Remark. Theorem 12 is the version of the recursive continuity theorem given in Čeitin's [1]. As Čeitin mentions there, it implies directly that a recursive operator on a recursively separable subspace F_1 of F into $N = (N, \lambda xy\ x = y)$ is the restriction of some *partial recursive functional* (in the sense of Kleene [4], § 63) whose domain contains F_1 to F_1. This problem was proposed by Myhill and Shepherdson in [12] and was solved independently of Čeitin by Kreisel, Lacombe and Schoenfield in [5].

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Reçu par la Rédaction le 16. 4. 1963

A remark on Sikorski's extension theorem for homomorphisms in the theory of Boolean algebras

by

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1. Introduction. In [1], Sikorski proved the following important extension theorem for Boolean homomorphisms.

THEOREM (R. Sikorski). *Let $\mathfrak{B}_0$ be a subalgebra of a Boolean algebra $\mathfrak{B}$, and let $\mathfrak{B}'$ be a complete Boolean algebra. Then every homomorphism of $\mathfrak{B}_0$ into $\mathfrak{B}'$ can be extended to a homomorphism of $\mathfrak{B}$ into $\mathfrak{B}'$.*

Sikorski's proof of this theorem consists of two parts: (i) First the following fundamental lemma is proved.

LEMMA. *Let $\mathfrak{B}_0$ be a subalgebra of a Boolean algebra $\mathfrak{B}$, and let $\mathfrak{B}'$ be a complete Boolean algebra. If $a_1, \ldots, a_n$ are a finite number of elements of $\mathfrak{B}$ and if $\mathfrak{B}_n$ is the subalgebra of $\mathfrak{B}$ generated by $\mathfrak{B}_0$ and the elements $a_1, \ldots, a_n$, then every homomorphism of $\mathfrak{B}_0$ into $\mathfrak{B}'$ can be extended to a homomorphism of $\mathfrak{B}_n$.*

(ii) Using Zorn's lemma or transfinite induction in conjunction with the preceding Lemma, it is shown, in a standard fashion, that every homomorphism of $\mathfrak{B}_0$ into $\mathfrak{B}'$ can be extended to all of $\mathfrak{B}$.

By specialization we obtain that Sikorski's theorem implies the prime ideal theorem for Boolean algebras (see p. 114 in [2]), i.e., every proper ideal in a Boolean algebra can be extended to a prime (= maximal) ideal. It was shown, however, by J. D. Halpern (see [3]) that the axiom of choice is independent from the Boolean prime ideal theorem in a set theory which will be made more explicit in due course. It seems therefore natural to ask whether may be Sikorski's extension theorem follows already from the Boolean prime ideal theorem rather than from the axiom of choice?

In the present paper we shall report on some results which were obtained in trying to settle this question. The present investigations seem to indicate that Sikorski's theorem is independent from the Boolean

* Work on this paper was supported in part by National Science Foundation Grant G-19914.

prime ideal theorem and that the axiom of choice is independent from Sikorski's theorem. We shall prove, however, that Sikorski's extension theorem for complete and atomic Boolean algebras is logically equivalent to the Boolean prime ideal theorem. This result can be proved in a number of ways. The proof we shall present in this paper is based on the method of construction by reduced powers. We have chosen this method as it seems to be particularly suitable for attacking extension problems. For instance, as a by-product of our investigations, we obtain that Sikorski's theorem is logically equivalent to one of its immediate consequences, namely, that every complete Boolean algebra is a retract of every Boolean algebra of which it is a subalgebra. Furthermore, we are also able to discuss with this method the case that the Boolean algebra $\mathfrak{B}'$ is not complete.

The remainder of this paper consists of four sections. In section 2, we first of all state which axioms of set theory are supposed to hold throughout this paper. Then we give a formulation of the result of Halpern quoted earlier. Section 2 is concluded with the enumeration of a few other axioms which will be used in this paper and which are known to be equivalent to the Boolean prime ideal theorem. In section 3, we recall some definitions of the theory of Boolean algebras in connection with the notion of a reduced power of a Boolean algebra. It concludes with a result which is the crucial result in proving the main theorem of this paper. The main result of this paper concerning Sikorski's extension theorem is given in section 4. Finally, in section 5 we shall discuss briefly what can be said about the extension of Boolean homomorphisms if we do not assume that the range space is complete.

The author is pleased to acknowledge that his thinking on the subject of the present paper was greatly stimulated by conversations with J. D. Halpern.

