On Finitely Additive Measures in Boolean Algebras

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1. Introduction

The present paper is concerned with the theory of positive real and finite measures on Boolean algebras which are finitely additive but not necessarily countably additive. The main object of this paper is a recent result of S. Koshi which, stated roughly, reads: For a special class of Boolean algebras, which includes the class of all measure algebras of countably additive finite measures, every finitely additive measure is countably additive if restricted to some suitable ideal. This result was obtained by Koshi by using topological methods. We shall show, however, that a measure theoretic approach to this problem is possible and that it will yield a slightly stronger result. Furthermore, the measure theoretic approach seems to be more direct and somewhat simpler.

The paper is divided into 6 sections. Sections 2 and 3 give some preliminaries about the theory of Boolean algebras and the theory of measures of Boolean algebras respectively. In section 4, the class of Boolean algebras for which Koshi's result holds is discussed extensively. Section 5 is devoted to Koshi's theorem and some related theorems. Finally, in section 6 we apply the results of section 5 in order to obtain, among other things, a new proof of a theorem by Kelley which deals with the problem of the existence of strictly positive countably additive measures on Boolean algebras.

In the summer of 1961, some of the results of the present paper were presented at a meeting in Oberwolfach on Boolean Algebras and Measure Theory.

2. Some notation and terminology concerning Boolean algebras

For notation and terminology which is not explained in this section and in the remainder of this paper, we refer the reader to either [3] or [12].

In this paper, $\mathfrak{B}$ will always denote a *non-degenerate Boolean algebra.* The elements of a Boolean algebra will be denoted by $a, b, \ldots$; the *zero element* by 0 and the *unit element* by 1. The Boolean operations of *join* and *meet* will be denoted by $\vee$ and $\wedge$ respectively. The unique *complement* of an element a will be denoted by $\bar{a}$, Furthermore, $a \leqq b$ means $a \wedge b = a$ which is equivalent to $a \vee b = b$.

If A is a subset of $\mathfrak{B}$, then $\sup A = a$ and $\inf A = b$ will always mean that A has a least upper bound equal to a and A has a greatest lower bound equal to b respectively. If $\emptyset$ is the empty subset of $\mathfrak{B}$, then $\sup \emptyset = 0$ and $\inf \emptyset = 1$. If A is not empty, then $\sup A \geqq \inf A$.

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A Boolean algebra $\mathfrak{B}$ is said to be (σ-) *complete* if every (countable) subset of $\mathfrak{B}$ has a least upperbound. A Boolean algebra $\mathfrak{B}$ is said to be *supercomplete* if every subset A of $\mathfrak{B}$ contains a countable subset A' such that $\sup A = \sup A'$. It is clear that every supercomplete Boolean algebra is complete and that every complete Boolean algebra is σ-complete.

A subset $\mathfrak{J}$ of $\mathfrak{B}$ is called an *ideal* if $a, b \in \mathfrak{J}$ implies that $a \vee b \in \mathfrak{J}$ and $a \in \mathfrak{J}$ and $b \in \mathfrak{B}$ implies that $a \wedge b \in \mathfrak{J}$. If $\mathfrak{J}$ is an ideal, then $0 \in \mathfrak{J}$. An ideal $\mathfrak{J}$ of a Boolean algebra is called *dense* in $\mathfrak{B}$ if for every $a \in \mathfrak{B}$ and $a \neq 0$, there exists an element $b \in \mathfrak{J}$ such that $b \neq 0$ and $b \leqq a$. An ideal $\mathfrak{J}$ of a Boolean algebra is called *superdense* if for every $a \in \mathfrak{B}$ there exists an increasing sequence $\{a_k : k = 1, 2, \ldots\}$ of elements of $\mathfrak{J}$ such that $a = \sup_k a_k$.

Two elements $a, b \in \mathfrak{B}$ are called *disjoint* if $a \wedge b = 0$. A subset A of $\mathfrak{B}$ is called *disjointed* if every pair of different elements of A are disjoint.

A Boolean algebra $\mathfrak{B}$ is said to satisfy the *σ-chain condition* if every disjointed subset of $\mathfrak{B}$ is at most countable.

We shall conclude this section with the following useful theorem. Although this theorem is not new we shall, for the sake of completeness, include a proof.

Theorem 2. 1. *If $\mathfrak{B}$ is a σ-complete Boolean algebra, then $\mathfrak{B}$ is supercomplete if and only if $\mathfrak{B}$ satisfies the σ-chain condition.*

Proof. We shall first prove that if $\mathfrak{B}$ is supercomplete, then $\mathfrak{B}$ satisfies the σ-chain condition. For this purpose, we assume that A is a disjointed subset of $\mathfrak{B}$. Since $\mathfrak{B}$ is supercomplete, there exists a countable subset A' of A such that $\sup A = \sup A'$. If A is uncountable, then there exists an element $a \in A$ such that $a \neq 0$ and $a \notin A'$. Hence, $\sup A = \sup A' \vee a > \sup A'$ and a contradiction is obtained.

