

A Combinatorial Methodology for Optimizing Non-Binary Graph-Based Codes: Theoretical Analysis and Applications in Data Storage

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Abstract

Non-binary (NB) low-density parity-check (LDPC) codes are graph-based codes that are increasingly being considered as a powerful error correction tool for modern dense storage devices. Optimizing NB-LDPC codes to overcome their error floor is one of the main code design challenges facing storage engineers upon deploying such codes in practice. Furthermore, the increasing levels of asymmetry incorporated by the channels underlying modern dense storage systems, e.g., multi-level Flash systems, exacerbates the error floor problem by widening the spectrum of problematic objects that contributes to the error floor of an NB-LDPC code. In a recent research, the weight consistency matrix (WCM) framework was introduced as an effective combinatorial NB-LDPC code optimization methodology that is suitable for modern Flash memory and magnetic recording (MR) systems. The WCM framework was used to optimize codes for asymmetric Flash channels, MR channels that have intrinsic memory, in addition to canonical symmetric additive white Gaussian noise channels. In this paper, we provide an in-depth theoretical analysis needed to understand and properly apply the WCM framework. We focus on general absorbing sets of type two (GASTs). In particular, we introduce a novel tree representation of a GAST called the unlabeled GAST tree, using which we prove that the WCM framework is optimal in the sense that it operates on the minimum number of matrices to remove a GAST. Then, we enumerate these WCMs and demonstrate the significance of the savings achieved by the WCM framework in the number of matrices processed to remove a GAST. Moreover, we provide a linear-algebraic analysis of the null spaces of WCMs associated with a certain GAST. We derive the minimum number of edge weight changes¹ needed to remove a GAST via its WCMs, along with how to choose these changes. Additionally, we propose a new set of problematic objects, namely the oscillating sets of type two (OSTs), which contribute to the error floor of NB-LDPC codes with even column weights on asymmetric channels, and we show how to customize the WCM framework to remove OSTs. We also extend the domain of the WCM framework applications by demonstrating its benefits in optimizing column weight 5 codes, codes used over Flash channels with soft information, and spatially-coupled codes. The performance gains achieved via the WCM framework range between 1 and nearly 2.5 orders of magnitude in the error floor region over interesting channels.

I. INTRODUCTION

Modern dense storage devices, e.g., multi-level Flash and magnetic-recording (MR) devices, operate at very low frame error rate (FER) values, motivating the need for strong error correction techniques. Because of their capacity approaching performance, low-density parity-check (LDPC)

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¹In the WCM framework, a GAST is removed via careful processing of the weights of its edges (the original and the new weights are not zeros). Throughout this paper, the edge weight changes are always with respect to the original configuration.

codes [1]–[4] are becoming the first choice for many of the modern storage systems [5]–[13]. Under iterative quantized decoding, LDPC codes suffer from the error floor problem, which is a change in the FER slope that undermines the chances of reaching desirable very low FER levels [14]–[18]. It was demonstrated in the literature that absorbing sets (ASs), which are detrimental subgraphs in the Tanner graph of the LDPC code, are the root cause of the error floor problem [19], [20]. There are other works that studied different classes of detrimental objects, specifically, stopping sets [21] and trapping sets [22], [23]. Research works investigating the error floor problem of LDPC codes include [16], [19], [22]–[30].

Particularly, for non-binary LDPC (NB-LDPC) codes, the authors in [20] used concepts from [21] to study non-binary elementary absorbing sets (EASs), and showed that EASs are the detrimental objects which contribute the most to the error floor of NB-LDPC codes over the canonical additive white Gaussian noise (AWGN) channel. The observation that the combinatorial structure of the dominant detrimental objects critically depends on the characteristics of the channel of interest was first introduced in [31] and then in [10]; we introduced balanced absorbing sets (BASs) and demonstrated their dominance in the error floor of NB-LDPC codes over partial-response (PR) channels, which exemplify 1-D MR channels [32], [33]. Motivated by the asymmetry possessed by practical Flash channels [34], [35], in a recent research [5], we introduced general absorbing sets (GASs) and general absorbing sets of type two (GASTs) to capture the dominant problematic objects over realistic Flash channels. GASs and GASTs subsume previously introduced AS subclasses (namely EASs and BASs).

In [20] and [10], NB-LDPC code optimization algorithms tailored to AWGN and PR channels, respectively, were proposed. While the weight consistency matrix (WCM) framework introduced in [5] was originally motivated by the need to optimize NB-LDPC codes for asymmetric Flash channels, we customized this methodology to fit for channels with memory (e.g., PR channels), symmetric channels (e.g., AWGN channels), as well as practical Flash channels, achieving at least 1 order of magnitude performance gain over all these channels. The core idea of the WCM framework is representing a problematic object, e.g., a GAST, using a small set of matrices, called the WCMs. Since problematic objects in an NB-LDPC code are described in terms of both their weight conditions as well as their topological conditions, there are explicit weight properties associated with the WCMs of an object. By changing the null spaces of all the WCMs associated with an object such that the weight conditions of these WCMs are broken [5], this problematic object is removed from the Tanner graph of the code. A key feature of the WCM framework is that the GASTs removal process is performed solely via manipulating the edge weights of the Tanner graph of the NB-LDPC code, which consequently preserves all the structural topological properties of the code being optimized.

For NB-LDPC codes with fixed column weights, our contributions in this paper are:

- 1) We characterize GASTs via their WCMs. In particular, we define the unlabeled GAST tree to describe the underlying topology of a GAST, where the leaves of this tree represent the WCMs of the GAST. Using this tree, we prove the optimality of the WCM framework by demonstrating that the framework indeed operates on the minimum possible number of matrices to remove the detrimental object. We also deploy concepts from graph theory and combinatorics to compute the exact number of WCMs associated with a GAST in different cases. We further compare the number of matrices the WCM framework operates on with the number of matrices an alternative idea works with, showing the significant reduction (reaches over 80%) achieved by the WCM framework in the cases of interest.

- 2) Based on tools from graph theory and linear algebra, we propose the first comprehensive analysis of the removal process of GASTs. We start off with discussing the dimensions of the null spaces of WCMs; these null spaces play the central role in the detection and removal of a GAST. Then, we derive the best that can be done to process a short WCM (a WCM that has fewer rows than columns) during the GAST removal process. Finally, we provide the minimum number of edge weight changes needed to remove a GAST, along with how to select the edges and the new weights to guarantee appropriate removal of the GAST through its WCMs.
- 3) We introduce new combinatorial objects that capture the majority of the non-GAST detrimental objects in the error floor of NB-LDPC codes that have even column weights over asymmetric Flash channels. We define oscillating sets (OSs) and oscillating sets of type two (OSTs). Furthermore, we expand the analysis of GASTs in [5] to cover OSTs, describing how the WCM framework can be customized to remove OSTs, after GASTs have been removed, to achieve additional performance gains.
- 4) We extend the scope of the WCM framework by using it to optimize codes with different properties and for various applications. Specifically, we show that despite the good error floor performance of NB-LDPC codes with column weight 5 before optimization, more than 1 order of magnitude gain in the uncorrectable bit error rate (UBER) over practical Flash channels is achievable via the WCM framework. We further apply the theoretical concepts in item 3 for NB-LDPC codes with column weight 4 over practical Flash channels to achieve overall UBER gains up to nearly 2.5 orders of magnitude. Additionally, we optimize NB-LDPC codes for practical Flash channels with more soft information (6 reads). We also use the WCM framework to optimize NB-LDPC codes with irregular check node (CN) degrees and fixed variable node (VN) degrees; we show that more than 1 order of magnitude performance gain in the FER is achievable by optimizing spatially-coupled (SC) codes [36]–[41] used over AWGN and PR channels.

