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EXPONENTIAL STABILIZATION OF DRIFTLESS NONLINEAR CONTROL SYSTEMS VIA TIME-VARYING, HOMOGENEOUS FEEDBACK

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ABSTRACT. This paper brings together results from a number of different areas in control theory to provide an algorithm for the synthesis of locally exponentially stabilizing control laws for a large class of driftless nonlinear control systems. The stability is defined with respect to a non-standard dilation and is termed "δ-exponential" stability. The δ-exponential stabilization relies on the use of feedbacks which render the closed loop vector field homogeneous with respect to a dilation. These feedbacks are generated from a modification of Pomet's algorithm for smooth feedbacks. Converse Lyapunov theorems for time-periodic homogeneous vector fields guarantee that local exponential stability is maintained in the presence of higher order (with respect to the dilation) perturbing terms.

1. Introduction

This paper develops an algorithm for generating stabilizing control laws for systems of the form

$$\dot{x} = \sum_{i=1}^{m} X_i(x)u_i(x,t) \qquad x \in \mathbb{R}^n, \tag{1}$$

where each X_i is an "input" vector field on $\mathbb{R}^n$ and the controls, $u_i(x,t)$, depend on the system state and time. Systems of this form arise in the study of mechanical systems with velocity constraints and have received renewed attention as an example of strongly nonlinear systems. See [14] for an introduction and more detailed motivation. For such systems, control methods based on linearization cannot be applied and nonlinear techniques must be utilized. Other approaches to the stabilization problem may be found in [3, 1, 19].

The type of stabilization achieved by the feedbacks considered in this paper is termed δ-exponential stability since the "norm" used to measure the size of signals is a positive definite function that is homogeneous with respect to a non-standard dilation. This idea was first introduced by Kawski [8]. The definition of δ-exponential stability and its relation to the usual notion of exponential stability are established in Section 2.

The main result of the paper is an algorithm which generates locally δ-exponentially stabilizing feedbacks for (1) under some mild assumptions on the vector fields. The heart of the algorithm is an extension of Pomet's method [16]. In particular, we assume that the input vector fields are analytic and that in coordinates adapted to a suitable filtration (defined below) the homogeneous degree one approximation of the original vector fields, denoted X_i^1, satisfy

$$\begin{aligned} \operatorname{rank}\Big\{ & X_1^1, X_2^1, \ldots, X_m^1, \\ & [X_1^1, X_2^1], \ldots, [X_1^1, X_m^1], \ldots, \\ & \operatorname{ad}^j_{X_1^1} X_2^1, \ldots, \operatorname{ad}^j_{X_1^1} X_m^1, \ldots \Big\}(x_0) = n. \end{aligned} \tag{2}$$

The point x_0 is the desired equilibrium point.

The existence of continuous exponentially stabilizing feedbacks has been resolved by Coron in [2]. How one obtains such feedbacks for a general driftless system is not known. Pomet and Samson [17] have explicitly designed a class of homogeneous controllers which result in δ-exponential stabilization for "chained form" systems. The contributions of this paper lie in setting up a natural framework for studying the δ-exponential stabilization of systems satisfying (2) via an extension of Pomet's synthesis algorithm. The set of systems satisfying (2) contain chained form systems. The results of this paper may be summarized as:

Main Result. Suppose the degree one approximation of the input vector fields of (1) satisfy equation (2). Then the algorithm described in this paper produces continuous time-periodic control functions u_i, that are smooth on $\mathbb{R}^n \setminus \{0\}$ and locally δ-exponentially stabilize $x = 0$.

The primary motivation for obtaining exponential stabilizers comes from the fact that there is significant evidence that more "practical" feedbacks are required in actual applications than are currently available. The synthesis methods in [16] and [20] and elsewhere produce smooth stabilizing feedbacks which always result in algebraic (non-exponential) rates of convergence (see [15]). We have implemented a number of non-exponential and δ-exponential stabilizers on a mobile robot [12]. Our experiments demonstrate that smooth feedbacks do *not* return the robot to a small neighborhood of the origin in a reasonable amount of time and that δ-exponential stabilizers provide an effective alternative. It is important to note that smooth stabilizers cannot transfer the state to a neighborhood of the origin in an arbitrarily small amount of time when constraints are placed on the control signal. Increasing the "gain" of the feedback increases the magnitude of the control effort and decreasing the period of the time dependence effectively increases the *rate* of the signal input into the actuators. Both of these "methods" have been suggested as a way of improving the convergence of the closed loop error. However, upper limits on the control magnitude and rate exist in any physical system. The δ-exponential stabilizers provide practical convergence for low dimensional systems while addressing the issues noted above.

