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Sony J. Akkarakaran, Palghat P. Vaidyanathan, "Role of principal component filter banks in noise reduction," Proc. SPIE 3813, Wavelet Applications in Signal and Image Processing VII, (26 October 1999); doi: 10.1117/12.366792

SPIE.

Event: SPIE's International Symposium on Optical Science, Engineering, and Instrumentation, 1999, Denver, CO, United States

Invited Paper

The Role of Principal Component Filter Banks in Noise Reduction

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ABSTRACT

The purpose of this paper is to demonstrate the optimality properties of principal component filter-banks for various noise reduction schemes. Optimization of filter-banks (FB's) for coding gain maximization has been carried out in the literature, and the optimized solutions have been observed to satisfy the principal component property, which has independently been studied. Here we show a strong connection between the optimality and the principal component property; which allows us to optimize FB's for many other objectives. Thus, we consider the noise-reduction scheme where a noisy signal is analyzed using a FB and the subband signals are processed either using a hard-threshold operation or a zeroth order Wiener filter. For these situations, we show that a principal component FB is again optimal in the sense of minimizing the expected mean-square error.

Keywords: Principal Component Filter-Banks, noise-reduction

1. INTRODUCTION

A generic filter-bank based signal processing scheme is shown in Fig. 1. Suppose we are allowed to choose any filter-bank (FB) from a class $\mathcal{C}$ of uniform orthonormal M-channel FB's. For example, $\mathcal{C}$ could be the class of ideal FB's, or that of FB's having FIR filters with a certain bound on their order. This paper is concerned with the **problem of finding the best FB from the class** $\mathcal{C}$ for a particular kind of subband processing, for given statistics of the input. In particular we focus on the situation when the FB input is a noisy signal, and the subband processors are aimed at removing the noise.

To explain our usage of the term 'best FB', consider the situation where the FB is used for data compression, and so the processors P_i in Fig. 1 are quantizers. Under the standard high bit-rate quantization noise models and assuming optimal bit allocation among the subband quantizers, minimizing the mean-square reconstruction error is equivalent to minimizing the product of the variances of the subband signals.[1] Thus, for this situation, the best FB is the one that minimizes this product of subband variances.

When the class $\mathcal{C}$ consists of all M-channel orthogonal transform coders, the optimum FB in $\mathcal{C}$ for the above situation is the KLT.[5] It produces subband signals $v_i^{(x)}(n)$ in Fig. 1 such that the vector process $(v_0^{(x)}, v_1^{(x)}, \ldots, v_{M-1}^{(x)})^T$ has a diagonal autocorrelation matrix. When $\mathcal{C}$ is the class of all (unconstrained) M-channel orthonormal FB's, the optimum FB has been obtained in,[3,1] It produces a vector process $(v_0^{(x)}, v_1^{(x)}, \ldots, v_{M-1}^{(x)})^T$ (see Fig. 1) that has a diagonal power-spectrum (psd) matrix, with the diagonal elements (i.e. the subband spectra) ordered according to a condition referred to as spectral majorization.[1] In both these cases, the optimum FB turns out to be a *principal component filter bank* (PCFB) for the class $\mathcal{C}$. PCFB's were first propounded in,[2] and are defined in Section 3.

There is a stronger connection between optimality of the FB and the principal component property. This connection, which we believe is the precise reason for the optimality of PCFB's, does not seem to have been observed in the literature. The main result is that the PCFB is optimal whenever the objective to be minimized is a concave function of the subband variances produced by the FB. In the above-mentioned coding problem, the objective was the product of the subband variances. Minimizing it is equivalent to minimizing its logarithm, which is a concave function of the subband variances. Thus, the PCFB is optimal. The subsequent sections elaborate on this result,

Work supported in part by the National Science Foundation under Grant MIP 0703755.

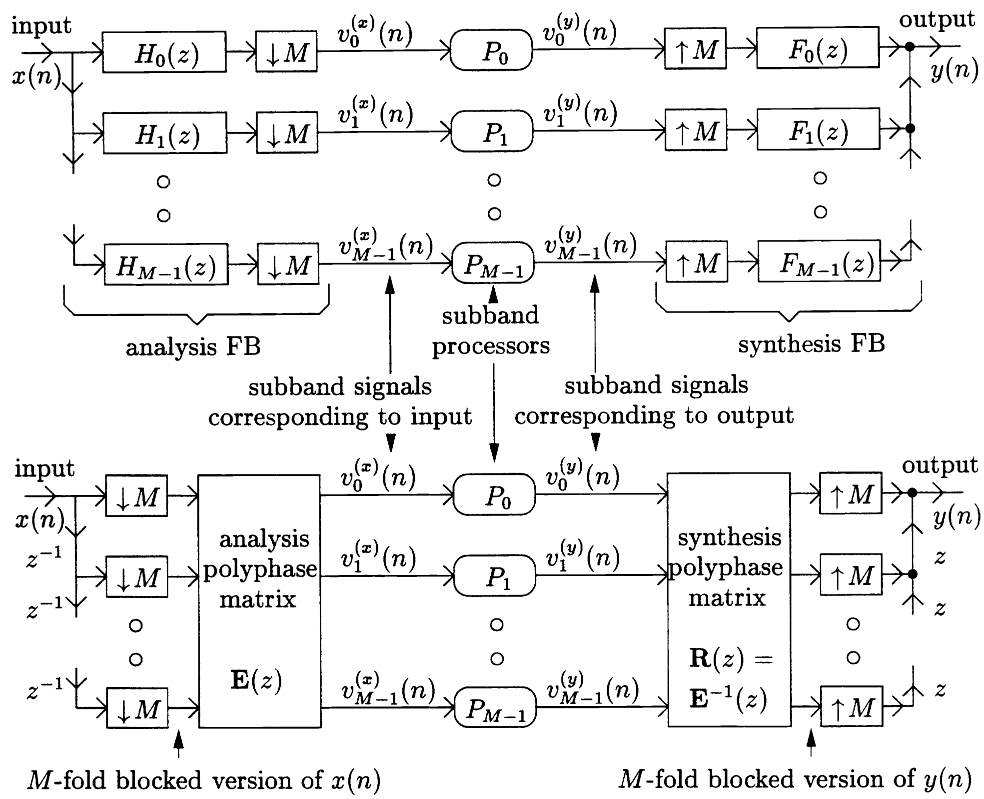


Figure 1. Generic FB based signal processing scheme.

and illustrate various other FB based signal processing schemes for which the FB optimization involves minimizing a concave function of the subband variances. For example, this happens in the noise suppression system where the FB input $x(n)$ in Fig. 1 is a signal corrupted by zero mean additive white noise, and the processors P_i are either zeroth order Wiener filters or hard-thresholders. Thus a PCFB is optimal for all these schemes as well.