We shall assume now that $\mathfrak{B}$ is σ-complete and satisfies the σ-chain condition. Let A be a non-empty subset of $\mathfrak{B}$ and let $\mathfrak{S}$ be the set of all ordered pairs (D, d), where D is a disjointed subset of $\mathfrak{B}$ and where d is a mapping of D into A such that $d(a) \geqq a$ for all $a \in D$. Let $\mathfrak{S}$ be ordered as follows: $(D, d) \leqq (D', d')$ whenever $D \subset D'$ and d' is an extension of d. Then $\mathfrak{S}$ is a non-empty inductively ordered set. Indeed, since $a \in A$ implies that (D, d), where $D = \{a\}$ and $d(a) = a$, is an element of $\mathfrak{S}$ we have that $\mathfrak{S}$ is not empty. Furthermore, if $\{(D_\nu, d_\nu) : \nu \in \mathsf{N}\}$ is a chain of elements of $\mathfrak{S}$, then (D, d), where $D = \bigcup(D_\nu : \nu \in \mathsf{N})$ and where d is defined by $d(a) = d_\nu(a)$ whenever $a \in D_\nu$, is easily seen to be an element of $\mathfrak{S}$. Hence, by Zorn's lemma $\mathfrak{S}$ contains a maximal element, say, (D_0, d_0). Since D_0 is disjointed and $\mathfrak{B}$ has the σ-chain condition we obtain that D_0 is countable. Then $\sup D_0 = a_0$ exists for $\mathfrak{B}$ is σ-complete. Then for all $a \in A$ we have that $a_0 \geqq a$. Indeed, if not, then there exists an element $a \in A$ such that not $(a \leqq a_0)$ holds. We conclude that $\bar{a}_0 \wedge a \neq 0$. Then $0 < \bar{a}_0 \wedge a \leqq a$. Hence, if $D' = D_0 \cup \{\bar{a}_0 \vee a\}$ and d' is a mapping of D' into A defined by $d' = d_0$ on D_0 and $d'(\bar{a}_0 \vee a) = a$, then (D', d') is strictly larger than (D_0, d_0) which contradicts the definition of (D_0, d_0). Thus $a_0 = \sup D_0 \leqq \sup d_0(D_0) \leqq a_0$, i. e., $\sup A$ exists and is equal to $\sup d_0(D_0)$. Since $d_0(D_0)$ is countable and $d_0(D_0) \subset A$ we have shown that $\mathfrak{B}$ is supercomplete. This completes the proof of the theorem.

As an immediate corollary we have the following result:

Theorem 2. 2. *A σ-complete and atomic Boolean algebra is supercomplete if and only if the set of its atoms is at most countable.*

Remark. There exist, however, many non-atomic and supercomplete Boolean algebras. Indeed, all measure algebras of finite non-atomic and countably additive measures are supercomplete (see Theorem 3. 1 below).

3. Definitions and some simple properties of measures

A real function m on a Boolean algebra $\mathfrak{B}$ is called a *finitely additive measure* if m satisfies the following conditions: (i) $m(a) \geqq 0$ for all $a \in \mathfrak{B}$; (ii) $m(a \vee b) = m(a) + m(b)$, whenever $a \wedge b = 0$; (iii) $m(1) \neq 0$. If m is a finitely additive measure on $\mathfrak{B}$, then $m(0) = 0$; $m(a) + m(b) = m(a \wedge b) + m(a \vee b)$, $a, b, \in \mathfrak{B}$; m is monotone, i. e., $a \leqq b$ implies $m(a) \leqq m(b)$; $m(a_1 \vee \cdots \vee a_n) = \sum_{i=1}^{n} m(a_i)$, whenever the set $\{a_1, \ldots, a_n\}$ is disjointed.

A finitely additive measure m is called *strictly positive* or *effective* if $m(a) = 0$ implies $a = 0$. Not every Boolean algebra admits a strictly positive finitely additive measure (see [9] for further information about this statement). If a Boolean algebra $\mathfrak{B}$ admits a strictly positive finitely additive measure m, then it satisfies the σ-chain condition. Indeed, if A is a disjointed subset of $\mathfrak{B}$, then for every $n = 1, 2, \ldots$, the set $A_n = \left\{a : a \in A \text{ and } m(a) \geqq \frac{1}{n}\right\}$ is at most finite. In view of Theorem 2. 1, we have obtained the following result of F. Wecken [14].