The rest of the paper is organized as follows. Section II summarizes the main concepts of the WCM framework. Then, Section III discusses the characterization of GASTs via their WCMs, in addition to the optimality proof and the WCMs enumeration. In Section IV we detail our analysis for the process of the GAST removal through WCMs. Afterwards, Section V discusses OSTs and how to customize the WCM framework to remove them. The simulation results are presented in Section VI. Finally, the paper is concluded in Section VII.

II. SUMMARY OF THE WCM FRAMEWORK

In this section, along with Appendices A and B, we provide a brief summary of the main concepts and ideas of the WCM framework for the sake of clarity and completeness. Details of the WCM framework were introduced in [5].

Asymmetry or/and intrinsic memory in the channel of interest can result in VN errors having high magnitudes. Consequently, the possibility to have absorbing set errors with degree-2 unsatisfied CNs (non-elementary absorbing set errors) becomes higher. This was the motivation behind introducing GASs and GASTs in [5] to capture the objects that dominate the error floor region of NB-LDPC codes over asymmetric channels (e.g., Flash channels). We start off with the definitions of a GAS and an unlabeled GAS.

Definition 1. Consider a subgraph induced by a subset \mathcal{V} of VNs in the Tanner graph of an NB-LDPC code. Set all the VNs in \mathcal{V} to values $\in GF(q) \setminus \{0\}$ and set all other VNs to 0. The

set \mathcal{V} is said to be an $(a, b, b_2, d_1, d_2, d_3)$ **general absorbing set (GAS)** over $GF(q)$ if and only if the size of \mathcal{V} is a , the number of unsatisfied (resp., degree-2 unsatisfied) CNs connected to \mathcal{V} is b (resp., b_2), the number of degree-1 (resp., 2 and > 2) CNs connected to \mathcal{V} is d_1 (resp., d_2 and d_3), and each VN in \mathcal{V} is connected to strictly more satisfied than unsatisfied neighboring CNs (for some set of given VN values).

GF refers to Galois field, and q is the GF size. We focus here on the case of $q = 2^\lambda$, where λ is a positive integer ≥ 2 . Furthermore, when we say in this paper that nodes are “connected”, we mean they are “directly connected” or they are “neighbors”, unless otherwise stated. The same applies conceptually when we say an edge is “connected” to a node or vice versa.

Definition 2. Let \mathcal{V} be a subset of VNs in the unlabeled Tanner graph of an NB-LDPC code. Let \mathcal{O} (resp., \mathcal{T} and \mathcal{H}) be the set of degree-1 (resp., 2 and > 2) CNs connected to \mathcal{V} . This graphical configuration is an (a, d_1, d_2, d_3) **unlabeled GAS** if it satisfies the following two conditions:

- 1) $|\mathcal{V}| = a$, $|\mathcal{O}| = d_1$, $|\mathcal{T}| = d_2$, and $|\mathcal{H}| = d_3$.
- 2) Each VN in \mathcal{V} is connected to more neighbors in $\{\mathcal{T} \cup \mathcal{H}\}$ than in \mathcal{O} .

Let \mathbf{H} denote the parity-check matrix of an NB-LDPC code defined over $GF(q)$. Consider an $(a, b, b_2, d_1, d_2, d_3)$ GAS in the Tanner graph of this code. Let \mathbf{A} be the $\ell \times a$ submatrix of \mathbf{H} that consists of $\ell = d_1 + d_2 + d_3$ rows of \mathbf{H} , corresponding to the CNs participating in this GAS, and a columns of \mathbf{H} , corresponding to the VNs participating in this GAS. From [5, Lemma 1], an $(a, b, b_2, d_1, d_2, d_3)$ GAS must satisfy:

- **Topological conditions:** Its unlabeled configuration must satisfy the unlabeled GAS conditions stated in Definition 2.
- **Weight conditions:** The set is an $(a, b, b_2, d_1, d_2, d_3)$ GAS over $GF(q)$ if and only if there exists an $(\ell - b) \times a$ submatrix \mathbf{W} of column rank $\tau_{\mathbf{W}} < a$, with elements $\psi_{e,f}$, $1 \leq e \leq (\ell - b)$, $1 \leq f \leq a$, of the GAS adjacency matrix \mathbf{A} , that satisfies the following two conditions:

- 1) Let $\mathcal{N}(\mathbf{W})$ be the null space of the submatrix \mathbf{W} , and let \mathbf{d}_k^T , $1 \leq k \leq b$, be the k th row of the matrix \mathbf{D} obtained by removing the rows of \mathbf{W} from \mathbf{A} . Let \mathbf{v} be a vector of VN values and \mathbf{R} be an $\ell \times \ell$ permutation matrix. Then,

$$\begin{aligned} \exists \mathbf{v} &= [v_1 \ v_2 \ \dots \ v_a]^T \in \mathcal{N}(\mathbf{W}) \text{ s.t. } v_f \neq 0, \forall f \in \{1, 2, \dots, a\}, \\ &\text{and } \mathbf{d}_k^T \mathbf{v} = m_k \neq 0, \forall k \in \{1, 2, \dots, b\}, \mathbf{m} = [m_1 \ m_2 \ \dots \ m_b]^T, \\ \text{i.e., } \mathbf{R}\mathbf{A}\mathbf{v} &= \begin{bmatrix} \mathbf{W}^{(\ell-b) \times a} \\ \mathbf{D}_{b \times a} \end{bmatrix} \mathbf{v}_{a \times 1} = \begin{bmatrix} \mathbf{0}^{(\ell-b) \times 1} \\ \mathbf{m}_{b \times 1} \end{bmatrix}. \end{aligned} \quad (1)$$

- 2) Let $\theta_{k,f}$, $1 \leq k \leq b$, $1 \leq f \leq a$, be the elements of the matrix \mathbf{D} . Then, $\forall f \in \{1, 2, \dots, a\}$,

$$\left(\sum_{e=1}^{\ell-b} F(\psi_{e,f}) \right) > \left(\sum_{k=1}^b F(\theta_{k,f}) \right), \quad (2)$$

where $F(\beta) = 0$ if $\beta = 0$, and $F(\beta) = 1$ otherwise.

Computations are performed over $GF(q)$.

In words, \mathbf{W} is the submatrix of satisfied CNs, and \mathbf{D} is the submatrix of unsatisfied CNs. Then, we define an important subclass of GASs, which are GASTs.

Definition 3. A GAS that has $d_2 > d_3$ and all the unsatisfied CNs connected to it (if any) belong to $\{\mathcal{O} \cup \mathcal{T}\}$ (i.e., having either degree 1 or degree 2) is defined as an (a, b, d_1, d_2, d_3) **general absorbing set of type two (GAST)**. Similar to the unlabeled GAS definition (Definition 2), we also define the (a, d_1, d_2, d_3) **unlabeled GAST**.

The three theorems essential for understanding the WCM framework are in [5]. Let γ be the column weight (VN degree) of the NB-LDPC code. We recall from [5, Theorem 2] that, given an (a, d_1, d_2, d_3) unlabeled GAST, the maximum number of unsatisfied CNs, b_{\max} , in the resulting GAST after edge labeling is upper bounded by:

$$b_{\max} \leq d_1 + b_{\text{ut}}, \quad \text{where} \quad (3)$$

$$b_{\text{ut}} = \left\lfloor \frac{1}{2} \left(a \left\lfloor \frac{\gamma - 1}{2} \right\rfloor - d_1 \right) \right\rfloor. \quad (4)$$

Here, b_{ut} is the upper bound on the maximum number of degree-2 unsatisfied CNs the resulting GAST can have. Because of the structure of the underlying unlabeled configuration, sometimes the exact maximum (obtained by [5, Algorithm 1], see Appendix A) is a quantity smaller than b_{ut} . We refer to this exact maximum as b_{et} . Thus,

$$b_{\max} = d_1 + b_{\text{et}}. \quad (5)$$

Throughout this paper, the notation “ut” (resp., “et”) in the subscript of b refers to the *upper bound on the* (resp., *exact*) maximum number of *degree-2* unsatisfied CNs.

