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Bosonization in three spatial dimensions and a 2-form gauge theory

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Abstract

We describe a 3d analog of the Jordan-Wigner transformation which maps an arbitrary fermionic system on a 3d spatial lattice to a 2-form $\mathbb{Z}_2$ gauge theory with an unusual Gauss law. An important property of this map is that it preserves the locality of the Hamiltonian. We give examples of 3d bosonic systems dual to free fermions and describe the corresponding Euclidean lattice models.

1 Introduction and summary

It is well known that every lattice fermionic system in 1d is dual to a lattice system of spins with a $\mathbb{Z}_2$ global symmetry (and vice versa). The duality is kinematic (independent of a particular Hamiltonian) and arises from the Jordan-Wigner transformation. Recently it has been shown that any lattice fermionic system in 2d is dual to a $\mathbb{Z}_2$ gauge theory with an unusual Gauss law [1]. The fermion can be identified with the flux excitation of the gauge theory. The 2d duality is also kinematic. In this paper we extend these results to 3d systems. We show that every lattice fermionic system in 3d is dual to a $\mathbb{Z}_2$ 2-form gauge theory with an unusual Gauss law. Here "2-form gauge theory" means that the $\mathbb{Z}_2$ variables live on plaquettes, while the parameters of the gauge symmetry live on links. 2-form gauge theories in 3+1D

have local flux excitations, and the unusual Gauss law ensures that these excitations are fermions.[1]

The form of the modified Gauss law is largely dictated by the observation first made in [2] that a bosonization of fermionic systems in d dimensions must have a global $(d-1)$-form $\mathbb{Z}_2$ symmetry with a particular 't Hooft anomaly. The standard Gauss law leads to a trivial 't Hooft anomaly, so bosonization requires us to modify it in a particular way. The precise form of the modified Gauss law and the bosonization map depends on the choice of the lattice. We describe them in two cases: the cubic lattice and a 3d triangulation.

Our 3d bosonization map is kinematic and local in the same sense as the Jordan-Wigner map: every local bosonic observable on the fermionic side, including the Hamiltonian density, is mapped to a local gauge-invariant observable on the gauge theory side.

In the literature, there are examples of specific bosonic models in 3d with emergent fermions. Our general construction is reminiscent of the work by Levin and Wen [3]. These authors constructed systems of rotors which have emergent fermions. In our approach rotors are replaced with $\mathbb{Z}_2$ spins. There are also several proposals for an analog of the Jordan-Wigner map in arbitrary dimensions [4, 5, 6]. Our construction is most similar to that of Bravyi and Kitaev [4]. One advantage of our construction is that we can clearly identify the kind of 3d bosonic systems that are dual to fermionic systems: they possess global 2-form $\mathbb{Z}_2$ symmetry with a specific 't Hooft anomaly, as proposed in [2]. It is also manifest in our approach that the bosonization map depends on a choice of spin structure.

Our bosonization method allows for an easy construction of bosonic systems dual to free fermions with an arbitrary dispersion law. As an illustration, we describe a bosonic model on a cubic lattice whose dual fermionic description involves Dirac cones. It can be regarded as a 3d analog of the Kitaev honeycomb model. Other 3d analogs of the honeycomb model have been proposed in [7, 8]. We also identify some Euclidean bosonic 4D models which are dual to free fermions. These models can be understood as 2-form $\mathbb{Z}_2$ gauge theories whose action involves a topological term.

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[1]In contrast, a 2-form $\mathbb{Z}_2$ gauge theory with the standard Gauss law is mapped, by a 3d version of the Kramers-Wannier duality, to a theory of bosonic spins.
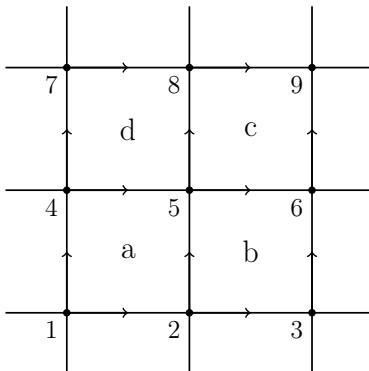
Figure 1: Bosonization on a square lattice requires constraints on vertices.

2 Bosonization on a three-dimensional lattice

2.1 Cubic lattice

We begin by reviewing the 2d bosonization on a square lattice following [1]. The set of vertices, edges, and faces are denoted V, E, F, and their elements v, e, f. On each face f of the lattice we place a single pair of fermionic creation-annihilation operators $c_f, c_f^\dagger$, or equivalently a pair of Majorana fermions γ_f, γ'_f. The even fermionic algebra consists of local observables with a trivial fermionic parity (i.e. $(-1)^F = 1$). It is generated by $(-1)^{F_f} = -i\gamma_f\gamma'_f$ and $S_e = i\gamma_{L(e)}\gamma'_{R(e)}$, where $L(e)$ and $R(e)$ are faces to the left and right of e, with respect to some orientation of e.

The bosonic dual of this system involves $\mathbb{Z}_2$-valued spins on the edges of the square lattice. For every edge e we define a unitary operator U_e which squares to 1. Labeling the faces and vertices as in Fig. 1, we define:

$$U_{56} = X_{56}Z_{25}$$
$$U_{58} = X_{58}Z_{45} \tag{1}$$

where X, Z are Pauli matrices acting on a spin at each edge. Operators U_e for other edges are defined by using translation symmetry.

It has been shown in [1] that U_e and S_e satisfy the same commutation relations. We also identify fermionic parity $(-1)^{F_f}$ at each face

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with the "flux operator" $W_f \equiv \prod_{e \subset f} Z_e$. The bosonization map is

$$
\begin{aligned}
(-1)^{F_f} &= -i\gamma_f \gamma_f' \longleftrightarrow W_f \\
S_e &= i\gamma_{L(e)} \gamma_{R(e)}' \longleftrightarrow U_e.
\end{aligned}
\tag{2}
$$

The condition $(-1)^{F_a}(-1)^{F_c} S_{58} S_{56} S_{25} S_{45} = 1$ on fermionic operators gives gauge constraints $W_{f_c} \prod_{e \supset v_5} X_e = 1$ for bosonic operators, or generally

$$
W_{\mathrm{NE}(v)} \prod_{e \supset v} X_e = 1
\tag{3}
$$

where $\mathrm{NE}(v)$ is the face northeast of v. Eq. (3) is the modified Gauss law for a 2d gauge theory.

Next, we introduce our bosonization method on an infinite 3d cubic lattice. Suppose that we have a model with fermions living at the centers of cubes. Let us describe the generators and relations in the algebra of local observables with trivial fermion parity (the even fermionic algebra for short).

On each cube t we have a single fermionic creation operator c_t and a single fermionic annihilation operator $c_t^\dagger$ with the usual anti-commutation relations. The fermionic parity operator on cube t is $(-1)^{F_t} = (-1)^{c_t^\dagger c_t}$. It is a "$\mathbb{Z}_2$ operator" (i.e. it squares to 1). All operators $(-1)^{F_t}$ commute with each other. The even fermionic algebra is generated by these operators and the operators $c_t^\dagger c_{t'}$, $c_t c_{t'}$, and their Hermitean conjugates, where t and t' are two cubes which share a face. Overall, we get four generators for each face and one generator for each cube. It is easier to work in the Majorana basis

$$
\gamma_t = c_t + c_t^\dagger, \quad \gamma_t' = (c_t - c_t^\dagger)/i.
\tag{4}
$$

The even fermionic algebra is generated by $i\gamma_{L(f)} \gamma_{R(f)}'$ and $-i\gamma_t \gamma_t'$ where each face is assigned an orientation from cube $L(f)$ to cube $R(f)$.

