case will be the focus of this paper, being the most basic form of competition, and often an approximation to reality. Specifically, a two-team zero-sum game is a multiplayer game in which the players are partitioned into two sets A and B, and a real-valued payoff tensor (of dimension equal to the number of players) specifies the payoff conditional on player actions; this payoff accrues positively to each player of Team B and negatively to each player of Team A.

If perfect coordination within each team can be achieved, then a zero-sum interaction between two teams is nothing but a zero-sum interaction between two "virtual" players. In the biological setting, the opposite extreme is relevant: an important model in evolutionary theory is the *weak selection* model (see [11, 1, 13, 12, 17, 18, 4, 10]), in which a species is a team, the genes are the players, the alleles of the gene are the possible actions of a player, and the allele frequencies are independent across genes. Likewise, the difficulty of coordination has long been treated in the economic literature as one of the forces limiting the size of firms [5].

[1] Marschak, 1955: "We define a team as a group of persons each of whom takes decisions about something different but who receive a common reward as the joint result of all those decisions."

This raises a natural question: does von Neumann's minimax theorem for zero-sum games continue to hold for a zero-sum team game where the individual members of the team play independently, and if not, by how much it is violated? Formally, such violation is expressed through a *duality gap* between the values of the game (always expressed as the payoff to Team B) under two scenarios: when all members of Team A must first commit to their randomized strategies, and then Team B gets to respond; and when all members of Team B must commit, and then Team A gets to respond.

Our work in Section 3 is the first quantification of the range of the duality gap. For two teams each of size k, we determine, for action spaces of size 2, the exact range; for action spaces of any size $m > 2$ we determine, as a function of k, the asymptotics of the range.

The key lemma in the lower bound on the duality gap for $m > 2$ may be of independent interest: fix a random set S of $g(m)$ m-ary strings of any length k. Then with high probability, any product distribution may place probability more than $m^{-k(1-o(1))}$ on at most $O(\log g(m))$ strings in S. For comparison, m^{-k} is the probability assigned by the uniform distribution.

In Section 4 we go on to investigate how the value of a game can be affected by more incremental changes, specifically, by exchanging the order of two players of opposing teams who were committing to their randomized strategies in immediate succession. There are games with duality gap bounded away from 0, in which all these value changes tend to 0 in k; whereas there are other games, including games symmetric within each team, in which the largest such value change is bounded away from 0 as a function of k.

The duality gap is the sum of two other quantities, the *defensive gaps* of the two teams. The defensive gap of Team B is the difference between the payoff to Team B if Team A must play a product strategy, and the payoff to Team B if Team A can use a joint source of randomness (hidden of course from B). Symmetrically we have the defensive gap of Team A. The defensive gap quantifies the reduction in effectiveness of a team when its members are unable to use a joint source of randomness.

Two decades ago von Stengel and Koller studied the special case of a single player A playing against a team B [16]. They focused on what they called team-maxmin strategies of Team B: product strategies which maximize the minimum payoff to B over responses of A. Their main result was that any team-maxmin strategy can be completed to a Nash equilibrium of the game by a suitable response distribution for A. To show that this was a novel prediction (and not merely about two-player zero-sum games) they gave an example of a game with two players on Team B in which the defensive gap of Team B is positive.

The defensive gap may be compared to two other notions in the literature. Assume Team A goes first, and think just of the multiplayer game being played by the k players of Team A. (Since Team B can respond optimally, even deterministically, once the strategies of Team A have been fixed, we may ignore the players of Team B and consider their response merely part of the definition of the game being played by Team A.) Then, since the payoffs to all players in Team A are identical, the defensive gap is the difference between the value of the best correlated equilibrium [3] and the best Nash equilibrium [15]. It quantifies, if you will, the penalty for not coordinating. Again, since the payoffs are identical within the team, social welfare agrees with individual welfare, and so the defensive gap is, conceptually, a Price of Stability [14] (although that "price" is normally defined as a ratio and not a difference).

2. Preliminaries

We consider two-team zero-sum games in which Team A has k players each with m choices in its action space; likewise for Team B. The payoff (to Team B) is specified by a tensor T in $(\mathbb{R}^m)^{\otimes 2k}$. (In the biological setting, each of A and B is a species with a genome of k genes, each taking on one of m possible alleles.) More formally to each player A_n, $n = 1, \ldots, k$ of Team A corresponds a vector space $U_n \cong \mathbb{R}^m$, and to each player B_n, $n = 1, \ldots, k$ of Team B, a vector space $V_n \cong \mathbb{R}^m$. Space U_n is spanned by a basis which we denote $u_{n,0}, \ldots, u_{n,m-1}$. Similarly for V_n. In this setting $T \in U_1^* \otimes \ldots \otimes U_k^* \otimes V_1^* \otimes \ldots \otimes V_k^*$. (With $*$ denoting dualization.) Letting $\boldsymbol{I} = (i_1, \ldots, i_k) \in \{0, \ldots, m-1\}^k$ represent an action of the players of Team A, and $\boldsymbol{J} = (j_1, \ldots, j_k) \in \{0, \ldots, m-1\}^k$ an action of the players of Team B, $T_{\boldsymbol{I}}^{\boldsymbol{J}}$ is the payoff to players of Team B (and minus the payoff to players of Team A).

In the case $m = 2$, the strategy of player A_n is specified by a parameter $0 \le p_n \le 1$ which is the probability with which he plays choice 0, i.e., vector $u_{n,0}$. Likewise player B_n has a parameter $0 \le q_n \le 1$ which is the probability with which he plays choice 0, i.e., vector $v_{n,0}$. For $m > 2$, the strategy of player A_n is specified by a probability distribution $p_n = (p_{n,0}, \ldots, p_{n,m-1})$ and the strategy of player B_n is specified by a probability distribution $q_n = (q_{n,0}, \ldots, q_{n,m-1})$. (Thus for $m = 2$, p_n is shorthand for $(p_n, 1-p_n) = (p_{n,0}, p_{n,1})$.)

We let T_p^q denote the expected payoff to Team B when the players of Team A use distributions p_n and those of Team B use distributions q_j. This notation generalizes the notation $T_{\boldsymbol{I}}^{\boldsymbol{J}}$, if one interprets $\boldsymbol{I}$ as the probability distribution on $\{0, \ldots, m-1\}^k$ supported solely on $\boldsymbol{I}$ (and similarly for $\boldsymbol{J}$). Equivalently, T_p^q equals the scalar given by contracting T with the tensor product of the vectors $(p_{n,0}, \ldots, p_{n,m-1})$ (ranging over n) and $(q_{n,0}, \ldots, q_{n,m-1})$ (ranging over j).

