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SPACE CHARGE EFFECTS IN BEAM TYPE MAGNETRONS

by

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Space Charge Effects in Beam-Type Magnetrons*

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A theoretical treatment of space charge effects in beam-type magnetron amplifiers and oscillators is given. It is assumed that the beam is relatively thin and that the magnetic field is large. The "cyclotron waves" are not treated. A space charge parameter appears in this theory of magnetron-type traveling-wave interaction in a manner which is analogous to the manner in which QC appears in ordinary traveling-wave interaction. A distinctive feature of the space charge waves in the magnetron case is that one increases along the beam and the other decreases along the beam. A simple physical explanation of this effect is given.

This theory is then used to determine the starting conditions of an M-type backward wave oscillator. It is found that when the tube is long in space charge wavelengths there is an appreciable reduction of starting current. When the space charge parameter approaches zero, the solutions found here reduce to the usual two-wave solutions.

I. INTRODUCTION

IN the last few years the interest in magnetron-type traveling-wave tubes has increased considerably because of the possibility of combining the wide-band characteristics of the traveling-wave tube with the high efficiency characteristics of the magnetron. Success in this direction with the M-type backward wave oscillator has been outstanding.[1] There remain, however, a few characteristics of these tubes which are not so well understood, such as negative electrode current, reduced starting current, and a tendency toward noisiness and the generation of spurious signals. It is now generally believed that the growing space charge wave propagated by a slipping stream of electrons (diocotron effect)[2] plays an important role in these phenomena. This paper presents a simplified small signal theory of the space charge waves on a relatively thin electron beam focused by crossed electric and magnetic fields and the interaction of such a beam with a nearby circuit which supports a slow electromagnetic wave. The slow wave circuit is represented by an admittance wall whose admittance depends on the propagation constant, an extension of Fletcher's method.[3] The end result is similar to that obtained by Pierce[4] in his analysis of the magnetron amplifier, except that here the mutual interaction between various electrons of the beam is included. The theory is then used to calculate the effect of space charge on the starting conditions of the M-type backward wave oscillator.

II. THIN BEAM DYNAMICS

A number of assumptions have been made which simplify the analysis. Perhaps the most restrictive is that the electron beam is taken to be very thin so that the same fields act on all electrons and all electrons are assumed to have the same unperturbed velocity, u_0, in the z direction. The latter assumption is clearly in violation of slipping stream steady flow condition, but as we shall see in Sec. IV, this apparently crude model does give a good description of the space charge waves which propagate on a thin *slipping stream.* The width of the interaction region is taken as w, and all quantities are assumed to be independent of the x coordinate, so that the problem is essentially two dimensional. The state of the electron beam shown in Fig. 1 may then be described by giving simply its surface charge density, $\sigma=\sigma_0+\sigma_1(\sigma_1\ll\sigma_0)$, and its displacement from the equilibrium position, $y_1(\beta y_1\ll1)$. A subscript zero will denote the steady or dc part of a quantity, and the subscript 1 will denote the small ac perturbation from the steady value. Waves whose dependence on time and the z coordinate is given by $e^{j(\omega t-\beta z)}$ will be assumed.

The linearized equations for the y and z components of ac electron velocity are

$$j(\omega-\beta u_0)v_{1y}=-\frac{e}{m}(\bar{E}_{1y}+v_{1z}B_0), \quad (1)$$

$$j(\omega-\beta u_0)v_{1z}=-\frac{e}{m}(\bar{E}_{1z}-v_{1y}B_0). \quad (2)$$

The electric field $(\bar{E}_{1y},\bar{E}_{1z})$ which acts on the electrons is taken to be the average of the field above the beam and the field below the beam

$$\bar{E}_{1y}=\tfrac{1}{2}[(E_{1y})_++(E_{1y})_-], \quad (3)$$

$$\bar{E}_{1z}=\tfrac{1}{2}[(E_{1z})_++(E_{1z})_-], \quad (4)$$

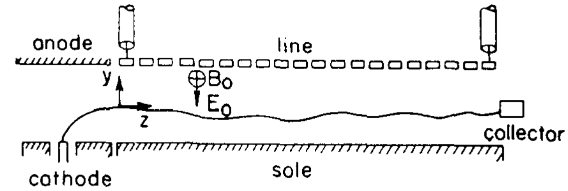

FIG. 1. Schematic diagram of interaction region of thin beam magnetron amplifier or oscillator.

