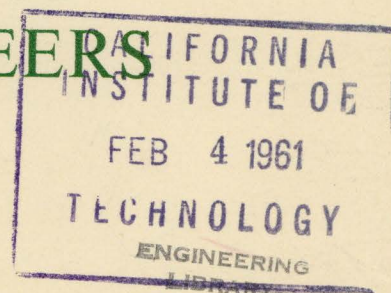




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ELECTRO-MECHANICAL MODES IN PLASMA WAVEGUIDES

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SUMMARY

In addition to increasing the cut-off frequencies of TM-modes in a waveguide, the introduction of a plasma column into the waveguide also introduces new modes of propagation. The properties of these modes, including the effect of an axial d.c. magnetic field but neglecting ion motion, have been studied by solving the field equations considering the electron plasma as a dielectric. The new modes generally have phase velocities much less than the velocity of light; one type exists down to zero frequency and another type is a backward wave. Neither the metallic conductor nor the axial magnetic field is essential to the existence of slow modes. Angular-dependent modes can exhibit Faraday rotation of polarization.

A qualitative explanation of these modes is given in terms of an equivalent electrical circuit for the transmission line. Many of the properties of these modes have been verified experimentally by measuring phase velocity of waves along a mercury-arc discharge in an axial magnetic field. These modes are closely related to space-charge waves in electron beams, and several interesting microwave applications are suggested.

(1) INTRODUCTION

The solution of Maxwell's equations in plasma-filled waveguides has received considerable attention,¹ but generally speaking, this attention has been directed toward finding how the electromagnetic waves are modified by the presence of the plasma. Space-charge waves on electron beams surrounded by a conducting drift tube have been studied extensively,² but these waves are generally associated with the drifting motion of the beam. It is not generally known that they also exist and carry energy in the absence of any drift motion or in the plasma-filled waveguide. The paper deals with the characteristics of these and related waves in an ideal ion-neutralized electron plasma.

These waves are to be distinguished from magnetohydrodynamic waves which exist in a plasma³ in which the collision frequency is much greater than the frequencies of interest, so that displacement currents may be neglected.

(2) INFINITE MAGNETIC FIELD

Let us consider first a cylindrical waveguide which is completely filled with the ideal plasma with a very large axial magnetic field. A rigorous small-signal analysis of the TM-waves* yields an expression for the propagation coefficient

$$\gamma^2 = \omega^2 \mu_0 \epsilon_0 - \frac{T^2}{1 - \frac{\omega_p^2}{\omega^2}}, \quad T = \frac{P_{nl}}{b} \quad \dots \quad (1)$$

where P_{nl} is the l th root of $J_n(x) = 0$, $\omega_p = \sqrt{\frac{\rho_0 e}{\epsilon_0 m}}$ is the electron plasma frequency, ρ_0 is the average electron charge density, and b is the waveguide radius. Fig. 1 shows the resulting ω/γ

* TE-waves are unaffected by the plasma.

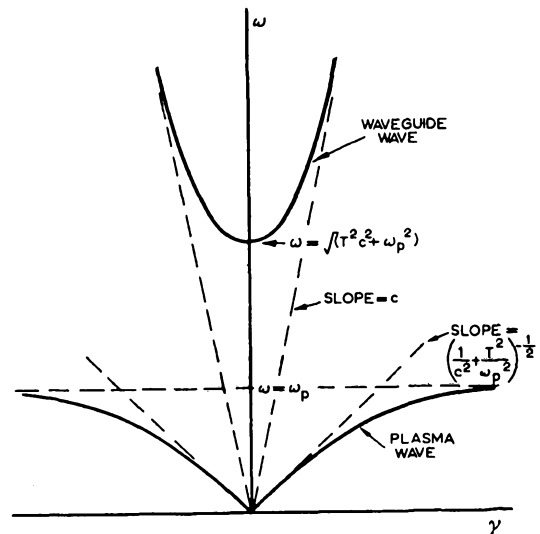


Fig. 1.—Phase characteristics of waves in a plasma-filled waveguide with an infinite axial magnetic field.