To illustrate the definition of these operators, we draw the dual lattice of the original lattice. In Fig. 2, fermions live on vertices and the orientations of each dual edge (face of the original lattice) are taken to be along $+x$, $+y$, and $+z$ directions. The Majorana hopping operator is defined by $S_f = i\gamma_{L(f)} \gamma_{R(f)}'$ where $L(f)$ and $R(f)$ are source and sink (starting and ending points) of dual edge f in the dual lattice. S_{f_i} and S_{f_j} anti-commute only when both dual edges f_i and f_j start from the same point or both end at the same point.
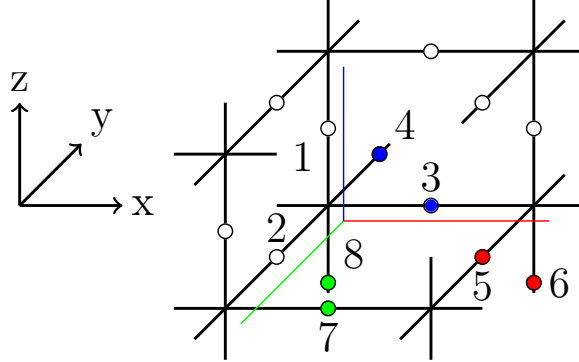
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Figure 2: (Color online) For edges in the dual lattice, the "framing" is defined by green, red, and blue edges, which is a small shift of duel edges [3]. Given a dual edge f, the operator U_f is defined as X_f times $Z_{f'}$ for those f' which intersect the framing of f when projected to the plane (i.e. $U_{f_1} = X_1 Z_3 Z_4$, $U_{f_2} = X_2 Z_7 Z_8$, and $U_{f_3} = X_3 Z_5 Z_6$).

The dual bosonic system has $\mathbb{Z}_2$ spins living on faces of the original lattice (or equivalently, on edges of the dual lattice). To define bosonic hopping operators U_f, we need to choose a framing for each edge of the dual lattice, i.e. a small shift of each dual edge along some orthogonal direction. We also assume that when projected on some generic plane (such as the plane of the page) a shifted dual edge intersects all dual edges transversally. For example, in Fig. 2 such a framing is indicated by red, green and blue lines (for dual edges along x, y and z directions, respectively), and the shift of the dual edge 1 intersects dual edges 3 and 4.[2] Now we define U_f as a product of X_f with all $Z_{f'}$ such that f' intersects the framing of f when projected to the plane of the page. For example, the hopping operator for the dual edge 1 is $U_1 = X_1 Z_3 Z_4$. Notice that U_1, U_3, and U_4 anti-commute with each other and U_3, U_5, and U_6 anti-commute with each other, while U_2 and U_3 commute, and U_1 and U_8 commute.

One can check that S_f and U_f have the same commutation relations. Therefore, the bosonization map in 3D can be defined as follows:

1. For any cube t let $W_t \equiv \prod_{f \subset t} Z_f$. We identify the fermionic states $|F_t = 0\rangle$ and $|F_t = 1\rangle$ with bosonic states for which $W_t = 1$

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[2]There are many choices of framing, and accordingly many versions of the bosonization map. By construction, they are related by automorphisms of the algebra of observables.

and $W_t = -1$, respectively. Thus

$$(-1)^{F_t} = -i\gamma_t\gamma'_t \longleftrightarrow W_t. \tag{5}$$

2. The fermionic hopping operator S_f is identified with U_f defined above:

$$S_f = i\gamma_{L(f)}\gamma'_{R(f)} \longleftrightarrow U_f. \tag{6}$$
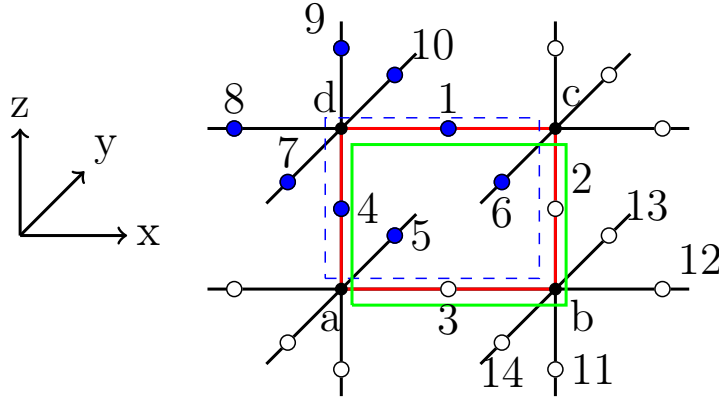

Figure 3: (Color online) The framing of the hopping term defined previously is indicated by the green square, while the gauge constraint involves the Z operators in the opposite framing (blue dashed square).

As in 2d, the bosonic operators satisfy some constraints. In Fig. 3, we calculate the product of S_f around the red square on the dual lattice:

$$\begin{aligned}
S_{f_1}S_{f_2}S_{f_3}S_{f_4} &= (i\gamma_d\gamma'_c)(i\gamma_b\gamma'_c)(i\gamma_a\gamma'_b)(i\gamma_a\gamma'_d) \\
&= -(-i\gamma_b\gamma'_b)(-i\gamma_d\gamma'_d) \\
&= -(-1)^{F_b}(-1)^{F_d} \longleftrightarrow -W_bW_d.
\end{aligned} \tag{7}$$

Its bosonic dual defined by (6) is the product of the corresponding operators U_f. Their definition involves a framing of the red square given by the green square:

$$\begin{aligned}
U_{f_1}U_{f_2}U_{f_3}U_{f_4} &= (X_1Z_2Z_6)(X_2Z_{12}Z_{13})(X_3Z_{11}Z_{14})(X_4Z_3Z_5) \\
&= -X_1X_2X_3X_4Z_5Z_6(Z_2Z_3Z_{11}Z_{12}Z_{13}Z_{14}) \\
&= -X_1X_2X_3X_4Z_5Z_6W_b.
\end{aligned} \tag{8}$$

Comparing (7) and (8), we get the constraint

$$
\begin{aligned}
1 =& X_1 X_2 X_3 X_4 Z_5 Z_6 W_d \\
=& X_1 X_2 X_3 X_4 Z_1 Z_4 Z_5 Z_6 Z_7 Z_8 Z_9 Z_{10}
\end{aligned}
\tag{9}
$$

The operators Z's are the edges crossed by dashed square in Fig. 3. The framing for gauge constraints is opposite to the framing used to define hopping operators. We have a gauge constraint for each face of dual lattice. In terms of the original lattice, there is one gauge constraint for each edge. All these constraints commute and thus define a $\mathbb{Z}_2$ 2-form gauge theory with an unusual Gauss law.

2.2 Examples

2.2.1 Soluble 3+1D lattice gauge theories

The standard Gauss law for a 2-form $\mathbb{Z}_2$ gauge theory is $\prod_{f \supset e} X_f = 1$. Such a bosonic gauge theory is dual to a theory of bosonic spins living on the vertices of the dual lattice. In particular, the quantum Ising model can be described by a $\mathbb{Z}_2$ 2-form gauge theory with the Hamiltonian

$$
H_{Ising} = g^2 \sum_f X_f + \frac{1}{g^2} \sum_t W_t.
\tag{10}
$$

This model is not soluble.