2. PROBLEM FORMULATION

We are given a class $\mathcal{C}$ of M-channel orthonormal FB's, and a set of M subband processors P_i, $i = 0, 1, \ldots, M-1$ (numbered arbitrarily). A processor is simply a well-defined function that maps input sequences to output sequences. The specification of this function might be independent of any statistical properties that the input sequences are assumed to have; or on the other hand it might not. Examples of the former kind of processors are fixed LTI systems and memoryless squaring devices. Examples of the latter kind are Wiener filters and pdf-optimized quantizers. The signal processing system consists of a FB from $\mathcal{C}$ and the processors P_i used in its subbands as shown in Fig. 1.

For this system, we define the **subband variance vector** as $\mathbf{v} = (\sigma_0^2, \sigma_1^2, \ldots, \sigma_{M-1}^2)^T$ whose i-th entry is the variance of the subband signal input to the processor P_i, for $i = 0, 1, \ldots, M-1$. It can be computed for each FB given the psd matrix of the M-fold blocked version of the scalar process $x(n)$ input to the FB. The optimization **search space** is defined as the set S consisting of all subband variance vectors associated with all FB's in the given class $\mathcal{C}$. We do not assume any constraint as to which processor P_i to use in which subband of the FB. The set S is therefore 'permutation-symmetric': If $\mathbf{v}$ is in S then all vectors obtained from $\mathbf{v}$ by permuting its entries are also in S. The problem at hand is to find the FB from $\mathcal{C}$ that minimizes an objective function that is well-defined on the class $\mathcal{C}$. The assumption we make on this objective is that it can be fully evaluated at each FB in $\mathcal{C}$ given the variances of the subband signals that the FB produces, and the information as to which variance enters which processor P_i. Thus the objective can be represented by a real-valued function g defined on the search space S. This happens for a number of FB based signal processing schemes, as will be seen later.

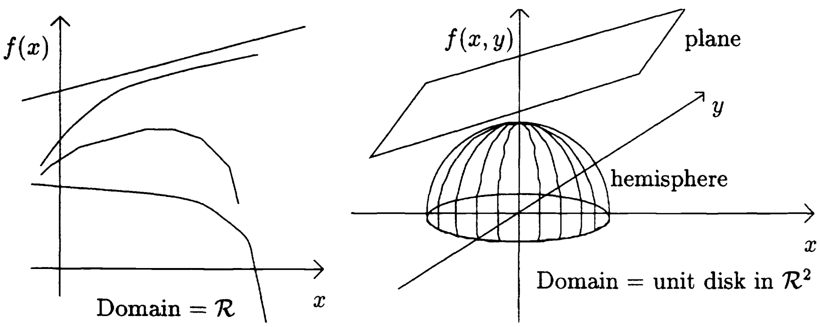


Figure 2. Concave functions on different domains.

Notice that the objective need not be symmetric in its arguments, i.e. g could have different values at two different vectors in S which are permutations of each other. This usually happens because the subband processors P_i are not identical. To find the best FB, we find the vector $\mathbf{v}_{opt} \in S$ that minimizes g over S. The optimum FB is then identified as any FB in $\mathcal{C}$ whose subband variances are the entries in $\mathbf{v}_{opt}$, provided the subbands of this FB are coupled to the subband processors in the order corresponding to $\mathbf{v}_{opt}$.

3. PCFB'S AND THEIR OPTIMALITY

3.1. Definitions and statement of result

Majorization : Given two sets A, B each having M real numbers (not necessarily distinct), A is defined to **majorize** B if the elements $a_i \in A$ and $b_i \in B$ arranged in descending order $a_0 \geq a_1 \geq \ldots \geq a_{M-1}$, and $b_0 \geq b_1 \geq \ldots \geq b_{M-1}$, obey the property that

$$\sum_{i=0}^{P} a_i \geq \sum_{i=0}^{P} b_i \quad \text{for all} \quad P = 0, 1, \ldots, M-1, \qquad \text{with equality holding when } P = M-1. \tag{1}$$

We say that a vector $\mathbf{a}$ majorizes another vector $\mathbf{b}$ if the set of entries of $\mathbf{a}$ majorizes that of $\mathbf{b}$.

PCFB's : Let us be given a class $\mathcal{C}$ of uniform orthonormal M-channel FB's, and the power-spectrum of the input to the FB. A PCFB for the class $\mathcal{C}$ is defined to be a FB in $\mathcal{C}$ whose set of subband variances majorizes the set of subband variances of *any* FB in $\mathcal{C}$. Alternatively, a PCFB may be defined as a FB that minimizes (over all FB's in $\mathcal{C}$) the mean-square error caused by dropping the P weakest (lowest variance) subbands, for any $P = 0, 1, \ldots, M$. The equivalence of these two definitions is due to the fact that dropping subbands results in a mean-square reconstruction error that is the sum of the variances of the dropped subband signals (upto a constant scale-factor of $\frac{1}{M}$). The PCFB and its existence depends on both the class $\mathcal{C}$ and the input spectrum.

Main result on PCFB optimality. Let $\mathcal{C}$ be a *perfectly arbitrary* class of uniform M-channel orthonormal FB's, such that a PCFB exists for this class. Then the search space S has the property that its *convex hull* co(S) is a *polytope* (defined in Section 3.2 below). All the *corners* of this polytope are permutations of each other, and are elements of S that correspond to the PCFB. The objective g to be minimized is a real-valued function on S. If it has an extension to co(S) on which it is *concave*, then at least one of the corners of the polytope is a minimum of g. Thus, **the PCFB is always optimal**. Further if g is strictly concave, then its minimum is necessarily at some corner of the polytope, i.e. **the optimum FB is necessarily a PCFB**.

