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Full Information and Full Control in a Behavioral Context

Raffaello D'Andrea*
raff@hot.caltech.edu

John C. Doyle
doyle@hot.caltech.edu

Division of Engineering and Applied Science
California Institute of Technology
Pasadena, California 91125

Abstract

In this paper, the concepts of Full Information and Full Control which arise in standard $\mathcal{H}_\infty$ theory are extended to the behavioral framework.

1 Introduction

In [1] and [2], a behavioral version of the $\mathcal{H}_\infty$ optimal control problem is solved. The solution consists ot two coupled Riccati equations, closely mirroring the standard $\mathcal{H}_\infty$ solution in [5] and [4]. In this paper the concepts of Full Information (FI) and Full Control (FC) are extended to the behavioral framework, and the implications of these definitions are explored. It is shown that these definitions are more fundamental than those given for the standard input/output (IO) case; in particular, the concept of state is not required and no a priori partition of the system variables into inputs and outputs needs to be performed.

The paper is organized as follows: After introducing the notation and providing background relevant to the paper in Section 2, the notions of FI and FC in the IO framework are reviewed in Section 3. In Section 4, the $\mathcal{H}_\infty$ Optimal Interconnection problem formulation is outlined and the solution presented. In Section 5 the behavioral versions of the FI and FC problems are introduced, followed by connections with the IO versions of the FI and FC problems and the associated Riccati equations in Section 6. An illustrative example is presented in Section 7, followed by the conclusions in Section 8.

2 Background and Notation

2.1 Basic Definitions

What follows is a brief summary of the notions introduced in [11] relevant to this paper. Systems for which the allowable trajectories are the solution set of the following set of differential equations will be considered:

$$R_L \frac{d^L w}{dt^L} + \cdots + R_0 w = 0, \tag{1}$$

where $R_0, \cdots, R_L$ are constant matrices. Defining

$$R(s) := R_L s^L + \cdots + R_0 \tag{2}$$

results in the shorthand notation $R(\frac{d}{dt})w = 0$ for equation (1). The above is referred to as an *autoregressive (AR)* representation. Elementary properties of polynomial matrix representations are outlined in Appendix A.

It is assumed that $w \in C^\infty(\mathbb{R}, \mathbb{R}^q)$, the set of infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}^q$. The shorthand notation C^∞ is used when the spatial dimension q is clear from context.

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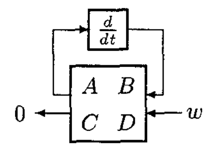

Figure 1: LFT representation

A system is denoted by $\Sigma := \{\mathbb{R}, \mathbb{R}^q, \mathcal{B}\}$, where $\mathbb{R}$ and $\mathbb{R}^q$ correspond to $\mathbb{R}^q$ valued, bi-infinite, continuous time, trajectories, and $\mathcal{B}$ is the *behavior*, or the allowable trajectories:

$$\mathcal{B} := \left\{ w \in C^\infty(\mathbb{R}, \mathbb{R}^q) \mid R(\frac{d}{dt})w = 0 \right\}. \tag{3}$$

The reader is referred to Appendix B for a review of the notion of interconnection for behavioral representations.

2.2 State Space Descriptions

The behavior $\mathcal{B}$ of $\Sigma := \{\mathbb{R}, \mathbb{R}^q, \mathcal{B}\}$ can always be captured in the following state space form [2]:

$$\begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \dot{x} \\ w \end{bmatrix} =: M \begin{bmatrix} \dot{x} \\ w \end{bmatrix}, \tag{4}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times q}$, $C \in \mathbb{R}^{r \times n}$, and $D \in \mathbb{R}^{r \times q}$. The above representation is a special case of a dual pencil representation, extensively studied in [7]. Because of the similarity of the above state space descriptions to the output nulling descriptions studied in [10], they will be referred to as *dual output nulling* (DON) representations. A DON representation matrix M is *minimal* if no other representation matrix exists of smaller dimension (since q is fixed, this means that no other representation exists with less number of states n or less number of equations r). It can be shown (see [7], [3]) that a DON representation matrix is minimal if and only if (C, A) is an observable pair and M is full row rank.

A DON representation is, in fact, a convenient way of capturing AR representations in the linear fractional transformation (LFT) framework, as shown in Figure 1. As will be demonstrated, in the context of optimal control it is more natural to view behavioral representations which are in the above form as opposed to standard state space representations (i.e., where the LFT is on an integrator, not a differentiator).