* Work supported by the Office of Naval Research.
[1] Warnecke, Guenard, Doehler, and Epsztein, Proc. Inst. Radio Engrs. 43, 413 (1955).
[2] MacFarlane and Hay, Proc. Phys. Soc. (London) B63, 409 (1950).
[3] R. C. Fletcher, Proc. Inst. Radio Engrs. 38, 413 (1950).
[4] J. R. Pierce, *Traveling Wave Tubes* (D. Van Nostrand Company, Inc., New York, 1950), Chap. XV.

the + sign denoting the field at $y=0+\epsilon$ and the − sign denoting the field at $y=0-\epsilon(\epsilon\rightarrow 0)$. It is necessary to use this average field since, because of the space charge, the electric field above the beam is different from the electric field below the beam.

Completing the dynamic equations is a one-dimensional continuity equation,[5]

$$\omega\sigma_1-\beta(u_0\sigma_1+\sigma_0 v_{1z})=0, \tag{5}$$

and the relation between the displacement y_1 and the y component of ac velocity, v_{1y},[5]

$$v_{1y}=j(\omega-\beta u_0)y_1. \tag{6}$$

These relations are sufficient to determine the motion of the electron beam when the electric field is given. The determination of the ac electric field from y_1 and σ_1 is discussed in the next section.

First, one more simplifying assumption is made. If the above equations of motion are used, we find four waves in the absence of the slow wave circuit. In low current beams, two of these waves have a phase velocity approximately equal to the electron velocity

$$\beta_{2,3}\approx\omega/u_0,$$

while the other two "cyclotron waves" have propagation constants given by

$$\beta_4\approx(\omega+\omega_c)/u_0,\quad \beta_5\approx(\omega-\omega_c)/u_0.$$

The presence of the circuit adds another wave,

$$\beta\approx\beta_1.$$

The additional simplifying assumption is one which eliminates waves 4 and 5 from the problem, and it is made simply to reduce the complexity of the expressions which will be obtained. The approximation should be reasonably good for most beam-type tubes. The simplest way in which to make this approximation is to neglect $j(\omega-\beta u_0)v_{1y}$ in comparison with $(e/m)B_0v_{1z}$ in Eq. (1) and $j(\omega-\beta u_0)v_{1z}$ in comparison with $(e/m)B_0v_{1y}$ in Eq. (2). Since v_{1y} and v_{1z} are of the same magnitude, this is equivalent to assuming that

$$\omega-\beta u_0\ll\omega_c. \tag{7}$$

The resulting equations of motion

$$v_{1y}=\bar{E}_{1z}/B_0, \tag{8}$$

$$v_{1z}=-\bar{E}_{1y}/B_0 \tag{9}$$

say that the electrons drift at *right angles* to the electric field, the rotational component of velocity which usually accompanies the drift being neglected. This is generally permissible when the magnetic field is large. With this approximation, one can easily solve for y_1 and σ_1 in terms of $\bar{E}_{1y}$ and $\bar{E}_{1z}$ from Eqs. (5), (6), (8), and (9):

$$y_1=\frac{1}{B_0}\frac{\bar{E}_{1z}}{j(\omega-\beta u_0)} \tag{10}$$

$$\sigma_1=-\frac{\sigma_0\beta}{B_0}\frac{\bar{E}_{1y}}{(\omega-\beta u_0)}. \tag{11}$$

Note that transverse displacements are produced by the longitudinal electric field and longitudinal bunching is produced by the transverse electric field.

III. DETERMINATION OF THE ELECTRIC FIELD

We have the problem of computing the electric field in the regions above and below the beam, given σ_1 and y_1. When the displacement of the beam from equilibrium is given by $y_1e^{j(\omega t-\beta z)}$ and the surface charge density is given by $\sigma=\sigma_0+\sigma_1e^{j(\omega t-\beta z)}$, the potential below the beam may be expanded in harmonics:

$$\phi=A_0\frac{y}{d}+\sum_{n\neq 0}A_n\sinh n\beta(y-d)e^{jn(\omega t-\beta z)}. \tag{12}$$

The first term is the dc part of the potential, while the remaining terms make up the ac part and are proportional to βy_1, σ_1/σ_0 or various powers thereof. When βy_1 is small compared with unity, only the $n=0$ and $n=1$ terms need to be considered. The $n=0$ or dc part of the potential is conveniently eliminated by superimposing a charge distribution which is just the negative of charge distribution of the unperturbed beam. Thus, to find the ac fields we consider the charge distribution shown in Fig. 2. When $\beta y_1\ll 1$, this charge distribution is equivalent to a surface charge density $\sigma_1e^{j(\omega t-\beta z)}$ and a double layer whose dipole moment per unit area is σ_0y_1. In passing through such a charge distribution, the potential is discontinuous by an amount $(\sigma_0/\epsilon_0)y_1$, and the normal derivative is discontinuous by $-\sigma_1/\epsilon_0$. Hence, the discontinuities in the two components of of electric field may be written

$$(E_{1y})_+-(E_{1y})_-=\sigma_1/\epsilon_0, \tag{13}$$

$$(E_{1z})_+-(E_{1z})_-=j\beta y_1\sigma_0/\epsilon_0. \tag{14}$$

These two equations tell us how to match the ac fields at the electron beam.