The paper is organized as follows. Section 2 motivates the use of homogeneous feedbacks and reviews

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properties of dilations and homogeneous functions. An approximation theorem by Hermes is also recalled. Section 3 briefly discusses converse Lyapunov theorems for time-varying homogeneous vector fields. These results are simple extensions of the work of Hermes [5] and Rosier [18] and are necessary in order to ensure local δ-exponential stability is not destroyed by higher order perturbations which may be present from the initial approximation process. Section 4 establishes an extension of Pomet's synthesis algorithm adapted to the homogeneous framework. The last section discusses the feedbacks and their implementation.

2. Homogeneous approximations of driftless systems

This section reviews dilations and homogeneous vector fields. We will review only those details necessary for the paper. Previous application of homogeneous approximations and homogeneous feedback are found in the references [6] and [8]. The first application of dilations and homogeneous feedbacks to driftless control systems is contained in the papers [10, 11]. Denote an element of $\mathbb{R}^n$ as $x = (x_1, x_2, \ldots, x_n)$. A dilation on $\mathbb{R}^n$ is defined by assigning n positive rationals $r = (r_1 = 1 \leq r_2 \leq \cdots \leq r_n)$ and the following map $\delta_\lambda^r : \mathbb{R}^n \to \mathbb{R}^n$,

$$\delta_\lambda^r x = (\lambda^{r_1} x_1, \ldots, \lambda^{r_n} x_n), \quad \lambda > 0.$$

We usually write δ_λ in place of δ_λ^r.

Definition 1. A continuous function $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is *homogeneous of degree $l \geq 0$ with respect to ${\delta_\lambda}^r$*, denoted $f \in H_l$ if $f({\delta_\lambda}^r x, t) = \lambda^l f(x,t)$. A continuous vector field $X(x,t)$ on $\mathbb{R}^n \times \mathbb{R}$ is *homogeneous of degree $m \leq r_n$ with respect to δ_λ* if $Xf = \sum_i f_i(x,t)\partial X/\partial x_i(x,t) \in H_{j-m}$ whenever f is smooth and $f \in H_j$. In coordinates, the i^{th} component of a homogeneous vector field of degree m is a homogeneous function of degree $r_i - m$. The parameter t is a time variable in our applications and is never scaled.

Definition 2. A continuous map from $\mathbb{R}^n \to \mathbb{R}$, $x \mapsto \rho(x)$, is called a *homogeneous norm* with respect to the dilation δ_λ when

(1) $\rho(x) \geq 0, \quad \rho(x) = 0 \Leftrightarrow x = 0,$
(2) $\rho(\delta_\lambda x) = \lambda \rho(x) \quad \forall \lambda > 0.$

For example, a homogeneous norm which is smooth on $\mathbb{R}^n \setminus \{0\}$ may always be defined as,

$$\rho(x) = (x_1^{c/r_1} + x_2^{c/r_2} + \cdots + r_n^{c/r_n})^{1/c}, \tag{3}$$

where c is some positive number evenly divizable by r_i. The usual vector p-norms are homogeneous with respect to the standard dilation ($r_i = 1$).

In the sequel we will define continuous homogeneous functions which are differentiable everywhere except the origin. We state some properties of these functions.

Property 2.1. Suppose $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is continuous and differentiable with respect to x on $\mathbb{R}^n \setminus \{0\}$, homogeneous of degree m with respect to the dilation δ_λ. Then $\frac{\partial}{\partial x_i}(f)(x,t)$ is a homogeneous function of degree $m - r_i$ with respect to δ_λ. If $m - r_i > 0$ then we define $\frac{\partial}{\partial x_i}(f)(0,t) = 0$ in order to make the new function continuous on $\mathbb{R}^n$.