If we impose the modified Gauss law (9) instead, the simplest analogous gauge-invariant Hamiltonian is

$$
H_b = g^2 \sum_f U_f + \frac{1}{g^2} \sum_t W_t.
\tag{11}
$$

The first and second term can be thought of as the kinetic and potential energies, respectively. This is dual to the fermionic Hamiltonian

$$
H_f = t \sum_f (c_{L(f)} c_{R(f)} - c_{L(f)}^\dagger c_{R(f)}^\dagger + c_{L(f)}^\dagger c_{R(f)} + c_{R(f)}^\dagger c_{L(f)}) + \mu \sum_t c_t^\dagger c_t
\tag{12}
$$

where $t = g^2$ and $\mu = \frac{2}{g^2}$. The fermionic Hamiltonian is free and thus soluble. By Fourier transform $c_{\vec{x}} = \frac{1}{\sqrt{N}} \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} c_{\vec{k}}$, the fermionic Hamiltonian becomes

$$
H_f = \sum_{\vec{k}} \epsilon_{\vec{k}} c_{\vec{k}}^\dagger c_{\vec{k}} + \sum_{\vec{k}} (\Delta_{\vec{k}} c_{\vec{k}} c_{-\vec{k}} + \text{h.c.})
\tag{13}
$$

with $\epsilon_{\vec{k}} = \mu + 2t(\cos k_x + \cos k_y + \cos k_z)$ and $\Delta_{\vec{k}} = t(e^{-ik_x} + e^{-ik_y} + e^{-ik_z})$. The Hamiltonian (13) can be written in the Bogoliubov-de-Gennes (BdG) formalism as

$$H_f = \frac{1}{2} \sum_{\vec{k}} \Psi_{\vec{k}}^{\dagger} H_{\text{BDG}}(\vec{k}) \Psi_{\vec{k}} \tag{14}$$

with

$$H_{\text{BDG}}(\vec{k}) = \begin{bmatrix} \epsilon_{\vec{k}} & -\Delta_{\vec{k}}^* \\ -\Delta_{\vec{k}} & -\epsilon_{\vec{k}} \end{bmatrix}, \quad \Psi_{\vec{k}} = \begin{bmatrix} c_{\vec{k}} \\ c_{-\vec{k}}^{\dagger} \end{bmatrix}. \tag{15}$$

The spectrum is

$$\begin{aligned} E^2 = &\, t^2(3 + 2\cos(k_x - k_y) + 2\cos(k_x - k_z) + 2\cos(k_y - k_z)) \\ &+ [\mu + 2t(\cos k_x + \cos k_y + \cos k_z)]^2. \end{aligned} \tag{16}$$

Notice that for $\mu = 0$ the gap closes for $\vec{k} = (q, q + \frac{2\pi}{3}, q + \frac{4\pi}{3})$ for arbitrary q.

2.2.2 Bosonic model with Dirac cones

Using the bosonization map (5) and (6), we can construct an equivalent bosonic model for any arbitrary fermionic model. For instance, Ref. [9] constructs a fermionic model on a cubic lattice with Dirac cones:

$$H = -t \sum_{\vec{r}} (s_x(\vec{r}) c_{\vec{r}+\hat{x}}^{\dagger} c_{\vec{r}} + s_y(\vec{r}) c_{\vec{r}+\hat{y}}^{\dagger} c_{\vec{r}} + s_z(\vec{r}) c_{\vec{r}+\hat{z}}^{\dagger} c_{\vec{r}} + \text{h.c.}) \tag{17}$$

with $s_x(\vec{r})$, $s_y(\vec{r})$, and $s_z(\vec{r})$ defined as

$$\begin{aligned} s_x(i, j, k) &= 1 \\ s_y(i, j, k) &= (-1)^i \\ s_z(i, j, k) &= (-1)^{i+j}. \end{aligned} \tag{18}$$

It is a model with nearest neighbor hopping. The spectrum is

$$E = \pm 2t \sqrt{\cos^2 k_x + \cos^2 k_y + \cos^2 k_z} \tag{19}$$

with two Dirac cones at $\vec{k} = (\pi/2, \pi/2, \pi/2)$ and $\vec{k} = (3\pi/2, \pi/2, \pi/2)$. Applying the bosonization map, the corresponding bosonic Hamilto-

nian is

$$H = -\frac{t}{2} \sum_{f_x} s_x(L(f_x)) U_{f_x} (1 - W_{L(f_x)} W_{R(f_x)})$$

$$-\frac{t}{2} \sum_{f_y} s_y(L(f_y)) U_{f_y} (1 - W_{L(f_y)} W_{R(f_y)}) \qquad (20)$$

$$-\frac{t}{2} \sum_{f_z} s_z(L(f_z)) U_{f_z} (1 - W_{L(f_z)} W_{R(f_z)}),$$

where f_x, f_y, f_z refer to faces normal to x, y, z-directions, with gauge constraints (9). On the bosonic side, it is very nontrivial to see that the model describes Dirac cones.
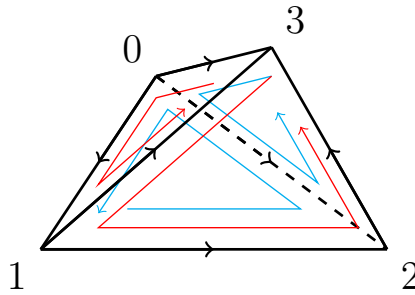
2.3   Triangulation

Figure 4: (Color online) A branching structure on a tetrahedron. The orientation of each face is determined by the right-hand rule. We defined this as the "+" tetrahedron, the directions of faces 012 and 023 are inward (blue) while the directions of faces 123 and 013 are outward (red). The directions of faces are reversed in the "−" tetrahedron (mirror image of this tetrahedron).

The bosonization method described above also works for any triangulation. For an arbitrary triangulation T of a 3d manifold M, we choose a branching structure. A branching structure is a choice of an orientation on each edge such that there is no oriented loop on any triangle. One simple way is to label vertices by different numbers and assign the direction of an edge from the vertex with smaller number to the vertex with larger number (see Fig. 4). Each tetrahedron has two inward faces and two outward faces (by right-hand rule). We place

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fermions at the centers of tetrahedrons. Each tetrahedron t contains Majorana operators γ_t and γ'_t. We define the fermionic hopping operator on each face f as

$$S_f = i\gamma_{L(f)}\gamma'_{R(f)} \tag{21}$$

where $L(f)/R(f)$ is the tetrahedron with f as a outward/inward face. Notice that S_f and $S_{f'}$ anti-commute only when f and f' share a tetrahedron with both f and f' inward or outward. To express this property mathematically, we introduce (higher) cup product used in algebraic topology. The definition and properties of the (higher) cup products are reviewed in Appendix A. If β_1 and β_2 are 2-cochains, then

$$\beta_1 \cup_1 \beta_2(0123) = \beta_1(023)\beta_2(012) + \beta_1(013)\beta_2(123). \tag{22}$$

Therefore, the commutation relations can be expressed as

$$S_f S_{f'} = (-1)^{\int f \cup_1 f' + f' \cup_1 f} S_{f'} S_f \tag{23}$$

where we abuse the notation $f \in C^2(T, \mathbb{Z}_2)$ as a 2-cochain with value 1 on face f and 0 otherwise. The even fermionic algebra is generated by the operators S_f for all faces and the fermionic parity operators $(-1)^{F_t}$ for all tetrahedra.

The dual bosonic variables are $\mathbb{Z}_2$ spins which live on faces of the triangulation. As before, the flux operator

$$W_t = \prod_{f \supset t} X_f$$

corresponds to $(-1)^{F_t}$ under the bosonization map.

Next we need to find bosonic operators U_f which have the same commutation relation as fermionic operators S_f. We should define U_f as X_f times $Z_{f'}$ for some faces f' which share a tetrahedron with f and have the same orientation with respect to the tetrahedron. One way to define U_f is

$$U_f = X_f \prod_{t \in \{L(f), R(f)\}} Z_{t_{023}}^{f(t_{012})} Z_{t_{013}}^{f(t_{123})} = X_f \prod_{f'} Z_{f'}^{\int f' \cup_1 f}. \tag{24}$$

U_f satisfy the commutation relation

$$U_f U_{f'} = (-1)^{\int f \cup_1 f' + f' \cup_1 f} U_{f'} U_f \tag{25}$$

which is the same as (23).