3 FI and FC in the IO Setting

In the standard IO $\mathcal{H}_\infty$ control problem of Figure 2, it is required to find a stabilizing controller $\mathbf{K}$ such that the energy gain from d to e is less than 1 [4]. The solution reduces to solving two Riccati equations and checking a coupling condition.

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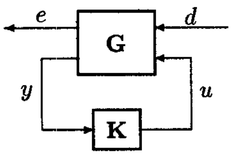

Figure 2: IO problem

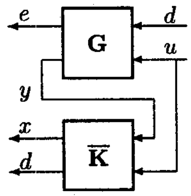

Figure 3: FI observer

Associated with each of the two Riccati equations are two special problems, which are constructed from a state space description for **G**: the FI and FC problems. In the FI problem, it is assumed that the controller has full access to the system state (denoted x) and the disturbances d. In the FC problem, it is assumed that the controller can influence the state equations (the ones involving $\dot{x}$) and the output error equations (the ones involving e) independently.

Given a system **G**, it can readily be shown that if the controller has access to x and d, the associated FC problem has a trivial solution; similarly, if the controller can influence the state and output error equations independently, the associated FI problem has a trivial solution. In each of the above two cases, only one Riccati equation needs to be solved, and the coupling condition is trivially satisfied.

In the behavioral framework, a system is described as the set of allowable trajectories; there is no distinction between inputs and outputs, and the concept of state is not an inherent property of the system. Thus one may ask the following question; is there a natural notion of FI and FC in the behavioral framework? If such a notion exists, it must not depend on IO partitions, and be state-space independent. We motivate below how the concept of state may be removed from the FI problem in the IO setting, as a prelude to the results of Section 5.

3.1 Stateless FI

A standard $\mathcal{H}_\infty$ problem reduces to a FI problem if an observer can be constructed which yields x and d, as shown in Figure 3. In this case, only one Riccati equation needs to be solved, since the associated FC problem is trivially satisfied, as previously discussed. As will be shown, the following is an equivalent condition: can an observer be constructed which yields d and e? We have the following proposition:

Proposition 1 *An observer can be constructed which yields x and d if and only if an observer (possibly improper) can be constructed which yields e and d.*

Proof: Let

$$\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]$$

be a minimal state space representation for **G**, with inputs d and u, and outputs e and y. Let an observer which yields x and d exist. Since $e = C_1x + D_{11}d + D_{12}u$, one can recover e as well. Now assume that an observer which yields e and d

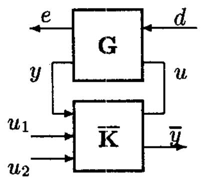

Figure 4: FC pre-compensator

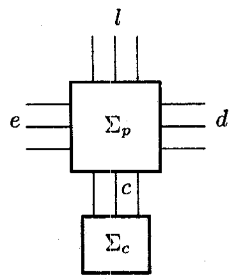

Figure 5: Problem formulation

exists. Note that

$$Cx = \begin{bmatrix} e \\ y \end{bmatrix} - D\begin{bmatrix} d \\ u \end{bmatrix}, \tag{5}$$

where all the variables on the right hand side are know. Thus by repeatedly differentiating the above equation and substituting for $\dot{x}$, and since (C, A) is an observable pair, one may recover state x. ∎

3.2 FC

A problem reduces to a FC problem if a pre-compensator can be constructed such that u_1 can be injected into the state equations and u_2 can be injected into the output error equations, as shown in Figure 4. Unlike the FI problem, however, there is no simple definition of FC which does not involve the state; this is a shortcoming of the IO framework, as will be shown in Section 5.

4 $\mathcal{H}_\infty$ Optimal Interconnections

The material in this section is a review of the problem formulation in [1] and [2]. Let $\mathcal{L}_2$ be the Hilbert space of square integrable functions, and denote the norm of an element $v \in \mathcal{L}_2$ as $\|v\|$. Let system $\Sigma_p = \{\mathbb{R}, \mathbb{R}^{q_e+q_d+q_c+q_l}, \mathcal{B}_p\}$ be given, i.e., w is partitioned into four parts, $w = (e, d, c, l)$:

e: error signals which are required to be small.

d: exogenous disturbances, unexplained by the given model.

c: variables which are accessible for control purposes.

l: latent variables, auxiliary variables used when constructing system Σ_p.