3.2. Discussion of the result

Recall that a function $f : D \rightarrow \mathcal{R}$ is defined to be concave if given any $\mathbf{x}, \mathbf{y} \in D$ and $\mu \in [0, 1]$,

$$f(\mu\mathbf{x} + (1-\mu)\mathbf{y}) \geq \mu f(\mathbf{x}) + (1-\mu) f(\mathbf{y}) \tag{2}$$

Graphically, this means that the function is always above its chord, as is seen from the examples in Fig. 2. Here the domain D of f is some subset of $\mathcal{R}^M$, however the definition makes sense only if D is a *convex* set. D is defined

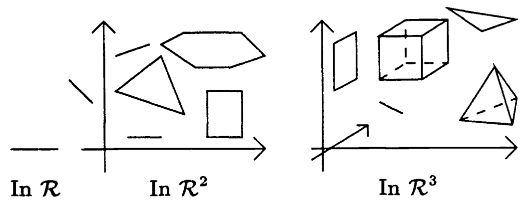


Figure 3. Polytopes in $\mathcal{R}$, $\mathcal{R}^2$ and $\mathcal{R}^3$.

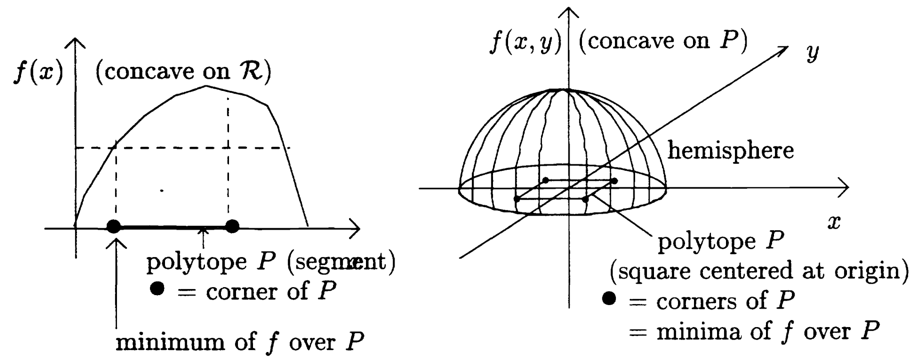


Figure 4. Optimality of corners of polytopes.

to be convex if any convex combination of any finite set of elements from D is also in D. A convex combination of the vectors $\mathbf{x}_i, i = 1, 2, \ldots, N$ is a vector of the form $\sum_{i=1}^{N} \alpha_i \mathbf{x}_i$ for some $\alpha_i \in [0, 1]$ that satisfy $\sum_{i=1}^{N} \alpha_i = 1$. The *convex hull* of a set E is defined as the set of all possible convex combinations using vectors from E, and is denoted by co(E). A convex *polytope* is defined as the convex hull of a *finite* set of points. Given such a polytope co(E), we can assume that no element of E is a convex combination of other elements of E. This is because any such element can be deleted from E without changing co(E). Under this condition, the elements of the finite set E are called *corners* of the polytope. The reason for these names is clear from examples of polytopes embedded in 1, 2 or 3 - dimensional space as shown in Fig. 3.

Now if the function $f : D \to \mathcal{R}$ is concave and D is a polytope, then at least one of the corners of D is a minimum of f over D. This fact is illustrated in Fig. 4, which makes it intuitively clear. Indeed it is a standard result in convex function theory, provable directly from the definitions of polytopes and concave functions.

In our problem, $f = g$, the objective function; and $D = \text{co}(S)$ where S is the optimization search-space (defined in Section 2). Further, if a PCFB exists then it can be shown that co(S) is a polytope whose corners correspond to the PCFB. This proves the main result on PCFB optimality (Section 3.1). The crucial fact that co(S) is a polytope when a PCFB exists, follows from the geometrical meaning of majorization.[10] It is proved in detail in.[12] The proof essentially is in two steps, each using a theorem from[10] : (1) If $\mathbf{a}, \mathbf{b}$ are two vectors such that $\mathbf{a}$ majorizes $\mathbf{b}$, then $\mathbf{b} = \mathbf{Qa}$ for some doubly stochastic matrix $\mathbf{Q}$ (i.e. a matrix with non-negative entries such that the sum of all entries in any row or column is unity). (2) Any doubly stochastic matrix can be written as a convex combination of permutation matrices. Thus, $\mathbf{b}$ is a convex combination of permutations of $\mathbf{a}$, and hence lies inside the polytope with the permutations of $\mathbf{a}$ as corners.

4. PROBLEMS WITH CONCAVE OBJECTIVES

This section shows a number of filter-bank based signal processing schemes for which the FB optimization objective is a concave function of the subband variances of the FB. Thus, from Section 3, if a PCFB exists then it is optimal for all these schemes.

4.1. General features and structure of the problems

Consider the generic FB based signal processing scheme shown in Fig. 1. We denote by $v_i^{(s)}(n)$ the i-th subband signal generated by feeding the signal $s(n)$ as input to the FB, for $i = 0, 1, \ldots, M-1$ (where the subbands are numbered according to the subband processors they are associated with). The system of Fig. 1 is aimed at producing a certain desired signal $d(n)$ at the FB output. It is deemed to be optimized if the actual FB output $y(n)$ is 'as close to' $d(n)$ as possible, i.e. some measure of the error signal $e(n) = d(n) - y(n)$ is minimized. To formulate this measure, we assume that the signals $x(n)$ and $d(n)$ are jointly CWSS(M) (wide sense cyclostationary with period M). Often the subband processors P_i are such that the error $e(n)$ is also a CWSS(M) process – this happens whenever the P_i are LTI systems for instance. The error measure is then the variance of the process $e(n)$ averaged over the period of cyclostationarity M. If the FB is orthonormal, this measure takes the form

$$\frac{1}{M}\sum_{i=0}^{M-1} E[|v_i^{(e)}|^2], \quad \text{where} \tag{3}$$

$$v_i^{(e)}(n) = v_i^{(d)}(n) - v_i^{(y)}(n), \quad \text{for } i = 0, 1, \ldots, M-1 \tag{4}$$

Thus $v_i^{(d)}(n)$ serves as the desired response that the processor P_i must try to approximate at its output as best as possible in the sense of minimizing $E[|v_i^{(e)}|^2]$.