By a standard argument, $\min_p \max_q T_p^q \ge \max_q \min_p T_p^q$. (We write everywhere min or max rather than inf or sup since the spaces are compact.) However, apart from the linear ($k = 1$) case, the gap $\min_p \max_q T_p^q - \max_q \min_p T_p^q$ can be positive.

Our purpose is to quantify this gap relative to the uniform norm $\|T\|_\infty = \max_{\boldsymbol{I},\boldsymbol{J}} |T_{\boldsymbol{I}}^{\boldsymbol{J}}|$. We define the duality gap of tensor T:

$$\operatorname{gap}(T) = \frac{\min_p \max_q T_p^q - \max_q \min_p T_p^q}{\|T\|_\infty} = \frac{\min_p \max_{\boldsymbol{J}} T_p^{\boldsymbol{J}} - \max_q \min_{\boldsymbol{I}} T_{\boldsymbol{I}}^q}{\|T\|_\infty} \tag{2.1}$$

where as above, $\boldsymbol{I}$ or $\boldsymbol{J}$ represent the pure strategy choosing that action.

Let $\mathrm{Team}_{m,k}$ denote the collection of two-team games with teams of size k and action spaces of size m. The principal quantity of interest is

$$\operatorname{gap}_{m,k} = \max_{T \in \mathrm{Team}_{m,k}} \operatorname{gap}(T) \tag{2.2}$$

It is trivial that $\operatorname{gap}_{m,k} \le 2$. Moreover $\operatorname{gap}_{m,k}$ is nondecreasing in k (because one may ignore the actions of players after the k'th player on each team), and in m (because one may map all actions $\ge m-1$ to action $m-1$).

Here and throughout the paper, upper-case P and Q denote mixed strategies of virtual players; that is to say, each is a general probability tensor (a tensor with nonnegative entries summing to 1), $P \in U_1 \otimes \ldots \otimes U_k$ and $Q \in V_1 \otimes \ldots \otimes V_k$. Lower-case p and q denote product distributions, i.e., rank-one probability tensors.

Extending the existing notation, T_P^Q is the expected payoff to B when A (as a virtual player) uses distribution P and B uses distribution Q. It is also useful to employ the standard convention that a repeated index indicates tensor contraction over that index, so $P^{\boldsymbol{I}} T_{\boldsymbol{I}}^{\boldsymbol{J}} = T_P^{\boldsymbol{J}} \in V_1^* \otimes \ldots \otimes V_k^*$ and $Q_{\boldsymbol{J}} T_{\boldsymbol{I}}^{\boldsymbol{J}} = T_{\boldsymbol{I}}^Q \in U_1^* \otimes \ldots \otimes U_k^*$.

By strong LP duality we can define the value of the virtual player game by

$$\operatorname{Val} T = \min_P \max_{\boldsymbol{J}} \{P^{\boldsymbol{I}} T_{\boldsymbol{I}}^{\boldsymbol{J}}\} = \max_Q \min_{\boldsymbol{I}} \{Q_{\boldsymbol{J}} T_{\boldsymbol{I}}^{\boldsymbol{J}}\}. \tag{2.3}$$

Let P and Q be strategies achieving equality in (2.3). We can usefully refine the study of $\operatorname{gap}(T)$ by defining the *defensive gap* of Team A in tensor T as

$$\operatorname{gap}_A(T) = \frac{\min_p \max_{\boldsymbol{J}} \{p^{\boldsymbol{I}} T_{\boldsymbol{I}}^{\boldsymbol{J}}\} - \max_{\boldsymbol{J}} \{P^{\boldsymbol{I}} T_{\boldsymbol{I}}^{\boldsymbol{J}}\}}{\|T\|_\infty} = \frac{\min_p \max_{\boldsymbol{J}} \{p^{\boldsymbol{I}} T_{\boldsymbol{I}}^{\boldsymbol{J}}\} - \operatorname{Val} T}{\|T\|_\infty}$$

where, of course, p ranges over product distributions. Likewise the defensive gap of Team B is

$$\operatorname{gap}_B(T) = \frac{\min_{\boldsymbol{I}} \{Q_{\boldsymbol{J}} T_{\boldsymbol{I}}^{\boldsymbol{J}}\} - \max_q \min_{\boldsymbol{I}} \{q_{\boldsymbol{J}} T_{\boldsymbol{I}}^{\boldsymbol{J}}\}}{\|T\|_\infty} = \frac{\operatorname{Val} T - \max_q \min_{\boldsymbol{I}} \{q_{\boldsymbol{J}} T_{\boldsymbol{I}}^{\boldsymbol{J}}\}}{\|T\|_\infty}$$

The defensive gap quantifies the reduced effectiveness of a team of players (when forced to commit to a mixed strategy to which the other team has a chance to respond), as compared with a virtual player (in the same situation).

3. The Defensive Gaps and the Duality Gap

In case the two teams have sizes k_A, k_B, and the players of the two teams have action spaces of various sizes, let $(m_{A,1}, \ldots, m_{A,k})$ be the numbers of actions available to the respective players of team A and $(m_{B,1}, \ldots, m_{B,k})$ the numbers of actions available to the respective players of team B. Without loss of generality every $m_{A,n}, m_{B,n} \geq 2$. Let $k = \min\{k_A, k_B\}$. Let $\hat{m}_A, \hat{m}_B$ be the geometric means $\hat{m}_A = \prod_n m_{A,n}^{1/k_A}$ and $\hat{m}_B = \prod_n m_{B,n}^{1/k_B}$; let $\overline{m}_A = \max m_{A,n}$ and $\overline{m}_B = \max m_{B,n}$. (Naturally, when all action spaces are the same size, $m = \hat{m}_A = \overline{m}_A = \hat{m}_B = \overline{m}_B$.) Let $\mathrm{Team}_{\hat{m}_A, \overline{m}_A, \hat{m}_B, \overline{m}_B, k}$ be the set of two-team games consistent with the indicated parameters and let $\operatorname{gap}_{\hat{m}_A, \overline{m}_A, \hat{m}_B, \overline{m}_B, k}$ denote the maximum duality gap for games in $\mathrm{Team}_{\hat{m}_A, \overline{m}_A, \hat{m}_B, \overline{m}_B, k}$.

