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DIFFERENTIAL FLATNESS AND ABSOLUTE EQUIVALENCE

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ABSTRACT

In this paper we give a formulation of differential flatness—a concept originally introduced by Fliess, Lévine, Martin, and Rouchon—in terms of absolute equivalence between exterior differential systems. Systems which are differentially flat have several useful properties which can be exploited to generate effective control strategies for nonlinear systems. The original definition of flatness was given in the context of differential algebra, and required that all mappings be meromorphic functions. Our formulation of flatness does not require any algebraic structure and allows one to use tools from exterior differential systems to help characterize differentially flat systems. In particular, we show that in the case of single input control systems (i.e., codimension 2 Pfaffian systems), a system is differentially flat if and only if it is feedback linearizable via static state feedback. However, in higher codimensions feedback linearizability and flatness are *not* equivalent: one must be careful with the role of time as well the use of prolongations which may not be realizable as dynamic feedbacks in a control setting. Applications of differential flatness to nonlinear control systems and open questions are also discussed.

1. INTRODUCTION

The problem of feedback linearization is traditionally approached in the context of differential geometry [10, 15]. A complete characterization of static feedback linearizability in the multi-input case is available, and for single input systems it has been shown that static and dynamic feedback linearizability are equivalent [4]. Some special results have been obtained for dynamic feedback linearizability of multi-input systems, but the general problem remains unsolved. Typically, the conditions for feedback linearizability are expressed in terms of the involutivity of distributions on a manifold.

More recently it has been shown that the conditions on distributions have a natural interpretation in terms of exterior differential systems [7, 16]. In exterior differential systems, a control system is viewed as a Pfaffian module. Some of the advantages of this approach are the wealth of tools available and the fact that implicit equations and non-affine systems can be treated in a unified framework. For an extensive treatment of exterior differential systems we refer to [1].

Fliess and coworkers [5, 11] studied the feedback linearization problem in the context of differential algebra and introduced the concept of *differential flatness*. In differential algebra, a system is viewed as a differential field generated by a set of variables (states and inputs). The system is said to be differentially flat if one can find a set of variables, called the flat outputs, such that the system is (non-differentially) algebraic over the differential field generated by the set of flat outputs. Roughly speaking, a system is flat if we can find a set of outputs (equal in number to the number of inputs) such that all states and inputs can be determined from these outputs without integration. More precisely, if the system has states $x \in \mathbb{R}^n$, and inputs $u \in \mathbb{R}^m$ then the system is flat if we can find outputs $y \in \mathbb{R}^m$ of the form

$$y = y(x, u, \dot{u}, \ldots, u^{(p)}) \tag{1}$$

such that,

$$\begin{aligned} x &= x(y, \dot{y}, \ldots, y^{(q)}) \\ u &= y(y, \dot{y}, \ldots, y^{(q)}). \end{aligned} \tag{2}$$

Differentially flat systems are useful in situations where explicit trajectory generation is required. Since the behaviour of flat system is determined by the flat outputs, we can plan trajectories in output space, and then map these to appropriate inputs. A common example is the kinematic car with trailers, where the xy position of the last trailer provides flat outputs [13]. This implies that all feasible trajectories of the system can be determined by specifying only the trajectory of the last trailer. Unlike other approaches in the literature (such as converting the kinematics into a normal form), this technique works globally.

A limitation of the differential algebraic setting is that it does not provide tools for regularity analysis. The results are given in terms of differential polynomials in the variables, without characterizing the solutions. In particular, solutions to the differential polynomials may not exist. For example, the system :

$$\begin{aligned} \dot{x}_1 &= u \\ \dot{x}_2 &= x_1^2, \end{aligned} \tag{3}$$

is flat in the differentially algebraic sense with flat output $y = x_2$. However, it is clear that the derivative of x_2 always has to be positive, and therefore we cannot follow an arbitrary trajectory in y space.

In differential algebra the coefficients of the polynomials are allowed to be meromorphic functions of time. However, to treat time as a special variable in the relations (2), one needs to resort to Lie-Bäcklund transformations on infinite dimensional spaces [6]. Also, the notion of flatness is more general than (dynamic) feedback linearizability, as is shown by the example of a rolling penny, and its promising applications in trajectory generation justify a deeper study.

In the beginning of this century, the French geometer E. Cartan developed a set of powerful tools for the study of equivalence of systems of differential equations [2, 3, 16]. Equivalence need not be restricted to systems of equal

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dimensions. In particular a system can be *prolonged* to a bigger system on a bigger manifold, and equivalence between these prolongations can be studied. This is the concept of *absolute equivalence* of systems. Prolonging a system corresponds to dynamic feedback, and it is clear that we can benefit from the tools developed by Cartan to study the feedback linearization problem.