In carrying out this matching procedure it is convenient to introduce the concept of normalized E-mode

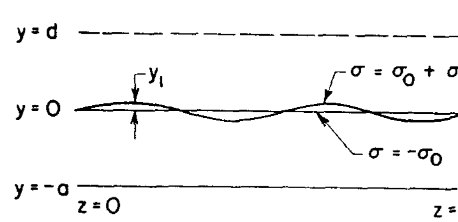

FIG. 2. Charge distribution giving rise to ac electric fields. The charge distribution and field are assumed to be independent of the x coordinate.

[5] Reference 4, Eqs. 15.14 and 15.17.

surface admittance, $Y_E{}^0 = E_{1y}/E_{1z}$. $Y_E{}^0$ is proportional to the usual E-mode surface admittance $Y_E = -H_{1x}/E_{1z}$,[6] since $H_{1x} = -(\omega\epsilon_0/\beta)E_{1y}$. The normalized admittance of free space is easily shown to be $\pm j$, and the admittance just below the electron beam of the space between the beam and the conducting plane at $y=-a$ is $j\coth\beta a$. Eliminating σ_1 and y_1 from (13) and (14), through the use of (10) and (11), and using (3) and (4) for $\bar{E}_{1y}$ and $\bar{E}_{1z}$ yields

$$(E_{1y})_+ = \left[\frac{\omega-\beta u_0-\sigma_0\beta/2\epsilon_0 B_0}{\omega-\beta u_0+\sigma_0\beta/2\epsilon_0 B_0}\right](E_{1y})_-, \tag{14a}$$

$$(E_{1z})_+ = \left[\frac{\omega-\beta u_0+\sigma_0\beta/2\epsilon_0 B_0}{\omega-\beta u_0-\sigma_0\beta/2\epsilon_0 B_0}\right](E_{1z})_-. \tag{15}$$

Thus,

$$Y_E{}^0(0^+) = \left[\frac{\omega-\beta u_0-\sigma_0\beta/2\epsilon_0 B_0}{\omega-\beta u_0+\sigma_0\beta/2\epsilon_0 B_0}\right]^2 Y_E{}^0(0^-). \tag{16}$$

The normalized surface admittance of the slow wave circuit is a function of the propagation constant, β. Since the propagation constants of interest in this problem do not differ much from the circuit propagation constant, β_1, the circuit admittance at the plane of the slow wave circuit may be expanded in a Taylor series.

$$Y_E{}^0 = Y_E{}^0\Big|_{\beta_1} + \frac{\partial Y_E{}^0}{\partial\beta}\Big|_{\beta_1}(\beta-\beta_1) + \frac{1}{2}\frac{\partial^2 Y_E{}^0}{\partial\beta^2}\Big|_{\beta_1}(\beta-\beta_1)^2 + \cdots. \tag{17}$$

This is a useful representation of the circuit admittance since generally only the first two terms are required. It has been shown that[7]

$$Y_E{}^0\Big|_{\beta_1} = j\coth\beta_1(a+d) \tag{18}$$

$$\frac{\partial Y_E{}^0}{\partial\beta}\Big|_{\beta_1} = \mp j\frac{2}{\omega\epsilon_0 w}\frac{\beta-\beta_1}{\beta_1}\frac{1}{K} \tag{19}$$

where the upper sign is for forward wave circuits and the lower sign is for backward wave circuits K is the interaction impedance, $E_{1z}{}^2/2\beta^2 P$ (taken to be positive), and w is the width of the circuit. This is essentially an extension of the result of Fletcher[3] and it applies to space harmonic structures if E_{1z} is taken to be the amplitude of the appropriate space harmonic field component and P is taken to be the total power flow of the wave.