Henceforth scale any $T \neq 0$ so that $\|T\|_\infty = 1$.

The implicit variable in all "$o(1)$" is k, thus, $\varepsilon(k) \in o(1)$ means that $\lim_{k \to \infty} \varepsilon(k) = 0$.

Theorem 1. $\operatorname{gap}_{2,k} = 2(1 - 2^{1-k})$, *and for every* $m > 2$, $2(1 - m^{-(1-o(1))k}) \leq \operatorname{gap}_{m,k} \leq 2(1 - m^{1-k})$. *More specifically:*

(1) *Upper bounds:*
$\operatorname{gap}_A(T) \leq (1 - \operatorname{Val} T)(1 - \hat{m}_A^{1-k_A})$ *and* $\operatorname{gap}_B(T) \leq (1 + \operatorname{Val} T)(1 - \hat{m}_B^{1-k_B})$.

(2) *Lower bounds:*
   (a) $\operatorname{gap}_{2,k} \geq 2(1 - 2^{1-k})$.
   (b) *For every* $m > 2$ *there is a function* $\varepsilon(k) \in o(1)$ *such that* $\operatorname{gap}_{m,k} \geq 2(1 - m^{-(1-\varepsilon(k))k})$.
   (c) *For every* $1 < \hat{m}_A \leq \overline{m}_A$ *and* $1 < \hat{m}_B \leq \overline{m}_B$ *there is a function* $\varepsilon(k) \in o(1)$ *such that* $\operatorname{gap}_{\hat{m}_A, \overline{m}_A, \hat{m}_B, \overline{m}_B, k} \geq 2 - \hat{m}_A^{-(1-\varepsilon(k))k_A} - \hat{m}_B^{-(1-\varepsilon(k))k_B}$.

Proof.

Part (1): Upper Bounds on the Defensive and Duality Gaps.

It suffices to show the claim for gap_A. The claim for gap_B follows by negating all entries of T, reversing the roles of the teams and applying the claim for gap_A. Recall also that $\mathrm{gap}(T) = \mathrm{gap}_A(T) + \mathrm{gap}_B(T)$; if $\hat{m}_A = \hat{m}_B$ and $k_A = k_B$ then this means that $\mathrm{gap}(T) \leq 2(1 - \hat{m}_A^{1-k})$.

Given an arbitrary coordinated mixed strategy P for team A, we wish to convert it to a rank-one probability tensor (a product distribution) that does no worse than the claimed defensive gap. The natural candidate would be by independent random variables having the same marginals as P. That is, for any player $1 \leq n \leq k_A$ set p_n to be the marginal distribution of P at player n, specifically, for any action $0 \leq i \leq m_{A,n} - 1$

$$p_n(i) = \sum_{\boldsymbol{I} \text{ such that } i_n = i} P^{\boldsymbol{I}}$$

and, letting

$$p = (p_1(0), \ldots, p_1(m_{A,1} - 1)) \otimes \ldots \otimes (p_{k_A}(0), \ldots, p_{k_A}(m_{A,k_A} - 1)), \tag{3.1}$$

use p as the rank-one strategy replacing P. But it turns out that this approach cannot be used to prove any bound on the defensive gap (see Appendix B).

Surprisingly, there is a less obvious rank-one strategy which can be obtained from P, and which provides a tight bound on the defensive gap.

For this purpose we show the existence of a distribution $\beta_n(0), \ldots, \beta_n(m_{A,n} - 1)$ for player n such that for every i, $\beta_n(i)^{k_A} \geq m_{A,n}^{1-k_A} p_n(i)$, which is to say,

$$\beta_n(i) \geq m_{A,\ell}^{1/k_A - 1} p_n(i)^{1/k_A}. \tag{3.2}$$

Such a distribution exists due to the inequality

$$\sum_i \frac{(m_{A,n} p_n(i))^{1/k_A}}{m_{A,n}} \leq 1,$$

which holds because by the power-mean inequality

$$\left(\sum_i \frac{(m_{A,n} p_n(i))^{1/k_A}}{m_{A,n}} \right)^{k_A} \leq \sum_i \frac{m_{A,n} p_n(i)}{m_{A,n}} = 1.$$

We now define the product distribution β which demonstrates the defensive gap by $\beta^{\boldsymbol{I}} = \prod_1^{k_A} \beta_n(i_n)$. We claim that for every $\boldsymbol{I}$, $\beta^{\boldsymbol{I}} \geq \hat{m}_A^{1-k_A} P^{\boldsymbol{I}}$. Note that for every n and i_n, $p_n(i_n) \geq P^{\boldsymbol{I}}$. Applying 3.2 we have

$$\beta^{\boldsymbol{I}} \geq \prod_n m_{A,n}^{1/k_A - 1} p_n(i)^{1/k_A} \geq P^{\boldsymbol{I}} \prod_n m_{A,n}^{1/k_A - 1} = P^{\boldsymbol{I}} \hat{m}_A^{1-k_A}$$

as required. Recalling that $\|T\|_\infty \leq 1$, we have that for every $\boldsymbol{J}$:

$$\beta^{\boldsymbol{I}} T_{\boldsymbol{I}}^{\boldsymbol{J}} \leq 1 - \hat{m}_A^{1-k_A} + \hat{m}_A^{1-k_A} P^{\boldsymbol{I}} T_{\boldsymbol{I}}^{\boldsymbol{J}}$$

which (upper bounding $P^{\boldsymbol{I}} T_{\boldsymbol{I}}^{\boldsymbol{J}}$ by $\mathrm{Val}\, T$, and subtracting $\mathrm{Val}\, T$ from each side) completes the proof of Part (1) of the theorem.

Part (2a): Lower Bound on $\mathrm{gap}_{2,k}$.

Write $\mathbf{0} = (0, \ldots, 0)$ and $\mathbf{1} = (1, \ldots, 1)$.

Consider the following tensor (explicitly written out for the case $k = 2$ in Appendix A).