2. Set theory. The axioms of set theory which shall be used throughout this paper consists first of all of the axioms of groups A, B and C of Gödel (see [4]). In place of Gödel's axiom D, the axiom of regularity, which states that for every class X there exists an element $x \in X$ such that $x \cap X = \emptyset$, where $\emptyset$ is the empty set, we shall assume that the following weaker form of the axiom of regularity holds:

AXIOM D'. *There exists a non-finite* (in the sense of Tarski) *set S of reflexive sets* (a set x is called *reflexive* whenever $x = \{x\}$) *such that every class X has an element $x \in X$ such that $x \cap X \subset S$.*

Axiom D' still implies the non-existence of a non-finite descending sequence of sets (i.e., $x_{i+1} \in x_i$, $i = 1, 2, \ldots$).

In this paper, *we shall assume that the system of axioms of groups* A, B, C *and the axiom* D' *hold.* Thus we do not assume that the axiom of choice holds. The system of axioms of groups A, B and C will be denoted by G; and G + D' will be denoted by Σ'. If a theorem in this paper is stated without further specification it means that it holds in Σ'.

The result of Halpern (see [3]) quoted in the introduction can now be stated more precisely as follows:

THEOREM (J. D. Halpern). *If G is consistent, then in Σ' the axiom of choice is independent from the Boolean prime ideal theorem.*

In the remainder of this paper we shall also use the following axioms which are known in Σ' to be equivalent to the prime ideal theorem for Boolean algebras. (See section 47 in [2].)

(i) *The Stone representation theorem for Boolean algebras*, which states that every Boolean algebra is isomorphic to an algebra of both open and closed subsets of a totally disconnected compact Hausdorff space.

(ii) *The Tychonoff theorem for compact Hausdorff spaces*, which states that every Cartesian product of a family of compact Hausdorff spaces is compact and Hausdorff in its product topology.

(iii) *The ultrafilter theorem*, which states that every non-empty family of non-empty subsets of a set which has the finite intersection property is contained in an ultrafilter.

3. Reduced powers of Boolean algebras. For terminology and notation about Boolean algebras which is not explained in this paper we refer the reader to [2].

We shall only consider *non-degenerate* Boolean algebras; and Boolean algebras will always be denoted by $\mathfrak{B}$ with or without superscripts or subscripts. The elements of a Boolean algebra will be denoted by $a, b, \ldots$ and sometimes by $A, B, \ldots$ (do not confuse this use of the letters A, B, C, which they were used in the preceding section); the zero element and unit element of a Boolean algebra will always be denoted by 0 and 1 respectively. The Boolean operations of join and meet will be denoted by $\vee$ and $\wedge$ respectively. The unique complement of an element a will be denoted by $-a$.

We shall now briefly recall the definition of a *reduced power of a Boolean algebra*. The definition of reduced power we shall adopt here was first given by Frayne, Scott and Tarski in [5] and has its origin in earlier work of Łoś [6].

Let $\mathfrak{B}$ be a Boolean algebra, and let D be any set. The set $\mathfrak{B}^D$ of all mappings of D into $\mathfrak{B}$ can obviously be made into a Boolean algebra by defining the Boolean operations pointwise. Thus, for instance, if $A, B, C \in \mathfrak{B}^D$, then $C = A \vee B$ means that $C(n) = A(n) \vee B(n)$ for all $n \in D$. Similarly for join and complementation.

Let $\mathfrak{F}$ be a *filter* on D. If $A, B \in \mathfrak{B}^D$, then we write $A \equiv_{\mathfrak{F}} B$ if and only if $\{n\colon A(n) = B(n)$ and $n \in D\} \in \mathfrak{F}$. It is easy to see that the relation $A \equiv_{\mathfrak{F}} B$ defines an equivalence relation between the elements of $\mathfrak{B}^D$. The set of all classes of equivalent elements defined by this equivalence relation will be denoted by $\mathfrak{B}^D/\mathfrak{F}$. It is convenient to denote the equivalence class determined by an element $A \in \mathfrak{B}^D$ by a. Thus with this convention, $A \in a$.