In this paper we reinterpret flatness in a differential geometric setting. We make extensive use of the tools offered by exterior differential systems, and the ideas of Cartan. This approach allows us to study some of the regularity issues, and also to give a more explicit treatment of time dependence. Moreover, we can easily make connections to the extensive body of theory that exists in differential geometry. We show how to recover the differentially algebraic definition, and give an exterior differential systems proof for a result proven by Martin [11, 12] in differential algebra: a flat system can be put into Brunovsky normal form by dynamic feedback in an open and dense set. This set need not contain an equilibrium point.

We also give a complete characterization of flatness for systems with a single input. In this case, flatness in the neighborhood of an equilibrium point is equivalent to linearizability by static state feedback around that point. This result is stronger than linearizability by endogenous feedback as indicated by Martin, since the latter only holds in an open and dense set. We also treat the case of time varying versus time invariant flat outputs, and show that in the case of a single input, autonomous system the flat output can always be chosen time independent. In exterior differential systems, the special role of the time coordinate is expressed as an independence condition, i.e., a one-form that is not allowed to vanish on any of the solution curves. A fundamental problem with exterior differential systems is that most results only hold on open dense sets. See for example [8]. It requires some special effort to obtain results in the neighborhood of a point, see for example [14]

The organization of the paper is as follows. In Section 2 we introduce the definitions pertaining to absolute equivalence and their interpretation in control theory. In Section 3 we introduce our definition of differential flatness and show how to recover the differential algebraic results. In Section 4 we present our main theorems characterizing flatness for single input systems, and in Section 5 we summarize our results and point out some open questions.

2. Prolongations and Control Theory

This section introduces the concept of prolongations, and states some basic theorems. It relates these concepts to control theory. Proofs of most of these results can be found in Sluis [16].

Definition 1. A *Pfaffian system I* on a manifold M is a submodule of the module of differential one-forms $\Omega^1(M)$ over the commutative ring of smooth functions $C^\infty(M)$. The Pfaffian system is generated by a set of one-forms $\{\omega^1, \dots, \omega^n\}$, and $I = \{\Sigma f_k \omega^k | f_k \in C^\infty(M)\}$.

People are often careless about this definition and call the set of generators, or the ideal $\mathcal{I}$ in $\Lambda(M)$ generated by I, a Pfaffian system. Since we are only dealing with Pfaffian systems the term *system* will henceforth mean a Pfaffian system.

Assumption 1. We will assume throughout this paper that the system is regular, i.e., that both the system and the set of exterior derivatives of all generators in the system have constant dimension.

For a Pfaffian system we can define its *derived system* $I^{(1)}$ as $I^{(1)} = \{\omega \in I | d\omega \equiv 0 \mod I\}$. The derived system is itself a Pfaffian system, so we can define the sequence $I, I^{(1)}, I^{(2)}, \dots$ which is called the *derived flag* of I. If the system is regular this sequence is decreasing, so there will be an N such that $I^{(N)} = I^{(N+1)}$. This $I^{(N)}$ is called the *bottom derived system.*

Definition 2. Let I be a Pfaffian system on a manifold M. Let B be a manifold such that $\pi : B \to M$ is a fiber bundle. A Pfaffian system J on B is a *Cartan prolongation* of the system I if the following hold.

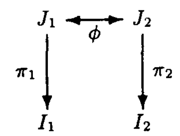
(1) $\pi^*(I) \subset J$

(2) For every integral curve $c : (-\epsilon, \epsilon) \to M$, there is a unique lift $\bar{c} : (-\epsilon, \epsilon) \to B$ with $\pi \circ \bar{c} = c$.

Note that the above definition implies that there is a smooth 1-1 correspondence between solutions of a system and its Cartan prolongation. Cartan prolongations are useful to study equivalence between systems of differential equations that are defined on manifolds of different dimensions. This occurs in dynamic feedback extensions of control systems. We increase the dimension of the state by adding dynamic feedback, but the extended system is still in some sense equivalent to the original system.

This allows us to define the concept of absolute equivalence introduced by Elie Cartan:

Definition 3. Two systems I_1, I_2 are called *absolutely equivalent* if they have Cartan prolongations J_1, J_2 respectively that are equivalent in the usual sense, i.e., there exists a diffeomorphism ϕ such that $\phi^*(J_2) = J_1$. This is illustrated in the following diagram:



When one studies the system of one-forms corresponding to a system of differential equations, the independent variable time becomes just another coordinate on the manifold along with the dependent variables. Hence the notion of an independent variable is lost. If x denotes the dependent variables, a solution to such a system $c : s \to (t(s), x(s))$ is a curve on the manifold. But we are only interested in solution curves which correspond to graphs of functions $x(t)$. Hence we need to reject solutions for which $\frac{dt}{ds}$ vanishes at some point. This is done by introducing dt as an *independence condition*, i.e., a one-form that is not allowed to vanish on any of the solution curves. An independence condition is well defined only up to a nonvanishing multiple and modulo I. We will write a system with independence condition τ as (I, τ).