Theorem 3. 1. *If a σ-complete Boolean algebra $\mathfrak{B}$ admits a strictly positive finitely additive measure, then $\mathfrak{B}$ is supercomplete.*

If m is a finitely additive measure on $\mathfrak{B}$, then the set $\mathfrak{J}_m = \{a : m(a) = 0\}$ is an ideal of $\mathfrak{B}$. Furthermore, m defines in a natural way a strictly positive finitely additive measure on $\mathfrak{B}/\mathfrak{J}_m$. The theorem of Wecken has been generalized to this case by Smith and Tarski [13] and again in view of Theorem 2. 1 can be stated as follows:

Theorem 3. 2. *If $\mathfrak{B}$ is a σ-complete Boolean algebra and m is a finitely additive measure on $\mathfrak{B}$, then $\mathfrak{B}/\mathfrak{J}_m$, where $\mathfrak{J}_m = \{a : a \in \mathfrak{B}$ and $m(a) = 0\}$ is supercomplete.*

A finitely additive measure m on a Boolean algebra $\mathfrak{B}$ is called *a countably additive* or *σ-additive measure* if for every countable disjointed set A of $\mathfrak{B}$, $m(\sup A) = \Sigma(m(a) : a \in A)$. From this definition it follows immediately that for every finitely additive measure m the following conditions are mutually equivalent: (i) m is σ-additive; (ii) if $a_i (i = 1, 2, \ldots)$ is decreasing and $\inf_i a_i = 0$, then $\inf_i m(a_i) = 0$; if $a_i (i = 1, 2, \ldots)$ is increasing and $a = \sup_i a_i$, then $m(a) = \sup_i m(a_i)$.

Following Yosida and Hewitt [16] we say that an finitely additive measure m is purely finitely additive if every countably additive measure m' such that $0 \leqq m' \leqq m$ is identically equal to zero.

The most important result in the theory of finitely additive measures in Boolean algebras is the following result of K. Yosida and E. Hewitt (see Theorem 1. 24 of [16]).

Theorem 3. 3. *Every finitely additive measure m in a Boolean algebra $\mathfrak{B}$ can be uniquely written as the sum of a σ-measure m_σ and a purely finitely additive measure m_p.*

We shall call m_σ the *σ-additive part* of m and m_p the *purely finitely additive part* of m.

The following result, which is of importance for section 5, was formulated by M. A. Woodbury [15], without proof, for set algebras and proved in general for Boolean algebras by H. Bauer (see "Satz 1" in [2]). It gives a construction of the σ-additive part of a measure.

Theorem 3. 4. *If m is a finitely additive measure on a Boolean algebra $\mathfrak{B}$, then for the σ-additive part m_σ of m the following formula holds: For all $a \in \mathfrak{B}$*

$$m_\sigma(a) = \inf\left(\lim m(a_i) : \{a_i : i = 1, 2, \ldots\} \textit{ increasing and } a = \sup_i a_i\right)$$

Since m is finitely additive it is obvious that, equivalently, we have $m_\sigma(a) = \inf\left(\Sigma\, m(a_i) : \{a_i : i = 1, 2, \ldots\} \text{ disjointed and } a = \sup_i (a_1 \vee \cdots \vee a_i)\right)$. It is this formula for m_σ which was proved by H. Bauer. In order to prove that the two formulas are the same observe that if $\{a_i : i = 1, 2, \ldots\}$ is disjointed and $\sup_i (a_1 \vee \cdots \vee a_i) = a$, then $a_1 \vee \cdots \vee a_i$ $(i = 1, 2, \ldots)$ is increasing in i and $m(a_1 \vee \cdots \vee a_i) = m(a_1) + \cdots + m(a_i)$. Conversely, if a_i $(i = 1, 2, \ldots)$ is increasing in i such that $\sup_i a_i = a$, then the set $A = \{a_1, a_2 \wedge \bar{a}_1, a_3 \wedge \bar{a}_2, \ldots, a_i \wedge \bar{a}_{i-1}, \ldots\}$ is disjointed, $\sup A = a$ and $m(a_1) + m(a_2 \wedge \bar{a}_1) + \cdots + m(a_i \wedge \bar{a}_{i-1}) = m(a_i)$.

4. The Egorov property

In this section we shall discuss the class of Boolean algebras which plays an important role in Koshi's theorem. This class of Boolean algebras is singled out by a property, which we have called the *Egorov property*. The reason for calling it the Egorov property shall be explained in due course. The property itself was introduced for the first time by H. Nakano (see [10] page 40) in the theory of semi-ordered linear spaces.