The objective is to find system $\Sigma_c = \{\mathbb{R}, \mathbb{R}^{q_c}, \mathcal{B}_c\}$ acting on the variables c (see Figure 5), such that $\Sigma := \Sigma_p \wedge \Sigma_c = \{\mathbb{R}, \mathbb{R}^{q_e+q_d+q_c+q_l}, \mathcal{B}\}$ satisfies the following:

(P1) **Unrestricted Disturbance:** For the interconnected system, d is free:

$$\forall d \in C^\infty,\ \exists\, e, c, l \in C^\infty s.t.\ w \in \mathcal{B}. \tag{6}$$

Equivalently, system Σ_c does not provide any additional information about the disturbance.

(P2) **Stability:**

$$d = 0, w \in \mathcal{B} \Longrightarrow \lim_{t\to\infty} e(t), c(t) = 0. \tag{7}$$

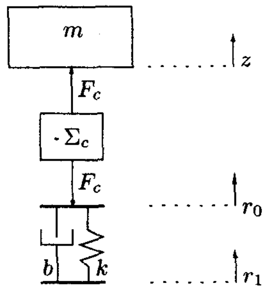

Figure 6: Suspension design

Thus if one stops exciting the system, the error and control signals decay to 0. Note that there is no such restriction on latent variables l; this will be motivated by the simple example in Section 4.1.

(P3) **Performance:**

$$\sup_{d\in(\mathcal{C}^\infty\cap\mathcal{L}_2),\ \|d\|\leq 1, w\in\mathcal{B}} \|e\| < 1. \tag{8}$$

Note that the general performance specification $\|e\| < \gamma$ can be imposed by appropriately scaling variable e.

In general, a system Σ_c which only has access to variables c will be referred to as a *compensator.* If in addition Σ satisfies constraints P1, P2, and P3, Σ_c will be referred to as an *allowable* compensator.

4.1 Example

The following simple example can be used to illustrate the problem formulation. It consists of a one degree of freedom suspension design. Consider the setup of Figure 6. The goal is to design system Σ_c, the suspension, in order to achieve certain performance objectives which will be described shortly. Variable m denotes the sprung mass, or the mass of the cab where the passengers will ride. Σ_c is the mechanism which is to be designed; it is restricted to be a relation between F_c and $z - r_0$. The spring and the damper model a tire, which is in contact with the road.

The equations describing the system and the performance objectives are as follows:

$$\begin{aligned} 0 &= F_c - m\ddot{z} \\ 0 &= F_c + b(\dot{r_0} - \dot{r_1}) + k(r_0 - r_1) \\ c_1 &= F_c \\ c_2 &= z - r_0 \\ e_1 &= z - r_1 \quad \text{(tracking)} \\ e_2 &= \dddot{z} \qquad\quad \text{(comfort)} \\ d &= \ddot{r}_1. \end{aligned} \tag{9}$$

The first two equations are the equations of motion about an equilibrium point. The second two equations dictate which variables system Σ_c has access to. The next two equations describe the performance objectives; the sprung mass is required to track the road, while simultaneously be subjected to small values of jerk (the jerk, or third derivative of position, is to a first approximation a good measure of passenger discomfort, and is in general a quantity which should be kept small in the design of mechanical systems [8]). The last equation models the allowable road disturbances; restricting d to be an $\mathcal{L}_2$ disturbance of unit norm restricts r_1 to be small at high frequencies and allows r_1 to be large at low frequencies.

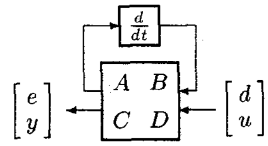

Figure 7: Input/output representation

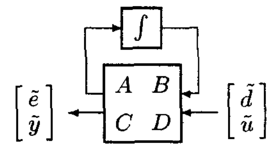

Figure 8: Dual problem

This corresponds to restricting large amplitude road disturbances to be gradual (hills), while allowing smaller amplitude disturbances to be sharper (potholes and speed bumps). Also note that when $d = 0$, $r_1(t) = C_0 + C_1 t$ for some constants C_0 and C_1; this corresponds to a constant climb, which should be allowed in the equations of motion.

It is clear from this example why the definition of stability should not encompass the latent variables: r_1 should not be restricted to decay to 0 when $d = 0$. In general, if one is concerned about the size of a latent variable, it could be penalized and be made a part of e.