Property 2.2. The magnitude of homogeneous functions may be estimated with a given homogeneous norm, ρ. If $f(x,t)$ is a continuous degree m (possibly < 0) function on $\mathbb{R}^n \setminus \{0\}$ then there exists a continuous function $M_1(\cdot)$ such that

$$|f(x,t)| \leq M_1(t)\rho^m(x),$$

When $f(x,t)$ is continuously differentiable with respect to x on $\mathbb{R}^n \setminus \{0\}$ then

$$|\frac{\partial f}{\partial x_i}| \leq M_2(t)\rho^{m-r_i}(x) \quad i = 1, \ldots, n,$$

where $M_2(\cdot)$ is continuous.

Property 2.3. Let $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be continuous and homogeneous of degree $m > 0$ with respect to δ_λ. Let $g : \mathbb{R}^n \setminus \{0\} \times \mathbb{R} \to \mathbb{R}$ be continuous and homogeneous of degree $l > -m$ (in particular, g may be unbounded at the origin), then the function h defined by

$$h(x,t) = \begin{cases} f(x,t)g(x,t) & x \in \mathbb{R}^n \setminus \{0\} \\ 0 & x = 0 \end{cases}$$

is homogeneous of degree $m + l$ and continuous.

The preceding properties are useful when defining a new function as the Lie derivative of a homogeneous function with respect to a homogeneous vector field. Suppose $X : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous homogeneous vector field of degree l and f is a continuous homogeneous function of degree m differentiable on $\mathbb{R}^n \setminus \{0\}$. Then $L_X f$ is a homogeneous function of degree $m - l$. If m is greater than l then the new function is continuous on $\mathbb{R}^n$ if it is defined to be zero at the origin.

Property 2.4. If $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous positive definite homogeneous degree l function, differentiable on $\mathbb{R}^n \setminus \{0\}$, then $\nabla f \neq 0$ for all $x \neq 0$.

These properties are easily proven using simple bounds on homogeneous functions in terms of homogeneous norms. The concept of δ-exponential stability of a vector field is now introduced in the context of a homogeneous norm. This definition was introduced by Kawski [8]. Let $f(t,x)$ be a continuous time-varying vector field,

$$f(t,x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n. \tag{4}$$

We assume that $x = 0$ is an isolated equilibrium point of the system, $f(t,0) = 0, \forall t$. A solution of (4) at time τ passing through x at time t is represented by $\psi(\tau, x, t)$.

Definition 3. The equilibrium point $x = 0$ is *locally exponentially stable with respect to the homogeneous norm* $\rho(\cdot)$ if for some neighborhood, U, of 0 there exist two strictly positive numbers α and β such that

$$\rho(\psi(t, x_0, t_0)) \leq \alpha \rho(x_0) e^{-\beta(t-t_0)} \quad \forall t \geq t_0, x_0 \in U.$$

This notion of exponential stability is called *δ-exponential stability* to distinguish it from the usual definition of exponential stability.

This definition is not equivalent to the usual definition of exponential stability except when the dilation is the standard dilation. However, it is possible to bound the Euclidean norm on the unit cube, $\{x : |x_i| \leq 1\}$,

$$\rho^{r_n}(x) \leq \|x\|_2 \leq K_r \rho^{r_1}(x) \quad \text{for some} \quad K_r > 0,$$

where ρ is taken to be the homogeneous norm in equation (3). Thus for any initial condition in the unit cube, the solutions of a δ-exponentially stable system satisfy,

$$\|\psi(t, x_0, t_0)\|_2 \leq K\|x_0\|_2^{r_1/r_n} \exp(-r_1\beta(t-t_0)),$$

for some $K, \beta > 0$. Thus, each state may be bounded by a decaying exponential envelope. The work in this paper relies on an important property of degree zero vector fields.

Property 2.5. If the vector field (4) is homogeneous of degree zero then local uniform asymptotic stability is equivalent to global exponential stability with respect to the homogeneous norm $\rho(x)$.