All prolongations are required to preserve the independence condition, i.e., τ can never become a one-form in the prolonged system.

An interesting subclass of Cartan prolongations is formed by *prolongations by differentiation*: If (I, τ) is a system with independence condition on M, and du an exact one-form on M that is independent of $\{I, \tau\}$, and if y is a fiber coordinate of B, then $\{I, du - y\tau\}$ is called a *prolongation by differentiation* of I. Note that we have omitted writing $\pi^*(du - y\tau)$ where $\pi : B \to M$ is the surjective submersion. We will make this abuse in the rest of the paper for notational convenience. Prolongations by differentiation correspond to adding integrators to a system. The coordinate u is the input that is differentiated.

If we add integrators to all controls, we obtain a *total prolongation*: Let (I, dt) be a system with independence condition, where dim $I = n$. Let $\dim M = n + p + 1$. Let $u_1, \ldots, u_p$ be coordinates such that $du_1, \ldots, du_p$ are independent of $\{I, dt\}$, and let $y_1, \ldots, y_p$ be fiber coordinates of B, then $\{I, du_1 - y_1 dt, \ldots, du_p - y_p dt\}$ is called a *total prolongation* of I. Total prolongations can be defined independent of coordinates, and are therefore intrinsic geometric objects. It can be shown that in codimension 2 (i.e.. a system with n generators on an $n+2$ dimensional manifold), all Cartan prolongations are locally equivalent to total prolongations.

Cartan prolongations provide an intrinsic geometric way to study dynamic feedbacks. We shall show that Cartan prolongations that extend a control system to another control system can be expressed as dynamic feedbacks in local coordinates.

We can view a control system as a Pfaffian system

$$I = \{dx_1 - f_1(x, u, t)dt, \ldots, dx_n - f_n(x, u, t)dt\} \quad (4)$$

with states $\{x_1, \ldots, x_n\}$ and inputs $\{u_1, \ldots, u_p\}$. Note that a control system is always assumed to have independence condition dt. If the functions f are independent of time then we speak of an *autonomous* control system. Clearly, $\{I, dt\}$ is integrable. The converse also holds; i.e., if $\{I, dt\}$ is integrable, then I can locally be written as a control system (see [16]).

We will call *dynamic feedback* a feedback of the form

$$\begin{aligned} \dot{z} &= a(x, z, v, t) \\ u &= b(x, z, v, t). \end{aligned} \quad (5)$$

If t does not appear in (a, b) we call (a, b) an *autonomous* dynamic feedback. An important question is what type of dynamic feedback corresponds to what type of prolongation. Clearly, prolongations by differentiation correspond to dynamic extension (adding integrators to the inputs). The following example shows that not every dynamic feedback corresponds to a Cartan prolongation:

Example 1. Consider the control system

$$\dot{x}_1 = u,$$

with feedback

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= -z_1 \\ u &= g(z)v. \end{aligned}$$

This dynamic feedback introduces harmonic components which can be used to asymptotically stabilize nonholonomic systems [9]. It is not a Cartan prolongation since (z, v) cannot be uniquely determined from (x, u).

It must be said that the feedback in Example 1 is somewhat unusual, in that most theorems concerning dynamic feedback are restricted to adding some type of integrator to the inputs of the system.

Definition 4. Let $\dot{x} = f(x, u, t)$ be a control system. The dynamic feedback in equation (5) is said to be *endogenous* if z and v can be expressed as functions of x, u, t and a finite number of their derivatives:

$$\begin{aligned} z &= \alpha(x, u, \ldots, u^{(p)}, t) \\ v &= \beta(x, u, \ldots, u^{(p)}, t). \end{aligned} \quad (6)$$

Note that this differs slightly from the definition given in [11] due to the explicit time dependence used here. The relationship between Cartan prolongations and endogenous dynamic feedback is given by the following two theorems. The first says that endogenous feedback with b a submersion corresponds to Cartan prolongation.

Theorem 1. *Let I be a control system on an open set $T \times X \times U$ which in coordinates (t, x, u) is given by $\dot{x} = f(x, u, t)$. Let J denote the control system on the open set $T \times X \times Z \times V$ which is obtained from the above system by including a dynamic feedback given by equation (5). Suppose further that the feedback is endogenous and that $\partial b/\partial(z, v)$ is full rank. Then J is a Cartan prolongation of I.*

Proof. Define the mapping $F : T \times X \times Z \times V \to T \times X \times U$ by $F(t, x, z, v) = (t, x, b(x, z, v, t))$. Since b is a submersion so is F. Furthermore b is surjective since the feedback is endogenous. Therefore F is surjective too. Since F is a surjective submersion $T \times X \times Z \times V$ is fibered over $T \times X \times U$. Hence we have that solutions $(t, x(t), z(t), v(t))$ of J project down to solutions $(t, x(t), b(x(t), z(t), v(t), t))$ of I. Therefore the first requirement of being a Cartan prolongation is satisfied. The second requirement of unique lifting property is trivially satisfied by the fact that z and v are obtained uniquely by equation (6). □