The final step is to determine the constraints on bosonic variables. There is one such constraint for each edge e. In the product $\prod_{f \supset e} S_f$, the only surviving terms are $-i\gamma_t \gamma_t'$ with one face going inward and one face going outward of t. The term $-i\gamma_t \gamma_t'$ is bosonized to $W_t \equiv \prod_{f \subset t} Z_f$. Therefore, the product can be written as

$$\prod_{f \supset e} S_f \sim \prod_{t|e=t_{01},t_{03},t_{12},t_{23}} W_t \tag{26}$$

where $\sim$ means that it is equal up to a sign, which will be treated carefully in the next paragraph. For a tetrahedron t containing an edge e with adjacent faces f_1 and f_2, consider the following product which gives W_t for $e = t_{01}, t_{03}, t_{12}, t_{23}$ and 1 otherwise:

$$Z_{f_1} Z_{f_2} \prod_{f' \subset t} Z_{f'}^{(f_1+f_2)\cup_1 f' + f' \cup_1 (f_1+f_2)} = \begin{cases} W_t, & \text{if } e = t_{01}, t_{03}, t_{12}, t_{23} \\ 1, & \text{otherwise} \end{cases} \tag{27}$$

Substituting this into (26), we have

$$\prod_{f \supset e} S_f \sim \prod_{f \supset e} \prod_{f'} Z_{f'}^{\int f' \cup_1 f + f \cup_1 f'} = \prod_{f'} Z_{f'}^{\int f' \cup_1 \delta e + \delta e \cup_1 f'}. \tag{28}$$

On the other hand, the product $\prod_{f \supset e} U_f$ is

$$\prod_{f \supset e} U_f \sim \prod_{f \supset e} X_f \prod_{f'} Z_{f'}^{\int f' \cup_1 f} = \left(\prod_{f \supset e} X_f \right) \prod_{f'} Z_{f'}^{\int f' \cup_1 \delta e}. \tag{29}$$

Identifying (28) and (29) gives

$$\left(\prod_{f \supset e} X_f \right) \prod_{f'} Z_{f'}^{\int \delta e \cup_1 f'} = 1 \tag{30}$$

This is the modified Gauss law (gauge constraint) on each edge e. One can check that constraints for different edges e_1 and e_2 commute since

$$\int (\delta e_1 \cup_1 \delta e_2 + \delta e_2 \cup_1 \delta e_1) =$$

$$= \int (e_1 \cup \delta e_2 + \delta e_2 \cup e_1 + e_2 \cup \delta e_1 + \delta e_1 \cup e_2) = 0 \tag{31}$$

where we have used the property $\int \delta e_1 \cup_1 \delta e_2 = \int (e_1 \cup \delta e_2 + \delta e_2 \cup e_1)$.

To be more precise about the signs in (28) and (29), we give the definition of S_β for $\beta \in C^2(T, \mathbb{Z}_2)$:

$$S_\beta S_{\beta'} = (-1)^{\int \beta' \cup_1 \beta} S_{\beta+\beta'} \tag{32}$$

or equivalently

$$S_\beta = (-1)^{\sum_{f < f' \in \beta} f \cup_1 f'} \prod_{f \in \beta} S_f \tag{33}$$

where the order of f in β doesn't affect the product due to its property (23). Note that the convention for the product $\prod$ is $\prod_{f \in \{f_1, f_2, \ldots, f_n\}} S_f = S_{f_n} \cdots S_{f_2} S_{f_1}$. We can also define U_β in the same way. It can be checked that

$$U_\beta = (-1)^{\sum_{f < f' \in \beta} f \cup_1 f'} \prod_{f \in \beta} U_f$$
$$=: \prod_{f \in \beta} U_f : \tag{34}$$

where $: \cdots :$ is the normal ordering which places all X operators to the left of Z operators. For example, we have

$$U_{\delta e} = (\prod_{f \supset e} X_f) \prod_{f'} Z_{f'}^{\int f' \cup_1 \delta e}. \tag{35}$$

On the other hand, we can show that

$$S_{\delta e} = (-1)^{\int_{w_2} e} Z_{f'}^{\int \delta e \cup_1 f' + f' \cup_1 \delta e} \tag{36}$$

where the 1-chain w_2 consists of all edges of the triangulation, together with the (02) edge for all "+" tetrahedra and the (13) edge for all "−" tetrahedra:

$$w_2 = \sum_e e + \sum_{t \in +\text{tetrahedra}} t_{02} + \sum_{t \in -\text{tetrahedra}} t_{13} \tag{37}$$

This is exactly the 1-chain representing the second Stiefel-Whitney class [2]. It is a 1-cycle, and therefore exact in a topologically-trivial situation.[3] Thus we can define

$$S_\beta^E = (-1)^{\int_E \beta} S_\beta \tag{38}$$

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[3] Actually, it is also exact if the space is any oriented 3-manifold, but in this paper we limit ourselves to topologically-trivial lattices.

where E is a spin structure (i.e. a 2-chain such that $\partial E = w_2$). The identification of $S_{\delta e}^E$ and $U_{\delta e}$ gives us the gauge constraint (30). Notice that our bosonization map depends on the choice of a spin structure E.

The modified Gauss law looks complicated, but it can be written down more concisely if we describe the spin configurations by a 2-cochain $B \in C^2(T, \mathbb{Z}_2)$. Our convention is that $B(f) = 1$ if $Z_f = -1$ and $B(f) = 0$ if $Z_f = 1$. Thus the unconstrained Hilbert space is spanned by vectors $|B\rangle$ for all B. A 2-form gauge transformation has a 1-cochain Λ as a parameter and acts by $B \mapsto B + \delta\Lambda$. For a general Λ, the Gauss law constraint is given by

$$\left(\prod_{f \in \delta\Lambda} X_f \right) \left(\prod_{f'} Z_{f'}^{\int \delta\Lambda \cup_1 f'} \right) (-1)^{\int \Lambda \cup \delta\Lambda} = 1 \qquad (39)$$

This formula is proved by $e \cup \delta e = 0$ and induction:

$$(\prod_{f_1 \in \delta\Lambda_1} X_{f_1})(\prod_{f_1'} Z_{f_1'}^{\delta\Lambda_1 \cup_1 f_1'})(-1)^{\int \Lambda_1 \cup \delta\Lambda_1}(\prod_{f_2 \in \delta\Lambda_2} X_{f_2})(\prod_{f_2'} Z_{f_2'}^{\delta\Lambda_2 \cup_1 f_2'})(-1)^{\int \Lambda_2 \cup \delta\Lambda_2}$$

$$= \left(\prod_{f \in \delta(\Lambda_1 + \Lambda_2)} X_f \right) \left(\prod_{f'} Z_{f'}^{\delta(\Lambda_1 + \Lambda_2) \cup_1 f'} \right) (-1)^{\int \Lambda_1 \cup \delta\Lambda_1 + \Lambda_2 \cup \delta\Lambda_2}(-1)^{\int \delta\Lambda_1 \cup_1 \delta\Lambda_2}$$

$$= \left(\prod_{f \in \delta(\Lambda_1 + \Lambda_2)} X_f \right) \left(\prod_{f'} Z_{f'}^{\delta(\Lambda_1 + \Lambda_2) \cup_1 f'} \right) (-1)^{\int (\Lambda_1 + \Lambda_2) \cup \delta(\Lambda_1 + \Lambda_2)}$$

$$(40)$$

where we use the identity $\int \delta\Lambda_1 \cup_1 \delta\Lambda_2 = \int \Lambda_1 \cup \delta\Lambda_2 + \delta\Lambda_2 \cup \Lambda_1$ in the last equality. Eq. (39) can be rewritten as

$$\left(\prod_{f \in \delta\Lambda} X_f \right) (-1)^{\int \Lambda \cup \delta\Lambda + \delta\Lambda \cup_1 B} = 1. \qquad (41)$$

Consider now the following 2-form gauge theory defined on a general triangulated 4D manifold Y:

$$S(B) = \sum_t |\delta B(t)| + i\pi \int_Y (B \cup B + B \cup_1 \delta B). \qquad (42)$$

Here $B \in C^2(Y, \mathbb{Z}_2)$, and the gauge symmetry acts by $B \to B + \delta\Lambda$. The action is gauge-invariant up to a boundary term:

$$S(B + \delta\Lambda) - S(B) = \int_{\partial Y} (\Lambda \cup \delta\Lambda + \delta\Lambda \cup_1 B). \qquad (43)$$

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This boundary term determines the Gauss law for the wave-function $\Psi(B)$ on the spatial slice $X = \partial Y$:

$$\Psi(B + \delta\Lambda) = (-1)^{\omega(\Lambda, B)} \Psi(B) \tag{44}$$

where $\omega(\Lambda, B) = \int_X (\Lambda \cup \delta\Lambda + \delta\Lambda \cup_1 B)$. The Gauss law is the same as the gauge constraint (41). In the following section we use this observation to construct a 4D lattice action for particular Hamiltonian gauge theories with the modified Gauss law.