diagram with two pass bands. The upper branch represents the modified electromagnetic wave and the lower branch represents the space charge or electro-mechanical mode. The latter has a high-frequency cut-off $\omega = \omega_p$ and no low-frequency cut-off. When the waveguide radius is small ($b \ll c/\omega_p$) the low-frequency phase velocity is approximately ω_p/T and much less than the velocity of light. The velocity is thus determined by the waveguide size and plasma frequency. Actually there are an infinite number of these modes, each with different r and θ variations in the fields, corresponding to different integers l and n . However, all have ω_p as the upper cut-off frequency. The non-zero group velocity of these waves implies that there is the possibility of signalling and energy transmission.

A qualitative explanation of the slow, low-frequency modes can be given as follows. Electrons can move freely in the axial direction but are prevented from moving in the transverse direction by the strong magnetic field. Thus the medium can be described by longitudinal³ and transverse permittivities

$$\epsilon_l = \epsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2}\right), \quad \epsilon_t = \epsilon_0$$

Transverse displacement currents $j\omega\epsilon_t E_t$ lead the transverse electric field by 90° , whereas longitudinal displacement currents $j\omega\epsilon_l E_l$ lag the longitudinal electric field by 90° when ω is less than ω_p . Thus we may think of the space as capacitive in the transverse direction and inductive in the longitudinal direction. Carrying this idea a little further yields the equivalent electrical network of Fig. 2(a), which represents the relations between the field (or voltage) and displacement current. Propagation is possible for $\omega < \omega_p$, i.e. when the series branch is inductive, but is not possible for $\omega > \omega_p$, i.e. when the series branch is capacitive.

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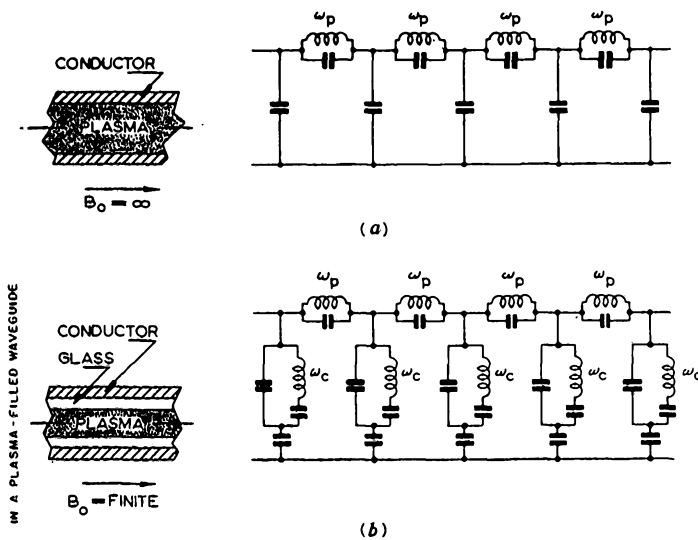


Fig. 2.—Equivalent circuits for plasma-filled waveguides.

(3) FINITE MAGNETIC FIELD

When the d.c. magnetic field B_0 is finite, electron motion perpendicular to the magnetic field is possible. The displacement in the plasma medium may be related to the electric field by a dyad

$$D = \epsilon \cdot E \quad (2)$$

where

$$\epsilon = \epsilon_0 \begin{vmatrix} e_r e_r \left(1 - \frac{\omega_p^2}{\omega^2 - \omega_c^2}\right) & e_r e_\theta \frac{-j\omega_c \omega_p^2}{\omega(\omega^2 - \omega_c^2)} & 0 \\ e_\theta e_r \frac{+j\omega_c \omega_p^2}{\omega(\omega^2 - \omega_c^2)} & e_\theta e_\theta \left(1 - \frac{\omega_p^2}{\omega^2 - \omega_c^2}\right) & 0 \\ 0 & 0 & e_z e_z \left(1 - \frac{\omega_p^2}{\omega^2}\right) \end{vmatrix} \quad (3)$$