The set $\mathfrak{B}^D/\mathfrak{F}$ can be made into a Boolean algebra in the following way: $a \vee b = c$ means there exist elements $A \in a$, $B \in b$ and $C \in c$ such that $\{n\colon n \in D$ and $A(n) \vee B(n) = C(n)\} \in \mathfrak{F}$ and similarly for the Boolean operations of join and complementation. Since $\mathfrak{F}$ is a filter, it follows immediately that the preceding definitions for the Boolean operations in $\mathfrak{B}^D/\mathfrak{F}$ are independent from the particular choice of the elements A, B and C in a, b and c, respectively. This justifies the definitions of these operations and it is easy to verify that with these definitions for join, meet and complementation, $\mathfrak{B}^D/\mathfrak{F}$ is a Boolean algebra. The Boolean algebra $\mathfrak{B}^D/\mathfrak{F}$ is called a *reduced power of* $\mathfrak{B}$. In fact, $\mathfrak{B}^D/\mathfrak{F}$ is a factor algebra of the Boolean algebra $\mathfrak{B}^D$. Indeed, if $\mathfrak{I}_{\mathfrak{F}} = \{A\colon A \in \mathfrak{B}^D$ and $\{n\colon A(n) = 0$ and $n \in D\} \in \mathfrak{F}\}$, then since $\mathfrak{F}$ is a filter, we obtain immediately that $\mathfrak{I}_{\mathfrak{F}}$ is an ideal in $\mathfrak{B}^D$, and the factor algebra $\mathfrak{B}^D/\mathfrak{I}_{\mathfrak{F}}$ is isomorphic to the reduced power $\mathfrak{B}^D/\mathfrak{F}$ of $\mathfrak{B}$.

If $\mathfrak{B}^D/\mathfrak{F}$ is a reduced power of $\mathfrak{B}$, then it is obvious that $\mathfrak{B}$ is isomorphic to the subalgebra of $\mathfrak{B}^D/\mathfrak{F}$ determined by the constant mappings of D into $\mathfrak{B}$. We shall always assume that $\mathfrak{B}$ is identified with this subalgebra of $\mathfrak{B}^D/\mathfrak{F}$ and we shall write, without hesitation, $\mathfrak{B} \subset \mathfrak{B}^D/\mathfrak{F}$.

If $\mathfrak{F}$ is an *ultrafilter* $\mathfrak{U}$ on D, then $\mathfrak{B}^D/\mathfrak{U}$ is called an *ultrapower of* B. The Boolean algebras $\mathfrak{B}^D/\mathfrak{U}$ and $\mathfrak{B}$ are isomorphic whenever $\mathfrak{U}$ is a fixed ultrafilter, i.e., an ultrafilter on D consisting of all subsets of D which contain a fixed element of D.

Sikorski's extension theorem implies the interesting result that if $\mathfrak{B}$ is complete and if $\mathfrak{B} \subset \mathfrak{B}_1$, then $\mathfrak{B}$ is a retract of $\mathfrak{B}_1$. In particular, if $\mathfrak{B}$ is complete, then $\mathfrak{B}$ is a retract of its reduced powers. In view of Theorem 4.4, it seems to be questionable whether this last result holds in Σ'. If we assume, however, that the Boolean prime ideal theorem holds, then we can prove at least that we have the following result which plays a fundamental role in the proof of the main result of this paper.

THEOREM 3.1. *Under the assumption that the Boolean prime ideal theorem holds, we have that every complete and atomic Boolean algebra is a retract of its ultrapowers.*

Proof. It is well known that a complete and atomic Boolean algebra $\mathfrak{B}$ is isomorphic to the algebra of all subsets of the set of all atoms of $\mathfrak{B}$, and hence, is isomorphic to a Cartesian product of two-element

Boolean algebras. Thus, by the Boolean prime ideal theorem, $\mathfrak{B}$ is a compact and Hausdorff space in the product topology. From the definition of this topology it follows immediately that the Boolean operations are continuous and that this topology is the order topology of $\mathfrak{B}$. Now, let $\mathfrak{B}^D/\mathfrak{U}$ be an ultrapower of $\mathfrak{B}$. Since $\mathfrak{B}$ is compact and Hausdorff, it follows that for every $A \in \mathfrak{B}^D$, $\lim_{\mathfrak{U}} A$ exists uniquely. From the definition of convergence of functions relative to filters we have that if $A \equiv_{\mathfrak{U}} B$, then $\lim_{\mathfrak{U}} A = \lim_{\mathfrak{U}} B$. This justifies the following definition $u(a) = \lim_{\mathfrak{U}} A$ whenever $A \in a$. Then u is a mapping of $\mathfrak{B}^D/\mathfrak{U}$ into $\mathfrak{B}$. Since the Boolean operations are continuous, we obtain that u is a homomorphism of $\mathfrak{B}^D/\mathfrak{U}$ into $\mathfrak{B}$. Furthermore, by its very definition, u leaves $\mathfrak{B} \subset \mathfrak{B}^D/\mathfrak{U}$ pointwise fixed. Hence, u is a homomorphism of $\mathfrak{B}^D/\mathfrak{U}$ onto $\mathfrak{B}$ which leaves $\mathfrak{B}$ pointwise fixed, i.e., $\mathfrak{B}$ is a retract of $\mathfrak{B}^D/\mathfrak{U}$, and the proof is finished.