Proof. The proof is analogous to the one in Hahn [4] except one uses a nonstandard dilation and corresponding homogeneous norm. □

Now we motivate the use of dilations in the study of driftless control systems and how they appear in the δ-exponential stabilization problem. The interest in applying dilations and their corresponding homogeneous feedbacks lies in the following approximation of the control vector fields in the driftless system (1). From now on, we assume that the equilibrium point of interest is the origin. Let $\mathcal{L}$ be the Lie algebra generated by the set $\{X_1, \ldots, X_m\}$. Define the following increasing filtration at zero, $\mathcal{F}^X$, of $\mathcal{L}$,

$$\begin{aligned} \mathcal{F}_0^X &= \{0\}, \\ \mathcal{F}_1^X &= \text{span}\{X_1, \ldots, X_m\}, \\ &\vdots \\ \mathcal{F}_k^X &= \text{span}\{\text{all products of i-tuples from} \\ &\qquad \{X_1, \ldots, X_m\}\} \quad i \leq k, \\ &\vdots \end{aligned} \tag{5}$$

and $\mathcal{F}^X = \{\mathcal{F}_j^X\}_{j\geq 0}$. We set $n_i = \dim\mathcal{F}_i^X(0)$. The dilation *adapted* to this filtration has the scalings $(r_1, \ldots, r_n)$ where $r_i = 1$ for $1 \leq i \leq n_1$, $r_i = 2$ for $n_1 + 1 \leq i \leq n_2$, etc. The system is controllable so $n_k = n$ for k greater than some finite integer. The local coordinates adapted to $\mathcal{F}^X$ locally express the original vector fields as a sum of vector fields homogeneous with respect to δ_λ^r. Since $X_i(x) \in \mathcal{F}_1^X$ then in these new coordinates (denoted by y),

$$X_i(y) = X_i^1(y) + X_i^0(y) + X_i^{-1}(y) + \cdots, \quad i = 1, \ldots, m,$$

where the $X_i^j(y)$ are vector fields of degree j with respect to δ_λ. Refer to [7] for the local coordinates computation and additional properites of the approximation. The new filtration, denoted $\mathcal{F}^{\hat{X}}$, of the set $\{X_1^1, \ldots, X_m^1\}$ is

$$\begin{aligned} \mathcal{F}_0^{\hat{X}} &= \{0\}, \\ \mathcal{F}_1^{\hat{X}} &= \text{span}\{X_1^1, \ldots, X_m^1\}, \\ &\vdots \\ \mathcal{F}_k^{\hat{X}} &= \text{span}\{\text{all products of i-tuples from} \\ &\qquad \{X_1^1, \ldots, X_m^1\}\} \quad i \leq k, \\ &\vdots \end{aligned}$$

and has the property that

$$\dim\mathcal{F}_i^{\hat{X}}(0) = \dim\mathcal{F}_i^X(0), \quad i = 0, 1, \ldots.$$

In other words, controllability of (1) (i.e. $\dim\mathcal{F}^X(0) = n$) is transferred to the approximating driftless system,

$$\dot{y} = X_1^1(y)u_1 + \cdots + X_m^1(y)u_m. \tag{6}$$

We assume in this paper that the approximate system (6) satisfies equation (2). The use of homogeneous feedback is strongly motivated by the existence of a controllable homogeneous approximating system (6). The natural dilation associated with the driftless control problem is the dilation adapted to the filtration defined above. If homogeneous degree one control functions u_i can be found such that $y = 0$ is a uniformly asymptotically stable equilibrium point of the closed loop system then $y = 0$ is δ-exponentially stable since the closed loop vector field is degree zero (Property 2.5). In the next section we show that the higher order perturbing terms, present when one considers the full set of equations in $y-$coordinates, do not locally change the stability type of the origin. In other words the original control system with feedback,

$$\begin{aligned} \dot{y} =& (X_1^1(y) + X_1^0(y) + \cdots)u_1(y,t) + \cdots + \\ & (X_m^1(y) + X_m^0(y) + \cdots)u_m(y,t), \end{aligned}$$

is still locally δ-exponentially stable.

3. Converse Lyapunov Theory

This section reviews converse Lyapunov stability theory for homogeneous systems and notes a simple extension for degree zero periodic vector fields. Stability theorems for perturbed homogeneous systems were first studied by Hermes [5]. However we use the converse Lyapunov results of Rosier [18] in this paper. The main theorem in [18] states that given an autonomous continuous homogeneous (with respect to some dilation δ_λ) vector field $\dot{x} = f(x)$ with asymptotically stable equilibrium point $x = 0$, there exists a homogeneous (with respect to the same dilation) Lyapunov function, smooth on $\mathbb{R}^n \setminus \{0\}$, and differentiable as many times as desired at the origin.