3 Euclidean 3+1D gauge theories with fermionic duals

In this section we write down Euclidean formulations of some of the gauge theories which are dual to free fermions. We will make use of cup products, and thus will assume that the 3d space is triangulated. Accordingly, the 3+1d lattice will be the product of the 3d triangulation and discrete time. As explained in the Appendix, (higher) cup products can also be defined on the 3d cubic lattice, thus similar considerations can be used to find the Euclidean formulation of gauge theories constructed in Section 2.1.

Consider the simplest gauge-invariant Hamiltonian compatible with the modified Gauss law:

$$H = -A \sum_f U_f - B \sum_t W_t. \tag{45}$$

The gauge constraint is

$$G_e \equiv \left(\prod_{f \supset e} X_f \right) \prod_{f'} Z_{f'}^{\int \delta e \cup_1 f'} = 1. \tag{46}$$

The partition function is

$$Z = \text{Tr } e^{-\beta H} = \text{Tr } T^M \tag{47}$$

where T is the transfer matrix defined as

$$T = \left(\prod_e \delta_{G_e, 1} \right) e^{-\delta\tau H}. \tag{48}$$

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The first factor arises from the gauge constraints on the Hilbert space. For calculation purposes, we can rewrite it as

$$\delta_{G_e,1} = \frac{1}{2}(1+G_e) = \frac{1}{2}\sum_{\lambda_e=\pm 1}(-1)^{\frac{1-\lambda_e}{2}\sum_{f'\in NE(e)}\frac{1-Z_{f'}}{2}}\cdot(-1)^{\frac{1-\lambda_e}{2}\sum_{f\supset e}\frac{1-X_f}{2}}$$

(49)

with $NE(e) \equiv \{f| \int \delta e \cup_1 f = 1\}$. Define $|m(\tau)\rangle = |\{S_f\}\rangle$ as the configuration of spins (in Z_f basis). Then the matrix elements of T are

$$\langle m'(\tau + \delta\tau)|T|m(\tau)\rangle = \langle m'(\tau+\delta\tau)|\left(\prod_v \delta_{G_e,1}\right)e^{-\delta\tau H}|m(\tau)\rangle \quad (50)$$

Next we need to use an identity

$$\langle S^{z'}|f(\sigma^x,\sigma^z)|S^z\rangle = \frac{1}{2}\sum_{S^x=\pm 1}f(S^x,S^z)(-1)^{\frac{1-S^x}{2}(\frac{1-S^{z'}}{2}+\frac{1-S^z}{2})} \quad (51)$$

where we assume that σ^x is to the right of σ^z in the function $f(\sigma^x,\sigma^z)$. Plugging this into (50), we get

$$\langle m'(\tau+\delta\tau)|\left(\prod_e \delta_{G_e,1}\right)e^{-\delta\tau H}|m(\tau)\rangle$$

$$= \langle m'(\tau+\delta\tau)|\left(\prod_e \delta_{G_e,1}\right)\left(\prod_f \sum_{S_f^x=\pm 1}|S_f^x\rangle\langle S_f^x|\right)e^{-\delta\tau H}|m(\tau)\rangle$$

$$\propto \left[\sum_{\lambda_e=\pm 1}(-1)^{\frac{1-\lambda_e}{2}\left(\sum_{f_2\supset e}\frac{1-S_{f_2}^x}{2}+\sum_{f_3\in NE(e)}\frac{1-S_{f_3}^{z}{}'}{2}\right)}(-1)^{\sum_{\lambda_e,\lambda_{e'}=-1}\int e\cup\delta e'}\right]$$

$$\left[\prod_f \sum_{S_f^x=\pm 1}(-1)^{\frac{1-S_f^x}{2}\left(\frac{1-S_f^{z}{}'}{2}+\frac{1-S_f^z}{2}\right)}e^{A\delta\tau S_e^x\prod_{f_1\in\Delta(f)}S_{f_1}^z}\right]\left(\prod_t e^{B\delta\tau\prod_{f_4\subset t}S_{f_4}^z}\right)$$

(52)

where $\Delta(f) \equiv \{f'|f'\cup_1 f = 1\}$ and the term $(-1)^{\sum_{\lambda_e,\lambda_{e'}=-1}\int e\cup\delta e'}$ comes from pushing all X_f operators to the right, which is the same as the last factor of Eq. (39). This term can be expressed as

$$i\pi\sum_i \int a_i \cup \delta a_i \quad (53)$$

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if we define a_i as 1-cochain on the ith layer with $a_i(e) = 1$ for $\lambda_e = -1$ and $a_i(e) = 0$ for $\lambda_e = 1$. We can interpret a_i at the ith layer as a 2-cochain which lives on the "temporal" faces between the ith and $i + 1^{\text{th}}$ layers.

After extracting this factor, the remaining terms are

$$
\sum_{\lambda_e = \pm 1} \left(\prod_e (-1)^{\frac{1-\lambda_e}{2} \sum_{f_3 \subset \text{NE}(e)} \frac{1-S_{f_3}^z{}'}{2}} \right) \left(\prod_t e^{B\delta\tau \prod_{f_4 \subset t} S_{f_4}^z} \right)
$$

$$
\left[\prod_f \sum_{S_f^x = \pm 1} (-1)^{\frac{1-S_f^x}{2} \left(\frac{1-S_f^{z'}}{2} + \frac{1-S_f^z}{2} + \sum_{e \subset f} \frac{1-\lambda_e}{2} \right)} \cdot e^{A\delta\tau S_e^x \prod_{f_1 \in \Delta(f)} S_{f_1}^z} \right]
$$

$$
= \sum_{\lambda_e = \pm 1} \left(\prod_e (-1)^{\frac{1-\lambda_e}{2} \sum_{f_3 \subset \text{NE}(e)} \frac{1-S_{f_3}^z{}'}{2}} \right) \left(\prod_t e^{B\delta\tau \prod_{f_4 \subset t} S_{f_4}^z} \right)
$$

$$
\left[\prod_e (e^{A\delta\tau \prod_{f_1 \in \Delta(f)} S_{f_1}^z} + e^{-A\delta\tau \prod_{f_1 \in \Delta(f)} S_{f_1}^z} S_f^{z'} S_f^z \prod_{e \subset f} \lambda_e) \right]
$$

$$
\sim \sum_{\lambda_e = \pm 1} \left(\prod_e (-1)^{\frac{1-\lambda_e}{2} \sum_{f_3 \subset \text{NE}(e)} \frac{1-S_{f_3}^z{}'}{2}} \right) \left(\prod_t e^{B\delta\tau \prod_{f_4 \subset t} S_{f_4}^z} \right)
$$

$$
\left[\prod_f e^{J S_f^{z'} S_f^z \prod_{e \subset f} \lambda_e} (-1)^{\left(\sum_{f_1 \in \Delta(f)} \frac{1-S_{f_1}^z}{2} \right) \left(\frac{1-S_f^{z'}}{2} + \frac{1-S_f^z}{2} + \sum_{e \subset f} \frac{1-\lambda_e}{2} \right)} \right]
$$

$$
= \sum_{\lambda_e = \pm 1} (-1)^{\sum_e \left(\frac{1-\lambda_e}{2} \sum_{f_3 \subset \text{NE}(e)} \frac{1-S_{f_3}^z{}'}{2} \right) + \sum_f \left(\sum_{f_1 \in \Delta(f)} \frac{1-S_{f_1}^z}{2} \right) \left(\frac{1-S_f^{z'}}{2} + \frac{1-S_f^z}{2} + \sum_{e_1 \subset f} \frac{1-\lambda_{e_1}}{2} \right)}
$$