Remark. In connection with the method used to prove the preceding theorem it may be of interest to point out that *if a complete Boolean admits a compact Hausdorff topology such that the Boolean operations are continuous, then the Boolean algebra is atomic and this topology is the order topology.*

4. Sikorski's extension theorem.

We shall now prove the main theorem of this paper.

THEOREM 4.1. *In Σ', we have that the Boolean prime ideal theorem is logically equivalent to Sikorski's extension theorem for complete and atomic Boolean algebras.*

Proof. For the sake of completeness, we shall include a proof that Sikorski's extension theorem for complete and atomic Boolean algebras implies the Boolean prime ideal theorem. For this purpose, let $\mathfrak{I}$ be a proper ideal in the Boolean algebra $\mathfrak{B}$, and let $\mathfrak{B}_0$ be the subalgebra of $\mathfrak{B}$ consisting of all elements of $\mathfrak{I}$ and their complements. We define a mapping h_0 of $\mathfrak{B}_0$ into the two-element Boolean algebra $\{0, 1\}$ as follows: $h_0(a) = 0$ whenever $a \in \mathfrak{I}$, and $h_0(a) = 1$ whenever $-a \in \mathfrak{I}$. Then h_0 is a homomorphism of $\mathfrak{B}_0$ into the complete and atomic Boolean algebra $\{0, 1\}$, and hence can be extended to all of $\mathfrak{B}$. The kernel of this extension is a prime (= maximal) ideal containing $\mathfrak{I}$.

Conversely, assume that the Boolean prime ideal theorem holds. Let $\mathfrak{B}_0$ be a subalgebra of $\mathfrak{B}$ and let h_0 be a homomorphism of $\mathfrak{B}_0$ into the complete and atomic Boolean algebra $\mathfrak{B}'$. Then we denote by D the set of all homomorphisms h of subalgebras of $\mathfrak{B}$, which contain $\mathfrak{B}_0$, into $\mathfrak{B}'$ and which are extensions of h_0. D is not empty since $h_0 \in D$. For every $a \in \mathfrak{B}$ we denote by D_a the set of all $h \in D$ such that a is contained in the domain of h. Since the Lemma, quoted in the introduction, holds even in Σ', it follows that the family $\{D_a\colon a \in \mathfrak{B}\}$ of subsets of D is a non-empty family of non-empty subsets of D which has the finite intersection property. The prime ideal theorem for Boolean algebras implies that there exists an ultrafilter $\mathfrak{U}$ on D which contains the family $\{D_a\colon a \in \mathfrak{B}\}$. We shall denote by $\mathfrak{B}''$ the ultrapower $\mathfrak{B}'^D/\mathfrak{U}$ of $\mathfrak{B}'$. We shall now construct a homomorphism h^* of $\mathfrak{B}$ into $\mathfrak{B}''$ which extends h_0. For this purpose, we assign to every element $a \in \mathfrak{B}$ the element $h^*(a)$ of $\mathfrak{B}''$ which is determined uniquely by the requirement: there exists an element $H \in h^*(a)$ such that $H(h) = h(a)$ for all $h \in D_a$. Then h^* is a mapping of $\mathfrak{B}$ into $\mathfrak{B}''$ which has the following properties: h^* is a Boolean homomorphism and $h^*(a) = h_0(a)$ for all $a \in \mathfrak{B}_0$, i.e., h^* is an extension of h_0 into $\mathfrak{B}''$. Of the first part of this statement we shall only prove that $h^*(a \vee b) = h^*(a) \vee h^*(b)$ since the proofs of $h^*(a \wedge b) = h^*(a) \wedge h^*(b)$ and $h^*(-a) = -h^*(a)$ are similar. To this end, let $H_{a\vee b} \in h^*(a \vee b)$ such that $H_{a\vee b}(h) = h(a \vee b)$ for all $h \in D_{a\vee b}$ and let H_a and H_b be defined similarly. Since $h(a \vee b) = h(a) \vee h(b)$ for all $h \in D_{a\vee b} \cap D_a \cap D_b \in \mathfrak{U}$, it follows that $H_{a\vee b}(h) = H_a(h) \vee H_b(h)$ on an element of $\mathfrak{U}$. From the definition of $\mathfrak{B}'^D/\mathfrak{U}$ we conclude that $h^*(a \vee b) = h^*(a) \vee h^*(b)$. If $a \in \mathfrak{B}_0$ and if $H \in h^*(a)$ such that $H(h) = h(a)$ for all $h \in D_a$, then, since every h is an extension of h_0, we have that $H(h) = h_0(a)$ for all $h \in D_a$. Hence, $h^*(a) = h_0(a)$ for all $a \in \mathfrak{B}_0$, i.e., h^* is a homomorphic extension of h_0 into $\mathfrak{B}''$. Since $\mathfrak{B}''$ is an ultrapower of the complete and atomic Boolean algebra $\mathfrak{B}'$, it follows from Theorem 3.1 that $\mathfrak{B}'$ is a retract of $\mathfrak{B}''$. Let u be a homomorphism of $\mathfrak{B}''$ into $\mathfrak{B}'$ which leaves $\mathfrak{B}'$ pointwise fixed. Then the composition $h = u \circ h^*$ is a homomorphism of $\mathfrak{B}$ into $\mathfrak{B}'$ which is an extension of h_0. This completes the proof of the theorem.