4.2 Solution

In [1], a solution to the $\mathcal{H}_\infty$ optimal interconnection problem is presented under the assumptions that the allowable compensator Σ_c forms a feedback interconnection with Σ_p, and that no latent variables l are present; as shown in [2], however, these assumptions are not restrictive.

Lemma 1 *Let Σ_p be given. If an allowable compensator Σ_c exists, then there exists a minimal DON representation for the behavior $\mathcal{B}_p$ of the following form*

$$\begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} A & 0 & B_1 & B_2W_2 \\ C_1 & -I & D_{11} & D_{12}W_2 \\ C_2 & 0 & D_{21} & -W_1 \end{bmatrix} \begin{bmatrix} \dot{x} \\ e \\ d \\ c \end{bmatrix}, \tag{10}$$

where

$$\begin{bmatrix} W_1 \\ W_2 \end{bmatrix}$$

is square and invertible.

Lemma 1 has a simple interpretation: for there to be a solution to the $\mathcal{H}_\infty$ Optimal Interconnection problem, there must exist a partition of control variables c into y and u such that the behavior of Σ_p can be captured by the possibly non-proper, non-standard IO map of Figure 7.

Given this representation, a solution to the $\mathcal{H}_\infty$ Optimal Interconnection problem may be obtained by applying a slightly modified version of the standard $\mathcal{H}_\infty$ solution in [5] to the system of Figure 8. Note that the system of Figure 8 is constructed purely for technical reasons; there is no natural physical interpretation for it.

Given the representation for the behavior of Σ_p in equation (10), the following assumptions are made on the problem data:

(A2) D_{12} full column rank with $[\, D_{12} \;\; D_\perp \,]$ unitary, D_{21} full row rank with $\begin{bmatrix} D_{21} \\ \tilde{D}_\perp \end{bmatrix}$ unitary.

(A3) $\begin{bmatrix} A - j\omega & B_2 \\ C_1 & D_{12} \end{bmatrix}$ full column rank $\forall\omega \neq 0$.

(A4) $\begin{bmatrix} A-j\omega & B_1 \\ C_2 & D_{21} \end{bmatrix}$ full row rank $\forall\omega \neq 0$.

Condition (A3) is equivalent to $(D_\perp^* C_1, A - B_2 D_{12}^* C_1)$ having no purely imaginary unobservable modes, except possibly at $s = 0$ (see [5]). A Kalman decomposition induces the following state transformation S:

$$A^{FI} = \begin{bmatrix} A_{11}^{FI} & A_{12}^{FI} \\ A_{21}^{FI} & A_{22}^{FI} \end{bmatrix} := S^{-1}AS, \tag{11}$$

$$\begin{bmatrix} B_1^{FI} & B_2^{FI} \end{bmatrix} = \begin{bmatrix} B_{11}^{FI} & B_{21}^{FI} \\ B_{12}^{FI} & B_{22}^{FI} \end{bmatrix} := S^{-1}\begin{bmatrix} B_1 & B_2 \end{bmatrix},$$

$$\begin{bmatrix} C_1^{FI} \\ C_2^{FI} \end{bmatrix} = \begin{bmatrix} C_{11}^{FI} & C_{12}^{FI} \\ C_{21}^{FI} & C_{22}^{FI} \end{bmatrix} := \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} S,$$

where in this co-ordinate system,

$$D_\perp^* C_1^{FI} = \begin{bmatrix} \tilde{C}^{FI} & 0 \end{bmatrix}, \; (A^{FI} - B_2^{FI} D_{12}^* C_1^{FI}) = \begin{bmatrix} \bar{A}^{FI} & 0 \\ * & \Sigma^{FI} \end{bmatrix}, \tag{12}$$

where Σ^{FI} is nilpotent and $(\tilde{C}^{FI}, \bar{A}^{FI})$ has no purely imaginary unobservable modes.

Similarly, condition (A4) is equivalent to $\left(A - B_1 D_{21}^* C_2, B_1 \tilde{D}_\perp^*\right)$ having no purely imaginary uncontrollable modes, except possibly at $s = 0$. State transformation T and A^{FC}, B^{FC}, and C^{FC} can be defined analogously.

The final assumption on the problem data follows, presented last since it is more natural to do so in the appropriate co-ordinate system:

(A1) $(A_{11}^{FI}, B_{21}^{FI})$ stabilizable, $(C_{21}^{FC}, A_{11}^{FC})$ detectable.