and $\omega_c = eB_0/m$ is the electron cyclotron frequency. Exact solution of Maxwell's equations in a waveguide filled with a dielectric with these properties is possible,¹ but the resulting determinantal equation for the propagation coefficients is not readily solved. For slow waves ($v_{phase}^2 \ll c^2$) the electric field can, to a good approximation, be derived from a scalar potential and the a.c. magnetic field neglected. An appropriate solution to the equation for this electrostatic potential is*

$$\phi = AJ_n(Tr)\epsilon^{j(\omega t - n\theta - \gamma z)} \quad (4)$$

$$\text{where } T^2 = -\gamma^2 \frac{\epsilon_{zz}}{\epsilon_{rr}} = -\gamma^2 \frac{(\omega^2 - \omega_p^2)(\omega - \omega_c^2)}{\omega^2(\omega^2 - \omega_p^2 - \omega_c^2)} \quad (5)$$

If the plasma completely fills the waveguide, $J_n(Ta)$ must be zero so that $T = P_{nl}/a$, where P_{nl} is the l th zero of the n th order Bessel function. If the plasma does not fill the waveguide, the equation determining the propagation coefficients is

$$\frac{\epsilon_{rr} Ta J_n'(Ta)}{\epsilon_0 J_n(Ta)} - jn \frac{\epsilon_{r\theta}}{\epsilon_0} = \gamma a \frac{I_n'(\gamma a) K_n(\gamma b) - I_n(\gamma b) K_n'(\gamma a)}{I_n(\gamma a) K_n(\gamma b) - I_n(\gamma b) K_n(\gamma a)} \quad (6)$$

where T is given by eqn. (5), a is the plasma radius and b is the waveguide radius.

The results are particularly simple when the plasma fills the waveguide ($b = a$). The propagation coefficient is given by

$$\gamma^2 = -\frac{\omega^2(\omega^2 - \omega_p^2 - \omega_c^2)}{(\omega^2 - \omega_p^2)(\omega^2 - \omega_c^2)} \left(\frac{P_{ln}}{a}\right)^2 \quad (7)$$

* The derivation of this result is outlined in Section 10.

The resulting ω/γ diagram is shown in Fig. 3 for two cases: $\omega_c > \omega_p$ and $\omega_c < \omega_p$. The perturbed waveguide modes are not predicted by the static analysis but are normally assumed to occur at much higher frequencies than shown in these Figures. When $\omega_c > \omega_p$, eqn. (7) agrees with the second term of eqn. (1). The first term of eqn. (1) is negligible for slow waves. Thus both

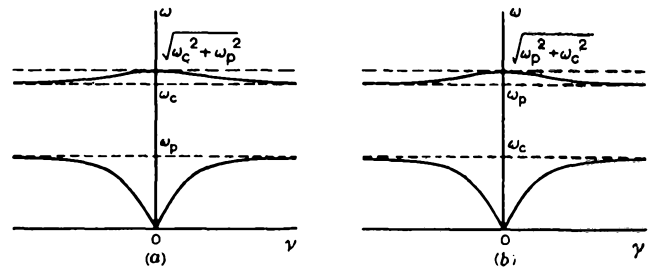


Fig. 3.—Phase characteristics of waves in a plasma-filled waveguide.

analyses predict the electro-mechanical or space-charge mode at low frequencies in this limiting case. At low frequencies the phase velocity is approximately constant and is given by

$$v_{ph} = \frac{a}{P_{nl}} \frac{\omega_p \omega_c}{\omega_p^2 + \omega_c^2} = \frac{a}{P_{nl}} \omega_p \text{ when } \omega_c^2 > \omega_p^2 \quad (8)$$

The static approximation furthermore predicts the existence of a backward wave, near the cyclotron frequency ω_c when $\omega_c > \omega_p$. The propagation band of this wave is normally rather narrow.