For a given (a, b, d_1, d_2, d_3) GAST, let \mathcal{Z} be the set of all (a, b', d_1, d_2, d_3) GASTs with $d_1 \leq b' \leq b_{\max}$, which have the same unlabeled GAST as the original (a, b, d_1, d_2, d_3) GAST. Here, b_{\max} is the largest allowable number of unsatisfied CNs for these configurations.

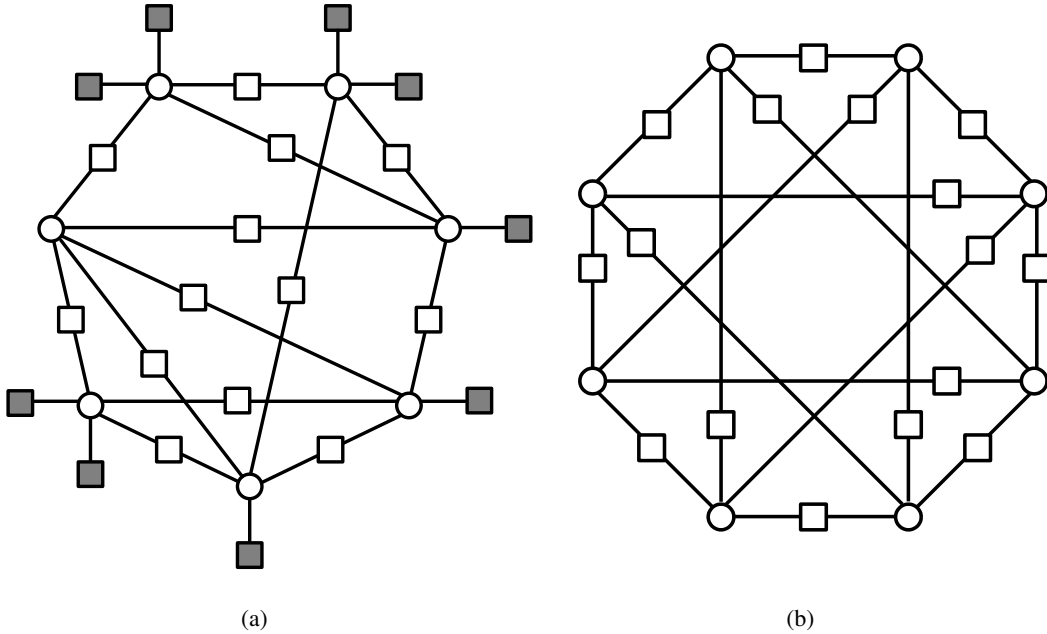


Fig. 1. (a) A (7, 9, 13, 0) unlabeled GAST ($\gamma = 5$). (b) An (8, 0, 16, 0) unlabeled GAST ($\gamma = 4$).

Definition 4. An (a, b, d_1, d_2, d_3) **GAST** is said to be **removed** from the Tanner graph of an NB-LDPC code if and only if the resulting object (after edge weight processing) $\notin \mathcal{Z}$.

Example 1. Fig. 1(a) shows a $(7, 9, 13, 0)$ unlabeled GAST ($\gamma = 5$). For this unlabeled GAST, $d_1 = 9$, and from (4), $b_{\text{ut}} = \lfloor \frac{1}{2} (7 \lfloor \frac{5-1}{2} \rfloor - 9) \rfloor = 2 = b_{\text{et}}$, which means the resulting GAST after edge labeling can have up to 2 degree-2 unsatisfied CNs. Thus, $b_{\text{max}} = 9 + 2 = 11$, and $9 \leq b' \leq 11$. Consequently, $\mathcal{Z} = \{(7, 9, 9, 13, 0), (7, 10, 9, 13, 0), (7, 11, 9, 13, 0)\}$. Fig. 1(b) shows an $(8, 0, 16, 0)$ unlabeled GAST ($\gamma = 4$). For this unlabeled GAST, $d_1 = 0$, and from (4), $b_{\text{ut}} = \lfloor \frac{1}{2} (8 \lfloor \frac{4-1}{2} \rfloor - 0) \rfloor = 4 = b_{\text{et}}$, which means the resulting GAST after edge labeling can have up to 4 degree-2 unsatisfied CNs. Thus, $b_{\text{max}} = 0 + 4 = 4$, and $0 \leq b' \leq 4$. Consequently, $\mathcal{Z} = \{(8, 0, 0, 16, 0), (8, 1, 0, 16, 0), (8, 2, 0, 16, 0), (8, 3, 0, 16, 0), (8, 4, 0, 16, 0)\}$.

For a given GAST, define a matrix \mathbf{W}^z to be the matrix obtained by removing b' , $d_1 \leq b' \leq b_{\text{max}}$, rows corresponding to CNs $\in \{\mathcal{O} \cup \mathcal{T}\}$ from the matrix \mathbf{A} , the GAST adjacency matrix. These b' CNs can simultaneously be unsatisfied under some edge labeling that produces a GAST which has the same unlabeled GAST as the given GAST. Let \mathcal{U} be the set of all such matrices \mathbf{W}^z . Each element in \mathcal{Z} has one or more matrices in \mathcal{U} .

Definition 5. For a given (a, b, d_1, d_2, d_3) GAST and its associated adjacency matrix \mathbf{A} and its associated set \mathcal{Z} , we construct a set of t matrices as follows:

- 1) Each matrix \mathbf{W}_h^{cm} , $1 \leq h \leq t$, in this set is an $(\ell - b_h^{\text{cm}}) \times a$ submatrix, $d_1 \leq b_h^{\text{cm}} \leq b_{\text{max}}$, formed by removing **different** b_h^{cm} rows from the $\ell \times a$ matrix \mathbf{A} of the GAST. These b_h^{cm} rows to be removed correspond to CNs $\in \{\mathcal{O} \cup \mathcal{T}\}$ that can simultaneously be unsatisfied under some edge labeling that produces a GAST which has the same unlabeled GAST as the given GAST.
- 2) Each matrix $\mathbf{W}^z \in \mathcal{U}$, for every element $\in \mathcal{Z}$, contains at least one element of the resultant set as its submatrix.
- 3) This resultant set has the **smallest cardinality**, which is t , among all the sets which satisfy conditions 1 and 2 stated above.

We refer to the matrices in this set as **weight consistency matrices (WCMs)**, and to this set itself as \mathcal{W} .

Throughout this paper, the notation “z” (resp., “cm”) in the superscript of a matrix means that the matrix is associated with an element in the set \mathcal{Z} (resp., a WCM).

Definition 6. Parameter b_{et} represents the **exact maximum number** of rows corresponding to degree-2 CNs that can be removed together from \mathbf{A} to extract a WCM. Similarly, we define b_{st} to be the **exact minimum number** of rows corresponding to degree-2 CNs that can be removed together from \mathbf{A} to extract a WCM. Remember that the rows corresponding to degree-1 CNs are always removed while extracting a WCM. Thus, $d_1 \leq d_1 + b_{\text{st}} \leq b_h^{\text{cm}} \leq d_1 + b_{\text{et}} = b_{\text{max}}$. Both b_{st} and b_{et} depend on the unlabeled GAST configuration.

Fig. 2 emphasizes the relation between a GAST and its associated WCMs, and roughly describes how the WCMs of this GAST are extracted.