$$
e^{J \sum_f S_f^{z'} S_f^z \prod_{e_1 \subset f} \lambda_{e_1} + B\delta\tau \sum_t \prod_{f_4 \subset t} S_{f_4}^z}
$$

$$(54)$$

where $\tanh J = e^{-2A\delta\tau}$. The last line is the usual action for a 4D $\mathbb{Z}_2$ gauge theory except for some sign factors. We regard these factors as coming from a topological action S_{top}. From (54), we see that S_{top} contains

$$
i\pi \left[\sum_e \frac{1-\lambda_e}{2} \sum_{f_3 \subset \text{NE}(e)} \frac{1-S_{f_3}^z{}'}{2} \right.
$$

$$
\left. + \sum_f \left(\sum_{f_1 \in \Delta(f)} \frac{1-S_{f_1}^z}{2} \right) \left(\frac{1-S_f^{z'}}{2} + \frac{1-S_f^z}{2} + \sum_{e_1 \subset f} \frac{1-\lambda_{e_1}}{2} \right) \right].
$$

$$(55)$$

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The first term is

$$\sum_e \frac{1-\lambda_e}{2} \sum_{f \supset e} \sum_{f \cup_1 f_3 = 1} \frac{1-S_{f_3}^z{}'}{2} = \sum_f \left(\sum_{e \subset f} \frac{1-\lambda_e}{2} \right) \left(\sum_{f \cup_1 f_3 = 1} \frac{1-S_{f_3}^z{}'}{2} \right) \tag{56}$$

which is equal to $\int \delta a_i \cup_1 b_{i+1}$ if we define b_i as a 2-cochain on the ith layer with $b_i(f) = \frac{1-S_f}{2}$. The second term is

$$\sum_f \left(\sum_{f_1 | \int f_1 \cup_1 f = 1} \frac{1-S_{f_1}^z}{2} \right) \left(\frac{1-S_f^{z\prime}}{2} + \frac{1-S_f^z}{2} + \sum_{e_1 \subset f} \frac{1-\lambda_{e_1}}{2} \right) \tag{57}$$

which is $\int b_i \cup_1 (b_i + b_{i+1} + \delta a_i)$. Collecting all terms, we get

$$S_{\text{top}}(\{a_i\}, \{b_i\}) = i\pi \sum_i \int a_i \cup \delta a_i + \delta a_i \cup_1 b_{i+1} + b_i \cup_1 (b_i + b_{i+1} + \delta a_i). \tag{58}$$

The usual term $e^{J \sum_f S_f^{z\prime} S_f^z \prod_{e_1 \subset f} \lambda_{e_1} + B\delta\tau \sum_T \prod_{f_4 \subset T} S_{f_4}^z}$ can be written as the exponential of (up to an unimportant constant)

$$S_{\text{4D gauge}}(\{a_i\}, \{b_i\})$$
$$= \sum_i \left(-2J \sum_f |b_i(f) + b_{i+1}(f) + \delta a_i(f)| - 2B\delta\tau \sum_t |\delta b_i(t)| \right) \tag{59}$$

where $|\cdots|$ gives the argument's parity 0 or 1. Combining (58) and (59), the Euclidean action becomes (up to an additive constant)

$$S(\{a_i\}, \{b_i\}) = S_{\text{top}}(\{a_i\}, \{b_i\}) + S_{\text{4D gauge}}(\{a_i\}, \{b_i\}), \tag{60}$$

which is analogous to (42). This action is gauge-invariant under gauge transformations

$$b_i \to b_i + \delta\lambda_i, \ a_i \to a_i + \delta\mu_i + \lambda_i + \lambda_{i+1}, \tag{61}$$

where λ_i are arbitrary 1-cochains and μ_i are arbitrary 0-cochains. In-

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deed, the change in the action is

$$
\begin{aligned}
\frac{\Delta S_{\text{top}}}{(i\pi)} =& \sum_i \int (a_i + \delta\mu_i^{0} + \lambda_i + \delta\lambda_{i+1}) \cup (\delta\lambda_i + \delta\lambda_{i+1}) \\
&+ (\delta\mu_i^{0} + \lambda_i + \lambda_{i+1}) \cup \delta a_i + (\delta\lambda_i + \delta\lambda_{i+1}) \cup_1 b_{i+1} + \delta a_i \cup_1 \delta\lambda_{i+1} \\
&+ (\delta\lambda_i + \delta\lambda_{i+1}^{0}) \cup_1 \delta\lambda_{i+1} + \delta\lambda_i \cup_1 (b_i + b_{i+1} + \delta a_i) \\
=& \sum_i \int a_i \cup (\delta\lambda_i + \delta\lambda_{i+1}) + (\lambda_i + \lambda_{i+1}) \cup (\delta\lambda_i + \delta\lambda_{i+1}) \\
&+ (\lambda_i + \lambda_{i+1}) \cup \delta a_i + a_i \cup \delta\lambda_{i+1} + \delta\lambda_{i+1} \cup a_i + \lambda_i \cup \delta\lambda_{i+1} + \delta\lambda_{i+1} \cup \lambda_i \\
&+ a_i \cup \delta\lambda_i + \delta\lambda_i \cup a_i \\
=& \sum_i \int \lambda_i \cup \delta\lambda_i + \lambda_{i+1} \cup \delta\lambda_{i+1} = 0
\end{aligned}
$$

$$(62)$$

where the terms with the same colors cancel out. In the above computation we assumed periodic time, so that there are no boundary terms. If we do not identify time periodically, the variation is a boundary term

$$
\int (\lambda_0 \cup \lambda_0 + \delta\lambda_0 \cup_1 b_0) + (\lambda_N \cup \lambda_N + \delta\lambda_N \cup_1 b_N), \qquad (63)
$$

which is the same as the boundary term (43) in the previous section.

We can also check that the action is invariant under a 2-form global symmetry

$$
B \to B + \beta \qquad (64)
$$

where a closed 2-cochain β can be represented by 2-cochains β_i (one for each time slice) and 1-cochains α_i satisfying $\beta_i + \beta_{i+1} + \delta\alpha_i = 0$. Using a gauge transformation (61) with

$$
\lambda_i = \sum_{j=0}^{i-1} \alpha_j, \quad \mu_i = 0 \qquad (65)
$$

for $i = 0, 1, ..., N-1$, we can see that $\beta_i' = \beta_0$, which is independent of i, and $\alpha_{N-1}' = \sum_{j=0}^{N-1} \alpha_j$ with other $\alpha_i' = 0$. Notice that α_{N-1}' is closed since $\beta_i' = \beta_{i+1}'$. Under this 2-form symmetry transformation

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β', the action changes by

$$\frac{\Delta S_{\text{top}}}{(i\pi)} = \int \cancel{\alpha'_{N-1} \cup \delta a_{N-1}}^{\;0} + \sum_i \delta a_i \cup_1 \beta_0 + \beta_0 \cup_1 (\sum_i \cancel{b_i + b_{i+1}}^{\;0}) + \sum_i \beta_0 \cup_1 \delta a_i$$

$$= \sum_i \int a_i \cup \beta_0 + \beta_0 \cup a_i + \beta_0 \cup a_i + a_i \cup \beta_0 = 0.$$

$$(66)$$

Thus the action is invariant under a global 2-form symmetry, as expected.

4   Gauging fermion parity

We have shown that a lattice fermionic system in 3d is dual to a bosonic spin system with the Gauss law constraints. In this section we show how to get rid of the constraints at the expense of coupling fermions to a $\mathbb{Z}_2$ gauge field.