Example 2.

$$\begin{cases} G^{\mathbf{0}}_{\boldsymbol{I}} &= \begin{cases} -1 & \text{if } \boldsymbol{I} = \mathbf{0} \\ 1 & \text{otherwise} \end{cases} \\ G^{\mathbf{1}}_{\boldsymbol{I}} &= \begin{cases} -1 & \text{if } \boldsymbol{I} = \mathbf{1} \\ 1 & \text{otherwise} \end{cases} \\ G^{\boldsymbol{J}}_{\mathbf{0}} &= \begin{cases} 1 & \text{if } \boldsymbol{J} = \mathbf{1} \\ -1 & \text{otherwise} \end{cases} \\ G^{\boldsymbol{J}}_{\mathbf{1}} &= \begin{cases} 1 & \text{if } \boldsymbol{J} = \mathbf{0} \\ -1 & \text{otherwise} \end{cases} \\ G^{\boldsymbol{J}}_{\boldsymbol{I}} &= 0 \text{ for all other } \boldsymbol{I}, \boldsymbol{J} \end{cases} \tag{3.3}$$

An informal description of this game is that if both Team A and Team B choose actions in $\{\mathbf{0}, \mathbf{1}\}$, then the outcome is as it would be in the "matching pennies" game. If just one of the teams chooses an action in $\{\mathbf{0}, \mathbf{1}\}$, then that team wins. If neither team chooses an action in $\{\mathbf{0}, \mathbf{1}\}$, then the game is a tie.

(We incidentally note that the proof of Part (2a) does not depend on setting entries to 0 in the last line of 3.3; the argument will go through with each entry taking any value in $[-1, 1]$.)

Now consider any strategy $p = (p_1, \ldots, p_k)$ for Team A (recall these are the probabilities of action 0). The expected payoff for action $\boldsymbol{J} = \mathbf{0}$ of Team B is

$$G^{\mathbf{0}}_p = 1 - 2p_1 \cdots p_k$$

The expected payoff for $\boldsymbol{J} = \mathbf{1}$ is

$$G^{\mathbf{1}}_p = 1 - 2(1 - p_1) \cdots (1 - p_k)$$

By the arithmetic-geometric mean inequality, $G^{\mathbf{0}}_p \geq 1 - 2(\frac{1}{k} \sum p_n)^k$ and $G^{\mathbf{1}}_p \geq 1 - 2(\frac{1}{k} \sum (1 - p_n))^k$. So

$$\frac{1 - \max\{G^{\mathbf{0}}_p, G^{\mathbf{1}}_p\}}{2} \leq \min\{(\frac{1}{k} \sum p_n)^k, (\frac{1}{k} \sum (1 - p_n))^k\}$$

equivalently

$$\left(\frac{1 - \max\{G^{\mathbf{0}}_p, G^{\mathbf{1}}_p\}}{2}\right)^{1/k} \leq \min\{\frac{1}{k} \sum p_n, \frac{1}{k} \sum (1 - p_n)\}$$

The RHS is at most $1/2$. So $\min_p \max\{G^{\mathbf{0}}_p, G^{\mathbf{1}}_p\} \geq 1 - 2^{1-k}$.

A similar argument applied to the strategy q of Team B establishes that $\max_q \min\{G^q_{\mathbf{0}}, G^q_{\mathbf{1}}\} \leq -1 + 2^{1-k}$. Adding the two contributions, Part (2a) of the theorem follows.

Part (2b): Lower Bound on $\text{gap}_{m,k}$**,** $m > 2$**.**

Proof: We non-constructively exhibit a game establishing the lower bound. Remember m is fixed while $k \to \infty$. We start by selecting, for a function $g(h(m))$ to be specified, $g(h(m))$ strings $S = (s^1, \ldots, s^{g(h(m))})$, each s^j chosen independently and uniformly in $\{0, \ldots, m-1\}^k$. A key part of the proof is the interesting fact that with high probability, this set (whose size is independent of k) has the property that for any product distribution, fewer than $h(m) \sim \log g(h(m))$ strings in S have probability more than $m^{-k(1-o(1))}$.

Formally, we argue that for a function $h(m)$ that is sufficiently large to satisfy conditions (*), (**) below, and for $g(h) = h^2 2^h$, there exists a function $\varepsilon_1(k)$, tending to 0 as $k \to \infty$, such that w.h.p. over the selection (as described above) of $g(h(m))$ strings S, for every list of player strategies $p_1, \ldots, p_k$ (p_n is a distribution on $\{0, \ldots, m-1\}$ and $p(s) = \prod_1^k p_n(s_n)$), the $h(m)$'th-largest $p(s^\ell)$ is at most $m^{-(1-\varepsilon_1(k))k}$. This is the same as saying that for every $R \subseteq S$, either $|R| < h(m)$ or $\min_{s \in R} p(s) \leq m^{-(1-\varepsilon_1(k))k}$.

To see this, consider selecting $h(m)$ strings $R = \{r^1, \ldots, r^{h(m)}\}$ independently and uniformly in $\{0, \ldots, m-1\}^k$. Denote by $w_n(j)$ the fraction of strings $r \in R$ s.t. $r_n = j$. Then by standard concentration theorems [2], there is a sufficiently large $h(m)$ such that:

(*) There exist functions $\varepsilon_2, \varepsilon_3, \varepsilon_4 \in o(1)$ such that with probability at least $1 - \varepsilon_2(k)$, for all but $\varepsilon_3(k) \cdot k$ coordinates $n \in \{1, \ldots, k\}$, for every $0 \leq j \leq m-1$,

$$\frac{1-\varepsilon_4(k)}{m} \leq w_n(j) \leq \frac{1+\varepsilon_4(k)}{m}. \tag{3.4}$$

Examine the geometric mean $(\prod_\ell p(r^\ell))^{1/h(m)} = \prod_{n=1}^k [\prod_{j=0}^{m-1} p_n(j)^{w_n(j)}]$. Upper bound this by ignoring coordinates n for which there is a j failing either of the inequalities in (3.4) and by noting that given $w_n(0), \ldots, w_n(m-1)$, the distribution p_n which maximizes the product in brackets is $p_n(j) = w_n(j)$. So

$$\begin{aligned}
(\prod_\ell p(r^\ell))^{1/h(m)} &\leq \left(\left(\frac{1+\varepsilon_4(k)}{m}\right)^{1-\varepsilon_4(k)}\right)^{k(1-\varepsilon_3(k))} \\
&\leq \left(\frac{1}{m} e^{\varepsilon_4(k)}\right)^{k(1-\varepsilon_3(k))(1-\varepsilon_4(k))} \\
&\leq m^{-k(1-\varepsilon_3(k)-\varepsilon_4(k))} e^{k\varepsilon_4(k)} \\
&\leq m^{-k(1-\varepsilon_3(k)-2\varepsilon_4(k))}.
\end{aligned}$$

We conclude that with probability at least $1 - \binom{g(h(m))}{h(m)} \varepsilon_2(k)$ over the selection of S, for every product strategy p, the $h(m)$'th-largest $p(s^\ell)$ is at most $m^{-(1-\varepsilon_1(k))k}$, where $\varepsilon_1 = \varepsilon_3 + 2\varepsilon_4$. Fix such a list S.