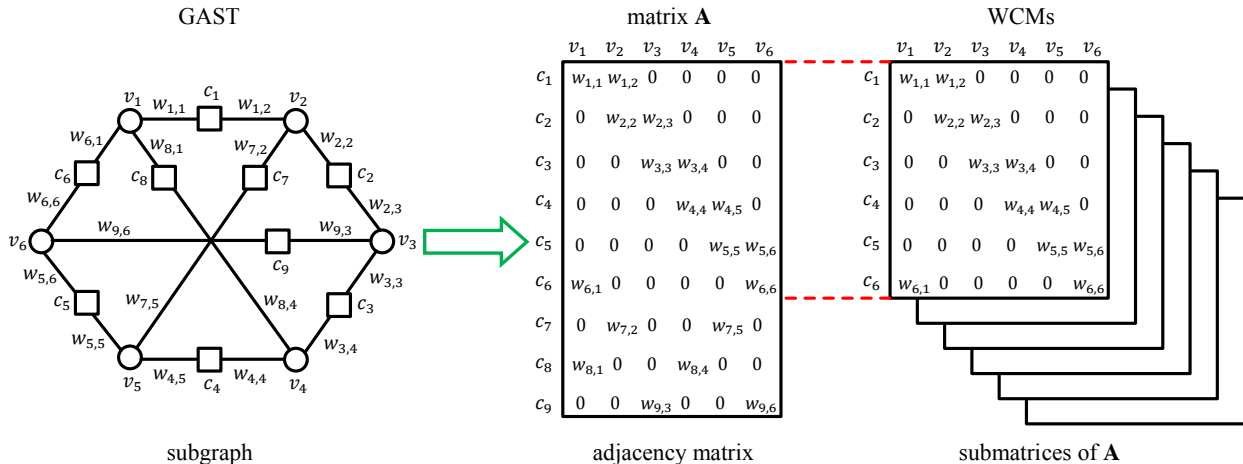


Fig. 2. An illustrative figure showing the process of extracting the WCMs of a $(6, 0, 0, 9, 0)$ GAST. Appropriate edge weights (w 's) $\in \text{GF}(q) \setminus \{0\}$ are assumed.

The core idea of the WCM framework is in [5, Theorem 3]. This theorem states that the necessary and sufficient processing needed to remove an (a, b, d_1, d_2, d_3) GAST, according to Definition 4, is to change the edge weights such that $\forall h$:

$$\text{If } \mathcal{N}(\mathbf{W}_h^{\text{cm}}) = \text{span}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{p_h}\}, \text{ then } \nexists \mathbf{r} = [r_1 \ r_2 \ \dots \ r_{p_h}]^T$$

$$\text{for } \mathbf{v} = r_1 \mathbf{x}_1 + r_2 \mathbf{x}_2 + \dots + r_{p_h} \mathbf{x}_{p_h} = [v_1 \ v_2 \ \dots \ v_a]^T \text{ s.t. } v_j \neq 0, \forall j \in \{1, 2, \dots, a\}, \quad (6)$$

where p_h is the dimension of $\mathcal{N}(\mathbf{W}_h^{\text{cm}})$. Computations are performed over $\text{GF}(q)$.

The WCM framework is easily adjusted to efficiently remove special subclasses of GASTs, namely EASs and BASs of type two (BASTs), by customizing the WCM definition. In particular, via replacing b_{\max} by $b_{e_{\max}} = d_1$ (resp., $b_{b_{\max}} = \lfloor \frac{ag}{2} \rfloor$) (see [10]), where $g = \lfloor \frac{\gamma-1}{2} \rfloor$, the WCM framework is customized to remove EASs (resp., BASTs), which are the dominant objects in the case of AWGN (resp., PR) channels. More details can be found in [5].

The two algorithms that constitute the WCM framework are [5, Algorithm 1], which is the WCM extraction algorithm, and [5, Algorithm 2], which is the code optimization algorithm. The steps of the two algorithms are listed in Appendices A and B for the reference of the reader.

A WCM that has (6) satisfied is said to be a WCM with **broken weight conditions**. A GAST is removed if and only if all its WCMs have broken weight conditions.

Note that the complexity of the process of removing a specific GAST using the WCM framework is mainly controlled by the number of WCMs, which is t , of that GAST (see the **for** loop in step 12 of [5, Algorithm 2]). Thus, the complexity of the WCM framework depends on the size of the set \mathcal{G} (see Appendix B) and the numbers of WCMs of the GASTs in \mathcal{G} .

III. CHARACTERIZING GASTS THROUGH THEIR WCMs

A. Proving the Optimality of the WCM Framework

The number of matrices needed to operate on for different GASTs controls the complexity of the code optimization process. In this subsection, we prove that the WCM framework is optimal in the sense that it works on the minimum possible number of matrices to remove a GAST. Recall that \mathbf{A} is the adjacency matrix of the GAST. Both \mathbf{W}^z and \mathcal{U} are defined in the paragraph before Definition 5. Our optimization problem is formulated as follows:

The optimization problem: We seek to find the set \mathcal{W} of matrices that has the minimum cardinality, with the matrices in \mathcal{W} representing submatrices of \mathbf{A} that can be used to remove the problematic GAST, without the need to work on other submatrices.

The optimization constraint: Each matrix in \mathcal{W} has to be a valid \mathbf{W}^z matrix in \mathcal{U} .

The optimization constraint is to ensure that we are performing not only sufficient, but also necessary processing to remove the object. Note that by definition, the set of WCMs is the solution of this optimization problem. Thus, the problem of proving the optimality of the WCM framework reduces to proving that the matrices we extract by [5, Algorithm 1], and operate on in [5, Algorithm 2] to remove the GAST, are indeed the WCMs.

Before we present the proof of optimality, we first introduce the definition of the GAST tree, which will simplify the proof. Since this tree does not depend on the edge weights of the configuration, we call it the unlabeled GAST tree. Recall that b_{et} is the maximum number of degree-2 CNs that can be unsatisfied simultaneously while the object remains a GAST. Define u^0 as the number of degree-2 CNs that can be unsatisfied individually while the object remains a GAST, and \mathbf{y}^0 as the vector in which the indices of such u^0 CNs are saved. Note that we always have $b_{\text{et}} \leq u^0$.

Definition 7. For a given (a, d_1, d_2, d_3) unlabeled GAST, we construct the **unlabeled GAST tree** of b_{et} levels (level 0 is not counted) as follows:

- Except the root node at level 0, each tree node represents a degree-2 CN in the unlabeled GAST. For any two CNs in the tree, being neighbors means that they can be unsatisfied simultaneously after labeling and the resulting object remains a GAST.
- Let $i_1, i_2, \dots, i_{b_{\text{et}}}$ be the running indices used to access nodes at different levels in the tree as follows. The index of a node at level j , $1 \leq j \leq b_{\text{et}}$, is saved in $\mathbf{y}_{i_1, i_2, \dots, i_{j-1}}^{j-1}$ and given by $\mathbf{y}_{i_1, i_2, \dots, i_{j-1}}^{j-1}(i_j)$. CN $c_{\mathbf{y}_{i_1, i_2, \dots, i_{j-1}}^{j-1}}(i_j)$ at level j is accessed via the path of nodes “root node – $c_{\mathbf{y}^0}(i_1) - c_{\mathbf{y}_{i_1}^1}(i_2) - c_{\mathbf{y}_{i_1, i_2}^2}(i_3) - \dots - c_{\mathbf{y}_{i_1, i_2, \dots, i_{j-1}}^{j-1}}(i_j)$ ”.
- At level 0, a virtual root node is assumed to be connected to the u^0 nodes with indices in \mathbf{y}^0 at level 1. Level j of the tree consists of all the nodes with indices in $\mathbf{y}_{i_1, i_2, \dots, i_{j-1}}^{j-1}$, $\forall i_1, i_2, \dots, i_{j-1}$. Level $j + 1$ of the tree is created as follows. Each CN $c_{\mathbf{y}_{i_1, i_2, \dots, i_{j-1}}^{j-1}}(i_j)$ at level j is connected to all the CNs with indices in $\mathbf{y}_{i_1, i_2, \dots, i_j}^j$ at level $j + 1$. These CNs can each be – simultaneously with the nodes on the path from the root node until $c_{\mathbf{y}_{i_1, i_2, \dots, i_{j-1}}^{j-1}}(i_j)$ – unsatisfied after labeling and the resulting object remains a GAST.
- The number of nodes at level $j + 1$ that are connected to $c_{\mathbf{y}_{i_1, i_2, \dots, i_{j-1}}^{j-1}}(i_j)$ at level j is $u_{i_1, i_2, \dots, i_j}^j$ (which is the size of the vector $\mathbf{y}_{i_1, i_2, \dots, i_j}^j$), with $u_{i_1, i_2, \dots, i_j}^j < u_{i_1, i_2, \dots, i_{j-1}}^{j-1}$, $\forall i_1, i_2, \dots, i_j$.
- The leaves of this tree are linked to the matrices extracted by [5, Algorithm 1] (before removing the repeated matrices).