Rosier's converse theorem extends to the class of continuous, time-periodic, homogeneous degree *zero* systems, $\dot{x} = f(x,t)$, with asymptotically stable equilibrium point $x = 0$. In this case the new function, $\overline{V}(x,t)$, is defined in the same manner as in [18] except that $V(x)$ is replaced with a smooth time-periodic Lyapunov function $\tilde{V}(x,t)$, the existence of which is proven in [9]. This extension is sketched in the appendix of [17]. However we wish to point out that Rosier's method does not necessarily yield a Lyapunov function in the case where the homogeneous vector field has degree different from zero. The important class for us is time-periodic degree zero vector fields and so Rosier's method succeeds in providing a homogeneous Lyapunov function. This is stated as a theorem.

Theorem 3.1 (Rosier). Let $x = 0$ be an asymptotically stable equilibrium point of the continuous time-periodic homogeneous degree zero vector field $\dot{x} = f(x,t)$. Then there exists a homogeneous (with respect to the same dilation as the equation), time-periodic Lyapunov function which is smooth on $\mathbb{R}^n \setminus \{0\}$ and differentiable at the origin as many times as desired.

As a final remark, it is not necessary that $\overline{V}$ be made differentiable at 0 in order for it to define a valid Lyapunov function. Indeed, $\frac{\partial V}{\partial x_i}$ need not be defined at $x = 0$, however $f_i \frac{\partial V}{\partial x_i}$ should always be continuous on $\mathbb{R}^n$. We end this section with a simple stability result for perturbed degree zero vector fields. The proof is a direct extension of the Rosier's results for time-periodic vector fields and so is not given.

Proposition 3.1. Let $x = 0$ be an δ-exponentially stable equilibrium point of the time-periodic continuous homogeneous degree zero vector field $\dot{x} = f(x,t)$. Consider the perturbed system

$$\dot{x} = f(x,t) + R(x,t), \tag{7}$$

where each component of $R(x,t)$ may be uniformly bounded:

$$|R_i(x,t)| \leq m\rho^{r_i+1}(x) \quad i = 1,\ldots,m, \quad x \in U.$$

U is an open neighborhood of the origin and $\rho(\cdot)$ is a homogeneous norm compatible with the dilation that leaves the unperturbed equation invariant. Then $x = 0$ remains a locally δ-exponentially stable equilibrium of the perturbed equation (7).

4. Local δ-Exponential Stabilization Algorithm

This section describes a local δ-exponential stabilization algorithm for driftless systems. It is based on an extension of Pomet's algorithm [16]. Using the approximation results in Section 2 we associate with the original system (in the new coordinates still denoted by x),

$$\dot{x} = \sum_{i=1}^{m} \left(X_i^1(x) + X_i^0(x) + X_i^{-1}(x) + \cdots \right) u_i, \tag{8}$$

the truncated system driftless control system,

$$\dot{x} = \sum_{i=1}^{m} X_i^1(x) u_i. \tag{9}$$

The X_i^1 are analytic vector fields, homogeneous degree 1 with respect to the dilation, δ_λ, defined in the approximation process. We now show how the algorithm in [16] may be modified to provide stabilizers for (9) when (9) satisfies equation (2). Many of the details are skipped in the proof of the extension since they are almost the same as in Pomet's proof. However, to remind the reader of the basic idea behind Pomet's algorithm we give a heuristic overview of how the algorithm works. Supposing the input vector fields satisfy (2), a 2π-periodic function of time, $\alpha(x,t)$, is chosen so that all nonzero solutions of $\alpha(x,t)X_1(x)$ are 2π-periodic and $x = 0$ is an equilibrium point. In order to define a positive definite function on the phase space, each closed periodic "loop" is assigned a positive number. This is accomplished by defining a positive definite function on a Poincaré map associated with the flow of αX_1. In other words, the flow is sampled at $t_0 \in [0, 2\pi)$ and then a positive definite function is applied to the value of the flow at this time. This resulting number is denoted $V(x,t)$. The feedback u_1 is defined to be the open loop part, α, *minus* the Lie derivative of $V(x,t)$ with respect to the vector field X_1. The remaining inputs $u_i, i = 1,\ldots,m$, are defined to be the minus the Lie derivative of $V(x,t)$ with respect to X_i. This choice of feedbacks guarantee that $x = 0$ is *stable*. Under some extra conditions the feedback can be shown to be uniformly asymptotically stabilizing.