Conversely, a Cartan prolongation can be realized by endogenous dynamic feedback, if the resulting prolongation is a control system:

Theorem 2. *Let I be a control system on a manifold M with p inputs, $\{u_1, \ldots, u_p\}$. Every Cartan prolongation $J = \{I, \omega_1, \ldots, \omega_r\}$ on B with independence condition dt such that J is again a control system is realizable by endogenous feedback.*

Proof. Let r denote the fiber dimension of B over M, and let $\{w_1, \ldots, w_r\}$ denote the fiber coordinates. Since I is a control system, (I, dt) is integrable, and we can find n first integrals $x_1, \ldots, x_n$. Integrability of $\{J, dt\}$ means that we can find r extra functions $a_1, \ldots, a_r$ such that $J = \{I, dz_1 - a_1 dt, \ldots, dz_r - a_r dt\}$. Pick p coordinates $v(u, w)$ such that $\{t, x, z, v\}$ form a set of coordinates of B. The v coordinates are the new control inputs. Clearly $a_i = a_i(x, z, v, t)$ since we have no other coordinates. Also since $\{t, x, z, v\}$ form coordinates for B, and u is defined on B, there has to be a function b such that $u = b(x, z, v, t)$. This recovers the form of equation (5). Since J is a Cartan prolongation, every (x, u, t) lifts to a unique (x, z, v, t). From Lemma 1 , to be presented in a later section, it then follows that we can express (z, v) as functions of x

and u and its derivatives. We thus obtain the form of equation (6). □

3. Differentially Flat Systems

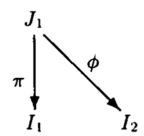
In this section we present a definition of flatness in terms of prolongations. This definition captures the spirit of the original definition in terms differential algebra [5]. Our definition makes use of the concept of an absolute morphism, introduced by Sluis [16].

Definition 5. An *absolute morphism* from a system (I_1, τ_1) on M_1 to a system (I_2, τ_2) on M_2 consists of a Cartan prolongation (J_1, τ_1) on $\pi : B_1 \to M_1$ together with surjective submersion $\phi : B_1 \to M_2$ such that

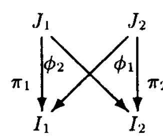
(1) $\phi^*(I_2) \subset J_1$,

(2) $\phi^*(\tau_2) = \lambda \tau_1 \bmod J_1$,

where λ is a smooth, nowhere vanishing function on B_1. This is illustrated below:



Definition 6. Two systems (I_1, τ_1) and (I_2, τ_2) are said to be *absolutely morphic* if there exist absolute morphisms from (I_1, τ_1) to (I_2, τ_2) and from (I_2, τ_2) to (I_1, τ_1). This is illustrated below:



Two systems (I_1, τ_1) and (I_2, τ_2) are said to be *invertibly absolutely morphic* if they are absolutely morphic and the following inversion property holds: let $c_1(t)$ be an integral curve of I_1 with $\tilde{c}_1$ the (unique) integral curve of J_1 such that $c_1 = \pi \circ \tilde{c}_1$, and let $\gamma(t) = \phi_2 \circ \tilde{c}_1(t)$ be the projection of $\tilde{c}_1$. Then we require that $c_1(t) = \phi_1 \circ \tilde{\gamma}(t)$, where $\tilde{\gamma}(t)$ is the lift of γ from I_2 to J_2. The same property must hold for solution curves of I_2.

If two systems are invertibly absolutely morphic, then the integral curves of one system map to the integral curves of the other and this process is invertible in the sense described above. If two systems are absolutely equivalent then they are also absolutely morphic, since they can both be prolonged to systems of the same dimension which are diffeomorphic to each other. However, for two systems to be absolutely morphic we do not require that any of the systems have the same dimension.

A differentially flat system is one in which the "flat outputs" completely specify the integral curves of the system. More precisely:

Definition 7. A system (I, dt) is *differentially flat* if it is invertibly absolutely morphic to the trivial system $I_t = (\{0\}, dt)$.

Notice that we require that the independence condition be preserved by the absolute morphisms, and hence our notion of time is the same for both systems. However, we do allow time to enter into the absolute morphisms which map one system onto the other.

If the system (I, dt) is defined on a manifold M, then we can restrict the system to a neighborhood around a point in M, which is again itself a manifold. We will call a system flat in that neighborhood if the restricted system is flat.