The equivalent transmission line of Fig. 2(a) can be extended to the finite-magnetic-field case, as shown in Fig. 2(b). Additional elements appear in the shunt branch. The transverse currents of the plasma are represented by a series inductance and capacitance resonant at frequency ω_c . In parallel with these elements is the capacitance of free space. When the plasma does not completely fill the waveguide an additional element is used to represent the capacitance from the surface of the plasma to the conducting cylinders. This equivalent transmission line yields phase characteristics similar to those of Fig. 3. In the backward wave band the series arm is capacitive and the shunt arm inductive. It is also of interest to note that, when there is no magnetic field, wave propagation is still possible at low frequencies by virtue of the capacitance of the space outside the plasma ($b \neq a$) and the inductance of the plasma.

The plasma waves have the following additional interesting properties:

- (a) The cut-off frequencies are determined by the dielectric properties and not the dimensions of the waveguide (when $b = a$). Only the phase velocities are influenced by the waveguide dimension.
- (b) Modes of different transverse dependence (different values of l and n) all have the same cut-off frequencies (when $a = b$).
- (c) Very slow waves are possible in small waveguides.
- (d) There is no Faraday rotation in the angular-dependent modes when $b = a$.

(4) FARADAY ROTATION

The phenomenon of Faraday rotation was an important consideration in the analysis of the perturbation of waveguide modes by the plasma,¹ and it is of interest to consider that possibility here. A superposition of the $n = +1$ and $n = -1$ modes of equal amplitudes yields a composite wave in which the transverse field is polarized in a certain direction. If the $n = +1$ and $n = -1$ modes have different phase velocities the direction of polarization of the composite wave will slowly rotate as the wave propagates down the waveguide. Higher-order angular-dependent modes may exhibit an analogous effect. When the plasma fills the waveguide eqn. (7) shows that modes having n of opposite signs propagate with exactly the same speed, and

thus there should be no rotation of polarization. On the other hand, when the plasma does not completely fill the waveguide, eqn. (6) must be solved. In that equation, n enters in such a way as to cause modes of opposite signs of n to have different phase velocities. Furthermore, reversing the direction of the magnetic field changes the sign of $\epsilon_{r\theta}$, and from eqn. (6) this has the same effect as changing the sign of n . Thus reversing the direction of the magnetic field should reverse the direction of rotation. Since the wave velocity is small, rather large rotations per unit length are expected.

(5) ATTENUATION AND INTERACTION IMPEDANCE

When the plasma electrons are produced by an electrical discharge in a gas, attenuation may arise because of collisions of the electrons with neutral gas molecules, positive ions, or the wall of the discharge tube. These collisions interrupt interaction with the wave and remove energy from it. An approximate way of including this effect is to define an average electron collision frequency ν_c and replace ω by $\omega - j\nu_c$ in the preceding equations. An approximate solution to these equations is obtained by writing

$$\alpha(\omega, \nu_c) + j\gamma(\omega, \nu_c) = j\gamma(\omega, 0) + j\frac{d\gamma(\omega, 0)}{d\omega}(-j\nu_c) + \dots$$

When ν_c is small the first two terms give a satisfactory approximation

$$\alpha(\omega, \nu_c) = \frac{d\gamma}{d\omega}\nu_c \dots \dots \dots (9)$$

$$\gamma(\omega, \nu_c) = \gamma(\omega, 0) \dots \dots \dots (10)$$

Thus, to a first approximation, the phase velocity is unaffected by collisions and the attenuation is proportional to the collision frequency and inversely proportional to group velocity, $d\omega/d\gamma$.