By a result of Erdös [6] there is an h_0 such that for every $h \geq h_0$, there is a tournament of size $g(h)$ that has no dominating set of size h. (A tournament is a digraph in which there is one directed edge between every pair of distinct vertices. A set of vertices U dominates a vertex j if some edge (u, j), $u \in U$, is present. A set of vertices U in a tournament is dominating if it dominates all vertices outside U.) That is, for every set R of size h, there is some vertex s outside R with edges pointing toward every vertex of R. (As an aside, Erdös's argument is existential but with slight loss in the numbers, tournaments without small dominating sets can be constructed explicitly [7].) Our second condition on $h(m)$ is that

(**) $h(m) \geq h_0$.

The game is as follows. Associate the strings of S with the vertices of the tournament in an arbitrary way. If neither team chooses a string in S, the game is a tie. If one team chooses a string in S and the other does not, the first team wins. If both teams choose strings in S, the winning vertex is that which points toward the other (with a tie for identical strings). In all cases a win means a payoff of 1 and a tie a payoff of 0.

Now we claim that whichever team goes second can achieve payoff $1 - m^{-(1-o(1))k}$. The argument is the same for both teams, so say Team A goes first and let p be its product strategy. Let $R = \{r \in S : p(r) > m^{-(1-\varepsilon_1(k))k}\}$. There is an $s \in S$ such that Team B wins against every $r \in R$. Team B deterministically responds with this s. Team B wins unless Team A selects an $s' \in S - R$ (and sometimes even then). The payoff to Team B is therefore at least $1 - (g(h(m)) - h(m))m^{-(1-\varepsilon_1(k))k} \geq 1 - m^{\log_m g(h(m)) - (1-\varepsilon_1(k))k}$ which, for $\varepsilon = \varepsilon_1 + \frac{1}{k}\log_m g(h(m))$, is $\geq 1 - m^{-(1-\varepsilon(k))k}$.

Part (2b) of the theorem follows.

Part (2c): Lower Bound on $\mathrm{gap}_{\hat{m}_A,\overline{m}_A,\hat{m}_B,\overline{m}_B,k}$**.**

The proof mimicks that of Part (2b), with a few more technicalities. As a first step we need a bipartite version of Erdös's theorem on tournaments. Let $T_{g,g'}$ be the set of bipartite tournaments on vertex sets V with $|V| = g$ and V' with $|V'| = g'$, that is to say directed graphs in which there are no edges within each of these sets, and with exactly one of the edges $(i,j),(j,i)$ present for $i \in V, j \in V'$. We say that U is a dominating set in a bipartite tournament if either $U \subseteq V$ and U dominates all $j \in V'$, or $U \subseteq V'$ and U dominates all $j \in V$.

Lemma 3. *There is an h_1 such that for every $h \geq h_1$ there is a bipartite tournament in $T_{g,g}$ for $g = g(h) = h^2 2^h$, that has no dominating set of size h.*

Proof. The proof follows that of Erdös. Select a bipartite tournament in $T_{g,g}$ u.a.r. Consider any $U \subseteq V$ of size $|U| = h$. For $j \in V'$, the probability that j is dominated by U is $1 - 2^{-h}$. The probability that every $j \in V'$ is dominated by U is $(1-2^{-h})^g$. The probability that there exists a U which dominates every $j \in V'$ is at most $\binom{g}{h}(1-2^{-h})^g$. We double this quantity to allow also for a dominating $U \subseteq V'$. Thus the probability a dominating set exists is $\leq 2\binom{g}{h}(1-2^{-h})^g \leq 2\binom{g}{h}e^{-g2^{-h}} \leq e^{h\log g - g2^{-h}} = e^{h(\log 2)(h+2\lg h)-h^2}$ (we have applied the inequality $1 - x \leq e^{-x}$), which is < 1 for sufficiently large h. Hence there exist in $T_{g,g}$ bipartite tournaments without dominating sets of size h. ☐

Consider selecting some $h(\hat{m}_A,\overline{m}_A)$ strings $R_A = (r_A^1,\ldots,r_A^{h(\hat{m}_A,\overline{m}_A)})$ in $\prod_n\{0,\ldots,m_{A,n}-1\}$ independently and uniformly. Let $w_{A,n}(j)$ be the fraction of strings in R_A whose n'th coordinate equals j.

For any functions $\varepsilon_2,\varepsilon_3,\varepsilon_4 \to 0$, by concentration theorems [2] the function h may be chosen sufficiently rapidly growing that for any coordinate n, $\Pr\left(|w_{A,n}(j) - \frac{1}{m_{A,n}}| > \frac{\varepsilon_4(k)}{m_{A,n}}\right) < \varepsilon_2(k)\varepsilon_3(k)/2$.

If we also select $h(\hat{m}_B,\overline{m}_B)$ strings R_B in $\prod_n\{0,\ldots,m_{B,n}-1\}$ independently and uniformly, and likewise define $w_{B,n}(j)$, then for any coordinate n, $\Pr\left(|w_{B,n}(j) - \frac{1}{m_{B,n}}| > \frac{\varepsilon_4(k)}{m_{B,n}}\right) < \varepsilon_2(k)\varepsilon_3(k)/2$.