It is straightforward to show that assumption (A2) is equivalent to requiring that D_{12} and D_{21} be full column rank and full row rank, respectively. In general, assumptions (A2), (A3), and (A4) are not necessary for an allowable compensator Σ_c to exist; (A1), however, must be satisfied.

Given that the above assumptions are satisfied, the solution to the $\mathcal{H}_\infty$ Optimal Interconnections problem consists of solving two Riccati equations and checking a coupling condition, as for the standard $\mathcal{H}_\infty$ solution in [5]. One Riccati equation arises from the following standard $\mathcal{H}_\infty$ FI problem:

$$\begin{bmatrix} \dot{\tilde{x}} \\ \tilde{e} \\ \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix} = \begin{bmatrix} A_{11}^{FI} & B_{11}^{FI} & B_{21}^{FI} \\ C_{11}^{FI} & D_{11} & D_{12} \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{d} \\ \tilde{u} \end{bmatrix}. \tag{13}$$

The other arises from the following standard $\mathcal{H}_\infty$ FC problem:

$$\begin{bmatrix} \dot{\tilde{x}} \\ \tilde{e} \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} A_{11}^{FC} & B_{11}^{FC} & I & 0 \\ C_{11}^{FC} & D_{11} & 0 & I \\ C_{21}^{FC} & D_{21} & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{d} \\ \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix} \tag{14}$$

If both of the above problems have solutions, and a coupling condition is satisfied, a DON representation for an allowable compensator may be constructed; the details are found in [2].

5 FI and FC in the Behavioral Setting

We proceed to define FI and FC in a behavioral context. Let Σ_p be given, and define $v := (e, d)$. The starting point is a minimal AR representation for the behavior $\mathcal{B}_p$:

$$R^v(\frac{d}{dt})v = R^c(\frac{d}{dt})c \tag{15}$$

Note that all the latent variables in Σ_p have been removed, as discussed in [2]. We have the following definitions:

Definition 1 *The $\mathcal{H}_\infty$ Optimal Interconnection problem is a FI problem if $R^v(s)$ is full column rank $\forall s \in \overline{\mathbb{C}}^+$.*

Definition 2 *The $\mathcal{H}_\infty$ Optimal Interconnection problem is a FC problem if $R^c(s)$ is full row rank $\forall s \in \overline{\mathbb{C}}^+$.*

We now discuss the ramifications of these definitions:

5.1 FI

Assume that $R^v(s)$ satisfies the FI rank condition. By Smith decomposing $R^v(s)$ and left multiplication by a unimodular matrix (which does not change the behavior, see [11]), it can be assumed that

$$R^v(s) = \begin{bmatrix} \Delta(s) \\ 0 \end{bmatrix} V(s), \tag{16}$$

$$R^c(s) = \begin{bmatrix} R_1^c(s) \\ R_2^c(s) \end{bmatrix}, \tag{17}$$

where $\Delta(s)$ is square and invertible for all s in $\overline{\mathbb{C}}^+$, $V(s)$ is unimodular, and $R_2^c(s)$ is full normal row rank. $R_2^c(s)$, in turn, can be decomposed as

$$R_2^c(s) = \begin{bmatrix} \Delta_2(s) & 0 \end{bmatrix} V_2(s) \tag{18}$$

where $\Delta_2(s)$ is square and of full normal rank, and $V_2(s)$ is unimodular. Define $\bar{c} := V_2(\frac{d}{dt})c =: (c_1, c_2)$. Thus $R_2^c(\frac{d}{dt})c = 0$ is equivalent to $\Delta_p(\frac{d}{dt})c_1 = 0$. Note that this change of co-ordinates in no way affects the feasibility of finding a solution; if $R_c^c(\frac{d}{dt})\bar{c} = 0$ is an allowable compensator in the new set of compensator co-ordinates, then $R_c^c(\frac{d}{dt})V_2(\frac{d}{dt})c = 0$ is an allowable compensator in the original compensator co-ordinates. Similarly if $R_c^c(\frac{d}{dt})c = 0$ is allowable in the original co-ordinates, then $R_c^c(\frac{d}{dt})V_c^{-1}(\frac{d}{dt})\bar{c} = 0$ is allowable in the new co-ordinates. As is argued in [2], a pre-compensator of the form $c_1 = 0$ can be applied to the system without changing the feasibility of finding an allowable compensator. Thus a FI problem can always be converted to the following form:

$$\Delta(\frac{d}{dt})V(\frac{d}{dt})v = R^c(\frac{d}{dt})c \tag{19}$$

after a possible change of compensator co-ordinates and applying an appropriate pre-compensator.