In order to establish the relationship between our definition and the differential algebraic notion of flatness, we need the following lemma on the nature of the dependence of the fiber coordinates of a Cartan prolongation on the coordinates of the base space:

Lemma 1. *Let (I, dt) be a system on a manifold M with local coordinates $(t, x) \in \mathbb{R}^1 \times \mathbb{R}^n$ and let (J, dt) be a Cartan prolongation on the manifold B with fiber coordinates $y \in \mathbb{R}^r$. Assume the regularity assumptions 1 hold. Then on an open dense set, each y_i can be uniquely determined from t, x and a finite number of derivatives of x.*

Proof. By Theorem 24 in [16] there is a prolongation by differentiation, on an open and dense set, say I_2, of J, with fiber coordinates z_i, that is also a prolongation by differentiation of the original system I, say with fiber coordinates w_i. This means that the (x, y, z, t) are diffeomorphic to (x, w, t): $y = y(x, w, t)$. The w are derivatives of x, and therefore the claim is proven. □

This lemma allows us to explicitly characterize differentially flat systems in a local coordinate chart. Let a system in local coordinates (t, x) be differentially flat and let the corresponding trivial system have local coordinates (t, y). Then there are surjective submersions h and g with the following property: Given any curve $y(t)$, then

$$x(t) = g(t, y(t), \ldots, y^{(q)}(t))$$

is a solution of the original system and furthermore the curve $y(t)$ can be obtained from $x(t)$ by

$$y(t) = h(t, x(t), \ldots, x^{(p)}(t)).$$

This follows from using definitions of absolute morphisms, the invertibility property, and Lemma 1, stating that fiber coordinates are functions of base coordinates and their derivatives and the independent coordinate.

This local characterization of differential flatness corresponds to the differential algebraic definition except that h and g need not be algebraic. Also, we do not require the system equations to be algebraic. The explicit time dependence corresponds to the differential algebraic setting where the differential ground field is a field of functions and not merely a field of constants. The functions g and h now being surjective submersions enables us to link the concept of flatness to geometric nonlinear control theory where we usually impose regularity.

Finally, the following theorem allows us to characterize the notion of flatness in terms of absolute equivalence.

Theorem 3. *Two systems are invertibly absolutely morphic if and only if they are absolutely equivalent.*

Proof. The 'if' part is trivial. We shall prove the 'only if' part. For convenience we shall not mention independence conditions. But they are assumed to be present and do not affect the proof. Let I_1 on M_1 and I_2 on M_2 be invertibly absolutely morphic. Let J_1 on B_1 be the prolongation of I_1 with $\pi_1 : B_1 \to M_1$ and similarly J_2 on B_2 be the prolongation of I_2 with $\pi_2 : B_2 \to M_2$. Let the absolute morphisms be $\phi_1 : B_2 \to M_1$ and $\phi_2 : B_1 \to M_2$.

We now argue that J_2 is a Cartan prolongation of I_1 (and hence I_1 and I_2 are absolutely equivalent). By assumption ϕ_1 is a surjective submersion and every solution $\tilde{c}_2$ of J_2 projects down to a solution c_1 of I_1 on M_1. The only extra requirement for J_2 on $\phi_1 : B_2 \to M_1$ to be a (Cartan) prolongation is that every solution c_1 of I_1 has a unique lift $\tilde{c}_2$ (on B_2) which is a solution of J_2.

To show existence of a lift, observe that for any given c_1 which is a solution of I_1, we can obtain its unique lift $\tilde{c}_1$ on B_1 (which solves J_1), and get its projection c_2 on M_2 (which solves I_2) and then consider its unique lift $\tilde{c}_2$ on B_2. Now it follows from the invertibility property that $\phi_1 \circ \tilde{c}_2 = c_1$. In other words, $\tilde{c}_2$ projects down to c_1.

To see the uniqueness of this lift, suppose $\tilde{c}_2$ and $\tilde{c}_3$ which are solutions of J_2 on B_2, both project down to c_1 on M_1. Consider their projections c_2 and c_3 (respectively) on M_2. When we lift c_2 or c_3 to B_2 and project down to M_1 we get c_1. Which when lifted to B_1 gives, say $\tilde{c}_1$. By the requirement of the absolute morphisms being invertible $\tilde{c}_1$ should project down to (via ϕ_2) c_2 as well as c_3. Then uniqueness of projection implies that c_2 and c_3 are the same. Which implies $\tilde{c}_2$ and $\tilde{c}_3$ are the same.

Hence J_2 is a Cartan prolongation of I_1 as well. Hence I_1 and I_2 are absolutely equivalent. □

Using this theorem we can completely characterize differential flatness in term of absolute equivalence:

Corollary 1. *A system (I, dt) is differentially flat if and only if it is absolutely equivalent to the trivial system $I_t = (\{0\}, dt)$.*

It is clear that all feedback linearizable systems are flat, since we can put them into Brunovsky normal form. The converse only holds in an open and dense set, as is shown by the following theorem. An analogous result was proven by Martin in a differentially algebraic setting.