The normalized admittance presented by the circuit to the upper surface of the electron beam may be expressed in terms of the admittance at the circuit plane $(y=d)$ by means of the admittance transformation formula[6]

$$Y_E{}^0(0^+) = \frac{Y_E{}^0(d) - j\tanh\beta d}{1 + j\tanh\beta d\, Y_E{}^0(d)}. \tag{20}$$

The normalized admittance presented by the space below the beam to the lower surface of the beam is

$$Y_E{}^0(0^-) = j\coth\beta a. \tag{21}$$

IV. CHARACTERISTIC WAVES OF THE SYSTEM

We may combine Eqs. (16), (20), and (21) of the previous section into a single equation, called the characteristic equation, which determines the values of the propagation constant β corresponding to the free waves of the system. In writing the characteristic equation we follow Pierce[8] and Muller[9] in introducing the incremental propagation constant δ by means of the definition

$$\beta = \frac{\omega}{u_0}(1+jD\delta). \tag{22}$$

D is an interaction parameter analogous to the parameter C of ordinary traveling wave interaction theory, and it is defined by

$$D^2 \equiv \frac{\omega}{\omega_c}\frac{I_0 K\alpha}{2V_0}, \quad \alpha = \frac{E_y}{E_z}, \tag{23}$$

where K and α are to be evaluated at the electron beam. Introducing a space charge parameter,

$$S \equiv \frac{-\sigma_0}{2\epsilon_0 B_0 u_0}\frac{1}{D}, \quad (\sigma_0 < 0), \tag{24}$$

and letting $\beta_1 = (\omega/u_0)(1+Db\mp jDd)$, the characteristic equation may be written

$$(\delta+jb\pm d)(\delta^2+2jgS\delta-S^2) = \pm\delta, \tag{25}$$

where the upper sign applies for forward wave interaction and the lower sign applies for backward wave interaction. D has been assumed to be small in comparison with unity, and

$$g \equiv \frac{\tanh(\omega d/u_0) - \tanh(\omega a/u_0)}{\tanh(\omega d/u_0) + \tanh(\omega a/u_0)}, \tag{26}$$

g is a purely geometrical parameter. It should be pointed out that the space charge parameter S, as defined here, is not analogous to Q since it is not independent of beam current.

When the electron beam is far from synchronism with the slow wave circuit ($b \gg 1$), Eq. (25) reduces to

$$(\delta^2+2jgS\delta-S^2) = 0. \tag{27}$$

[6] C. K. Birdsall and J. R. Whinnery, J. Appl. Phys. 24, 314 (1953).

[7] R. W. Gould, "A field analysis of the M-type backward wave oscillator," California Institute of Technology Electron Tube and Microwave Laboratory, Tech. Rept. No. 3 (September, 1955).

[8] Reference 4, Chap. 8.

[9] M. Muller, Proc. Inst. Radio Engrs. 42, 1651 (1954). Our notation coincides with the notation of this reference.

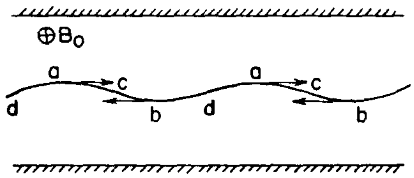

FIG. 3. Illustration of the growth mechanism.

The incremental propagation constants which are solutions of this equation describe the space charge waves which propagate as a beam between two conducting planes. Consider, for simplicity, that the beam is equidistant from both planes ($a=d$) so that $g=0$. The solutions of (27) are given by $\delta=\pm S$ so that the propagation constants are given by

$$\beta=\frac{\omega}{u_0}\left(1\pm j\frac{\sigma_0}{2\epsilon_0 B_0 u_0}\right)\equiv\frac{\omega}{u_0}\pm jh, \qquad (28)$$

where h plays the role of a plasma wave number. One space charge wave increases along the beam and the other decreases. Since the surface charge density, σ_0, is proportional to volume charge density, ρ_0, and the beam thickness, t, (of a beam of small but finite thickness), the rate of growth or decay of the space charge waves may also be written

$$h=\text{Imaginary Part }(\beta)=\pm\frac{\omega_p{}^2}{2\omega_c u_0}\left(\frac{\omega t}{u_0}\right). \qquad (29)$$

This rate of growth is in agreement with that predicted by the analysis of a thin ($\omega t/u_0<0.4$) slipping stream.[2,7] A careful study of slipping stream results, specialized to the thin beam limit, shows that the average transverse displacement of the beam, the linear charge density, and the components of electric field above and below the beam are related in exactly the way indicated by the analysis of this paper. Thus the two methods of analysis describe the same physical phenomenon.