Definition 4. 1 (*Egorov property*). *A Boolean algebra* $\mathfrak{B}$ *is said to have the Egorov property if for every double sequence* $a_{i,j}$ $(i = 1, 2, \ldots$ *and* $j = 1, 2, \ldots)$ *which is increasing in* j *for every* $i = 1, 2, \ldots$ *such that* $\sup_j a_{i,j} = a$ *for all* $i = 1, 2, \ldots$ *there exists an increasing sequence of elements* a_k $(k = 1, 2, \ldots)$ *and for every pair of indices* i, k $(i = 1, 2, \ldots$ *and* $k = 1, 2, \ldots)$ *there exists an index* $n = n(i, k)$ *such that* $a_k \leqq a_{i,n(i,k)}$ *for all* $i = 1, 2, \ldots$ *and* $k = 1, 2, \ldots$ *and* $\sup_k a_k = a$.

Of course there is no loss in generality to assume that $n = n(i, k)$ $(i = 1, 2, \ldots$ and $k = 1, 2, \ldots)$ is increasing in i and k separately. Furthermore, if the Boolean algebra $\mathfrak{B}$ is σ-complete, then we may take $a_k = \inf_i a_{i,n(i,k)}$ $(k = 1, 2, \ldots)$.

If a Boolean algebra has the Egorov property, then every ideal has the Egorov property too. In case that the Boolean algebra is σ-complete then every σ-complete subalgebra has also the Egorov property. The Egorov property is preserved under σ-isomorphisms.

Dually, we have the following theorem:

Theorem 4. 1. *A Boolean algebra* $\mathfrak{B}$ *has the Egorov property if and only if for every double sequence* $a_{i,j}$ $(i = 1, 2, \ldots$ *and* $j = 1, 2, \ldots)$ *which is decreasing in* j *for every* $i = 1, 2, \ldots$ *such that* $\inf_j a_{i,j} = a$ *for all* $i = 1, 2, \ldots$ *there exists a decreasing sequence* a_k $(k = 1, 2, \ldots)$ *and for every pair of indices* i, k $(i = 1, 2, \ldots$ *and* $k = 1, 2, \ldots)$ *an index* $n = n(i, k)$ *such that* $a_k \geqq a_{i,n(i,k)}$ $(i = 1, 2, \ldots$ *and* $k = 1, 2, \ldots)$ *and* $\inf_k a_k = a$.

Finally, we say that a Boolean algebra has the *weak Egorov property* if in Definition 4. 1 in place of $\sup_k a_k = a$ we have that $0 < \sup_k a_k \leqq a$ whenever $a > 0$. We shall concentrate our attention, however, on the Egorov property rather than on the weak Egorov property since we shall prove in section 6 that these two properties are equivalent for measure σ-algebras of finitely additive measures.

We shall now first give a sufficient condition for a Boolean algebra to have the Egorov property.

Theorem 4. 2. *A* σ*-complete Boolean algebra has the Egorov property if it admits a strictly positive countably additive measure.*

Proof. Let $a_{i,j}$ $(i = 1, 2, \ldots$ and $j = 1, 2, \ldots)$ be increasing in j for every $i = 1, 2, \ldots$ such that $\sup_j a_{i,j} = a$ for all $i = 1, 2, \ldots$. Then for every pair of indices i, k $(j = 1, 2, \ldots;$ $k = 1, 2, \ldots)$ there exists an index $n = n(i, k)$ such that $m(a \wedge \bar{a}_{i,n(i,k)}) \leqq \frac{1}{2^i k}$. Then, if we let $a_k = \inf_i a_{i,n(i,k)}$ $(k = 1, 2, \ldots)$ we obtain that

$a \wedge \bar{a}_k = \sup_i (a \wedge \bar{a}_{i,n(i,k)})$ $(k = 1, 2, \ldots)$. Hence, $m(a \wedge \bar{a}_k) \leqq \Sigma_i\, m(a \wedge \bar{a}_{i,n(i,k)}) \leqq \frac{1}{k}$, i. e., $\sup_k a_k = a$. This completes the proof of the theorem.

Remark. This simple result justifies the terminology used in Definition 4. 1. For the reason why Egorov's theorem holds in the theory of measures for the class of a. e. finite measurable functions relative to some finite countably additive measure can be easily traced back to the Egorov property of the measure algebra.

A property which is closely related to the Egorov property is *weak σ-distributivity*. A Boolean algebra $\mathfrak{B}$ is called weakly σ-distributive if for every double sequence $a_{i,j}$ $(i = 1, 2, \ldots$ and $j = 1, 2, \ldots)$ which is increasing in j for every $i = 1, 2, \ldots$ we have that $\inf_i \sup_j a_{i,j} = \sup_{\{n\}} \inf_i a_{i,n(i)}$, where the last sup is to be taken over all infinite sequences of indices $n = n(i)$ $(i = 1, 2, \ldots)$.