Let the variance of $v_i^{(x)}(n)$ be denoted by σ_i^2. The subband variance vector (defined in Section 2) is thus $\mathbf{v} = (\sigma_0^2, \sigma_1^2, \ldots, \sigma_{M-1}^2)^T$. In many situations, the processors P_i are such that

$$E[|v_i^{(e)}(n)|^2] = h_i(\sigma_i^2) \tag{5}$$

where h_i is some function that depends on the kind of processor P_i, and is independent of the FB. Thus, for such processors P_i, (3) and (5) show that the FB optimization objective g takes the form

$$g(\mathbf{v}) = \frac{1}{M}\sum_{i=0}^{M-1} h_i(\sigma_i^2) \tag{6}$$

If the h_i are concave on $[0, \infty)$ then g is concave on co(S) where S is the search space (defined in Section 2). Thus, from Section 3, PCFB's are optimal whenever the h_i are concave on $[0, \infty)$. We may note that often all the h_i are identical functions, the typical reason being that the processors P_i are identical. In this case g is symmetric in its arguments, i.e. it is not changed by permutations of the σ_i^2. Hence the subbands of optimum FB can be coupled to the subband processors in an arbitrary fashion. If the h_i are not identical, g loses this symmetry property, and then the coupling has to be done in a definite way to ensure optimality. In the high bit-rate coding problem with optimal bit allocation,[1] $h_i(x) = \log(x)$. At low bit-rates, let the i-th quantizer have a normalized quantizer function f_i. Under the assumption that f_i is independent of the FB (thus ruling out pdf-optimized quantizers), $h_i(x) = f_i(b_i)x$ where b_i is the number of bits alloted to the i-th subband.[6] Since all these h_i are concave (on $[0, \infty)$), this gives a direct proof of the results of.[1,6]

4.2. Denoising/Wiener filtering for white noise

Here the FB input in Fig. 1 is $x(n) = s(n) + \mu(n)$ where $s(n)$ is a pure signal and $\mu(n)$ is zero mean white noise. We assume that $\mu(n)$ is uncorrelated to $s(n)$, and has a fixed known variance $\eta^2 > 0$. The overall desired output signal is $d(n) = s(n)$. The i-th subband process $v_i^{(x)}(n)$ contains a signal component $v_i^{(s)}(n)$ and a zero mean additive noise component $v_i^{(\mu)}(n)$. Orthonormality of the FB ensures that the subband noise components are also white with variance η^2, and are uncorrelated to the signal components.

4.2.1. Subband processors as constant multipliers

Suppose each processor P_i is a fixed multiplier of value k_i (memoryless LTI system). Then

$$v_i^{(e)}(n) = v_i^{(d)}(n) - v_i^{(y)}(n) = (1-k_i)v_i^{(s)}(n) - k_i v_i^{(\mu)}(n) \tag{7}$$

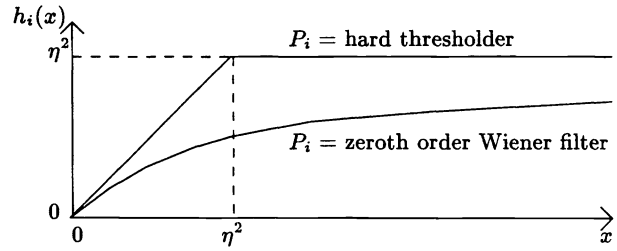


Figure 5. Subband error functions.

Thus, since $v_i^{(\mu)}(n)$ is zero mean and uncorrelated to $v_i^{(s)}(n)$,

$$E[|v_i^{(e)}(n)|^2] = |1-k_i|^2\,\sigma_i^2 + |k_i|^2\,\eta^2 \tag{8}$$

where σ_i^2 is the i-th subband variance corresponding to the signal $s(n)$, i.e. $\sigma_i^2 = E[|v_i^{(s)}(n)|^2]$. Comparison with (5) identifies the h_i in (5,6) as

$$h_i(x) = |1-k_i|^2\,x + |k_i|^2\,\eta^2 \tag{9}$$

which is linear in x, and is hence concave. Notice that while in (5), σ_i^2 was the variance of the subband signal $v_i^{(x)}(n)$ corresponding to the FB input $x(n)$, here it is the variance of $v_i^{(s)}(n) = v_i^{(x)}(n) - v_i^{(\mu)}(n)$. This distinction is not very serious here: It says that the optimal FB is a PCFB *for the signal* $s(n)$ (as opposed to the FB input $x(n)$). However in the present problem, because the noise is white, and $E[|v_i^{(s)}(n)|^2] = \sigma_i^2 = E[|v_i^{(x)}(n)|^2] - \eta^2$, we find that PCFB's for $s(n)$ are also PCFB's for $x(n)$ and vice verca. The situation when the noise is colored is more involved[12]: In *certain cases* it is possible to show optimality of a *simultaneous* PCFB for signal and noise (if it exists). This is discussed in detail in Section 6.

4.2.2. Using multipliers matched to input statistics.

If the processor P_i is a zeroth order Wiener filter, then it is a multiplier given by

$$k_i = \frac{\sigma_i^2}{\sigma_i^2+\eta^2} \tag{10}$$

where σ_i^2 is the variance of $v_i^{(s)}(n)$. On the other hand, if P_i is a hard-threshold operator, it keeps or kills the subband depending on whether the variance of the subband signal component is greater than or less than the variance of the noise component. In this case, it is a multiplier given by

$$k_i = \begin{cases} 1 & \text{if } \sigma_i^2 \geq \eta^2 \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

These schemes can be implemented in practice by estimating σ_i^2 from the subband process $v_i^{(x)}(n)$, which is possible since η^2 is known. Substituting these k_i in (8) and comparing with (5) shows that we have a new set of h_i, i.e.

$$h_i(x) = \begin{cases} \frac{x\eta^2}{x+\eta^2} & \text{if } P_i = 0^{\text{th}} \text{ order Wiener filter} \\ \min(x,\eta^2) & \text{if } P_i = \text{hard thresholder} \end{cases}$$

These functions are plotted in Fig. 5, and are concave on $[0,\infty)$. Thus the PCFB is optimal for any mixture of zeroth order Wiener filters and hard thresholders in the subbands.