In particular, if we select R_A and R_B independently, then with probability at least $1-\varepsilon_2(k)$ the following both hold:

$$|D_A| < \varepsilon_3(k)k_A \quad \text{and} \tag{3.5}$$
$$|D_B| < \varepsilon_3(k)k_B$$

where

$$D_A = \{n : |w_{A,n}(j) - \frac{1}{m_{A,n}}| > \frac{\varepsilon_4(k)}{m_{A,n}}\} \quad \text{and}$$
$$D_B = \{n : |w_{B,n}(j) - \frac{1}{m_{B,n}}| > \frac{\varepsilon_4(k)}{m_{B,n}}\}.$$

Examine the geometric mean $(\prod_\ell p(r_A^\ell))^{1/h(\hat{m}_A)} = \prod_{n=1}^{k_A}[\prod_{j=0}^{m_{A,n}-1} p_n(j)^{w_{A,n}(j)}]$. Conditional on (3.5) holding, we upper bound this mean by ignoring coordinates $n \in D_A$ and by noting that given $w_{A,n}(0), \ldots, w_{A,n}(m_{A,n}-1)$, the distribution p_n which maximizes the product in brackets is $p_n(j) = w_{A,n}(j)$. So

$$\begin{aligned} (\prod_\ell p(r_A^\ell))^{1/h(\hat{m}_A)} &\leq \prod_{n \notin D_A} \left(\frac{1+\varepsilon_4(k)}{m_{A,n}}\right)^{1-\varepsilon_4(k)} \\ &\leq (1+\varepsilon_4(k))^{k_A} \left(\prod_{n \notin D_A} \frac{1}{m_{A,n}}\right)^{1-\varepsilon_4(k)} \end{aligned}$$

Apply $(\prod_{n \notin D_A} \frac{1}{m_{A,n}})/(\prod_n \frac{1}{m_{A,n}}) = \prod_{n \in D_A} m_{A,n} \leq \overline{m}_A^{\varepsilon_3(k)k_A}$ to get

$$\begin{aligned} (\prod_\ell p(r_A^\ell))^{1/h(\hat{m}_A)} &\leq (1+\varepsilon_4(k))^{k_A} \hat{m}_A^{-(1-\varepsilon_4(k))k_A} \overline{m}_A^{\varepsilon_3(k)(1-\varepsilon_4(k))k_A} \\ &= \hat{m}_A^{-(1-\varepsilon_4(k))(1-\frac{\varepsilon_3(k)\log \overline{m}_A}{\log \hat{m}_A} - \frac{\log(1+\varepsilon_4(k))}{(1-\varepsilon_4(k))\log \hat{m}_A})k_A} \\ &= \hat{m}_A^{(1-\varepsilon_1(k))k_A} \qquad (3.6) \end{aligned}$$

for some $\varepsilon_1(k) \in o(1)$.

Now let $h = \max\{(h(\hat{m}_A, \overline{m}_A), (h(\hat{m}_B, \overline{m}_B)\}$. It follows from (3.6) that if we select $g(h)$ strings $S_A = (s_A^1, \ldots, s_A^{g(h)})$ in $\prod_n\{0, \ldots, m_{A,n}-1\}$ independently and uniformly, then with probability at least $1 - \binom{g(h)}{h}\varepsilon_2(k)$ over the selection of S_A, for every product strategy p, the h'th-largest $p(s_A^\ell)$ is at most $\hat{m}_A^{-(1-\varepsilon_1(k))k}$. The same claim holds for selection of a set $S_B = (s_B^1, \ldots, s_B^{g(h)})$ in $\prod_n\{0, \ldots, m_{B,n}-1\}$. Fix such lists S_A, S_B.

Now form a two-team game as follows. Associate S_A and S_B with the vertices of two sides of a bipartite tournament with the dominating-set-free property ensured by Lemma 3. If neither team chooses a string in S_A or S_B, the game is a tie. If Team A chooses a string in S_A and Team B does not choose a string in S_B, Team A wins. Similarly Team B wins if it selects a string in S_B and Team A does not select a string in S_A. If the teams choose strings in S_A and S_B respectively, the winner is determined by the orientation of the tournament edge. In all cases a win means a payoff of 1 and a tie a payoff of 0.

Now we claim that if Team B goes second it can achieve payoff $1 - \hat{m}_A^{-(1-o(1))k_A}$; likewise if Team A goes second it can achieve payoff $1 - \hat{m}_B^{-(1-o(1))k_B}$. The argument is the same for both teams, so say Team A goes first and let p be its product strategy. Let $R_A = \{r \in S_A : p(r) > \hat{m}_A^{-(1-\varepsilon_1(k))k_A}\}$. Since $|R_A| < h$, there is an $s \in S_B$ that dominates every $r \in R_A$. Team B deterministically responds with this s. Team B wins unless Team A selects an $s' \in S_A - R_A$ (and sometimes even then). The payoff to Team B is therefore at least $1 - (g(h) - h)\hat{m}_A^{-(1-\varepsilon_1(k))k_A}$ which, for $\varepsilon = \varepsilon_1 + \frac{1}{k_A}\log_{\hat{m}_A} g(h)$, is $\geq 1 - \hat{m}_A^{-(1-\varepsilon(k))k_A}$.

Part (2c) of the theorem follows. □

4. Order Refinements

It is natural to consider a more general scenario in which players of the two teams commit to their strategies in some (not necessarily strict) alternation. That is to say, let π be any bijection from $\{1, \ldots, 2k\}$ to $\{A_1, \ldots, A_k, B_1, \ldots, B_k\}$. If $\pi(\ell) = A_n$ for some n then let $M(\ell)$ be the quantifier $\min_{p_n}$ (minimization over the distribution p_n); if $\pi(\ell) = B_n$ for some n then let $M(\ell) = \max_{q_n}$. Then the value of game T with respect to order π is defined to be

$$V(T, \pi) = M(1) \ldots M(2k) T_p^q.$$

In particular, let π^{AB} be an order in which all the members of Team A go first, that is, $\pi^{AB}(\ell) = A_\ell$ for $\ell \leq k$, and $\pi^{AB}(\ell) = B_{\ell-k}$ for $\ell > k$. (Note that V is invariant under exchange of same-team players with adjacent quantifiers.) Likewise let π^{BA} be an order in which Team B goes first. Then the duality gap of game T is

$$\operatorname{gap}(T) = V(T, \pi^{AB}) - V(T, \pi^{BA}).$$

We now ask how much V may change when we change π by a single adjacent transposition.

The effect of an adjacent transposition can be large, even if there are many players; this phenomenon occurs for the following uninteresting reason. Consider games with k players on each team, yet which depend on the actions of only two players from each team. This already allows for games with a duality gap bounded away from 0 (specifically, as we saw, 1 in the case $m = 2$), yet the value of the game will change only under the four transpositions which exchange two opposing significant players. Necessarily, one of these increments is at least 1/4.

If we are interested in the possibility, then, that for games with many players, V may change only incrementally under adjacent transposition, then we must restrict the class of games under consideration. A very natural restriction, which eliminates the previous example, is to *symmetric games*, by which we mean games invariant under permutation of the actions taken by members of a team.