Dually, we have that a Boolean algebra is weakly σ-distributive if and only if for every double sequence $a_{i,j}$ $(i = 1, 2, \ldots$ and $j = 1, 2, \ldots)$ which is decreasing in j for every $i = 1, 2, \ldots$ we have that $\sup_i \inf_j a_{i,j} = \inf_{\{n\}} \sup_i a_{i,n(i)}$.

The notion of weak σ-distributivity derives its importance in the theory of measures from the following theorem of Horn and Tarski (see [5] and footnote 1 on page 104 in [12]) which is analogous to Theorem 4. 2: Every σ-complete Boolean algebra which admits a strictly positive finite countably additive measure is weakly σ-distributive.

In view of Theorem 4. 2, the following theorem is a generalization of the theorem of Horn and Tarski.

Theorem 4. 3. *If $\mathfrak{B}$ is σ-complete and if $\mathfrak{B}$ has the Egorov property, then $\mathfrak{B}$ is weakly σ-distributive.*

Proof. Let $a_{i,j}$ $(i = 1, 2, \ldots$ and $j = 1, 2, \ldots)$ be a double sequence of elements of $\mathfrak{B}$ which is increasing in j for every $i = 1, 2, \ldots$. If $\inf_i \sup_j a_{i,j} = a$, then we have to show that $\sup_{\{n\}} \inf_i a_{i,n(i)} = a$. Since for every sequence n of indices, $a_{i,n(i)} \leqq \sup_j a_{i,j}$ $(i = 1, 2, \ldots)$ we obtain that $\inf_i a_{i,n(i)} \leqq a$ for all sequences n of indices. Thus a is an upper bound of the family $\{\inf_i a_{i,n(i)} : \{n\}\}$. We shall prove that a is a least upper bound of this family. For this purpose we observe that $\bar{a} \vee \inf_i \sup_j a_{i,j} = 1$. Since $\bar{a} \vee \inf_i \sup_j a_{i,j} = \inf_i \sup_j (a_{i,j} \vee \bar{a})$ we obtain that $\sup_j (a_{i,j} \vee \bar{a}) = 1$ for all $i = 1, 2, \ldots$. Then it follows from the Egorov property of $\mathfrak{B}$ that for every pair of indices i, k $(i = 1, 2, \ldots$ and $k = 1, 2, \ldots)$ there exists an index $n = n(i, k)$ such that $\sup_k \inf_i (a_{i,n(i,k)} \vee \bar{a}) = 1$. Hence, $\bar{a} \vee \sup_k \inf_i a_{i,n(i,k)} = 1$. We conclude that for every element b such that $\inf_i a_{i,n(i)} \leqq b$ for all sequences n we have that $\bar{a} \vee b = 1$, or equivalently, $a \leqq b$. This finishes the proof of the theorem.

From Wecken's result (Theorem 3. 1) it follows that every σ-complete Boolean algebra which admits a strictly positive countably additive measure is supercomplete. Thus the following theorem shows that Theorem 4. 2 and the result of Horn and Tarski are in effect equivalent.

Theorem 4. 4. *If $\mathfrak{B}$ is supercomplete, then $\mathfrak{B}$ has the Egorov property if and only if $\mathfrak{B}$ is weakly σ-distributive.*

Proof. We have only to show that if $\mathfrak{B}$ is weakly σ-distributive, then $\mathfrak{B}$ has the Egorov property since the other half of the theorem is contained in the previous result. To this end, we assume that the double sequence $a_{i,j}$ $(i = 1, 2, \ldots$ and $j = 1, 2, \ldots)$ is increasing in j for every $i = 1, 2, \ldots$ and that $\sup_j a_{i,j} = a$ for all $i = 1, 2, \ldots$. Since $\mathfrak{B}$ is supercomplete there exists a countable collection of sequences $n_k = n(i, k)$

$(i = 1, 2, \ldots$ and $k = 1, 2, \ldots)$ such that $a = \sup_k \inf_i a_{i,n(i,k)}$. Hence, $\mathfrak{B}$ has the Egorov property and the proof is finished.

If $\mathfrak{B}$ is complete, then we cannot show that the property of weak σ-distributivity implies the Egorov property. For we shall prove in the following theorem that, under the assumption that the continuum hypothesis holds, this may be false. Since we know that the continuum hypothesis cannot be disproved (see [4]) we cannot show that for complete Boolean algebras the Egorov property follows from the property of being weakly σ-distributive.