Theorem 4. *Every differentially flat system can be put into Brunovsky normal form in an open and dense set through endogenous feedback.*

Proof. Let J, J_t be the Cartan prolongations of I, I_t respectively. Then from Theorem 24 in [16], on an open and dense set, there is a prolongation by differentiation of J_t that is also a prolongation by differentiation of I_t, say J_{t1}. Let J_1 be the corresponding Cartan prolongation of J. Then J_1 is equivalent to J_{t1}, which is in Brunovsky normal form. In particular, since J_1 is a Cartan prolongation, it can be realized by endogenous feedback. □

Example 2. Consider the motion of a rolling penny, as shown in Figure 1. Let (x_1, x_2) represent the xy position of the penny on the plane, x_3 represent the heading angle of the penny relative to a fixed line on the plane, and x_4 represent the rotational velocity of the angle of Lincoln's head, i.e., the rolling velocity. We restrict $x_3 \in [0, \pi)$ since we cannot distinguish between a positive rolling velocity x_4 at a heading angle x_3 and a negative rolling velocity at a heading angle $x_3 + \pi$.

The dynamics of the penny can be written as a Pfaffian system described by

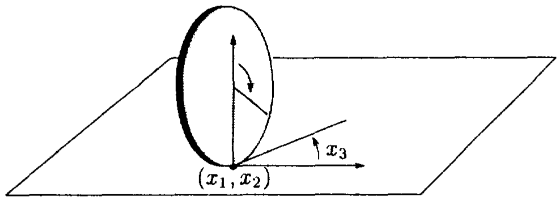


Figure 1: Rolling penny

$$
\begin{aligned}
\omega^1 &= \sin x_3\, dx_1 - \cos x_3\, dx_2 \\
\omega^2 &= \cos x_3\, dx_1 + \sin x_3\, dx_2 - x_4 dt \\
\omega^3 &= dx_3 - x_5 dt \\
\omega^4 &= dx_4 - u_1 dt \\
\omega^5 &= dx_5 - u_2 dt
\end{aligned}
\tag{7}
$$

where $x_5 = \dot{x}_3$ is the velocity of the heading angle. The controls u_1 and u_2 correspond the the torques around the rolling and heading axes. We take dt as the independence condition.

This system is differentially flat using the outputs x_1 and x_2 plus knowledge of time. Given x_1 and x_2, we can use ω_1 to solve uniquely for x_3. Then given these three variables plus time, we can solve for all other variables in the system by differentiation with respect to time. This argument also shows that the system is time independent differentially flat, since we only need to know $y = (x_1, x_2)$ and derivatives of y up to order three in order to solve for all of the states of the system. Moreover, there are no singularities in these equations, so we have a true equivalence.

Notice that this system is *not* equivalent to a chain of integrators. This is because x_3 is determined from x_1 and x_2 by a prolongation which is not a prolongation by differentiation relative to the independence condition dt (although it is still a Cartan prolongation). Once x_1, x_2 and x_3 are determined, the remaining coordinates are determined by differentiation and hence they correspond to a prolongation by differentiation of the system $(\{\omega_1\}, dt)$.

Often we will be interested in a more restricted form of flatness that eliminates the explicit appearance of time that appears in the general definition.

Definition 8. An absolute morphism between two autonomous control systems is a ***time-independent absolute morphism*** if maps the states and inputs of (I_1, dt) to the states and inputs of (I_2, dt) and time is also preserved. A system (I, dt) is ***time-independent differentially flat*** if it is differentially flat using time-independent absolute morphisms.

Note that the example given above is time-independent differentially flat. One might be tempted to think that if the control system I is autonomous and knowing that the trivial system is autonomous, we can assume that the absolute morphism $x = \phi(t, y, y^{(1)}, \ldots, y^{(q)})$ has to be time independent as well. That this is not true is illustrated by the following example.

Example 3. Consider the system $\dot{y} = ay$, and the coordinate transformation $y = x^2 e^{t+x}$. Then $\dot{x} = \frac{(a-1)x}{2+x}$. Both systems are autonomous, but the coordinate transformation depends on time.

4. Flatness for Single Input Systems

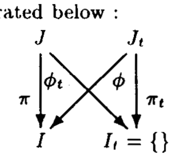
For single input control systems, the corresponding differential system has codimension 2. There are a number of results available in codimension 2 which allow us to give a complete characterization of differentially flat single input control systems. In codimension 2 every Cartan prolongation is a total prolongation around every point of the fibered manifold. This allows us to proof the following

Theorem 5. *Let I be an autonomous control system :*

$$I = \{dx_1 - f_1(x,u)dt, \dots, dx_n - f_n(x,u)dt\},$$

where u is a scalar control, i.e., the system has codimension 2. If I is time-independent differentially flat around an equilibrium point, then I is feedback linearizable by static autonomous feedback at that equilibrium point.