Notice that in Fig. 5, the Wiener filter curve lies fully below the hard threshold curve, i.e. the Wiener filter yields a lower mean-square error. This is expected since it is by definition the optimum choice of multiplier k_i in this sense. Use of hard thresholds is motivated by other considerations,[7,8] for example to effect a bias–variance tradeoff. Indeed, (7) shows that when $s(n)$ has nonzero mean and $k_i \in [0,1]$, the estimation bias decreases if k_i increases. The Wiener filter always produces bias, while the hard thresholder produces zero bias whenever it results in $k_i = 1$.

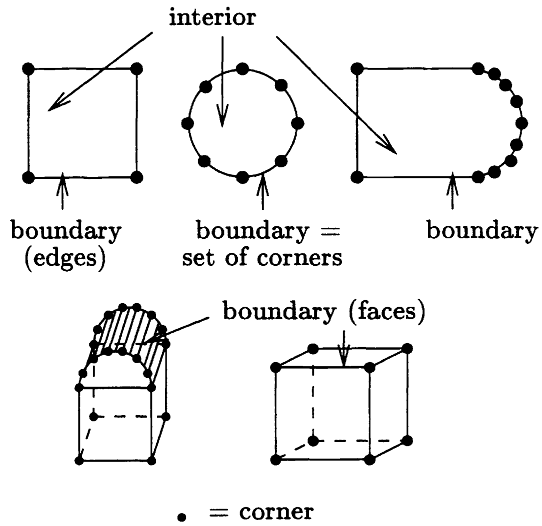


Figure 6. Corners and boundaries of compact convex sets.

5. WHAT IF THERE IS NO PCFB?

As seen earlier, existence of a PCFB implies that the set S of realizable subband variance vectors (the optimization search space) has a special structure: Its convex hull co(S) is a *polytope*. By definition this means that co(S) = co(E) where E is a *finite* set. Assuming that E is chosen to have as few elements as possible, the vectors in E are known as *corners* of the polytope. When a PCFB exists, in fact these corners are permutations of each other, and correspond to the PCFB. So the PCFB is always optimal whenever the minimization objective is concave over the polytope co(S), as illustrated in Fig. 4.

Thus whenever co(S) is a polytope, the optimization can be reduced to a search over the *finite* set of FB's that correspond to the corners of the polytope. When a PCFB exists, this set has exactly one element, namely the PCFB. If there is no PCFB, one could hope that if co(S) is indeed still a polytope, then it would not be very difficult to identify this finite set of FB's that corresponds to its corners (and thereby solve the optimization problem). However, a polytope is a fairly structured object. Given an input power spectrum and a class $\mathcal{C}$ of FB's, say the class of FIR FB's with a given bound on the filter orders, there is no apriori reason to believe that the corresponding set co(S) is a polytope. In general, co(S) would thus be a bounded convex set that is not necessarily a polytope. We shall assume that co(S) is closed (or compact), which will be true for most 'natural' classes $\mathcal{C}$ of FB's. We will next observe that corners can be defined for arbitrary convex sets (not necessarily polytopes), and note that they have optimality properties similar to those discussed above.

5.1. Arbitrary convex sets: corners and their optimality

Definition.[10] Let B be a convex subset of $\mathcal{R}^M$. A point $\mathbf{z} \in B$ is said to be an *extreme point*, or a *corner* of B if

$$\mathbf{z} = \alpha\mathbf{x} + (1-\alpha)\mathbf{y} \quad \text{with} \quad \alpha \in (0,1),\ \mathbf{x},\mathbf{y} \in B$$

$$\text{implies} \quad \mathbf{x} = \mathbf{y}\ (= \mathbf{z}).$$

Geometrically, we cannot draw a line-segment that contains $\mathbf{z}$ in its interior (i.e. not as an endpoint) and yet lies wholly within the set B. The interior of B cannot have any corners, because around each point in the interior we can draw a ball that lies wholly in B. So all the corners lie on the boundary (which is the set of points of B that are not in the interior). However, not all boundary points are necessarily corners. If B is a polytope, the above definition can be verified to coincide with the earlier definition of corners of a polytope. These points are illustrated in Fig. 6, which shows the corners of some closed and bounded (or compact) convex sets.

It can be shown without much effort, that every compact convex set is the convex hull of its boundary, and that it has at least one corner. The proof of the following result however is less obvious:

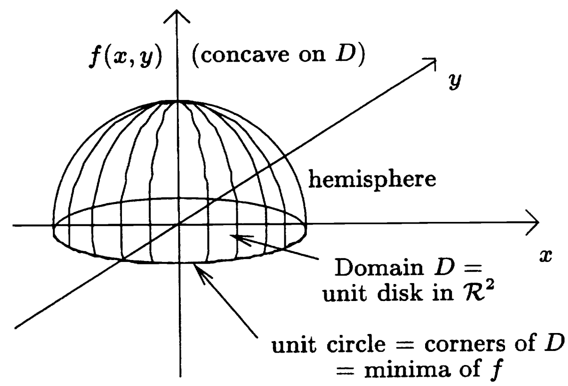


Figure 7. Optimality of corners of compact convex sets.

Krein-Milman theorem / Internal representation of convex sets[10,11]: Every compact convex set is the convex hull of its corners.

This result is evidently true for polytopes, and can be verified to be true in the examples shown in Fig. 6. The result is thus intuitively clear (although its formal proof might not be obvious). Its importance lies in the fact that it can be used to immediately prove :

Optimality of corners: Given any function g that is concave on a compact convex set D, at least one of the corners of D is a minimum of g. Further if g is strictly concave then its minimum is necessarily at a corner of the set.

For the special case when the compact convex set is a polytope, this result was discussed earlier and is illustrated in Fig. 4. Fig. 7 illustrates the result for a compact convex set that is *not* a polytope. In Fig. 7, all corners are 'equally good', i.e. all are minima, but this of course need not be true in general.