The travelling-wave-tube interaction impedance of the lowest circularly symmetrical mode has been estimated by finding the a.c. magnetic field H_θ to the first order (in finding the propagation coefficients H was neglected entirely) from the Maxwell equation $\nabla \times H = j\omega\epsilon \cdot E$. From this the Poynting vector may be evaluated, and the interaction impedance on the axis is found to be

$$K_{axis} = \frac{E_z^2}{2\gamma^2 P} = \frac{c}{a} \sqrt{\frac{\mu_0}{\epsilon_0}} \sqrt{\frac{(\omega_c^2 - \omega^2)}{(\omega^2 - \omega_p^2 - \omega_c^2)(\omega_p^2 - \omega^2)}} \frac{1}{\pi P_{01} J_1^2(P_{01})} \dots \dots \dots (11)$$

Thus the interaction impedance is zero when $\omega = \omega_c$ and infinite when $\omega = \omega_p$ and $\omega = \sqrt{(\omega_p^2 + \omega_c^2)}$.

(6) RELATION TO SPACE-CHARGE WAVES IN ELECTRON BEAMS

The waves in an ideal plasma are very closely related to space-charge waves in electron beams, the principal difference being that the electrons in the beam have a drift velocity along the axis of the system. The solutions already found are readily modified to include a drift velocity u_0 along the axis of symmetry by means of a transformation of co-ordinates. If ω and γ are the frequency and propagation coefficient, respectively, in the co-ordinate system in which the electrons are at rest, the frequency ω' and propagation coefficient γ' in a co-ordinate system in which the electrons drift with velocity u_0 ($u_0 \ll c$) are*

$$\omega' = \omega + \gamma' u_0 \dots \dots \dots (12)$$

$$\gamma' = \gamma \dots \dots \dots (13)$$

* A non-relativistic result.

The wavelengths are the same and ω' is just the Doppler shifted frequency. The solutions previously obtained can be easily carried over the drifting case. A typical ω/γ diagram constructed from Fig. 1 with the aid of eqns. (12) and (13) is shown in Fig. 4. The slanted dashed lines denote the space-charge-wave

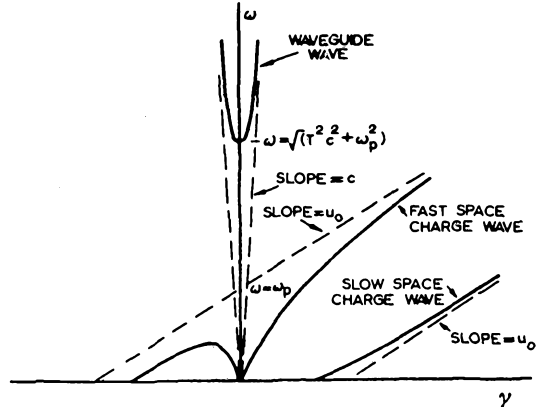


Fig. 4.—Phase characteristics of waves in an electron beam of velocity u_0 in an infinite magnetic field.

solutions in a beam of infinite radius, or the one-dimensional space-charge waves. The departure from these curves when the beam radius is small may be ascribed to the reduction of a.c. space-charge fields by the nearby conductor, and the lower branch of Fig. 1 is essentially a plot of the space-charge-wave reduction factor against propagation.

Of special interest in Fig. 4 are the two additional waves with negative propagation coefficients at low frequencies ($\omega < \omega_p$). One is a backward wave. Unless u_0 is small these waves are absent. They may be of importance in the understanding of noise propagation near the potential minimum of a diode where the drift velocity is low.

(7) EXPERIMENT

To verify some of the features of the theory described, the phase shift and attenuation of waves along a plasma column have been measured with the apparatus shown in Fig. 5. The

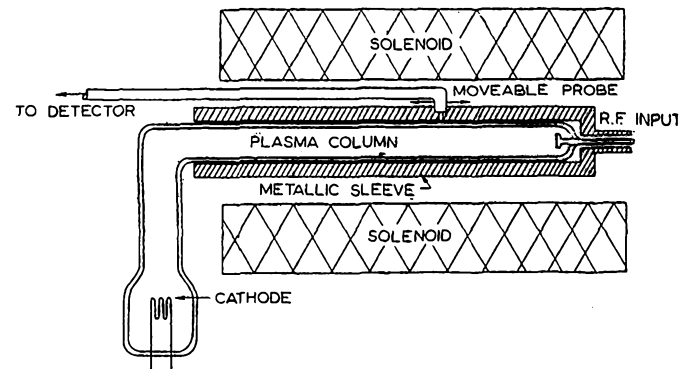


Fig. 5.—Schematic of apparatus for measuring wavelengths and attenuation in a plasma-filled waveguide.