Note that for the parameters u and \mathbf{y} , the superscript refers to the level prior to the level in which the nodes exist, and the subscript refers to the running indices used to access the nodes. Note also that [5, Algorithm 1] is designed to generate the unlabeled GAST tree.

Fig. 3 shows an unlabeled GAST tree for a configuration that has $b_{\text{et}} = 3$. The configuration has three levels after the root node. We say that each tree node at level j , $j \neq 0$, in the unlabeled GAST tree is **linked to** a matrix $\mathbf{W}^z \in \mathcal{U}$ extracted by removing $(d_1 + j)$ rows from the matrix

A. These rows correspond to all the d_1 degree-1 CNs, and the j degree-2 CNs on the path from the virtual root node to this tree node in the configuration. We also say that every valid matrix $\mathbf{W}^z \in \mathcal{U}$ is **linked to** multiple tree nodes.

It can be shown that b_{et} is the number of levels (nested loops in [5, Algorithm 1]), after which $u_{i_1, i_2, \dots, i_{b_{\text{et}}}}^{b_{\text{et}}} = 0, \forall i_1, i_2, \dots, i_{b_{\text{et}}}$. Moreover, because the WCMs do not necessarily have the same row dimension, [5, Algorithm 1] may stop at b_k levels, $b_k \leq b_{\text{et}}$, starting from some $c_{y^0(i_1)}$, which results in an $(\ell - b_h^{\text{cm}}) \times a$ WCM with $b_h^{\text{cm}} = d_1 + b_k \leq b_{\text{max}} = d_1 + b_{\text{et}}$. The smallest value of b_k is b_{st} , i.e., $b_{\text{st}} \leq b_k \leq b_{\text{et}}$.

Remark 1. Note that the unlabeled GAST tree is unique for a given unlabeled configuration. In other words, two non-isomorphic (a, d_1, d_2, d_3) configurations have two different unlabeled GAST trees even though they have the same $a, d_1, d_2,$ and d_3 parameters.

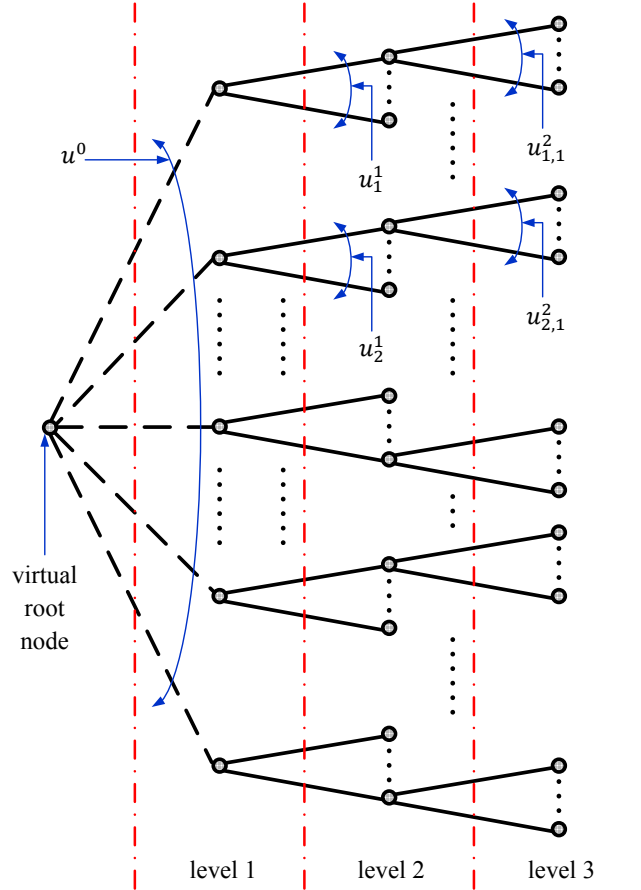


Fig. 3. An unlabeled GAST tree with $b_{\text{et}} = 3$.

Repetitions in tree nodes linked to matrices \mathbf{W}^z come from the fact that we are addressing the permutations and not the combinations in the tree. In other words, if we have a path from the root node at level 0 to a tree node at level 2 that has c_1 at level 1 then c_4 at level 2 on it, there must exist another path from the root node at level 0 to another tree node at level 2 that has c_4 at level 1 then c_1 at level 2 on it. Obviously, removing the row of c_1 then the row of c_4 , or first c_4 then c_1 (in addition to the rows of degree-1 CNs) from \mathbf{A} to extract a matrix is exactly the same.

Now, we are ready to present the optimality theorem and its proof.

Theorem 1. *Consider an (a, b, d_1, d_2, d_3) GAST with $b_{\text{et}} > 0$. After eliminating repetitions, the set of matrices which are linked to the leaves of the unlabeled GAST tree characterized by Definition 7 is the set of WCMs, i.e., the set \mathcal{W} of minimum cardinality.*

Proof: According to Definition 5, [5, Theorem 3], and its proof, each matrix $\mathbf{W}^z \in \mathcal{U}$ must have at least one matrix $\in \mathcal{W}$ as its submatrix (as \mathcal{W} is the set of WCMs). The relation between a matrix \mathbf{W}_1^z linked to tree node 1 at level j and a matrix \mathbf{W}_2^z linked to tree node 2 at level $j+1$, provided that tree node 2 is a child of tree node 1, is as follows. Matrix \mathbf{W}_2^z is a submatrix of matrix \mathbf{W}_1^z , extracted by removing one more row, that corresponds to a degree-2 CN, from \mathbf{W}_1^z . Following the same logic, the multiset of matrices, say \mathcal{W}_{rep} , linked to tree nodes with no children (the leaves of the tree) contains submatrices of every possible matrix \mathbf{W}^z . We let the set $\mathcal{W}_{\text{nrep}}$ be \mathcal{W}_{rep} after eliminating the repetitions.

Now, we prove the sufficiency and minimality, which implies the optimality of $\mathcal{W}_{\text{nrep}}$. The sufficiency is proved as follows. Any matrix \mathbf{W}^z that is linked to a tree node with a child will be redundant if added to the set $\mathcal{W}_{\text{nrep}}$ because in $\mathcal{W}_{\text{nrep}}$ there already exists a submatrix of this \mathbf{W}^z (from the analysis above). The minimality is proved as follows. If we eliminate any matrix from $\mathcal{W}_{\text{nrep}}$, there will be at least one matrix \mathbf{W}^z that has no submatrices in $\mathcal{W}_{\text{nrep}}$ (which is the eliminated matrix itself since it is linked to a node (nodes) with no children). Thus, we cannot further reduce the cardinality of $\mathcal{W}_{\text{nrep}}$. Hence, the set $\mathcal{W}_{\text{nrep}}$ is indeed the set \mathcal{W} of WCMs, which proves the optimality of the WCM framework. ■

B. Enumeration of WCMs Associated with a GAST

In this subsection, we provide the exact number of distinct WCMs associated with a GAST. Moreover, we present particular examples, where this number reduces to a combinatorial function of the column weight of the code. Since the number of WCMs (and also their sizes) associated with a GAST only depends on the unlabeled configuration and not on the edge weights, we associate the number of distinct WCMs, t , to the unlabeled GAST throughout this paper.