Proof of optimality of corners: Let $\mathbf{v}_{opt}$ be the minimum of g over D. (Its existence is either assumed or follows if g is assumed to be continuous.) By the Krein-Milman theorem, $\mathbf{v}_{opt}$ is a convex combination of some set of corners of D, i.e.

$$\mathbf{v}_{opt} = \sum_{j=1}^{J} \beta_j \mathbf{z}_j \quad \text{where} \quad \beta_j \in [0,1], \ \sum_{j=1}^{J} \beta_j = 1 \tag{12}$$

for some distinct corners $\mathbf{z}_j$ of D. Now at least one of the $\mathbf{z}_j$ has to be a minimum of g over D. If not, then $g(\mathbf{z}_j) > g(\mathbf{v}_{opt})$ for all $j = 1, 2, \ldots, J$, and hence

$$g(\mathbf{v}_{opt}) = g(\sum_{j=1}^{J} \beta_j \mathbf{z}_j) \geq \sum_{j=1}^{J} \beta_j g(\mathbf{z}_j) > \sum_{j=1}^{J} \beta_j g(\mathbf{v}_{opt}) = g(\mathbf{v}_{opt}), \tag{13}$$

i.e. $g(\mathbf{v}_{opt}) > g(\mathbf{v}_{opt})$ which is a contradiction. Hence at least one corner of D is a minimum of g over D. The first inequality above is the Jensen's inequality for concave functions. If g is strictly concave, then this inequality is strict unless one of the β_j is unity. Hence in this case $\mathbf{v}_{opt}$ equals the corresponding $\mathbf{z}_j$, i.e. the minimum is necessarily at a corner of D.

$\bigtriangledown \bigtriangledown \bigtriangledown$

In our FB optimization problem, $D = \text{co}(S)$ where S is the search space. Let E be the set of corners of D, so $E \subset S$ and $\text{co}(S) = \text{co}(E)$. From the above result, the optimization over $\text{co}(S)$ can be reduced to one over E. Thus the analytical tractability of the problems considered earlier can be traced to the fact that $\text{co}(S)$ is a polytope, i.e. that E is finite. In general, 'almost every' corner in E is associated with a concave (in fact linear) objective for which the corner is the unique minimum.* Thus, if a PCFB does not exist, or more precisely if E is not finite, it is not

*This can be proved using the concept of 'exposed points' or 'tangent hyperplanes' to convex sets.[11]

possible to make a general statement about the optimality of any single FB for a large class of objectives. It might be possible to avoid a suboptimum numerical search for a specific objective. However the analytical solution will have to exploit the specific structures of both the objective function and the set co(S). Thus we see that in absence of a PCFB, the problem of finding the optimum FB for a given concave objective usually becomes *analytically intractable*. So in such cases, a numerical procedure (that in general gives a suboptimum solution) such as a gradient-descent based algorithm is usually needed. It is enough to search for the minima over the set E (as opposed to S or co(S)); however it is not known to the authors at this time whether there are numerical search procedures that can exploit this fact.

5.2. The "sequence of compaction-filters" algorithm

This is an algorithm that has sometimes been proposed[1,4] to find a 'good' FB in classes $\mathcal{C}$ that need not necessarily have PCFB's. It involves a sequential maximization of subband variances. To be explicit, it is carried out by rearranging the elements of each vector in the optimization search space S in decreasing order, and then picking the 'greatest' of these vectors in the 'dictionary ordering' on $\mathcal{R}^M$. This vector can thus be shown to be a corner of co(S).

When the class $\mathcal{C}$ has a PCFB, all corners of co(S) correspond to the PCFB. Hence the algorithm always produces the PCFB, and is thus optimal for many problems as shown by the earlier sections. On the other hand, if a PCFB does not exist, then there will be at least two corners that are not equivalent, i.e. whose coordinates are not permutations of each other. The sequential algorithm produces one corner, but the minima of the concave objective could easily be at other non-equivalent corners. Thus the algorithm could be suboptimum.

To illustrate this point, consider the following hypothetical example with $M = 3$ channels: Let $\text{co}(S) = \text{co}(E)$ where the set E consists of vectors $\mathbf{v}_1 = (3, 2, 1)^T, \mathbf{v}_2 = (2.9, 2.2, 0.9)^T$ and their permutations. Since E is finite, co(S) is a polytope whose corners lie in E. Since neither of $\mathbf{v}_1, \mathbf{v}_2$ *majorizes* the other, in fact all elements of E are corners of co(S). A PCFB does not exist because $\mathbf{v}_1, \mathbf{v}_2$ are not permutations of each other. Now consider the high bit-rate coding problem of.[1] Here the objective to be minimized over S is $\pi(\mathbf{v})$, the product of the coordinates of $\mathbf{v} \in S$. (As noted earlier, this is equivalent to minimizing an objective that is concave on co(S).) Since $\pi(\mathbf{v}_1) = 6$ and $\pi(\mathbf{v}_2) = 5.742$, $\mathbf{v}_2$ is the minimum. However, the sequential algorithm produces $\mathbf{v}_\alpha = \mathbf{v}_1$, and is thus suboptimum.

More generally, let $P \subseteq \text{co}(S)$ be the polytope whose corners are permutations of the vector $\mathbf{v}_\alpha$ produced by the sequential algorithm. Then $P = \text{co}(S)$ iff a PCFB exists. Now consider the function $f(\mathbf{v}) = -d(\mathbf{v}, P)$, where $d(\mathbf{v}, P) = \min\{\|\mathbf{v} - \mathbf{x}\| : \mathbf{x} \in P\}$ is the minimum distance from $\mathbf{v}$ to P using any valid norm $\|.\|$ on $\mathcal{R}^M$. It can be shown that f is a well-defined continuous concave function on $\mathcal{R}^M$. From the definition it is clear that (1) f has a constant value (zero) on P, and (2) if a PCFB does not exist, then P is actually the set of *maxima* of f over co(S). Since the sequential algorithm produces subband variance vector $\mathbf{v}_\alpha \in P$, it leads to the worst possible choice of FB's for an infinite family of such concave objectives f.