Theorem 4. 5. *Under the assumption that the continuum hypothesis holds, a complete and atomic Boolean algebra has the Egorov property if and only if the set of its atoms is at most countable.*

Proof. Since every complete atomic Boolean algebra is completely isomorphic to the set algebra of all subsets of its set of atoms (see 25. 1 in [12]) we have to prove Theorem 4. 5 only for complete algebras of sets. To this end, assume that E is a set of cardinal $\aleph_1$. If the continuum hypothesis holds, i. e., $c = \aleph_1$, Sierpinski showed (see Prop. C_{11}, p. 53 of [11]) that there exists a double sequence $E_{i,j}$ $(i = 1, 2, \ldots$ and $j = 1, 2, \ldots)$ of subsets of E which is increasing in j for every $i = 1, 2, \ldots$ such that $\bigcup_{j=1}^{\infty} E_{i,j} = E$ for all $i = 1, 2, \ldots$ and for any double sequence of indices $n = n(i, k)$ $(i = 1, 2, \ldots$ and $k = 1, 2, \ldots)$ the set $\bigcap_{i=1}^{\infty} E_{i,n(i,k)}$ is countable for every $k = 1, 2, \ldots$. Hence, $\bigcup_{k=1}^{\infty} \bigcap_{i=1}^{\infty} E_{i,n(i,k)}$ is countable. We conclude that the algebra of subsets of E does not have the Egorov property.

Conversely, if E is countable, then the algebra of all its subsets satisfies obviously the σ-chain condition and hence, by Theorem 2. 1, is supercomplete. Furthermore, since every complete set algebra is completely distributive it follows from Theorem 4. 4 that the Boolean algebra of all subsets of E has the Egorov property. This completes the proof of the theorem.

In the following theorem we shall show that Theorem 4. 4 is best in a sense.

Theorem 4. 6. *If the continuum hypothesis holds, then every complete Boolean algebra which has the Egorov property is supercomplete.*

Proof. From Theorem 2. 1 it follows that we have to show that $\mathfrak{B}$ satisfies the σ-chain condition. For this purpose, we shall assume that E is a disjointed subset of $\mathfrak{B}$. There is no loss in generality to assume that $0 \notin E$. Then for every subset A of E we define $\tau(A) = \sup A$. It is obvious that $\tau(\emptyset) = 0$. Furthermore, we shall denote $\tau(E)$ by e. We shall prove that τ has the following properties:

(i) If $A \subset E$, then $\tau(E - A) = \tau(E) \wedge \overline{\tau(A)}$.

(ii) For every family of subsets $\{A_\nu : \nu \in N\}$ of E we have that $\tau(\bigcup_\nu A_\nu) = \sup_\nu \tau(A_\nu)$ and $\tau(\bigcup_\nu A_\nu) = \inf_\nu \tau(A_\nu)$.

(iii) The mapping τ is one-to-one.

In order to prove (i) we observe that $\tau(A) \vee \tau(E - A) = \tau(E) = e$. Furthermore, since E is disjointed we have that $a \in A$ implies $a \wedge b = 0$ for all $b \in E - A$. Hence, $a \in A$ implies that $a \wedge \tau(E - A) = 0$. In the same way, we obtain that $\tau(A) \wedge \tau(E - A) = 0$, which proves (i). For the proof of (ii) we observe that $\sup_\nu \tau(A_\nu) = \sup_\nu \sup (A_\nu) = \sup (\bigcup_\nu A_\nu) = \tau(\bigcup_\nu A_\nu)$. The second part of (ii) follows from the first part of (ii) and (i) in the following way:

$$\inf_\nu \tau(A_\nu) = e - \sup_\nu \tau(E - A_\nu) = \tau\big(E - \bigcup_\nu (E - A_\nu)\big) = \tau(\bigcap_\nu A_\nu).$$

Finally, (iii) follows immediately from (ii) and the fact that $0 \notin E$. Indeed, if $A, B < E$ and $A \neq B$, then $\tau(A \cup B) > \tau(A)$ and $\tau(B)$. This contradicts $\tau(A \cup B) = \tau(A) \vee \tau(B)$ whenever $\tau(A) = \tau(B)$.

From the above properties of τ it follows that the set algebra of all subsets of E is completely isomorphic to the Boolean algebra of elements $\tau(A)$, $A < E$ with unit e and with the Boolean operations defined in $\mathfrak{B}$. It is trivial that this Boolean algebra has the Egorov property. Hence, the algebra of all subsets of E has the Egorov property. Under the assumption of the continuum hypothesis, we conclude from Theorem 4. 5 that E is at most countable. This proves that $\mathfrak{B}$ satisfies the σ-chain condition and hence, by Theorem 2. 1, $\mathfrak{B}$ is supercomplete and the proof is finished.