We first identify the following two types of unlabeled GAST configurations according to the properties of their unlabeled GAST trees.

Definition 8. *An (a, d_1, d_2, d_3) same-size-WCMs unlabeled GAST satisfies one of the following two conditions:*

- 1) *It has $b_{\text{et}} = 0$ and $u^0 = 0$, which results in $|\mathcal{W}| = 1$.*
- 2) *It has $u^0 > 0$, and its tree has the property that $w_{i_1, i_2, \dots, i_j}^j = 0$ only if $j = b_{\text{et}}, \forall i_1, i_2, \dots, i_{b_{\text{et}}}$, which results in all the WCMs having the same $(\ell - b_{\text{max}}) \times a$ size, $b_{\text{max}} = d_1 + b_{\text{et}}$.*

Definition 9. *An (a, d_1, d_2, d_3) u-symmetric unlabeled GAST is a same-size-WCMs unlabeled GAST which satisfies the following condition. If $u^0 > 0$, its tree has the property that at any level j , $w_{i_1, i_2, \dots, i_{j-1}}^{j-1}$ is the same, $\forall i_1, i_2, \dots, i_{j-1}$.*

An example of a same-size-WCMs unlabeled GAST that is not u-symmetric is the $(7, 9, 13, 0)$ configuration shown in Fig. 5(a). The $(6, 0, 9, 0)$ and the $(8, 0, 16, 0)$ configurations shown in Fig. 7(a) are examples of u-symmetric unlabeled GASTs.

We start off with the count for the general case.

Theorem 2. *Given the unlabeled GAST tree, an (a, d_1, d_2, d_3) unlabeled GAST, with the parameters $b_{st} > 0$ and $b_{et} > 0$, results in the following number, t , of distinct WCMs (t is the size of the set \mathcal{W}) for the labeled configuration:*

$$t = \sum_{b_k=b_{st}}^{b_{et}} \frac{1}{b_k!} \sum_{i_1=1}^{u^0} \sum_{i_2=1}^{u_{i_1}^1} \sum_{i_3=1}^{u_{i_1, i_2}^2} \cdots \sum_{i_{b_k}=1}^{u_{i_1, i_2, \dots, i_{b_k-1}}^{b_k-1}} T\left(u_{i_1, i_2, \dots, i_{b_k}}^{b_k}\right), \quad (7)$$

where $b_{st} \leq b_k \leq b_{et}$. Here, $T\left(u_{i_1, i_2, \dots, i_{b_k}}^{b_k}\right) = 1$ if $u_{i_1, i_2, \dots, i_{b_k}}^{b_k} = 0$, and $T\left(u_{i_1, i_2, \dots, i_{b_k}}^{b_k}\right) = 0$ o/w.

Proof: To prove Theorem 2, we recall the unlabeled GAST tree. The number of nodes in this tree at any level $b_k > 0$ is given by:

$$\mu_{b_k} = \sum_{i_1=1}^{u^0} \sum_{i_2=1}^{u_{i_1}^1} \sum_{i_3=1}^{u_{i_1, i_2}^2} \cdots \sum_{i_{b_k}=1}^{u_{i_1, i_2, \dots, i_{b_k-1}}^{b_k-1}} (1). \quad (8)$$

From the previous subsection, the number, t_{rep, b_k} , of WCMs (not necessarily distinct) extracted by removing $b_h^{\text{cm}} = d_1 + b_k$ rows from \mathbf{A} equals the number of leaves at level b_k . Note that the leaves at level b_k do not have connections to level $b_k + 1$ (no children) in the tree. As a result, t_{rep, b_k} is given by:

$$t_{\text{rep}, b_k} = \sum_{i_1=1}^{u^0} \sum_{i_2=1}^{u_{i_1}^1} \sum_{i_3=1}^{u_{i_1, i_2}^2} \cdots \sum_{i_{b_k}=1}^{u_{i_1, i_2, \dots, i_{b_k-1}}^{b_k-1}} T\left(u_{i_1, i_2, \dots, i_{b_k}}^{b_k}\right). \quad (9)$$

To compute the number of distinct WCMs, we need to eliminate repeated WCMs. Since a WCM extracted by removing $(d_1 + b_k)$ rows from \mathbf{A} appears $b_k!$ times, we compute the number of distinct WCMs that are extracted by removing $(d_1 + b_k)$ rows from \mathbf{A} using (9) as follows:

$$t_{b_k} = \frac{1}{b_k!} \sum_{i_1=1}^{u^0} \sum_{i_2=1}^{u_{i_1}^1} \sum_{i_3=1}^{u_{i_1, i_2}^2} \cdots \sum_{i_{b_k}=1}^{u_{i_1, i_2, \dots, i_{b_k-1}}^{b_k-1}} T\left(u_{i_1, i_2, \dots, i_{b_k}}^{b_k}\right). \quad (10)$$

The total number of distinct WCMs is then obtained by summing t_{b_k} in (10) over all values of b_k , $b_{st} \leq b_k \leq b_{et}$, to reach t in (7). ■

Example 2. *Fig. 4(a) shows a $(6, 2, 5, 2)$ unlabeled GAST ($\gamma = 3$). As demonstrated by the unlabeled GAST tree in Fig. 4(b), the configuration has WCMs that are not of the same size. Since $b_{st} = 1$, $b_{et} = b_{ut} = 2$, and $u^0 = 3$ (that are c_2 , c_3 , and c_4), (7) reduces to:*

$$t = \sum_{b_k=1}^2 \frac{1}{b_k!} \sum_{i_1=1}^3 \sum_{i_2=1}^{u_{i_1}^1} T\left(u_{i_1, \dots, i_{b_k}}^{b_k}\right) = \frac{1}{1!}(0 + 1 + 0) + \frac{1}{2!}(1 + 0 + 1) = 2.$$

Thus, the configuration has only 2 WCMs, extracted by removing the rows of the following groups of CNs from \mathbf{A} : $\{(c_3, \mathcal{O}), (c_2, c_4, \mathcal{O})\}$, where $\mathcal{O} = \{c_8, c_9\}$. We explicitly list the set \mathcal{O} of degree-1 CNs to highlight the fact that the rows of these CNs are always removed, irrespective of the action on the remaining rows in \mathbf{A} .

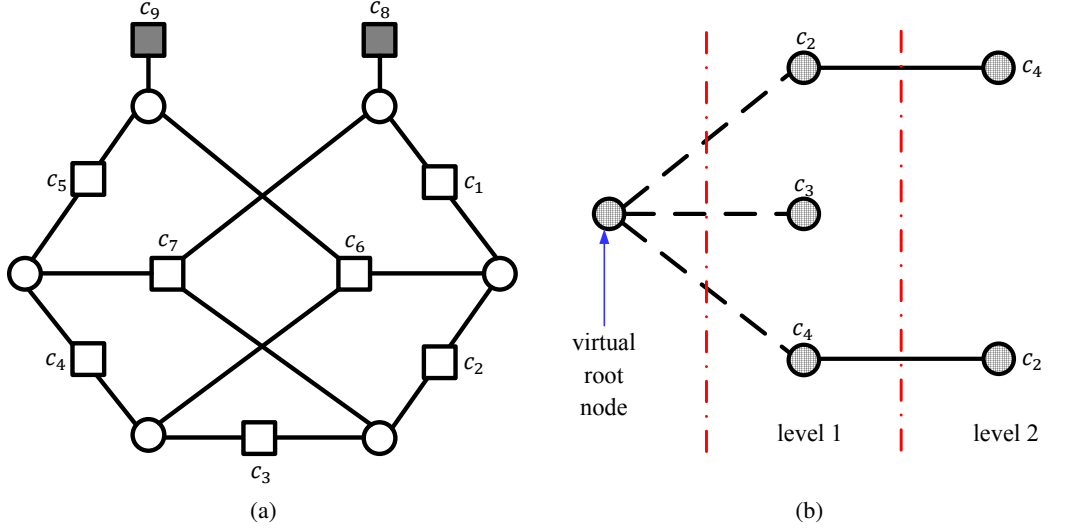


Fig. 4. (a) A $(6, 2, 5, 2)$ unlabeled GAST ($\gamma = 3$). (b) The associated unlabeled GAST tree ($b_{\text{et}} = 2$).