The extension of Pomet's algorithm to δ-exponentially stabilize systems of the form (9) is now developed. The following modification of Proposition 1 in [16] is made (as in [16], the vector field X_1^1 plays a particular role),

Proposition 4.1. Let $\alpha : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a time-periodic, smooth on $\mathbb{R}^n \setminus \{0\} \times \mathbb{R}$, homogeneous degree one function with respect to δ_λ. Assume α also satisfies the following conditions,

$$\begin{aligned} \alpha(x, t+2\pi) &= \alpha(x,t) \quad \forall(t,x) \\ \alpha(x,-t) &= -\alpha(x,t) \quad \forall(t,x) \\ \alpha(0,t) &= 0 \quad \forall t. \end{aligned} \tag{10}$$

Let $V : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a function defined as,

$$V(x,t) = \varrho(\psi(0,x,t)),$$

where $\varrho : \mathbb{R}^n \to \mathbb{R}$ is any positive definite homogeneous degree 2 function that is smooth on $\mathbb{R}^n \setminus \{0\}$. Here $\psi(\tau,t,x)$ represents the flow of the vector field $\alpha(x,t)X_1^1(x,t)$ evaluated at time τ and passing through x at time t. The function V has the following properties,

(1) V is smooth on $\mathbb{R}^n \setminus 0$,
(2) V is homogeneous degree 2 with respect to δ_λ,
(3) V is 2π-periodic with respect to t: $V(t+2\pi, x) = V(t,x)$,
(4) $V(x,t) = 0 \quad \Leftrightarrow \quad x = 0$,
(5) $\frac{\partial}{\partial x} V(t.x) \neq 0 \quad \forall x \neq 0$ (the gradient at 0 may not be defined),
(6) $V(x,t)$ is a proper map $\forall t \in [0, 2\pi)$.

Proof. The product of the scalar degree one *function* $\alpha(x,t)$ with the degree one *vector field* $X_1^1(x)$ defines a degree *zero* vector field $(\alpha X_1^1)(x,t)$, by the convention established in Definition 1. This new vector field is smooth on $\mathbb{R}^n \setminus \{0\}$ and its flow is complete. Completeness follows from the dilation scaling property enjoyed by solutions of degree zero vector fields and the exponential upper bound on the growth of solutions (this bound is established using the same techniques as the proof of Property 2.5; see the appendix of [12] for explicit computations). Hence, $\psi(\tau,x,t)$ is a homeomorphism $\forall \tau,t,x$ and a smooth diffeomorphism $\forall \tau, t$ and $x \neq 0$. The proof showing the flow is 2π-periodic is identical to the proof in [16]. Items (1) to (3) are easily shown using the properties of the flow and the function ϱ. Item (4) follows from the fact ϱ is positive definite and the origin and any nonzero x cannot lie on the same trajectory. Item (5) may be written for $x \neq 0$,

$$\frac{\partial}{\partial x} V(x,t) = \nabla\varrho(0,x,t) \cdot D_x\psi(0,x,t).$$

$\nabla\varrho(y) \neq 0$ for $y \neq 0$ from Property 2.4 and $D_x\psi$ is full rank for nonzero x. Lastly, $V(x,t)$ is proper for any $t \in [0, 2\pi)$ since it satisfies the bounds $c_1\rho^2(x) < V(x,t) < c_2\rho^2(x)$. □

The following choice of inputs u_i render (9) stable,

$$\begin{aligned} u_1(t,x) &= \alpha(t,x) - L_{X_1^1} V(x,t) \\ u_2(t,x) &= -L_{X_2^1} V(x,t) \\ &\vdots \\ u_m(t,x) &= -L_{X_m^1} V(x,t). \end{aligned} \tag{11}$$

Note that these control functions are smooth functions of t and $x \in \mathbb{R}^n \setminus \{0\}$. Under additional assumptions $x = 0$ is exponentially stable with respect the homogeneous norm.