Given c, let $\tilde{v}$ solve the following system of equations

$$\Delta(\frac{d}{dt})V(\frac{d}{dt})\tilde{v} = R^c(\frac{d}{dt})c. \tag{20}$$

Thus

$$\Delta(\frac{d}{dt})\left(V(\frac{d}{dt})(v - \tilde{v})\right) = 0, \tag{21}$$

and by the assumed structure of $\Delta(s)$, $V(\frac{d}{dt})\tilde{v} \overset{t\to\infty}{\longrightarrow} V(\frac{d}{dt})v$. Since $V(s)$ is unimodular, this implies that $\tilde{v} \overset{t\to\infty}{\longrightarrow} v$. Thus by having access to control variables c, one may infer what variables v are, resulting in knowledge of all the system variables. Note that this is exactly the interpretation given in Section 3.1.

There is a connection between the FI problem discussed here and the behavioral version of $\mathcal{H}_\infty$ explored in [9]. In [9], $R^v(s)$ is assumed to be the identity, which is a special case of the FI rank condition.

5.2 FC

Assume that $R^c(s)$ satisfies the FC rank condition. Using a Smith decomposition, it can be assumed that

$$R^c(s) = [\,\Delta(s)\ 0\,]\,V(s) \tag{22}$$

where $\Delta(s)$ is square and invertible for all s in $\overline{\mathbb{C}}^+$, and $V(s)$ is unimodular. Define $\bar{c} := V(\frac{d}{dt})c =: (c_1, c_2)$. Thus

$$R^v(\frac{d}{dt})v = \Delta(\frac{d}{dt})c_1 \tag{23}$$

and c_2 does not affect v.

Given $\tilde{c}_1$, let c_1 solve the following system of equations

$$\Delta(\frac{d}{dt})c_1 = \tilde{c}_1. \tag{24}$$

Thus

$$R^v(\frac{d}{dt})v = \tilde{c}_1, \tag{25}$$

and by the assumed structure of $\Delta(s)$, $\tilde{c}_1 \overset{t\longrightarrow\infty}{\longrightarrow} 0 \Rightarrow c_1 \overset{t\longrightarrow\infty}{\longrightarrow} 0$. Thus one can fully control all the equations which involve variables v, and control variables c_1 approach the desired values $\tilde{c}_1$. It is clear why there is no simple IO interpretation of the above result, as mentioned in Section 3.2; controlling the equations involving d and e has no simple counterpart in the IO framework. The duality is apparent, however; in the IO FI problem, it was shown that estimating x and d is equivalent to estimating e and d. In the FC problem, controlling x and e is equivalent to controlling d and e.

6 Connections with Riccati Solution

As outlined in Section 4, the solution to the $\mathcal{H}_\infty$ Optimal Interconnection problem consists of solving two Riccati equations and checking a coupling condition. We show in this section that if the FI rank condition of Definition 1 is satisfied, the IO FC problem of equation (14) has a trivial solution and only the Riccati equation associated with the IO FI problem of equation (13) needs to be solved. Similarly, if the FC rank condition of Definition 2 is satisfied, the IO FI problem of equation (13) has a trivial solution and only the Riccati equation associated with the IO FC problem of equation (14) needs to be solved.

6.1 FI

Let equation (19) be given, where $\Delta(s)$ is square and invertible for all s in $\overline{\mathbb{C}}^+$ and $V(s)$ is unimodular. By Lemma 1, there exists a DON representation for the behavior $\mathcal{B}_p$ as in equation (10). We have the following Theorem:

Theorem 1 *D_{21} is square and invertible, and the eigenvalues of $(A - B_1 D_{21}^{-1} C_2)$ have negative real part or are zero. Furthermore, the IO FC problem of equation (14) has a trivial solution.*