Now, we analyze the important special case of same-size-WCMs configurations.

Lemma 1. *Given the unlabeled GAST tree, a same-size-WCMs (a, d_1, d_2, d_3) unlabeled GAST, with the parameter $b_{\text{et}} > 0$ ($b_{\text{st}} = b_{\text{et}}$), results in the following number, t , of distinct WCMs (t is the size of the set \mathcal{W}) for the labeled configuration:*

$$t = \frac{1}{b_{\text{et}}!} \sum_{i_1=1}^{u^0} \sum_{i_2=1}^{u_{i_1}^1} \sum_{i_3=1}^{u_{i_1, i_2}^2} \cdots \sum_{i_{b_{\text{et}}}=1}^{u_{i_1, i_2, \dots, i_{b_{\text{et}}-1}}^{b_{\text{et}}-1}} (1). \quad (11)$$

Proof: We prove Lemma 1 by substituting $b_k = b_{\text{st}} = b_{\text{et}}$ in (7):

$$t = \frac{1}{b_{\text{et}}!} \sum_{i_1=1}^{u^0} \sum_{i_2=1}^{u_{i_1}^1} \sum_{i_3=1}^{u_{i_1, i_2}^2} \cdots \sum_{i_{b_{\text{et}}}=1}^{u_{i_1, i_2, \dots, i_{b_{\text{et}}-1}}^{b_{\text{et}}-1}} T(u_{i_1, i_2, \dots, i_{b_{\text{et}}}}^{b_{\text{et}}}) = \frac{1}{b_{\text{et}}!} \sum_{i_1=1}^{u^0} \sum_{i_2=1}^{u_{i_1}^1} \sum_{i_3=1}^{u_{i_1, i_2}^2} \cdots \sum_{i_{b_{\text{et}}}=1}^{u_{i_1, i_2, \dots, i_{b_{\text{et}}-1}}^{b_{\text{et}}-1}} (1). \quad (12)$$

The second equality in (12) follows from the fact that $T(u_{i_1, i_2, \dots, i_{b_{\text{et}}}}^{b_{\text{et}}}) = 1$ since $u_{i_1, i_2, \dots, i_{b_{\text{et}}}}^{b_{\text{et}}} = 0$, $\forall i_1, i_2, \dots, i_{b_{\text{et}}}$ by definition of b_{et} . ■

Example 3. *Fig. 5(a) shows a $(7, 9, 13, 0)$ unlabeled GAST ($\gamma = 5$). As demonstrated by the unlabeled GAST tree in Fig. 5(b), this is a same-size-WCMs configuration. Since $b_{\text{et}} = b_{\text{ut}} = 2$ and $u^0 = 5$ (that are c_3, c_4, c_9, c_{11} , and c_{12}), (11) reduces to:*

$$t = \frac{1}{2!} \sum_{i_1=1}^5 \sum_{i_2=1}^{u_{i_1}^1} (1) = \frac{1}{2} (1 + 1 + 3 + 2 + 3) = 5.$$

Thus, the configuration has only 5 WCMs of the same size (11×7) , extracted by removing the rows of the following groups of CNs from the matrix \mathbf{A} : $\{(c_3, c_{12}, \mathcal{O}), (c_4, c_9, \mathcal{O}), (c_9, c_{11}, \mathcal{O}), (c_9, c_{12}, \mathcal{O}), (c_{11}, c_{12}, \mathcal{O})\}$, where $\mathcal{O} = \{c_{14}, c_{15}, c_{16}, c_{17}, c_{18}, c_{19}, c_{20}, c_{21}, c_{22}\}$.

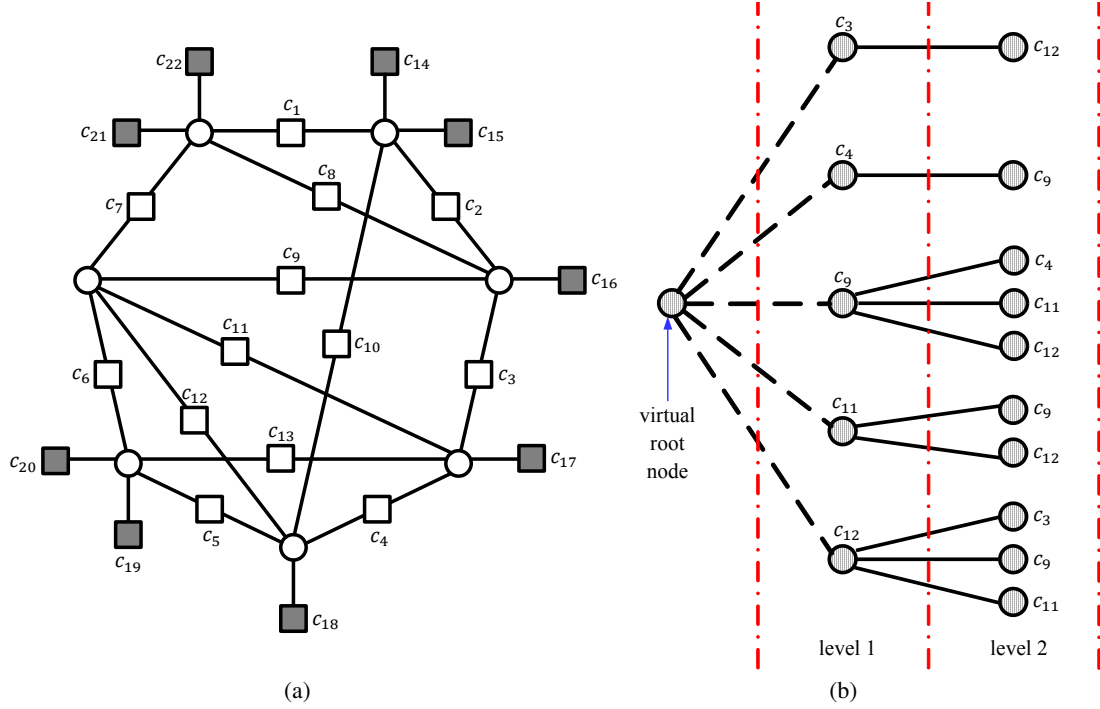


Fig. 5. (a) A $(7, 9, 13, 0)$ unlabeled GAST ($\gamma = 5$). (b) The associated unlabeled GAST tree ($b_{\text{et}} = 2$).

Another important special case to study is the case of u-symmetric configurations.