Theorem 4.1. Suppose the approximate system satisfies (2) and an α satisfying Proposition 4.1 is chosen. If the following conditions are satisfied,

$$\left.\begin{array}{r} L_{X_1}V(t,x) = \ldots = L_{X_m}V(t,x) = 0 \\ \alpha(t,x) = \frac{\partial\alpha}{\partial t}(t,x) = \frac{\partial^2\alpha}{\partial t^2}(t,x) = \ldots = 0 \end{array}\right\} \Rightarrow x = 0, \quad (12)$$

then $x = 0$ is a globally δ-exponentially stable equilibrium point of (9) with respect to the dilation when the feedback (11) is applied.

Proof. The proof that feedback (11) is asymptotically stabilizing when the hypothesis of Theorem 4.1 is satisfied is virtually identical to the one given in [16] and so the reader is referred to this reference for the details. The control functions defined by (11) are smooth on $\mathbb{R}^n \setminus \{0\}$ and degree one since $\alpha(x,t)$ is degree one and $\deg\left(L_{X_i^1}V(x,t)\right) = \deg(V(x,t)) - \deg\left(X_i^1(x)\right) = 2 - 1$. Applying these degree one feedbacks to the degree one vector fields (9) results in a degree zero closed loop vector field. The vector field is time-periodic so the asymptotic stability of the zero solution is uniform in time which, by Property 2.5, is δ-exponential. □

Remark. In practice it may be difficult verify the conditions in the theorem to conclude asymptotic stability. It is useful to choose α such that $\alpha(t,x) = 0 \Leftrightarrow x = 0$. For example, $\alpha(x,t) = \rho(x)\sin t$, where ρ is any smooth homogeneous norm, satisfies the hypothesis of the theorem.

We end this section by demonstrating that the terms neglected in the truncated system do not locally change the stability of the equilibrium point. We state this in a proposition.

Proposition 4.2. Suppose the conditions of Theorem 4.1 hold. Then the feedback (11) locally δ-exponentially stabilizes the original system (8).

Proof. Consider the feedback (11) applied to (8) but written as $\dot{x} = \sum_{i=1}^{n} X_i^1(x)u_i(x,t) + R(x,t)$, where $R(x,t) = \sum_{i=1}^{n}\left(\sum_{j=1}^{\infty} X_i^{1-j}(x)\right)u_i(x,t)$. The m vector fields $\sum_{j=1}^{\infty} X_i^{1-j}(x), i = 1,\ldots,m$, are analytic and the k^{th} component is a sum of homogeneous polynomials of degree greater than or equal to r_k so that the absolute value of k^{th} component is bounded by $c_i\rho^{r_k}(x)$ in a sufficiently small neighborhood of the origin. Since the u_i are homogeneous degree one functions then the absolute value of the k^{th} component of $R(x,t)$ may be bounded by a scalar times ρ^{r_k+1} in a neighborhood of the origin. The local stability result follows from application of Proposition 3.1. □

Remark. Certain driftless control systems may be transformed to exactly a nilpotent homogeneous form. Examples are the chained form or power form systems [14, 20]. In this case Theorem 4.1 provides a globally δ-exponentially stabilizing feedback since there are no "higher order" perturbing terms.

Finally the algorithm may be summarized as,

(1) Compute the local coordinate change which places the input vector fields in form

$$\dot{x} = \sum_{i=1}^{m}\left(X_i^1(x) + X_i^0(x) + X_i^{-1}(x) + \cdots\right)u_i.$$

(2) If the relation

$$\operatorname{rank}\Big\{X_1^1, X_2^1, \ldots, X_m^1, [X_1^1, X_2^1], \ldots, [X_1^1, X_m^1], \ldots, \operatorname{ad}^j_{X_1^1}X_2^1, \ldots, \operatorname{ad}^j_{X_1^1}X_m^1, \ldots\Big\}(0) = n.$$

is satisfied then continue with the procedure.

(3) Construct homogeneous degree one feedbacks, using the approximate control system,

$$\dot{x} = X_1^1(x)u_1 + \cdots + X_m^1(x)u_m,$$

according to Proposition 4.1 and equation (11).

(4) These feedback applied to the original system are still locally δ-exponentially stabilizing by Proposition 4.2.

5. Discussion

In general, it will be impossible to compute the feedbacks explicitly since they depend on the flow of a nonlinear ODE. This should not be considered a limitation of the method since the low cost of computer memory makes look-up tables a feasible solution.