It turns out, however, that even for symmetric games, adjacent transpositions can create large jumps in the value of the game. For an example we return to the game G of Example 2. We show that for any k, the order of the first three players can affect the outcome decisively.

Lemma 4. *If $\ell \geq 3$ is the first time that Team B plays in π, then $V(G, \pi) \geq 1/2$. Likewise if $\ell \geq 3$ is the first time that Team A plays in π, then $V(G, \pi) \leq -1/2$.*

Proof. We argue only the case that Team A goes first, the other case following similarly; we further suppose, and due to the symmetries of G this is without loss of generality, that players A_1, A_2 go first. Having selected distributions p_1, p_2, the probability that both actions are 0's is $p_1 p_2$, and the probability that both are 1's is $(1 - p_1)(1 - p_2)$. One of these quantities is at most 1/4. If (a) $p_1 p_2 \leq 1/4$, all players of Team B choose action 0, and if (b) $(1 - p_1)(1 - p_2) < 1/4$, all players of Team B choose action 1. In instance (a), the expected payoff is $E_p^{\mathbf{0}} G \geq (3/4) \cdot 1 + (1/4) \cdot (-1) \geq 1/2$; the same payoff bound follows similarly in instance (b). □

To quantify the effect of this lemma, let $\tau_{\ell,\ell+1}$ be transposition of the ℓ'th and $(\ell+1)$'th quantifiers. Since we can transition in four adjacent transpositions between a permutation which favors Team A by 1/2, and one which favors Team B by 1/2, we have:

Corollary 5. *There exists a permutation π and an ℓ such that $|V(G,\tau_{\ell,\ell+1}\pi) - V(G,\pi)| \geq 1/4$.*

Thus, despite the symmetry of the game, forcing a couple of members of one team to go first is enough to put that team at a significant disadvantage.

In view of the above, it is worthwhile showing that there even *exist* games in which the outcome is affected only incrementally by the order of play. Specifically:

Theorem 6. *There exist symmetric team games with any $k \geq 1$ players per team, with duality gap bounded away from 0 (as a function of k), but in which any adjacent transposition in the order of play changes the value of the game only by $O(1/k)$.*

Proof. We use the following game with $m = 2$.

Example 7. For an action $\boldsymbol{I} \in \{0,1\}^k$ by the players of Team A, and an action $\boldsymbol{J} \in \{0,1\}^k$ by the players of Team B, let $x =$ the number of 1's in $\boldsymbol{I}$, $y =$ the number of 1's in $\boldsymbol{J}$, and define game H by:

$$H_{\boldsymbol{I}}^{\boldsymbol{J}} = \mathcal{H}_x^y = \frac{-4(2x-y-k/2)(x+2y-3k/2)}{9k^2} \tag{4.1}$$

(where $\mathcal{H}$ is a $(k+1)\times(k+1)$ matrix). Observe that $\|H\|_\infty = 1$.

We first show that the duality gap of tensor H is bounded away from 0; this is for reasons similar to, although slightly more complicated than, those of our previous example G. Specifically, if Team A goes first in H, it can pick a distribution p on x with arbitrary mean $0 \leq \mu \leq k$, and a variance σ^2 that (because x is a sum of independent variables with variances $\leq 1/4$) is bounded by $k/4$. Now Team B can choose $y \in \{0,k\}$ so that $|y-\mu| \geq k/2$. Then

$$\begin{aligned}
H_p^y &= -\frac{4}{9k^2}E[(2(x-\mu)+2(\mu-k/2)-(y-k/2))((x-\mu)+(\mu-k/2)+2(y-k/2))] \\
&= -\frac{4}{9k^2}[2\sigma^2 - 2(y-k/2)^2 + 2(\mu-k/2)^2 + 3(\mu-k/2)(y-k/2)] \\
&\geq \frac{4}{9k^2}[-k/2 + k^2/2 + 3(k/2-\mu)(3y+2\mu-5k/2)]
\end{aligned}$$

The rule for selecting y ensures that the last term is nonnegative, so we have $H_p^y \geq \frac{2k(k-1)}{9k^2}$ and consequently, for $k \geq 2$, $H_p^y \geq 1/9$. That is to say, $\min_p \max_q H_p^q \geq 1/9$.

In order to complete this argument, note that under the rotation $x \to k-y, y \to x$, we have

$$-\mathcal{H}_x^y = \mathcal{H}_{k-y}^x.$$

Consequently, any distribution for x that Team A employs to ensure payoff for Team B of at most c can be translated by Team B into a distribution for y that ensures payoff at least $-c$; and any distribution for y that Team B employs to ensure a payoff of at least $-c$, can be translated by Team A into a distribution for x that ensures payoff at most c.

It follows that $\max_q \min_p H_p^q \leq -1/9$. (And incidentally that $\operatorname{Val} H = 0$.) Thus $\operatorname{gap}(H) \geq 2/9$.

We now show that in this game, adjacent transpositions of the operators have only an incremental effect, in the sense discussed earlier. The basic reason is that $\mathcal{H}_x^y$ has small Lipschitz constant (with respect to the coordinates x, y), and therefore $H_{\boldsymbol{I}}^{\boldsymbol{J}}$ has small Lipschitz constant (with respect to Hamming distance h on $\boldsymbol{I}, \boldsymbol{J}$). We employ the following lemma.

Lemma 8. *Let $m = 2$.*

(1) *For any T and S:*

$$|V(T+S,\pi) - V(T,\pi)| \leq \|S\|_\infty.$$

(2) *Let $\lambda \geq 0$ and T be such that if $h(\boldsymbol{I}, \boldsymbol{I}') = 1$ and $h(\boldsymbol{J}, \boldsymbol{J}') = 1$ hold for the tuple $\boldsymbol{I}, \boldsymbol{I}', \boldsymbol{J}, \boldsymbol{J}'$, then $|T_{\boldsymbol{I}}^{\boldsymbol{J}} - T_{\boldsymbol{I}'}^{\boldsymbol{J}}| \leq \lambda$ and $|T_{\boldsymbol{I}}^{\boldsymbol{J}} - T_{\boldsymbol{I}}^{\boldsymbol{J}'}| \leq \lambda$.*
Then

$$|V(T, \tau_{\ell,\ell+1}\pi) - V(T,\pi)| \leq \lambda/2.$$

Proof. (1) We show that Team B can ensure a payoff of at least $V(T,\pi) - \|S\|_\infty$ when playing game $T+S$, simply by pretending that it is playing game T. (The argument in the other direction is identical.) Specifically, upon reaching a quantifier $M(\ell)$ which is controlled by Team B, say $M(\ell) = \max_{q_n}$, player B_n chooses a distribution q_n which is optimal in game T against the distributions of the two teams which have already been fixed for $\ell' < \ell$. The payoff is now $(T+S)_p^q = T_p^q + S_p^q \geq V(T,\pi) + S_p^q \geq V(T,\pi) - \|S\|_\infty$.