Our bosonization map is

$$(-1)^{F_t} = -i\gamma_t \gamma'_t \longleftrightarrow W_t \equiv \prod_{f \subset t} Z_f$$

$$S_f^E = (-1)^{\int_E f}(i\gamma_{L(f)}\gamma'_{R(f)}) \longleftrightarrow U_f \equiv X_f \prod_{f'} Z_{f'}^{\int f' \cup_1 f}$$

$$(67)$$

with gauge constraints

$$\left(\prod_{f \supset e} X_f\right) \prod_{f'} Z_{f'}^{\int \delta e \cup_1 f'} = 1. \tag{68}$$

Now, we introduce new $\mathbb{Z}_2$ fields (spins), with operators $\tilde{X}$, $\tilde{Y}$, and $\tilde{Z}$, which live on faces and couple to fermions via a Gauss law constraint

$$(-1)^{F_t} = \prod_{f \subset t} \tilde{Z}_f. \tag{69}$$

The fermionic hopping operator must be modified to

$$S_f^E = (-1)^{\int_E f}(i\gamma_{L(f)}\gamma'_{R(f)})\tilde{X}_f \tag{70}$$

in order to commute with the Gauss law constraint (69). The bosonization map becomes

$$-i\gamma_t\gamma_t' = \prod_{f\subset t}\tilde{Z}_f \longleftrightarrow W_t \equiv \prod_{f\subset t}Z_f$$
$$S_f^E = (-1)^{\int_E f}(i\gamma_{L(f)}\gamma_{R(f)}')\tilde{X}_f \longleftrightarrow U_f \equiv X_f\prod_{f'}Z_{f'}^{\int f'\cup_1 f} \tag{71}$$

and, similar to (35) and (36), the identification of $U_{\delta e}$ and $S_{\delta e}$ gives

$$\prod_{f\supset e}\tilde{X}_f \longleftrightarrow \left(\prod_{f\supset e}X_f\right)\prod_{f'}Z_{f'}^{\int \delta e\cup_1 f'} \tag{72}$$

The equations (71) and (72) define a bosonization map for fermions coupled to a dynamical $\mathbb{Z}_2$ gauge field. In this case, their is no constraint on the bosonic variables.

We can apply this modified bosonization/fermionization map to a $\mathbb{Z}_2$ version of the Levin-Wen rotor model [3] on general triangulation:

$$H = -\sum_t Q_t - \sum_e B_e \tag{73}$$

with

$$Q_t = \prod_{f\subset t}Z_f$$
$$B_e = \prod_{f\supset e}\left(X_f\prod_{f'}Z_{f'}^{\int f\cup_1 f'}\right) \tag{74}$$
$$= \left(\prod_{f\supset e}X_f\right)\prod_{f'}Z_{f'}^{\int \delta e\cup_1 f'}$$

Since Q_t and B_e are just W_t and $(\prod_{f\supset e}X_f)\prod_{f'}Z_{f'}^{\int \delta e\cup_1 f'}$, the above bosonic model is equivalent to a model of a $\mathbb{Z}_2$ gauge field coupled to fermions and a Hamiltonian

$$H = -\sum_t\prod_{f\subset t}\tilde{Z}_f - \sum_e\prod_{f\supset e}\tilde{X}_f. \tag{75}$$

The fermions are static, since the above Hamiltonian does not include fermionic hopping terms. The only interaction between the fermions

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and the gauge field is via the Gauss law constraint

$$\prod_{f \subset t} \tilde{Z}_f = (-1)^{F_t}. \tag{76}$$

Thus a complicated model bosonic model is mapped to a simple $\mathbb{Z}_2$ lattice gauge theory coupled to static fermions.

As another application of the modified bosonization map, consider again the bosonic gauge theory on a cubic lattice with the Hamiltonian (20)

$$H = -\frac{t}{2} \sum_{i=x,y,z} \sum_{f_i} s_i(L(f_i)) U_{f_i} (1 - W_{L(f_i)} W_{R(f_i)}) \tag{77}$$

and a gauge constraint (9). This constrained model is dual a model of free fermions with Dirac cones. After coupling the fermions to a $\mathbb{Z}_2$ gauge field and applying the modified map, we find that the bosonic model (77) without any gauge constraints is equivalent to a fermionic model with the Hamiltonian

$$H = -t \sum_{\vec{r}} (s_x(\vec{r}) \tilde{X}_x(\vec{r}) c^\dagger_{\vec{r}+\hat{x}} c_{\vec{r}} + s_y(\vec{r}) \tilde{X}_y(\vec{r}) c^\dagger_{\vec{r}+\hat{y}} c_{\vec{r}} + s_z(\vec{r}) \tilde{X}_z(\vec{r}) c^\dagger_{\vec{r}+\hat{z}} c_{\vec{r}} + \text{h.c.})$$

$$\tag{78}$$

with $(-1)^{c^\dagger_t c_t} = \prod_{f \subset t} \tilde{Z}_f$. The operators $\tilde{W}_e \equiv \prod_{f \supset e} \tilde{X}_f$ commute with the Hamiltonian, so we can project the Hilbert space into sectors with fixed $\tilde{W}_e$ ($\tilde{W}_e$ is arbitrary ± 1 as long as it satisfies $\prod_{e \supset v} \tilde{W}_e = 1$). In the sector $\tilde{W}_e = 1$ for all e, the Hamiltonian (78) returns to (17). The model of unconstrained spins with the Hamiltonian (77) thus can be regarded as a 3d analog of Kitaev's honeycomb model.

5 Conclusions

In this paper we constructed a bosonization map for an arbitrary fermionic system on a 3d lattice. The lattice can be either a cubic one or a triangulation. The dual bosonic system is a 2-form gauge theory and thus has local constraints (the modified Gauss law). While we did not emphasize this point in the paper, the form of the constraints is largely determined by requiring the system to have a 2-form $\mathbb{Z}_2$ symmetry with a particular 't Hooft anomaly. As explained in the end of section 2, another way to understand the constraints is to note that they arise from a 4D 2-form gauge theory with a "topological"

term in the action

$$S_{top} = i\pi \int_Y B \cup B + B \cup_1 \delta B, \qquad (79)$$

where B is a 2-form $\mathbb{Z}_2$ gauge field (i.e. a 2-cochain with values in $\mathbb{Z}_2$). This action is invariant under a gauge symmetry $B \mapsto B + \delta\lambda$, where λ is a 1-cochain, up to a non-trivial boundary term, and it is this boundary term which leads to a modified Gauss law.

One can get rid of the constraint on the bosonic side at the expense of coupling the fermions to a $\mathbb{Z}_2$ gauge field (i.e. by gauging the fermion parity). We used this observation to construct a model with spins and no constraints which is dual to a model of free fermions coupled to a static gauge field, and thus is soluble.

The simplest Euclidean 4D 2-form gauge theory which leads to the correct form of the Gauss law for the wave-functions has the action (42). It is very likely that this model is dual to a model of free fermions for any triangulated 4D manifold Y. It would be very interesting to prove this. Our methods are insufficient here, since they are tied to the Hamiltonian formalism, while (42) makes sense only on a 4D triangulation, but not on a 3d triangulation times discrete time, and thus is intrinsically Euclidean. In this paper, instead of attacking this problem head-on, we showed that a complicated-looking 2-form gauge theory with an action (60) is dual to a theory of free fermions. This Euclidean theory leads to the same Gauss law as (42), but has the advantage that it is defined on a 3d triangulation times discrete time, and thus can be analyzed by our methods.

A (Higher) cup products on a triangulation and a cubic lattice

In the case of a general triangulation, our bosonization procedure is based on the properties of the cup product $\cup$ and the higher cup product $\cup_1$. These mathematical operations have been defined by Steenrod [11] (see also Appendix B in [12] for a review) for an arbitrary simplicial complex, but not for a cubic lattice. In this section, we will describe a version of these definitions for the cubic lattice and check that the usual properties of these products hold.

For a simplicial complex, the cup product of cochains $\cup$ is defined

as [10]

$$[A_p \cup B_p](0, 1, \cdots, p+q) = A_p(0, 1, \cdots p)B_q(p, p+1, \cdots, p+q), \quad (80)$$

while the higher cup product $\cup_1$ is defined as [11, 12]

$$[A_p \cup_1 B_q](0, \cdots, p+q-1)$$
$$= \sum_{i_0} A(0, \cdots, i_0, i_0 + q, \cdots, p+q-1)B(i_0, \cdots, i_0 + q). \quad (81)$$

Here A_p and B_q are arbitrary p-cochain and q-cochains with values in $\mathbb{Z}_2$. We will limit ourselves to the case of $\mathbb{Z}_2$ valued cochains, since this is all we need in this paper.