The proof of the preceding theorem shows that if the Boolean algebra $\mathfrak{B}'$ is complete but not necessarily atomic as in the case of Sikorski's extension theorem, then we can still prove the following theorem.

THEOREM 4.2. *Let $\mathfrak{B}_0$ be a subalgebra of $\mathfrak{B}$, and let h_0 be a homomorphism of $\mathfrak{B}_0$ into a complete Boolean algebra $\mathfrak{B}'$. Then under the assumption that the Boolean prime ideal theorem holds, we have that there exists an ultrapower $\mathfrak{B}''$ of $\mathfrak{B}'$ such that h_0 can be extended to a homomorphism of $\mathfrak{B}$ into $\mathfrak{B}''$.*

We indicated in the preceding section that Sikorski's extension theorem implies, in particular, that every complete Boolean algebra is a retract of its ultrapowers. Thus, in view of the preceding theorem, we have the following theorem.

THEOREM 4.3. *In Σ', Sikorski's extension theorem is logically equivalent to the conjunction of the following two axioms:* (i) *the Boolean prime ideal theorem and* (ii) *every complete Boolean algebra is a retract of its ultrapowers.*

We have not been able to determine whether (i) and (ii) imply the axiom of choice.

Our method will enable us to prove still another equivalent form of Sikorski's theorem which we believe to be more interesting than the preceding theorem.

THEOREM 4.4. *In Σ', Sikorski's extension theorem is logically equivalent to the statement that every complete Boolean algebra is a retract of its reduced powers.*

Proof. We have only to show that the axiom that every complete Boolean algebra is a retract of its reduced powers implies Sikorski's theorem. To this end, let $\mathfrak{B}_0$ be a subalgebra of $\mathfrak{B}$, and let h_0 be a homomorphism of $\mathfrak{B}_0$ into the complete Boolean algebra $\mathfrak{B}'$. Then with the notation used in the proof of Theorem 4.1, we still have that $\{D_a\colon a \in \mathfrak{B}\}$ is a non-empty family of non-empty subsets of D which has the finite intersection property since the Lemma quoted in the introduction holds in Σ'. Hence, there exists a filter $\mathfrak{F}$ on D containing the family $\{D_a\colon a \in \mathfrak{B}\}$. Then in the same way as in the proof of Theorem 4.1 we can show that h_0 can be extended to a homomorphism h^* of $\mathfrak{B}_0$ into the reduced power $\mathfrak{B}'^D/\mathfrak{F}$ of $\mathfrak{B}'$. Under the hypothesis that $\mathfrak{B}'$ is a retract of $\mathfrak{B}'^D/\mathfrak{F}$, let f be a homomorphism of $\mathfrak{B}'^D/\mathfrak{F}$ onto $\mathfrak{B}'$ which leaves $\mathfrak{B}'$ pointwise fixed, then the homomorphism $f \circ h^*$ is an extension of h_0 to all of $\mathfrak{B}$ into $\mathfrak{B}'$.