plasma is the positive column of a mercury-arc discharge. The radio-frequency signal is introduced on the anode and the signal along the column is sampled with a movable probe. Phase shift and attenuation are determined by comparing the probe signal with reference signal from the signal generator. A solenoid provides the axial magnetic field. Propagation has

been investigated in the region 20 Mc/s-4 Gc/s. The cyclotron and plasma frequencies are adjustable over about this same range. The existence of the two bands of propagation and the way in which the cut-off frequencies vary with magnetic field and arc current (the plasma frequency is approximately proportional to the square root of the arc current) have been verified.

Measurements in the lower band have been used to determine the collision frequency ν_c and electron plasma frequency ω_p . Certain small systematic variations have been interpreted in terms of non-uniform charge density, and they yield information about the variation of charge density with radius. Near the cut-off of the lower band and in the upper band a considerable noise modulation of the transmitted signal has prevented satisfactory measurements, although transmission is clearly discernible. The cause of this noise is not known, but it may be due to fluctuations of the plasma frequency along the column. Experiments are now in progress to verify the existence of the $n = \pm 1$ angular-dependent modes and to measure Faraday rotation.

(8) CONCLUSION

The existence of electro-mechanical modes of propagation in plasma waveguides has been demonstrated theoretically and experimentally. When the plasma diameter is small these modes have low phase velocity and hence high phase shift per unit length. Furthermore the phase shift is controllable by controlling the discharge current. Thus these modes may be of interest in the design of new low-frequency electrically-controlled phase-shifters. The angular-dependent modes should exhibit Faraday rotation, with higher rotation per unit length than is generally obtained with the perturbed waveguide modes. This could make possible relatively small low-frequency isolators provided that the ratio of attenuation to rotation can be made small enough.

The existence of the backward wave makes possible the design of a backward-wave oscillator in which a separate electron beam is allowed to interact with the backward wave of the plasma. In generating higher and higher microwave frequencies this shifts the emphasis from the fabrication of very small and delicate slow-wave circuits to obtaining high electron densities or large magnetic fields.

Finally, and probably most important, the techniques discussed here must also be regarded as a new and potentially valuable plasma diagnostic tool. One immediately apparent

application is to the determination of electron densities by measuring the velocity of a microwave signal which propagates along a cylindrical plasma column. The frequency at which the measurement can be made is much less than the electron plasma frequency. The usual cavity methods of measuring electron densities are limited to densities for which $\omega_p < \omega$. Thus the propagation method extends the range of densities which can be measured with microwaves to the very high densities (10^{16} per cm^3) required for thermonuclear reactions.⁴ Other applications to plasma diagnostics are being explored, particularly with regard to determining the variation in charge density with radius.

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(10) APPENDIX

For slow ($v_{phase}^2 \ll c^2$) waves the a.c. magnetic field is negligible and $\nabla \times E = -j\omega B$ is approximately zero. Thus the electric field may be derived from a scalar potential, $E = -\nabla\phi$. From eqn. (2), $D = \epsilon \cdot E$. Since we regard the plasma as a dielectric in which the electronic charge is bound to the ionic charge and there is no free charge, $\nabla \cdot D = 0$, or in terms of the potential ϕ ,

$$\nabla \cdot (\epsilon \cdot \nabla\phi) = 0 \dots \dots (14)$$

Using the dielectric tensor (3), this becomes

$$\epsilon_{rr} \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial\phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2\phi}{\partial\theta^2} \right] + \epsilon_{zz} \frac{\partial^2\phi}{\partial z^2} = 0 \dots (15)$$

Eqn. (4) is a solution of this equation.