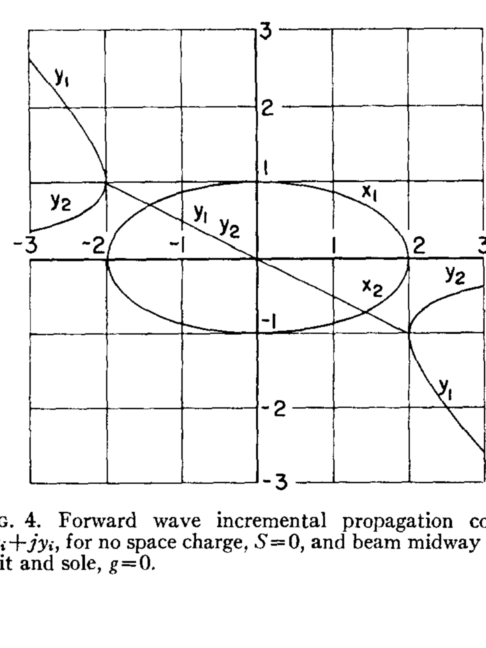

FIG. 4. Forward wave incremental propagation constants, $\delta_i=x_i+jy_i$, for no space charge, $S=0$, and beam midway between circuit and sole, $g=0$.

The growing space charge waves which have been found here have a simple physical explanation. Consider a perturbation of the beam of the type shown in Fig. 3. If this perturbation is viewed in a coordinate system moving with the electrons at velocity u_0, the dc electric field disappears, and the magnetic field is unaltered. Electrons at phase a experience an upward force caused by all other electrons (and image forces when the planes are nearby), and similarly electrons in phase b experience a downward force. Were it not for the strong magnetic field, these forces would immediately augment the original perturbation. Because of the magnetic field the electric field causes the electrons to move in the direction indicated by the arrows and thus become bunched in phase c and spread out in phase d. This bunching causes a longitudinal electric field which, because of the strong magnetic field, causes the original perturbation to be augmented.

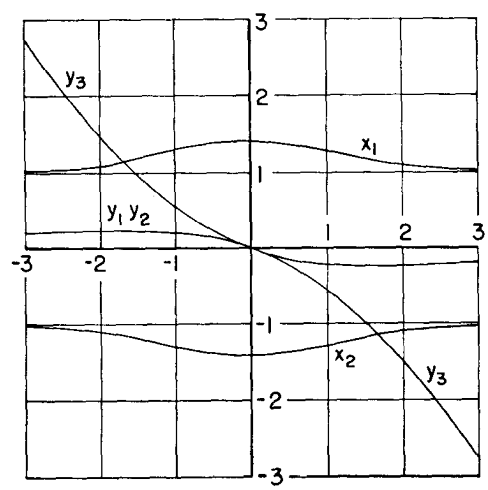

FIG. 5. Forward wave incremental propagation constants, $\delta_i=x_i+jy_i$, for moderate space charge, $S=1$, and beam midway between circuit and sole, $g=0$.

Thus, the growing and decaying space charge waves embody a combination of transverse displacement and longitudinal bunching. Because the drift velocities are inversely proportional to the magnetic field, the rate at which the perturbation builds up is decreased by increasing the magnetic field.

We have solved Eq. (25) for the three values of δ, as a function of b, for several values of S and g with $d=0$. Figure 4 shows the solution for forward wave interaction when $g=0$ and $S=0$ (negligible space charge). In this case Eq. (25) may be factored

$$\delta=0, \quad (\delta+jb)\delta=1. \qquad (30)$$

Figure 5 shows the solution for forward wave interaction when $g=0$ and $S=1$. A comparison of Figs. 4 and 5 shows that the effect of space charge is to increase the rate of gain in forward wave interaction, whereas exactly the opposite is true in ordinary traveling wave interaction. When $g=0$, maximum x_1 occurs for $b=0$

and $(x_1)_{max} = (1+S^2)^{\frac{1}{2}}$. Figure 5 also shows the growing space charge waves away from synchronism. Figure 6 shows the incremental propagation constants for backward wave interaction when $g=0$ and $S=0$ (negligible space charge). It may be seen that there are no growing waves in this case. Figure 7 is also for backward wave interaction but for $g=0$ and $S=1/\sqrt{3}$. Growing space charge waves are present away from synchronism, but they are suppressed by the strong interaction with the circuit near synchronism.

V. EXCITATION OF THE WAVES AND THE BACKWARD-WAVE OSCILLATOR STARTING CONDITIONS

The amplitude of each of the three waves is determined by specifying the z component of the electric field, E_{1z}; the beam displacement, y_1; and the ac charge density, σ_1; at the beginning of the interaction region ($z=0$). Since the approximations made have already eliminated the two cyclotron waves (as well as a circuit wave with phase velocity in the negative z direction), we have too few linearly independent solutions to be able to specify v_{1y} and v_{1z}, as well as the above three quantities. We could instead choose to specify v_{1y} and v_{1z} and not to specify y_1 and σ_1. This amounts to specifying the rate of change of y_1 and σ_1 with time (in the electron's coordinate system), and it appears to be the poorer of the two alternatives. In the case of the backward wave oscillator where the beam enters unmodulated, requiring $y_1(0)$ and $\sigma_1(0)$ to be zero does not guarantee that $v_{1y}(0)$ and $v_{1z}(0)$ will be zero. Conversely, requiring $v_{1y}(0)$ and $v_{1z}(0)$ to be zero does not guarantee that $y_1(0)$ and $\sigma_1(0)$ will be zero. Here it is perhaps clearer that we should specify $y_1(0)$ and $\sigma_1(0)$.