Proof: That D_{21} is square follows directly by setting $c = 0$ in equation (10) and by noting that $\Delta(s)V(s)$ is square. First assume that D_{21} is not invertible. Then there exists vector d_0 such that $D_{21}d_0 = 0$; then $(x, e, d, c) := (B_1 d_0, D_{11} d_0, d_0, 0)$ satisfies equation (10), and thus $\Delta(0)V(0)\begin{bmatrix} D_{11}d_0 \\ d_0 \end{bmatrix} = 0$, a contradiction. Now let $s_0 \neq 0$ be an eigenvalue of $(A - B_1 D_{21}^{-1} C_2)$ with positive or zero real part. By Schur complement arguments, this implies that there exist vectors x_0 and d_0 such that

$$\begin{bmatrix} A - s_0 I & B_1 \\ C_2 & D_{21} \end{bmatrix} \begin{bmatrix} x_0 \\ d_0 \end{bmatrix} = 0.$$

Thus $(x, e, d, c) = (s_0 x_0, C_1 x_0 + D_{11} d_0, d_0, 0)\exp(s_0^{-1} t)$ satisfies equation (10). Note that if $d_0 = 0$, then $C_1 x_0 \neq 0$ by the minimality of the DON representation in equation (10); thus $\Delta(s_0^{-1})V(s_0^{-1})\begin{bmatrix} C_1 x_0 + D_{11} d_0 \\ d_0 \end{bmatrix} = 0$, a contradiction.

Note that since D_{21} is invertible, it can be assumed to be unitary without loss of generality (pre-multiply the last row of equation (10) by D_{21}^{-1}, for example). The detectability assumption in (A1) and the rank condition in (A4) are satisfied as well. Since the eigenvalues of $(A - B_1 D_{21}^{-1} C_2)$ have negative real part or are zero, it follows that the eigenvalues of $(A_{11}^{FC} - B_{11}^{FC} D_{21}^* C_{21}^{FC})$ must have negative real part. Applying the following control strategy to the IO FC problem

$$\tilde{u}_1 = -B_{11} D_{21}^* \tilde{y}, \tag{26}$$
$$\tilde{u}_2 = -D_{11} D_{21}^* \tilde{y} \tag{27}$$

results in the following closed loop equations:

$$\dot{\tilde{x}} = (A_{11}^{FC} - B_{11}^{FC} D_{21}^* C_{21}^{FC})\tilde{x}, \tag{28}$$
$$\tilde{e} = (C_{11}^{FC} - D_{11}^{FC} D_{21}^* C_{21}^{FC})\tilde{x}. \tag{29}$$

Since the closed loop eigenvalues have negative real part, the above constant feedback law solves the IO FC problem, and results in perfect disturbance attenuation. ∎

Note that the Riccati equation associated with the above IO FC problem has zero as a solution (all the closed loop modes are stable and uncontrollable); thus only the IO FI problem needs to be solved, and the coupling condition between the two Riccati solutions is trivially satisfied.

6.2 FC

Let equation (23) be given, where $\Delta(s)$ is square and invertible for all s in $\overline{\mathbb{C}}^+$. To avoid extra notation, replace c_1 by c. By Lemma 1, there exists a DON representation for the behavior $\mathcal{B}_p$ as in equation (10). We have the following Theorem:

Theorem 2 *D_{12} is square and invertible, and the eigenvalues of $(A - B_2 D_{12}^{-1} C_1)$ have negative real part or are zero. Furthermore, the IO FI problem of equation (13) has a trivial solution.*

Proof: The proof is essentially the dual of the FI case and follows by setting $(d, e) = 0$ in equation (10); the details are omitted. ∎

Analogous to the previous case, the Riccati equation associated with the associated IO FI problem has zero as a solution (all the closed loop modes are stable and unobservable); thus only the IO FC problem needs to be solved, and the coupling condition between the two Riccati solutions is trivially satisfied.

7 Example

We return to the example of Section 4.1. For positive values of b and k, this is a FI problem; the solution presented in [2] reduces to one Riccati equation. This may also be verified by expressing variables e and d as functions of c:

$$b\dot{e}_1 + ke_1 = b\dot{c}_2 + kc_2 - c_1 \tag{30}$$
$$e_2 = \frac{1}{m}\dot{c}_1 \tag{31}$$
$$d = \frac{1}{m}c_1 - \ddot{e}_1 \tag{32}$$

Thus e_1 can be recovered using the first equation, e_2 from the second equation, and since e_1 is now known, d can be recovered from the third equation.

8 Conclusions

The concepts of FI and FC are naturally defined in the behavioral framework. As in standard $\mathcal{H}_\infty$ theory, a separation structure may be obtained for the $\mathcal{H}_\infty$ Optimal Interconnection problem by considering FI and FC problems. The interpretation of FI is to be able to reconstruct the output error and disturbance from the control variables. The interpretation of FC is to be able to fully affect all the equations involving the output error and the disturbance. The two Riccati equation solution in [2] can be interpreted in terms of coupled FI and FC problems.