Remark. Theorem 4. 6 is related to a result of I. Amemiya (see [1] and [7]) which states that under the continuous hypothesis every Dedekind-complete vector lattice (= Riesz space) R which has the Egorov property is super Dedekind complete. Since [1] is not easily accessible we shall indicate briefly a proof of this result. Let $\mathfrak{B} = \mathfrak{B}(R)$ be the Boolean algebra of all projectors of R (for terminology see [10]). Then $\mathfrak{B}$ is complete and has the Egorov property. Hence, according to Theorem 4. 5, $\mathfrak{B}$ is supercomplete. Then from Theorem 13. 2 of [10] it follows that R is super Dedekind complete. Concerning this and other results of Riesz spaces we shall report in a joint paper with A. C. Zaanen which is to appear in the Proceedings of the Royal Academy of the Netherlands.

5. A theorem of S. Koshi

We shall begin this section with the principal lemma on which Koshi's result will be based. It may be considered to be an improvement of Theorem 3. 4 in case the Boolean algebra has the Egorov property.

Lemma 5. 1. *Let $\mathfrak{B}$ be a Boolean algebra and let m be a finitely additive measure defined on $\mathfrak{B}$.*

(i) *If $\mathfrak{B}$ has the Egorov property, then for every $a \in \mathfrak{B}$ there exists an increasing sequence of elements $a_k (k = 1, 2, \ldots)$ such that $\sup_k a_k = a$ and $m_\sigma(a) = \sup_k m(a_k)$, where m_σ is the σ-part of m.*

(ii) *If $\mathfrak{B}$ has the weak Egorov property and $a \neq 0$, then there exists an increasing sequence of elements $a_k (k = 1, 2, \ldots)$ such that $0 < \sup_k a_k \leqq a$ and $\sup_k m(a_k) \leqq m_\sigma(a)$.*

Proof. (i) If $a \in \mathfrak{B}$, then, by Theorem 3. 4, for every $i = 1, 2, \ldots$ there exists an increasing sequence $a_{i,j}$ $(j = 1, 2, \ldots)$ such that $\sup_j a_{i,j} = a$ $(i = 1, 2, \ldots)$ and $\sup_j m(a_{i,j}) \leqq m_\sigma(a) + 1/i$ $(i = 1, 2, \ldots)$. Since $\mathfrak{B}$ has the Egorov property, it follows that there exists an increasing sequence a_k $(k = 1, 2, \ldots)$ and a double sequence of indices $n = n(i,k)$ $(i = 1, 2, \ldots$ and $k = 1, 2, \ldots)$ such that $\sup_k a_k = a$ and $a_k \leqq a_{i,n(i,k)}$ $(i = 1, 2, \ldots$ and $k = 1, 2, \ldots)$. Hence, $m(a_k) \leqq m(a_{i,n(i,k)}) \leqq \sup_j m(a_{i,j}) \leqq m_\sigma(a) + 1/i$ for all $i = 1, 2, \ldots$ and $k = 1, 2, \ldots$. We conclude that $m(a_k) \leqq m_\sigma(a)$ for all $k = 1, 2, \ldots$. Since $\sup_k a_k = a$ it follows from Theorem 3. 4 that $\sup_k m(a_k) = m_\sigma(a)$.

Since the proof of (ii) is similar to the proof of (i) we shall leave it to the reader.

We are in a position to formulate and to prove the following theorem which generalizes Theorem 1 of Koshi [8].

Theorem 5. 1. *Let $\mathfrak{B}$ be a Boolean algebra.*

(i) *If $\mathfrak{B}$ has the Egorov property, then to every finitely additive measure m there corresponds a superdense ideal $\mathfrak{J}$ such that m is countably additive on $\mathfrak{J}$, or equivalently, every purely finitely additive measure on $\mathfrak{B}$ vanishes on some superdense ideal of $\mathfrak{B}$.*

(ii) *If* $\mathfrak{B}$ *has the weak Egorov property, then to every finitely additive measure* m *there corresponds a dense ideal* $\mathfrak{J}$ *such that* m *is countably additive on* $\mathfrak{J}$, *or equivalently, every purely finitely additive measure* m *vanishes on some dense ideal of* $\mathfrak{B}$.

Proof. Since (ii) follows in the same way from (ii) of Lemma 5. 1 as (i) follows from (i) of Lemma 5. 1 we shall only proof (i). To this end, let m be a finitely additive measure on $\mathfrak{B}$. Then, by Theorem 3. 3, $m = m_\sigma + m_p$. Furthermore, $(m_p)_\sigma \equiv 0$. From (i) of Lemma 5.1 it follows that for every $a \in \mathfrak{B}$ there exists an increasing sequence a_k $(k = 1, 2, \ldots)$ such that $\sup_k m_p(a_k) = 0$. Hence, $\mathfrak{J}_{m_p} = \{a : m_p(a) = 0\}$ is a superdense ideal; and $m = m_\sigma$ on $\mathfrak{J}_{m_p}$. This finishes the proof of the theorem.