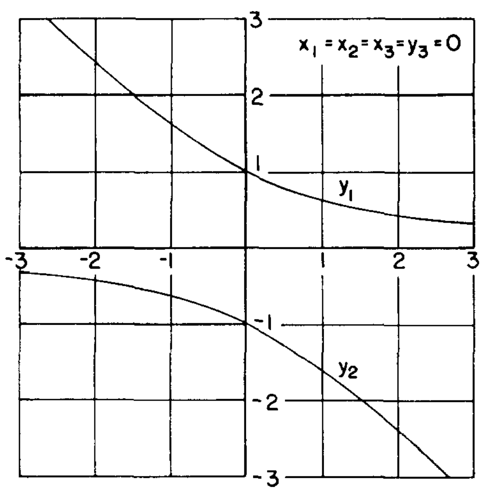

FIG. 6. Backward wave incremental propagation constants, $\delta_i = x_i + jy_i$, for no space charge, $S=0$, and beam midway between circuit and sole, $g=0$.

The field above the electron beam is the superposition of three waves:

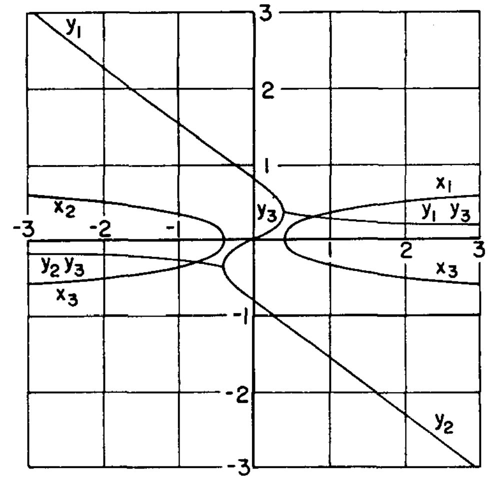

FIG. 7. Backward wave incremental propagation constants, $\delta_i = x_i + jy_i$, for moderate space charge, $S=1/\sqrt{3}$, and beam midway between circuit and sole, $g=0$.

$$E_{1z} = \sum_{i=1}^{3} E_i[\cosh\beta_i(y-d) - jY_{Ei}^0 \sinh\beta_i(y-d)]e^{-j\beta_i z}, \quad (31)$$

$$E_{1y} = \sum_{i=1}^{3} E_i[\sinh\beta_i(y-d) - jY_{Ei}^0 \cosh\beta_i(y-d)]e^{-j\beta_i z} \quad (32)$$

where Y_{Ei}^0 is normalized surface admittance for the ith wave at the slow wave circuit ($y=d$) and E_i is the amplitude of the z component of electric field of the ith wave at the circuit. Thus, at the circuit,

$$E_{1z} = \sum_{i=1}^{3} E_i e^{-j\beta_i z} = \exp\left(-j\frac{\omega}{u_0}z\right)\sum_{i=1}^{3} E_i \exp\left(\frac{\omega}{u_0}D\delta_i z\right). \quad (33)$$

Using the equations of the previous section and then making the small D approximation, the transverse displacement and the ac surface charge density can be written in terms of the wave amplitudes E_1, E_2, and E_3:

$$\frac{\omega}{u_0}y_1 = A\exp\left(-j\frac{\omega}{u_0}z\right) \times \sum_{i=1}^{3} \frac{\delta_i - jS}{\delta_i^2 + 2jgS\delta_i - S^2} E_i \exp\left(\frac{\omega}{u_0}D\delta_i z\right), \quad (34)$$

$$\frac{\sigma_1}{\sigma_0} = A\coth\frac{\omega a}{u_0}\exp\left(-j\frac{\omega}{u_0}z\right) \times \sum_{i=1}^{3} \frac{\delta_1 + jS}{\delta_i^2 + 2jgS\delta_i - S^2} E_i \exp\left(\frac{\omega}{u_0}D\delta_i z\right) \quad (35)$$

where

$$A=\frac{1}{D}\frac{1}{B_0u_0}\frac{\operatorname{sech}\frac{\omega}{u_0}d}{1+\tanh\frac{\omega}{u_0}d\coth\frac{\omega}{u_0}a}. \tag{36}$$