6. COLORED NOISE SUPPRESSION

In the denoising problems of Section 4.2, the noise $\mu(n)$ at the FB input was white with variance η^2. Hence the subbands of any orthonormal FB would have noise components that are white with variance η^2. If the input noise is colored, the variances of the subband noise components now *depend on choice of FB*. Thus all equations of Section 4.2 will have to be modified by replacing the fixed noise variance η^2 by the specific subband noise variance η_i^2. The objective function thus depends on two subband variance vectors $\mathbf{v}_\sigma$ and $\mathbf{v}_\eta$, respectively corresponding to signal and noise. In fact the objective has the form

$$f(\mathbf{v}_\sigma, \mathbf{v}_\eta) = \frac{1}{M} \sum_{i=0}^{M-1} f_i(\sigma_i^2, \eta_i^2), \quad \text{where} \tag{14}$$

$$f_i(x, y) = \begin{cases} |1 - k_i|^2 x + |k_i|^2 y & \text{for constant multiplier } k_i \text{ in } i\text{-th subband} \\ \frac{xy}{x+y} & \text{for 0-th order Wiener filter in } i\text{-th subband} \\ \min(x, y) & \text{for hard thresholder in } i\text{-th subband} \end{cases} \tag{15}$$

Thus the optimization search space here is S_v, the set of all realizable pairs $(\mathbf{v}_\sigma, \mathbf{v}_\eta)$. We denote by S_σ and S_η respectively the set of realizable signal and noise subband variance vectors. The above functions f_i can be verified to be concave over the non-negative quadrant of $\mathcal{R}^2$. So the objective f is concave over $T \triangleq \text{co}(S_\sigma) \times \text{co}(S_\eta) \supset S_v$.

When the noise is white, S_η has exactly one element, and a PCFB for the signal is optimum. Notice that this PCFB is also a *common* PCFB for both the signal $s(n)$ and the noise $\mu(n)$, since any FB is a PCFB for a white input. Such a common PCFB (if it exists) has certain optimality properties even if neither the signal nor the noise is white. We now state and outline proofs of these properties.

6.1. Statement of results

Result 1. If the power spectrum (psd) matrices of the M-fold blocked versions of the signal and noise are scaled versions of each other, then for any class $\mathcal{C}$, a common signal and noise PCFB (if it exists) is always optimal.

Result 2. If $\mathcal{C}$ is the class of all M-channel orthogonal transform coders, a common signal and noise PCFB is always optimal no matter what the psd matrices corresponding to $s(n)$ and $\mu(n)$ are (provided of course they are such that a common PCFB exists). Notice that in this case, the common PCFB is also a PCFB for the signal $x(n) = s(n) + \mu(n)$. This is not true in general, for example for the class of all (unconstrained) M-channel FB's.

Result 3. If $\mathcal{C}$ is the class of all M-channel orthonormal FB's (unconstrained), there are large classes of psd matrices of $s(n)$ and $\mu(n)$ for which a common signal and noise PCFB exists but is still not optimal for many concave objectives.

Result 4. As Result 3 shows, for arbitrary class $\mathcal{C}$ and input psd matrices, a common signal and noise PCFB is not necessarily optimal for *all* concave objectives. However, it is still always optimal for a certain nontrivial subset of these objectives. There is a simple finite procedure that decides whether or not a given concave objective falls in this subset.

6.2. Proofs of results

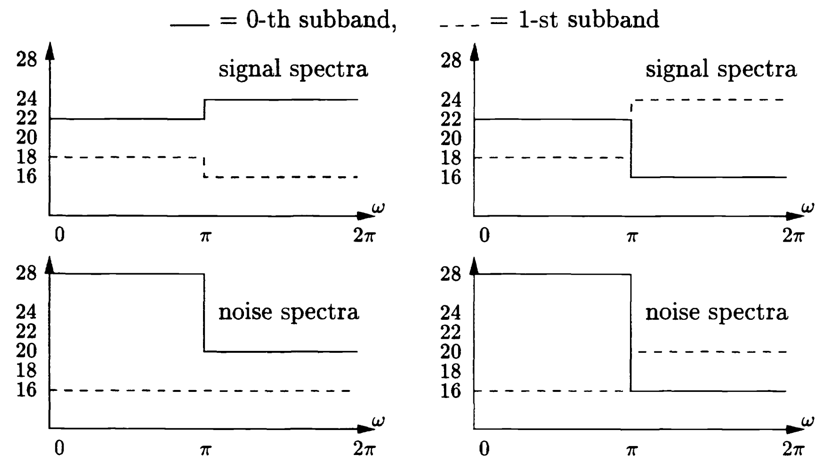
Result 1 is true because under the condition it imposes, the signal and noise subband variance vectors for any FB are obtainable from each other by a constant scaling. Thus, the objective can be rewritten so that it depends on only one of these subband variance vectors, and we can use the earlier results on PCFB optimality. Notice that in this case, a PCFB for the signal is also a PCFB for the noise and vice-verca. To prove the remaining results, we assume at the outset that a separate PCFB exists for both the signal $s(n)$ and the noise $\mu(n)$ (since this is a condition in all these results). As stated in Section 3.1, this implies $\text{co}(S_\sigma) = \text{co}(E_\sigma)$ where $E_\sigma \subset S_\sigma$ is a finite set of vectors all of which correspond to the signal PCFB (and are hence permutations of each other). Similarly $\text{co}(S_\eta) = \text{co}(E_\eta)$ where $E_\eta \subset S_\eta$ is a finite set corresponding to the noise PCFB. Thus $T = \text{co}(E_\sigma) \times \text{co}(E_\eta) = \text{co}(E_\sigma \times E_\eta)$. So it can be seen that T is a polytope whose set of corners is the (finite) Cartesian product $E_\sigma \times E_\eta$. Thus, $S_v \subset \text{co}(S_v) \subset T$, where T is a polytope.

All corners of $\text{co}(S_v)$ lie in S_v, and at least one of them is the optimum. So we try to examine the nature of the corners of $\text{co}(S_v)$. The first observation is that any corner of the polytope T that lies in S_v is also a corner of $\text{co}(S_v)$. To establish results 2,3,4 of Section 6.1, we now assume that a common signal and noise PCFB exists. Now any vector in S_v that corresponds to such a PCFB is a corner of T. Conversely, any corner of T that lies in S_v corresponds to such a PCFB. Let $E_v \subseteq E_\sigma \times E_\eta$ denote the set of all corners of T that lie in S_v. The points in E_v are hence corners of $\text{co}(S_v)$, but it is however not clear whether or not $\text{co}(S_v)$ has other corners. In the extreme situation when $E_v = E_\sigma \times E_\eta$, then in fact $\text{co}(S_v) = T$ and it does not have other corners. Since all its corners correspond to a common signal and noise PCFB, such a PCFB would always be optimal. However, $E_v = E_\sigma \times E_\eta$ is usually possible only in contrived cases, or with a degeneracy such as white noise.