Appendix

A Polynomial Matrices

Polynomial matrices are used extensively when describing the behavior of a system. What follows are some definitions and results pertaining to polynomial matrices used throughout the paper; the reader is referred to [6] for details.

A square polynomial matrix $R(s)$ is said to be *nonsingular* if $\det(P(s)) \neq 0$. A nonsingular polynomial matrix whose determinant is not a function of s is called *unimodular*. Equivalently, $R(s)$ is unimodular if and only if $R^{-1}(s)$ is a polynomial matrix.

$R(s)$ is said to have *full normal row rank* if $R(s)$ is full row rank for almost all $s \in \mathbb{C}$. Similarly, $R(s)$ is said to have *full normal column rank* if $R(s)$ is full column rank for almost all $s \in \mathbb{C}$.

$R(s)$ is said to be *right invertible* if there exists a polynomial matrix $M(s)$ such that $R(s)M(s) = I$. Equivalently, $R(s)$ is right invertible if and only if $R(s)$ is full row rank for all $s \in \mathbb{C}$. $R(s)$ is said to be *left invertible* if there exists a polynomial matrix $M(s)$ such that $M(s)R(s) = I$. Equivalently, $R(s)$ is left invertible if and only if $R(s)$ is full column rank for all $s \in \mathbb{C}$. If $R(s)$ is *right invertible*, then there exists a polynomial matrix $N(s)$ such that

$$\begin{bmatrix} R(s) \\ N(s) \end{bmatrix}$$

is unimodular. If $R(s)$ is *left invertible*, then there exists a polynomial matrix $N(s)$ such that $[\,R(s)\ \ N(s)\,]$ is unimodular.

Any polynomial matrix $R(s)$ can be *Smith* decomposed as

$$R(s) = U(s) \begin{bmatrix} \Delta(s) & 0 \\ 0 & 0 \end{bmatrix} V(s), \tag{33}$$

where $U(s)$ and $V(s)$ are unimodular and $\Delta(s)$ is square and nonsingular.

B Integer Invariants and Interconnection

The following definitions are from [11] and [12]. There are several integer invariants associated with a system Σ. One is $p^*(\Sigma)$, the number of outputs in any input-output map; given a minimal AR representation $R(s)$ (one that has full normal row rank, see [11]) for Σ, this integer invariant is equal to the number of rows of $R(s)$. Equivalently, given a minimal DON representation matrix, this integer invariant is equal to r. Another integer invariant is the minimum number of states required to describe Σ in state space form, $n^*(\Sigma)$; given a minimal DON representation matrix, this invariant is equal to n.

The *interconnection* of two systems $\Sigma_1 = \{\mathbb{R}, \mathbb{R}^q, \mathcal{B}_1\}$ and $\Sigma_2 = \{\mathbb{R}, \mathbb{R}^q, \mathcal{B}_2\}$, possessing the same variables w, is defined to be

$$\Sigma_1 \wedge \Sigma_2 := \{\mathbb{R}, \mathbb{R}^q, \mathcal{B}_1 \cap \mathcal{B}_2\}; \tag{34}$$

the resulting behavior is simply the intersection of the two behaviors. Thus an allowable trajectory must satisfy the governing equations of both systems. Note that each of Σ_1 and Σ_2 can be trivially augmented to posses the same variables w.

$\Sigma_1 \wedge \Sigma_2$ is termed a *feedback interconnection* if

$$p^*(\Sigma_1 \wedge \Sigma_2) = p^*(\Sigma_1) + p^*(\Sigma_2). \tag{35}$$

An interpretation of the above is that the laws of the systems can be viewed as independent. A feedback interconnection is termed *regular* if

$$n^*(\Sigma_1 \wedge \Sigma_2) = n^*(\Sigma_1) + n^*(\Sigma_2). \tag{36}$$

If $n^*(\Sigma_1 \wedge \Sigma_2) < n^*(\Sigma_1) + n^*(\Sigma_2)$, the interconnection is termed *singular*. Regular feedback interconnections are the standard ones considered in feedback control. Singular feedback interconnections differ in that the interconnection results in algebraic constraints on the states; thus the states of the individual systems must be matched before interconnection can take place.

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