In the case that $\mathfrak{B}$ has the Egorov property, Lemma 5. 1 can be improved upon in the following way:

Lemma 5. 2. *If the Boolean algebra* $\mathfrak{B}$ *has the Egorov property and if* $\{m_i : i = 1, 2, \ldots\}$ *is a countable family of finitely additive measures on* $\mathfrak{B}$, *then for every* $a \in \mathfrak{B}$ *there exists an increasing sequence* a_k $(k = 1, 2, \ldots)$ *such that* $\sup_k a_k = a$ *and* $\sup_k m_i(a_k) = m_{i,\sigma}(a)$ *for all* $i = 1, 2, \ldots$.

Proof. For every $i = 1, 2, \ldots$ there exists a sequence $a_{i,j}$ $(j = 1, 2, \ldots)$ which is increasing in j such that $\sup_j a_{i,j} = a$ for all $i = 1, 2, \ldots$ and $\sup_j m_i(a_{i,j}) = m_{i,\sigma}(a)$ ((i) of Lemma 5. 1). Since $\mathfrak{B}$ has the Egorov property there exists an increasing sequence a_k $(k = 1, 2, \ldots)$ and a double sequence of indices $n = n(i, k)$ $(i = 1, 2, \ldots$ and $k = 1, 2, \ldots)$ such that $\sup_k a_k = a$ and $a_k \leqq a_{i,n(i,k)}$ $(i = 1, 2, \ldots$ and $k = 1, 2, \ldots)$. It follows from $m_i(a_k) \leqq m_i(a_{i,n(i,k)})$ $(i = 1, 2, \ldots$ and $k = 1, 2, \ldots)$ that $\sup_k m_i(a_k) \leqq \sup_k m_i(a_{i,n(i,k)}) \leqq \sup_j m(a_{i,j}) = m_{i,\sigma}(a)$ for all $i = 1, 2, \ldots$. Hence, by Theorem 3. 4, we have that $\sup_k m_i(a_k) = m_{i,\sigma}(a)$ for all $i = 1, 2, \ldots$. This completes the proof of the Lemma.

In the same way as Theorem 5. 1 follows from Lemma 5. 1, Theorem 5. 2 follows from Lemma 5. 2. This theorem generalizes Corollary 1 of [8].

Theorem 5. 2. *If* $\mathfrak{B}$ *has the Egorov property, then to every countable family of finitely additive measures there corresponds a superdense ideal in* $\mathfrak{B}$ *on which they are simultaneously countably additive, or equivalently, every countable family of purely finitely additive measures vanish simultaneously on some superdense ideal in* $\mathfrak{B}$.

6. A theorem of Kelley

In general, a Boolean algebra does not admit a strictly positive finitely additive measure and to a lesser extent a strictly positive countably additive measure. In [6] Kelley showed that if a Boolean algebra satisfies certain conditions then the two problems are equivalent. We shall prove now the following generalization of Kelley's theorem.

Theorem 6. 1. *If a Boolean algebra* $\mathfrak{B}$ *admits a strictly positive finitely additive measure and has the weak Egorov property, then* $\mathfrak{B}$ *admits a strictly positive countably additive measure.*

Proof. Let m be a strictly positive finitely additive measure on $\mathfrak{B}$. By (ii) of Theorem 5. 1 it follows that there exists a dense ideal $\mathfrak{J}$ in $\mathfrak{B}$ such that $m = m_\sigma$ on $\mathfrak{J}$. If $m_\sigma(a) = 0$ for some $a \neq 0$ then there exists an element $b \in \mathfrak{J}$ such that $b \neq 0$, $b \leqq a$ and $m(b) = m_\sigma(b)$. Hence, $m(b) = 0$ which contradicts the fact that m is strictly positive and the proof is finished.

With this result and the results of [6] we have now the following form of Theorem 9 of Kelley's paper:

Theorem 6. 2. *If $\mathfrak{B}$ is a σ-complete Boolean algebra and if $\mathfrak{B}$ admits a strictly positive finitely additive measure, then the following conditions are mutually equivalent:*

(i) *$\mathfrak{B}$ has the weak Egorov property,*

(ii) *$\mathfrak{B}$ admits a strictly positive countably additive measure,*

(iii) *$\mathfrak{B}$ has the Egorov property,*

(iv) *$\mathfrak{B}$ is weakly σ-distributive,*

(v) *In the Stone representation space of $\mathfrak{B}$ is every set of the first category nowhere dense.*

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