(2) We must show that if $\pi(\ell) = A_{n_1}$ and $\pi(\ell+1) = B_{n_2}$, then Team B can ensure a payoff of at least $V(T,\pi) - \lambda$ when playing in the order $\tau_{\ell,\ell+1}\pi$, i.e., when A_{n_1} follows B_{n_2}. For any choice of p_n's, each player $B_{n'}$ has a response $q_{n'}^\pi$ which is a function only of the p_n's that are earlier in the order π (i.e., s.t. $\pi^{-1}(A_n) < \pi^{-1}(B_{n'})$), such that $T_p^{q^\pi} \geq V(T,\pi)$. Note that by observing the preceding p_n's chosen by Team A, a player $B_{n'}$ already implicitly knows how the preceding members of Team B have responded, and so the function $q_{n'}^\pi$ does not need to depend on those distribution choices.

The idea now is that for $n' \neq n_2$, player $B_{n'}$ simply makes the response $q_{n'}^\pi$. Player B_{n_2} chooses the uniform distribution.

The implication is that for n' s.t. $\pi^{-1}(B_{n'}) < \ell$, players $B_{n'}$ are responding optimally (w.r.t. the order π) to Team A's choices. For n' s.t. $\pi^{-1}(B_{n'}) > \ell + 1$, players $B_{n'}$ effectively pretend that player B_{n_2} also responded optimally (although it could not since it did not have available p_{n_1}), and continue to make their own optimal responses to the p_n's.

Eventually, Team A has chosen some distributions p_n, and Team B has chosen the distributions $q_{n'}$ that are optimal responses w.r.t. order of play π, except that q_{n_2} has been modified to be the uniform distribution, for which we write u_{n_2}. So, instead of the product distribution $p_1 \times \ldots \times p_k \times q_1^\pi \times \ldots \times q_k^\pi$, the actual distribution is $p_1 \times \ldots \times p_k \times q_1^\pi \times \ldots u_{n_2} \ldots \times q_k^\pi$. Conditional on any actions by the players other than B_{n_2}, then, the expected change in the payoff created by shifting between the two distributions, is at most $\lambda/2$. □

Now return to the game H. One may verify that for $0 \leq x \leq k$ and $0 \leq y \leq k-1$, $|\mathcal{H}_x^y - \mathcal{H}_x^{y+1}| \leq \frac{4(7k/2-2)}{9k^2}$; likewise for $0 \leq x \leq k-1$ and $0 \leq y \leq k$, $|\mathcal{H}_x^y - \mathcal{H}_{x+1}^y| \leq \frac{4(7k/2-2)}{9k^2}$. Consequently H has Lipschitz constant $\lambda < \frac{14}{9k}$.

Applying Lemma 8(2), we conclude that for any π and ℓ, $|V(H, \tau_{\ell,\ell+1}\pi) - V(H,\pi)| < \frac{7}{9k}$, completing the proof of the theorem. □

5. Discussion

We have characterized the possible range of the duality gap. The examples which achieved large gap were highly structured. It would be interesting to find natural conditions on a game (particularly a symmetric game) that ensure small duality gap.

It would be interesting to extend our inquiry to teams (possibly more than two) competing in non-zero-sum games.

The example of Theorem 6 was constructed specifically in order to demonstrate that there are games with large duality gap whose value is affected only incrementally by the order of play. It would be desirable to identify natural classes of, or good characterizations of, games with this property.

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Appendix A. The lower bound tensor for $m = 2$, $k = 2$

Here is the tensor G for $k = 2$, with Team A (action $\boldsymbol{I}$) controlling the high-order bits and Team B (action $\boldsymbol{J}$) the low-order bits.

$$G = \left(\begin{array}{cc|cc} -1 & -1 & 1 & 0 \\ -1 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & -1 \end{array}\right) \tag{A.1}$$

Incidentally note that for every $\boldsymbol{I}$, there is a $G_{\boldsymbol{I}}^{\boldsymbol{J}} = 1$, while for every $\boldsymbol{J}$, there is a $G_{\boldsymbol{I}}^{\boldsymbol{J}} = -1$; so the pure strategy duality gap of this tensor equals 2, i.e., as bad as the trivial bound.

Appendix B. No defensive gap for the marginal distributions

Here we show that the marginal distribution p in Eq. 3.1 does not provide any nontrivial defensive gap. This is because there is no $\gamma > 0$ for which an upper bound of the form

$$p^{\boldsymbol{I}} T_{\boldsymbol{I}}^{\boldsymbol{J}} \leq 1 - \gamma + \gamma P^{\boldsymbol{I}} T_{\boldsymbol{I}}^{\boldsymbol{J}} \tag{B.1}$$

holds for all T, P.

To see this, we may rewrite (B.1) as

$$(p - P)^{\boldsymbol{I}} T_{\boldsymbol{I}}^{\boldsymbol{J}} \leq (1 - \gamma)(1 - P^{\boldsymbol{I}} T_{\boldsymbol{I}}^{\boldsymbol{J}}) \tag{B.2}$$

Suppose that $T^{\boldsymbol{J}} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $P = \begin{pmatrix} 1-\delta & \\ & \delta \end{pmatrix}$. Then $p = \begin{pmatrix} (1-\delta)^2 & \delta(1-\delta) \\ \delta(1-\delta) & \delta^2 \end{pmatrix}$. So $P^{\boldsymbol{I}} T_{\boldsymbol{I}}^{\boldsymbol{J}} = 1 - 2\delta$ while $p^{\boldsymbol{I}} T_{\boldsymbol{I}}^{\boldsymbol{J}} = 1 - 2\delta^2$. Then $(p - P)^{\boldsymbol{I}} T_{\boldsymbol{I}}^{\boldsymbol{J}} = 2\delta(1-\delta)$ while $1 - P^{\boldsymbol{I}} T_{\boldsymbol{I}}^{\boldsymbol{J}} = 2\delta$, so that for any $\gamma > 0$ the inequality (B.2) fails for all $\delta < \gamma$.

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