Proof of Result 2

If $\mathcal{C}$ is the class of M-channel orthogonal transform coders, the common signal and noise PCFB is the common KLT, which is unique. So all vectors in E_v are corners of $\text{co}(S_v)$ that correspond to the common KLT, and it turns out that $\text{co}(S_v)$ *has no other corners.* Thus, $\text{co}(S_v) = \text{co}(E_v)$ is a polytope with set of corners E_v, and at least one of these corners minimizes the objective. So a common signal and noise PCFB (KLT) is always optimal. The crucial fact that $\text{co}(S_v) = \text{co}(E_v)$ is true no matter what the signal and noise spectra are (assuming of course that they are such that a common KLT exists). It is shown in detail in,[12] and thus completes the proof.

Proof of Result 3



FB - I: simultaneous signal and noise PCFB.
$\mathbf{v}_\sigma^I = (23, 17)^T, \quad \mathbf{v}_\eta^I = (24, 16)^T$

FB - II: permute band $\omega \in [\pi, 2\pi]$ in FB - I.
$\mathbf{v}_\sigma^{II} = (19, 21)^T, \quad \mathbf{v}_\eta^{II} = (22, 18)^T$

Figure 8. Suboptimality of signal and noise PCFB.

When $\mathcal{C}$ is the class of all M-channel orthonormal FB's (unconstrained) then the property $\text{co}(S_v) = \text{co}(E_v)$ still holds in certain restricted situations, for example when the signal and noise psd matrices are both constant. (In this case the PCFB's are the corresponding KLT's.) However, it does not hold for all signal and noise psd matrices. If it does not hold, it implies that $\text{co}(S_v)$ has other corners besides the points in E_v. There would then be concave objectives for which one of these other corners (which clearly does not correspond to a common signal and noise PCFB) is optimal. This is illustrated by the example of Fig. 8. Here the filter-bank FB-II is not a PCFB for either the signal or the noise. However it is better than FB-I, the common signal and noise PCFB for $\mathcal{C}$, for the denoising problem using either hard-thresholding or zeroth order Wiener filtering in both subbands. This can be verified by substituting the corresponding signal and noise subband variance vectors from Fig. 8 into the objectives for these problems. Note that many more such examples can be created, for instance by applying small perturbations on the spectra in Fig. 8. This proves Result 3 of Section 6.1. Fig. 9 shows the various geometries of S_v as a subset of T arising out of the situations discussed thus far. (The figure is only illustrative, since T actually lies in an even dimensional space and not in $\mathcal{R}^3$.)

Proof of Result 4

We know from Theorem 1 that at least one of the finitely many corners of T is a minimum of the objective *over* T. Such corners, though easy to find, may not lie in S_v, and are hence not useful in general since we are seeking for minima over S_v (or $\text{co}(S_v)$). However, there are always concave objectives with the property that their minimum over T is a corner of T that actually lies in S_v. This corner would hence be the minimum of such an objective over $S_v \subset T$ too. Thus for the subset $\mathcal{F}$ of concave objectives having this property, a common PCFB is optimal. ($\mathcal{F}$ is however not the complete set of concave objectives for which a common signal and noise PCFB is optimal.) Recall that the (finite) set $E_\sigma \times E_\eta$ of corners of T is fully specified given the signal and noise subband variance vectors generated by the common signal and noise PCFB. Thus we have a simple finite procedure to identify whether or not a given concave objective is in the subset $\mathcal{F}$: We evaluate the objective at each corner of T, and find whether or not the minimum over these corners (which is the minimum over T) lies in S_v. This establishes Result 4 of Section 6.1.

7. CONCLUDING REMARKS

We have pointed out a basic connection between FB optimization and the principal component property. We have shown that PCFB's are optimal for various signal processing schemes such as subband denoising using zeroth order Wiener filters and hard thresholders. We have discussed these optimization problems in situations where a PCFB

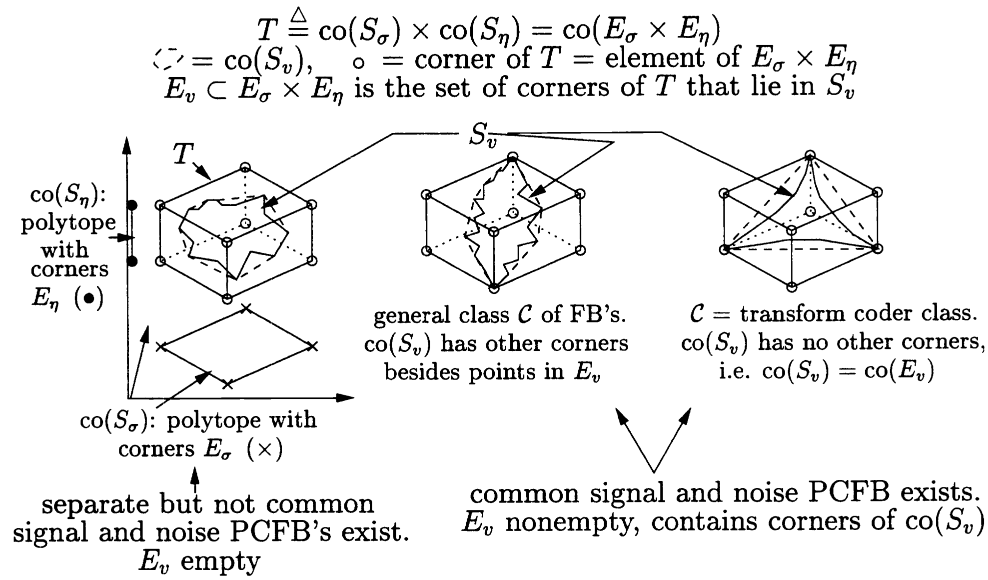


Figure 9. Geometry of the search-space.

does not exist. We have also considered the case of colored noise suppression, where PCFB optimality is somewhat more restricted. Certain extensions to *biorthogonal* and *nonuniform* FB's, can be found in.[12]

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