Proof. Let I be defined on M with coordinates (x, u, t), let the trivial system I_t be defined on M_t with coordinates (y, t), let the prolongation of I_t be J_t, and let J_t be defined on B_t. This is illustrated below :



First we show that J_t can be taken as a Goursat normal form around the equilibrium point. In codimension 2, every Cartan prolongation is a repeated total prolongation in a neighborhood of every point of the fibered manifold ([16], Theorem 5). Let $I_{t0} = I_t, I_{t1}, I_{t2}, \dots$ denote the total prolongations starting at I_t, defined on fibered manifolds $B_{t0} = B_t, B_{t1}, \dots$. If y_2 denotes the fiber coordinate of $B_{t,1}$ over B_{t0}, then I_{t1} has the form $\lambda dt + \mu dy_1$. Now, $\mu \neq 0$ at the equilibrium point, since $y_1 \equiv c$ is a solution curve to I_t, which would not have a lift to I_{t1} if $\mu = 0$, since dt is required to remain the independence condition of all Cartan prolongations. From continuity $\mu \neq 0$ around the equilibrium point. So we can define $y_2 := -\lambda/\mu$, and I_{t1} can be written as $dy_1 - y_2 dt$. We can continue this process for every Cartan prolongation, both of I_t and of I. This brings J_t in Goursat normal form in a neighborhood of the equilibrium point.

Now we will argue that we don't need to prolong I to establish equivalence. Since J is a Cartan prolongation, and therefore a total prolongation, its first derived system will be equivalent to the first derived system of J_t. Continuing this we establish equivalence between I and I_{tn}, where $I_{tn} = \{dy_1 - y_2 dt, \dots, dy_n - y_{n+1} dt\}$. So we have $y = (y_1, \dots, y_{n+1}) = y(x, u, t)$.

Next we will show that $y_1, \dots, y_{n+1}$ are independent of time, and that $y_1, \dots, y_n$ are independent of u. By assumption y_{n+1} is independent of time. Since the corresponding derived systems on each side are equivalent, $dy_{n+1} - y_n dt$ is equivalent to the last one-form in the derived flag of I. Since the differential du does not appear in this one-form, y_{n+1} is independent of u. Analogously, $y_i, i = 2, \dots, n$ are all independent of u. Since the $y_i, i = n, \dots, 1$ are repeated derivatives of y_{n+1}, and since I is autonomous, these coordinates are also independent of time.

Therefore $y_i = y_i(x), i = 2, \dots, n+1, y_1 = y_1(x, u)$ and the system J_t' is just a chain of integrators with input y_1. The original system I is equivalent to this linear system by a coordinate transformation on the states and a state dependent and autonomous feedback. This coordinate transformation is well defined around the equilibrium point. It is therefore feedback linearizable by a static feedback that is autonomous. Note that $\partial y_1/\partial u \neq 0$ because y_1 is the only of the y variables that depends on u. □

We will now show that in the case of an autonomous system, we don't need the assumption of time invariant flatness to conclude static feedback linearizability. We will require the following preliminary result :

Lemma 2. *Given a one-form $\alpha = A_i(x,u)dx_i - A_0(x,u)dt$ (using implicit summation) on a manifold R^{n+2} with coordinates (x, u, t), and suppose we can write $\alpha = dX(x,u,t) - U(x,u,t)dt$. Then we can also write α as $\alpha = dY(x) - V(x,u)dt$, i.e., we can take the function X independent of time and the input, and we can take U independent of time. If we know in addition that $\alpha = A_i(x)dx_i - A_0(x)dt$, then we can scale α as $\alpha = dY(x) - V(x)dt$, i.e., we can take V independent of u as well.*

Proof. (based on a suggestion by W. Sluis) Write $\alpha = \eta - A_0(x,u)dt$, where $\eta = A_i(x,u)dx_i$, then

$$\alpha \wedge d\alpha = \eta \wedge d\eta - A_0 dt \wedge d\eta - \eta \wedge dA_0 \wedge dt.$$

We also know

$$\alpha \wedge d\alpha = -dX \wedge dU \wedge dt,$$

and from

$$\alpha \wedge d\alpha \wedge dt = 0$$

it follows that

$$\eta \wedge d\eta \wedge dt = 0,$$

and since η has no t or dt dependence,

$$\eta \wedge d\eta = 0.$$

Hence, $\eta = N(x,u)dM(x)$ for some functions M, N, where $N \neq 0$ due to our regularity assumption. And so

$$\begin{aligned} \alpha &= N(x,u)dM(x) - A_0(x,u)dt \\ &= N(x,u)(dM(x) - A_0(x,u)N^{-1}(x,u)dt) \\ &\simeq dM(x) - A_0(x,u)N^{-1}(x,u)dt \\ &=: dY(x) - V(x,u)dt. \end{aligned} \tag{8}$$

Here $\simeq$ denotes equivalence of Pfaffian systems, in the sense that they generate the same ideal. The second part follows since both η, and therefore N, and A_0, are independent of u. □

Theorem 6. *Let I be a differentially flat, autonomous control system (with a possibly time varying flat output): $I = \{dx_1 - f_1(x,u)dt, \dots, dx_n - f_n(x,u)dt\}$, where u is a scalar control, i.e., the system has codimension 2. Then I is feedback linearizable by static autonomous feedback.*