The smooth nature of the feedbacks on $\mathbb{R}^n \setminus \{0\}$ is important. In practice, the control engineer would desire the feedbacks to be at least Lipschitz on this set. The reason for this is as follows. Suppose there is a submanifold on $\mathbb{R}^n \setminus \{0\}$, denoted A, such that the feedbacks are not Lipschitz in a neighborhood of A. If there are trajectories of the closed loop system which are transverse to A then the control signal does not posses a bounded time derivative. Since most driftless models are derived from kinematic constraints, the control inputs are velocities. An unbounded velocity derivative implies unbounded forces to effect the desired motion. A real actuator would find this control signal difficult to track. Along these same lines, we have shown [13] that the homogeneous "kinematic" inputs of a nonholonomic control system may be extended to homogeneous "torque" inputs which still provide δ-exponential stabilization.

An example illustrating the stabilization algorithm of Section 4 and its robustness to higher order perturbations is presented below.

Example 5.1. Consider the two input driftless system,

$$\dot{x} = \begin{pmatrix} \frac{1}{2}x_1 \\ 1 + \frac{1}{6}x_2 \\ \frac{1}{5}x_1x_2^2 \end{pmatrix}u_1 + \begin{pmatrix} 1 + \frac{1}{4}x_2 \\ \frac{1}{2}x_1x_2 \\ \frac{1}{2}x_2^2 + \frac{1}{3}x_1^3 \end{pmatrix}u_2. \quad (13)$$

(2) is satisfied since

$$\operatorname{rank}\{X_1(0), X_2(0), [X_1, X_2](0), \operatorname{ad}^2_{X_1}X_2(0)\} = 3$$

The dilation scalings are computed as $r_1 = r_2 = 1$ and $r_3 = 3$. The homogeneous norm which we will use is

$$\rho(x) = (x_1^6 + x_2^6 + x_3^2)^{(1/6)}. \quad (14)$$

The input vector fields are in the proper coordinates and so the degree one approximate system is obtained by truncating all terms in the i^{th} component of (13) which are

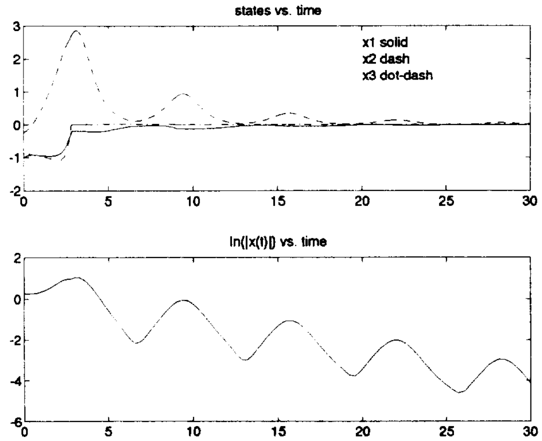

Figure 1: State versus time

degree r_i or greater. Thus the approximate system for which we design the feedback is

$$\begin{aligned} \dot{x} &= X_1^1(x)u_1 + X_2^1(x)u_2 \\ &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2}x_2^2 \end{pmatrix} u_2. \end{aligned} \tag{15}$$

Define the "open loop" input to be $\alpha(x,t) = \rho(x)\sin t$. The flow of the vector field

$$\begin{aligned} \dot{x} &= \alpha(x,t)X_1^1(x) \\ &= \rho(x)\sin t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \end{aligned}$$

is denoted $\psi(\tau, x, t)$. One choice for the positive definite degree 2 function ϱ is

$$\varrho(p) = \frac{1}{2}\left(p_1^2 + p_2^2 + \frac{p_3^2}{\rho^4(p)}\right).$$

Hence, the Lyapunov function V is defined as $V(x,t) = \varrho(\psi(0,x,t))$. The actual feedbacks are

$$\begin{aligned} u_1 &= \alpha(x,t) - L_{X_1^1(x)}V(x,t) \\ u_2 &= -L_{X_2^1(x)}V(x,t). \end{aligned} \tag{16}$$

These functions locally exponentially stabilize (13) with respect the norm (14) and can be numerically computed. Computer simulations of (13) with the feedbacks (16) are shown in Figure 1. The system is obviously asymptotically stable even in the presence of the perturbing terms. A plot of $\ln(\|x(t)\|)$ demonstrates the exponential convergence because of the linear upper bound with negative slope.

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