When the longitudinal electric field at the circuit, the transverse displacement, and ac surface charge density are known at $z=0$, the amplitude of the ith wave is given by

$$E_i=\frac{\delta_i^2+2jgS\delta_i-S^2}{(\delta_i-\delta_j)(\delta_i-\delta_k)}\left\{E_{1z}(0)-\frac{1}{2A\coth\frac{\omega}{u_0}a}\frac{\sigma_1(0)}{\sigma_0}\right.$$
$$\times\left[\delta_j+\delta_k+2jgS-\frac{\delta_j\delta_k-S^2}{jS}\right]-\frac{1}{2A}\frac{\omega y_1(0)}{u_0}$$
$$\left.\times\left[\delta_j+\delta_k+2jgS+\frac{\delta_j\delta_k-S^2}{jS}\right]\right\}, \tag{37}$$

where i, j, and k are cyclical permutations of 1, 2, and 3. With these equations it is possible to express the field at the circuit, the displacement of the beam, or the ac surface charge density of the beam at any z coordinate in terms of the values of these three quantities at $z=0$.

As an application of this result the start-oscillation conditions for an N-type backward wave oscillator have been found. In the backward wave oscillator the beam enters the interaction region with no modulation, hence $y_1(0)=\sigma_1(0)=0$. Using (33) and (37), the electric field at the circuit becomes

$$E_{1z}=E_{1z}(0)\exp\left(-j\frac{\omega z}{u_0}\right)$$
$$\times\sum_{i=1}^{3}\frac{\delta_i^2+2jgS\delta_i-S^2}{(\delta_i-\delta_j)(\delta_i-\delta_k)}\exp\left(\frac{\omega}{u_0}D\delta_i z\right). \tag{38}$$

In the backward wave oscillator, the collector end of the tube ($z=L$) is terminated in such a way that the electric field vanishes there and the power output is taken from the gun end of the tube ($z=0$). The starting conditions are found by solving

$$0=\sum_{i=1}^{3}\frac{\delta_i^2+2jgS\delta_i-S^2}{(\delta_i-\delta_j)(\delta_i-\delta_k)}\exp\left(\frac{\omega}{u_0}LD\delta_i\right) \tag{39}$$

for the length, L, and the velocity difference parameter, b. S and g are constants which are assumed to be known. Equations which are very similar to Eqs. (39) and (25) arise in the theory of the longitudinally focused backward wave oscillator, and a more detailed discussion of the interpretation and method of solution of these equations is to be found in the literature.[10–12]

An analytical solution of Eqs. (39) and (25) has been obtained for a special case, $g=d=0$, which corresponds to an electron beam halfway between a lossless slow wave circuit and the conducting plane at $y=-a$. Assume that $b=0$ will solve this pair of equations. Then the solutions of (25) are

$$\delta_1=+(S^2-1)^{\frac{1}{2}},\quad \delta_2=-(S^2-1)^{\frac{1}{2}},\quad \delta_3=0. \tag{40}$$

For $S>1$, one wave amplitude increases exponentially with distance, one decreases exponentially with distance, and the amplitude of the third is constant. When $S<1$, all three waves have constant amplitude. For this special case, the solution of Eq. (39) may be written

$$\begin{aligned}\frac{\omega}{u_0}LD&=\frac{\cosh^{-1}S^2}{(S^2-1)^{\frac{1}{2}}}\quad S^2>1\\&=\frac{\cos^{-1}S^2}{(1-S^2)^{\frac{1}{2}}}\quad S^2<1.\end{aligned} \tag{41}$$

Equations (25) and (39) reduce to the corresponding two-wave equations[9]:

$$(\delta_{1,2}+jb+d)\delta_{1,2}=\pm1 \quad\text{and}\quad \delta_3=0,$$
$$0=\delta_1\exp\left(\frac{\omega}{u_0}LD\delta_1\right)-\delta_2\exp\left(\frac{\omega}{u_0}LD\delta_2\right),$$

when the space charge parameter, S, is equal to zero.