Remark. We have mentioned earlier that Sikorski's theorem implies the following rather interesting result. *Every complete Boolean algebra is a retract of every Boolean algebra of which it is a subalgebra.* Hence, the preceding theorem implies that *this immediate consequence of of Sikorski's theorem is in Σ' logically equivalent to Sikorski's extension theorem.*

5. Sikorski's theorem for non-complete spaces. Sikorski showed (see D on p. 119 of [2]) that the condition that the Boolean algebra $\mathfrak{B}'$ is *complete* in his theorem (see introduction) is *essential*. It is therefore of interest to see what remains of Sikorski's theorem if we *do not assume that $\mathfrak{B}'$ is complete.* In this section we shall give a few theorems which deal with this question.

If the Boolean algebra $\mathfrak{B}'$ is not complete, then it follows from the Boolean prime ideal theorem that $\mathfrak{B}'$ can be embedded into a complete and atomic Boolean algebra, namely, the Boolean algebra of all subsets of its Stone representation space. Thus we have the following theorem.

THEOREM 5.1. *If $\mathfrak{B}_0$ is a subalgebra of a Boolean algebra $\mathfrak{B}$ and if h_0 is a homomorphism of $\mathfrak{B}_0$ into a Boolean algebra $\mathfrak{B}'$, then under the assumption that the Boolean prime ideal theorem holds, we have that h_0 can be extended to a homomorphism of $\mathfrak{B}$ into the Boolean algebra of all subsets of the Stone representation space of $\mathfrak{B}'$.*

If one recalls that every Boolean algebra can be embedded into a complete Bollean algebra, then Sikorski's theorem implies that if h_0 is a homomorphism of a subalgebra $\mathfrak{B}_0$ of $\mathfrak{B}$ into a Boolean algebra $\mathfrak{B}'$, then h_0 can be extended to a homomorphism of $\mathfrak{B}$ into any of the complete extensions of $\mathfrak{B}'$. If we do not assume that Sikorski's theorem holds, then by observing that in Σ' it can also be shown, by a method analogous to Dedekind's method of completion of the rationals by cuts, that every Boolean algebra can be embedded into a complete Boolean algebra (see foonote 1, p. 118 of [2]), the following result is evident.

THEOREM 5.2. *Let $\mathfrak{B}_0$ be a subalgebra of $\mathfrak{B}$, and let h_0 be a homomorphism of $\mathfrak{B}_0$ into a Boolean algebra $\mathfrak{B}'$. Then, under the assumption that the Boolean prime ideal theorem holds, we have that for every complete extension $\mathfrak{B}'_c$ of $\mathfrak{B}'$ there exists an ultrapower $\mathfrak{B}''$ of $\mathfrak{B}'_c$ such that h_0 can be extended to a homomorphism of $\mathfrak{B}$ into $\mathfrak{B}''$.*

We conclude this section with the following theorem.

THEOREM 5.3. *Let $\mathfrak{B}_0$ be a subalgebra of $\mathfrak{B}$, and let h_0 be a homomorphism of $\mathfrak{B}_0$ into a Boolean algebra $\mathfrak{B}'$. Then, if $\mathfrak{B}'_c$ is complete extension of $\mathfrak{B}'$, there exists a reduced power $\mathfrak{B}''$ of $\mathfrak{B}'_c$ such that h_0 can be extended to a homomorphism of $\mathfrak{B}$ into $\mathfrak{B}''$.*

Proof. Since h_0 is a homomorphism of $\mathfrak{B}_0$ into $\mathfrak{B}'_c$, it follows from the proof of Theorem 4.3 that h_0 can be extended to all of $\mathfrak{B}$ into a reduced power of $\mathfrak{B}'_c$.

Remark. Professor Ph. Dwinger kindly informed me that the following result, which is a consequence of Theorem 5.3, can be obtained by methods contained in a joint paper of F. Yaqub and himself concerning the theory of amalgamation of Boolean algebras:

Let $\mathfrak{B}_0$ be a subalgebra of $\mathfrak{B}$, and let h_0 be a homomorphism of $\mathfrak{B}_0$ into $\mathfrak{B}'$. Then there exists a Boolean algebra $\mathfrak{B}''$ which contains $\mathfrak{B}'$ as a subalgebra such that h_0 can be extended to a homomorphism of $\mathfrak{B}$ into $\mathfrak{B}''$.

Added in proof. Due to the recent developments in set theory, the author was kindly informed by Professor Halpern that his result concerning the independence of the axiom of choice from the prime ideal theorem for Boolean Algebras also holds in Gödel's set theory consisting of the axiom groups A, B and C and the axiom of regularity D.

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Reçu par la Rédaction le 27. 4. 1963