To generalize these formulas to the cubic lattice, we first develop an intuition for the cup product $\cup$. On a triangle Δ_{012}, the usual cup product for two 1-cochains λ and λ' is

$$\lambda \cup \lambda'(012) = \lambda(01)\lambda'(12). \quad (82)$$

We can think of it as starting from vertex 0, passing through edges 01 and 12 consecutively, and ending at vertex 2, all the while following the orientation of the edges. Following the same logic, it is intuitive to define the cup product on a square $\square_{0134}$ (the bottom face in Fig. 5) as

$$\lambda \cup \lambda'(0134) = \lambda(01)\lambda'(14) + \lambda(03)\lambda'(34). \quad (83)$$

The two terms come from two oriented paths from vertex 0 to vertex 4. If λ and β are a 1-cochain and a 2-cochain, the usual cup product is

$$\lambda \cup \beta(0123) = \lambda(01)\beta(123)$$
$$\beta \cup \lambda(0123) = \beta(012)\lambda(23). \quad (84)$$

On the cubic lattice, the corresponding cup products are defined as follows:

$$\lambda \cup \beta(c) = \lambda(01)\beta(\square_{1457}) + \lambda(02)\beta(\square_{2567}) + \lambda(03)\beta(\square_{3467})$$
$$\beta \cup \lambda(c) = \beta(\square_{0236})\lambda(67) + \beta(\square_{0125})\lambda(57) + \beta(\square_{0134})\lambda(47) \quad (85)$$

where c is a cube whose vertices are labeled in Fig. 5. For a cup product involving 0-cochains, the definition is trivial:

$$v \cup \beta(\square_{0134}) = v(0)\beta(\square_{0134})$$
$$\beta \cup v(\square_{0134}) = \beta(\square_{0134})v(4)$$
$$v \cup \lambda(01) = v(0)\lambda(01)$$
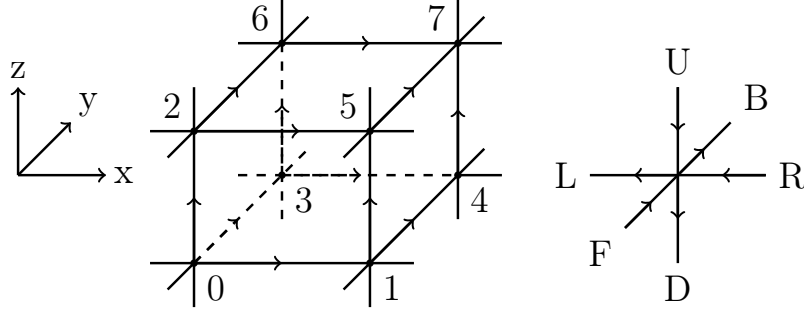$$\lambda \cup v(01) = \lambda(01)v(1). \quad (86)$$

Figure 5: There are six faces for each cube c. U,D,F,B,L,R stand for faces on direction "up","down","front","back","left","right". We assign the face U, F, R to be inward and D, B, L to be outward. The $\cup_1$ product on two 2-cochain is defined by $\beta \cup_1 \beta'(c) = \beta(L)\beta'(B) + \beta(L)\beta'(D) + \beta(B)\beta'(D) + \beta(U)\beta'(F) + \beta(U)\beta'(R) + \beta(F)\beta'(R)$

With the above definitions, it can be checked that the following equalities hold for cubic cochains of degrees 0, 1 and 2:

$$
\begin{aligned}
e_1 \cup \delta e_2 &= \delta e_1 \cup e_2 + \delta(e_1 \cup e_2) \\
v \cup \delta f &= \delta v \cup f + \delta(v \cup f).
\end{aligned}
\tag{87}
$$

The next step is to define the $\cup_1$ product on the cubic lattice. It need not satisfy all the properties that $\cup_1$ has on a triangulation. The only properties of $\cup_1$ that we need are the anti-commutativity for faces with the same direction and the identity we used in (31), (40), and (62):

$$
\int e_1 \cup \delta e_2 + \delta e_2 \cup e_1 = \int \delta e_1 \cup_1 \delta e_2 \ (\text{mod } 2).
\tag{88}
$$

Therefore, we only need to define $\cup_1$ product for two 2-cochains so that it satisfies (88). Our convention for $\cup_1$ is shown in Fig. 5:

$$
\begin{aligned}
\beta \cup_1 \beta'(c) = {} &\beta(L)\beta'(B) + \beta(L)\beta'(D) + \beta(B)\beta'(D) \\
&+ \beta(U)\beta'(F) + \beta(U)\beta'(R) + \beta(F)\beta'(R)
\end{aligned}
\tag{89}
$$

Once the $\cup$ and $\cup_1$ products are defined on the cubic lattice, the bosonization procedure on a general triangulation can be applied to the cubic lattice. (26) and (27) are modified as follows:

$$
S_{\delta e} = (-1)^{\sum_{f<f'\in\delta e} f\cup_1 f'} \prod_{f\in\delta e} S_f = \prod_{c|e\in\{01,14,02,47,67,26\}} W_c
\tag{90}
$$

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$$Z_{f_1} Z_{f_2} \prod_{f' \subset c} Z_{f'}^{(f_1+f_2)\cup_1 f' + f' \cup_1 (f_1+f_2)} = \begin{cases} W_c, & \text{if } e \in \{01, 14, 02, 47, 67, 26\} \\ 0, & \text{otherwise} \end{cases}$$

(91)

for faces f_1 and f_2 join at the edge e. We implicitly choose $w_2 = 0$ in (36). We can use the $\cup_1$ product defined above to reproduce the fermionic hopping terms defined by framing in Fig. 2. The hopping
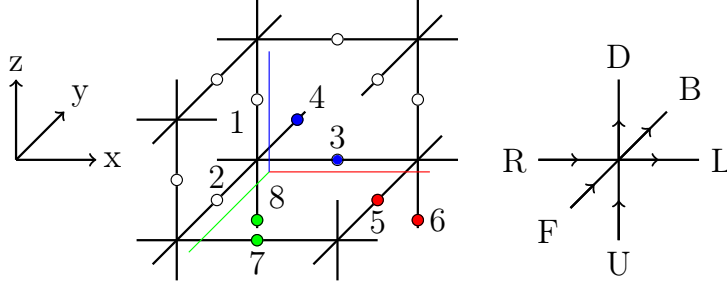

Figure 6: (Color online) We rotate the axis U,D,F,B,R,L in Fig. 5 to match the result in Fig. 2. Notice that the cube above is dual lattice and edges $1, 2, 3...$ in the dual lattice represent faces in the original lattice.

term defined by Eq. (24) is

$$U_f = X_f \prod_{f'} Z_{f'}^{\int f' \cup_1 f}.$$

(92)

Fig. 6 is dual to Fig. 5. Therefore, faces in Fig. 5 become edges in Fig. 6. Consider the hopping term along dual edge 3. On the vertex to the right, it represents the face R. From terms $\beta(F)\beta'(R)$ and $\beta(U)\beta'(R)$, the hopping term contains Z_5 (from F) and Z_6 (from U). On the vertex to the left, it represents the face L. Since there is no $\beta(D)\beta'(L)$ or $\beta(B)\beta'(L)$ term, it contributes nothing. So we have

$$U_3 = X_3 Z_5 Z_6.$$

(93)

Similarly, for edge 2, the hopping term has Z_7 (from $\beta(L)\beta'(B)$) and Z_8 (from $\beta(U)\beta'(F)$)

$$U_2 = X_2 Z_7 Z_8.$$

(94)

For edge 1, the hopping term has Z_3 (from $\beta(L)\beta'(D)$) and Z_4 (from $\beta(B)\beta'(D)$)

$$U_1 = X_1 Z_3 Z_4.$$

(95)

We get the exact same hopping terms defined by "framing" in Fig. 2.

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