The start-oscillation conditions for values of g other than zero have been found by solving Eqs. (25) and (39) on the Electrodata Datatron digital computer. In all cases, the circuit is assumed to be lossless ($d=0$). The results of these computations are summarized in Figs. 8 and 9 where $[(\omega LD/2\pi u_0)]_{\text{start}}$ and $(\beta_1-\beta_e)L_{\text{start}}$ are plotted *vs* the parameter $(\omega/u_0)LDS=hL$. The latter is essentially the length of the tube in space charge wavelengths, since an increase of this parameter

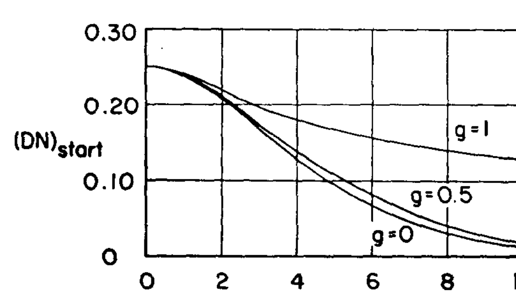

FIG. 8. M-type backward wave oscillator starting condition, $[(\omega LD/2\pi u_0)]_{\text{start}}$ *vs* hL. hL is the length of the tube in plasma wavelengths, as defined in the text.

[10] H. Heffner, Proc. Inst. Radio Engrs. 42, 930 (1954).
[11] H. R. Johnson, Proc. Inst. Radio Engrs. 43, 684 (1955).
[12] L. R. Walker, J. Appl. Phys. 24, 854 (1953).

by one unit corresponds to the distance in which the amplitude of the space charge waves of a free beam increase by a factor e, or 8.6*db*. Only the results for positive values of g are shown since $[(\omega LD/2\pi u_0)]_{\text{start}}$ is an even function of g and $[(\beta_1-\beta_e)L]_{\text{start}}$ is an odd function of g.

The results of this calculation may be summarized by saying that the effect of mutual interaction between various parts of the beam (commonly called the effect of space charge) is to decrease the starting length for a fixed current and hence the starting current for a fixed length, by an appreciable factor.

The theory presented here may also be used to find the effect of space charge on a thin beam magnetron amplifier. For the special case $g=b=d=0$, Eq. (38) becomes

$$E_{1z}(L)=E_{1z}(0)\exp\left(-j\frac{\omega}{u_0}L\right)\times\frac{\cosh\left[\frac{\omega L}{u_0}D(1+S^2)^{\frac{1}{2}}\right]+S^2}{1+S^2}. \quad (42)$$

In the notation of Pierce,[4]

$$A=-20\log_{10}[2(1+S^2)];\quad B=54.6(1+S^2)^{\frac{1}{2}}.$$

Thus, space charge increases the rate of growth of the wave along the beam but decreases the initial amplitude of the growing wave.

VI. CONCLUSIONS

A simple approximate theory of space charge effects in linear magnetron interaction has been developed. One explicit result is a prediction of the rate of growth and decay of the space charge waves which is in agreement with a more detailed analysis of wave propagation on a thin slipping stream of electrons. A comparison, not given in this paper, of the two methods of analysis shows that the one presented here gives other details in agreement with the slipping stream theory. Thus, both describe the same physical phenomenon. The simple theory has the advantage of leading to a clearer physical understanding of the growing space charge wave.

An application of the theory developed here to the M-type backward wave oscillator shows that the effect of space charge may reduce the starting length or current by an appreciable factor. The measured starting currents of M-type backward wave oscillators[1] are generally lower than predicted by the theory which neglects space charge effects, but it is not yet known whether the theory presented here satisfactorily accounts for the measurements. The feature of the experimental arrangement which is not taken into account here and which may also affect the results significantly is the lack of straight line trajectories.

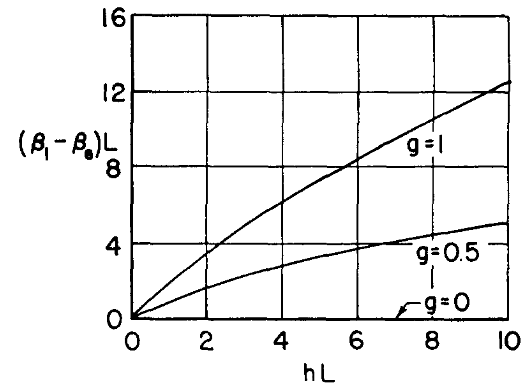

FIG. 9. M-type backward wave oscillator starting condition, $[(\beta_1-\beta_e)L]_{\text{start}}$ *vs* hL. hL is the length of the tube in plasma wavelengths as defined in the text.

In the future it may be of interest to apply this theory to the M-type backward wave amplifier with one or two circuits and to study the effect of loss on the starting current of a M-type backward wave oscillator. It would also be of interest to study the higher order modes of oscillation,[10,11] and it may be of interest to extend the theory to include all six waves.

VII. ACKNOWLEDGMENTS

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