Proof. Let $\{\alpha^i, i = 1, \dots, n\}$ and $\{\alpha_t^i, i = 1, \dots, n\}$ be one-forms adapted to the derived flag of I, I_t respectively. Thus, $I^{(i)} = \{\alpha^1, \dots, \alpha^{n-i}\}$ and $I_t^{(i)} = \{\alpha_t^1, \dots, \alpha_t^{n-i}\}$. Since I does not contain the differential du, the forms $\alpha^1, \dots, \alpha^{n-1}$ can be taken independent of u. Since I is autonomous, the forms $\alpha_1, \dots, \alpha_n$ can be chosen independent of time. We can thus invoke the second part of Lemma 2 for the forms $\alpha^1, \dots, \alpha^{n-1}$.

Assume $n \geq 2$. As in Theorem 5 we have equivalence between α^1 and $\alpha_t^1 = dy_{n+1}(x,t) - y_n(x,t)dt$ (if $n = 1$ we have $y_n = y_n(x,u,t)$, which we will reach eventually) Since I is autonomous we can choose α^1 time independent: $\alpha^1 = A_i(x)dx_i - A_0(x)dt$. From Lemma 2 we know that we can write α^1 as $dY_{n+1} - Y_n dt$ where Y_{n+1}, Y_n are functions of x only.

Again according to Lemma 2, we can write $\alpha^2 = dV(x) - W(x)dt$. Now from,

$$\begin{aligned} 0 &= d\alpha^1 \wedge \alpha^1 \wedge \alpha^2 \\ &= -dY_n \wedge dt \wedge dY_{n+1} \wedge dV \end{aligned}$$

we know $V = V(Y_n, Y_{n+1})$. And from

$$\begin{aligned} 0 &\neq d\alpha^2 \wedge \alpha^1 \wedge \alpha^2 \\ &= -dW \wedge dt \wedge dY_{n+1} \wedge dV \end{aligned}$$

we know that $\gamma_n := \partial V / \partial Y_n \neq 0$. Then, writing $\gamma_{n+1} := \partial V / \partial Y_{n+1}$, (and $\simeq$ denotes equivalence in the sense that both systems generate the same ideal),

$$\begin{aligned} \{\alpha^1, \alpha^2\} &\simeq \{dY_{n+1} - Y_n dt, \gamma_n dY_n + \gamma_{n+1} dY_{n+1} - W dt\} \\ &\simeq \{dY_{n+1} - Y_n dt, \gamma_n dY_n + \gamma_{n+1} Y_n dt - W dt\} \\ &\simeq \{dY_{n+1} - Y_n dt, dY_n - (-\gamma_{n+1} Y_n + W)/\gamma_n dt\} \\ &:= \{dY_{n+1} - Y_n dt, dY_n - Y_{n-1} dt\}. \qquad (9) \end{aligned}$$

Where Y_{n-1}, defined to be $Y_{n-1} = (-\gamma_{n+1} Y_n + W)/\gamma_n$, is independent of (t,u) since $(\gamma_n, \gamma_{n+1}, Y_n, W)$ are. One can continue this procedure, at each step defining a new coordinate Y_i. In the last step the variable $W = W(x,u)$ (this will also be the first step if $n = 1$), and therefore Y_1 depends on u nontrivially. Hence we obtain equivalence between I and $\{dY_{n+1} - Y_n dt, \dots, dY_2 - Y_1 dt\}$ with $Y_i = Y_i(x), i = 2, \dots, n+1$, and $Y_1 = Y_1(x,u)$, i.e., feedback linearizability by static autonomous feedback. □

Corollary 2. *A single input autonomous control system is differentially flat if and only if it is feedback linearizable by static, autonomous feedback.*

Corollary 3. *If a single input system is differentially flat we can always take the flat output as a function of the states only: $y = y(x)$.*

None of these results easily extend to higher codimensions. The reason for this is that only in codimension 2 every Cartan prolongation is a total prolongation. This is related to the well known fact that for SISO systems static linearizability is equivalent to dynamic linearizability.

5. Concluding remarks

We have presented a definition of flatness in terms of the language of exterior differential systems and prolongations. Our definition remains close to the original definition due to Fliess [5]. But it involves the notion of a preferred coordinate corresponding to the independent variable (usually time).

Using this framework we were able to recover all results in differential algebra. In particular we showed that flatness implies feedback linearizability in an open and dense set. This set need not contain an equilibrium point, and this linearizability is therefore of questionable utility. For a SISO flat system we resolved the regularity issue, and established feedback linearizability around an equilibrium point. We also resolved the time dependence of flat outputs in the SISO case.

The rolling penny is an example of a system that is flat but not linearizable by dynamic feedback. Therefore flatness is more general than feedback linearizability, and a further study is warranted. The most important open question is a characterization of flatness in codimension higher than 2.

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