Corollary 1. *Given the unlabeled GAST tree, a u-symmetric (a, d_1, d_2, d_3) unlabeled GAST, with the parameter $b_{\text{et}} > 0$, results in the following number, t , of distinct WCMs (t is the size of the set \mathcal{W}) for the labeled configuration:*

$$t = \frac{1}{b_{\text{et}}!} \prod_{j=1}^{b_{\text{et}}} u^{j-1}. \quad (13)$$

Proof: Since the u-symmetric case is a special case of the same-size-WCMs case, we use (11) to conclude:

$$t = \frac{1}{b_{\text{et}}!} \sum_{i_1=1}^{u^0} \sum_{i_2=1}^{u^1} \sum_{i_3=1}^{u^2} \cdots \sum_{i_{b_{\text{et}}}=1}^{u^{b_{\text{et}}-1}} (1) = \frac{1}{b_{\text{et}}!} \prod_{j=1}^{b_{\text{et}}} u^{j-1}. \quad (14)$$

Equation (14) follows from the fact that for a u-symmetric configuration, at any level j , $u_{i_1, i_2, \dots, i_{j-1}}^{j-1}$ is the same, $\forall i_1, i_2, \dots, i_{j-1}$. Thus, we can express $u_{i_1, i_2, \dots, i_{j-1}}^{j-1}$ in (11) as u^{j-1} , which is independent of $i_1, i_2, \dots, i_{b_{\text{et}}-1}$, $\forall j \in \{1, 2, \dots, b_{\text{et}}\}$. ■

Example 4. *Fig. 6(a) shows a $(6, 2, 11, 0)$ unlabeled GAST ($\gamma = 4$). As demonstrated by the unlabeled GAST tree in Fig. 6(b), the configuration is u-symmetric. Since $b_{\text{et}} = b_{\text{ut}} = 2$, $u^0 = 6$, and $u^1 = 1$, (13) reduces to:*

$$t = \frac{1}{2!} \prod_{j=1}^2 u^{j-1} = \frac{1}{2} (6)(1) = 3.$$

Thus, the configuration has only 3 WCMs of the same size (9×6) , extracted by removing the rows of the following groups of CNs from the matrix \mathbf{A} : $\{(c_1, c_4, \mathcal{O}), (c_7, c_8, \mathcal{O}), (c_9, c_{10}, \mathcal{O})\}$, where $\mathcal{O} = \{c_{12}, c_{13}\}$.

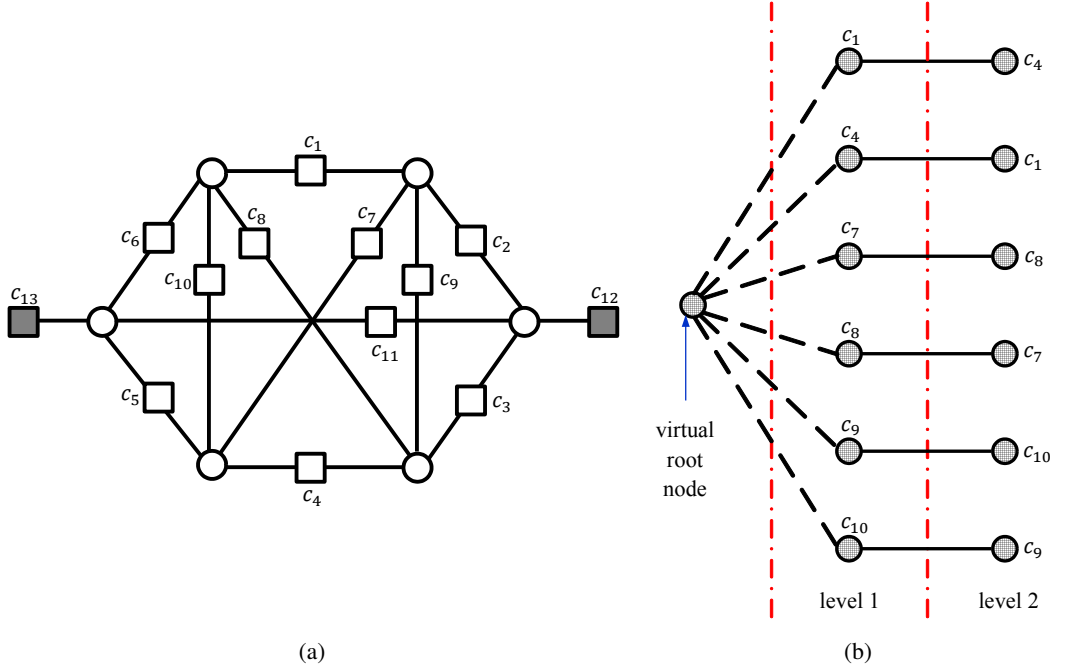


Fig. 6. (a) A $(6, 2, 11, 0)$ unlabeled GAST ($\gamma = 4$). (b) The associated unlabeled GAST tree ($b_{\text{et}} = 2$).

After providing the exact number of WCMs for different cases, we now study examples where the number of distinct WCMs associated with a configuration is proved to be a function only of the column weight γ (the VN degree). We study the u-symmetric version of the $(2\gamma, 0, \gamma^2, 0)$ unlabeled GASTs with $g = \lfloor \frac{\gamma-1}{2} \rfloor = 1$ (i.e., for $\gamma = 3$ or $\gamma = 4$).

Lemma 2. *A u-symmetric $(2\gamma, 0, \gamma^2, 0)$ unlabeled GAST, with $\gamma \in \{3, 4\}$ (see Fig. 7(a)), results in $t = \gamma!$ distinct WCMs for the labeled configuration.*

Proof: From (4), for a u-symmetric $(2\gamma, 0, \gamma^2, 0)$ unlabeled GAST, we have:

$$b_{\text{et}} = b_{\text{ut}} = \left\lfloor \frac{1}{2} \left(2\gamma \left\lfloor \frac{\gamma-1}{2} \right\rfloor - 0 \right) \right\rfloor = \gamma. \quad (15)$$

Notice that $\lfloor \frac{\gamma-1}{2} \rfloor = 1$ for $\gamma \in \{3, 4\}$. Thus, substituting (15) in (13) we reach:

$$t = \frac{1}{\gamma!} \prod_{j=1}^{\gamma} u^{j-1} = \frac{\gamma^2}{\gamma!} \prod_{j=2}^{\gamma} u^{j-1}, \quad (16)$$

where the second equality in (16) follows from the property that for a $(2\gamma, 0, \gamma^2, 0)$ unlabeled GAST, $u^0 = \gamma^2$.

Next, we compute u^{j-1} , $2 \leq j \leq b_{\text{et}} = \gamma$. At level 1, a degree-2 CN that has its index in \mathbf{y}^0 will be marked as unsatisfied resulting in:

$$u^1 = u^0 - 1 - 2(\gamma - 1) = \gamma^2 - 2\gamma + 1 = (\gamma - 1)^2. \quad (17)$$

Equation (17) follows from the fact that after such a degree-2 CN is selected to be marked as unsatisfied at level 1, all the remaining $(\gamma - 1)$ CNs connected to each of the two VNs sharing this

CN cannot be selected at level 2 (because $g = 1$ for $\gamma \in \{3, 4\}$). Thus, $u^1 = u^0 - (1 + 2(\gamma - 1))$, where the additional 1 represents the already selected CN itself. Furthermore:

$$u^2 = u^1 - 1 - 2(\gamma - 2) = (\gamma - 1)^2 - 2\gamma + 3 = (\gamma - 2)^2. \quad (18)$$

Note that the $2(\gamma - 1)$ CNs that cannot be selected at level 2 are connected to all the remaining $(2\gamma - 2)$ VNs in the configuration (after excluding the two VNs sharing the CN selected at level 1). Thus, any CN to be selected at level 2 results in $2(\gamma - 2)$ extra CNs² that cannot be selected at level 3. As a result, $u^2 = u^1 - (1 + 2(\gamma - 2))$, which is equation (18). By means of induction, we conclude that the same analysis applies for every u^{j-1} , $1 \leq j \leq b_{\text{et}} = \gamma$, yielding:

$$u^{j-1} = (\gamma - (j - 1))^2. \quad (19)$$

Substituting (19) into (16) gives:

$$t = \frac{1}{\gamma!} \gamma^2 (\gamma - 1)^2 (\gamma - 2)^2 \cdots 1^2 = \gamma!. \quad (20)$$

As a result, $t = \gamma!$, which completes the proof. ■

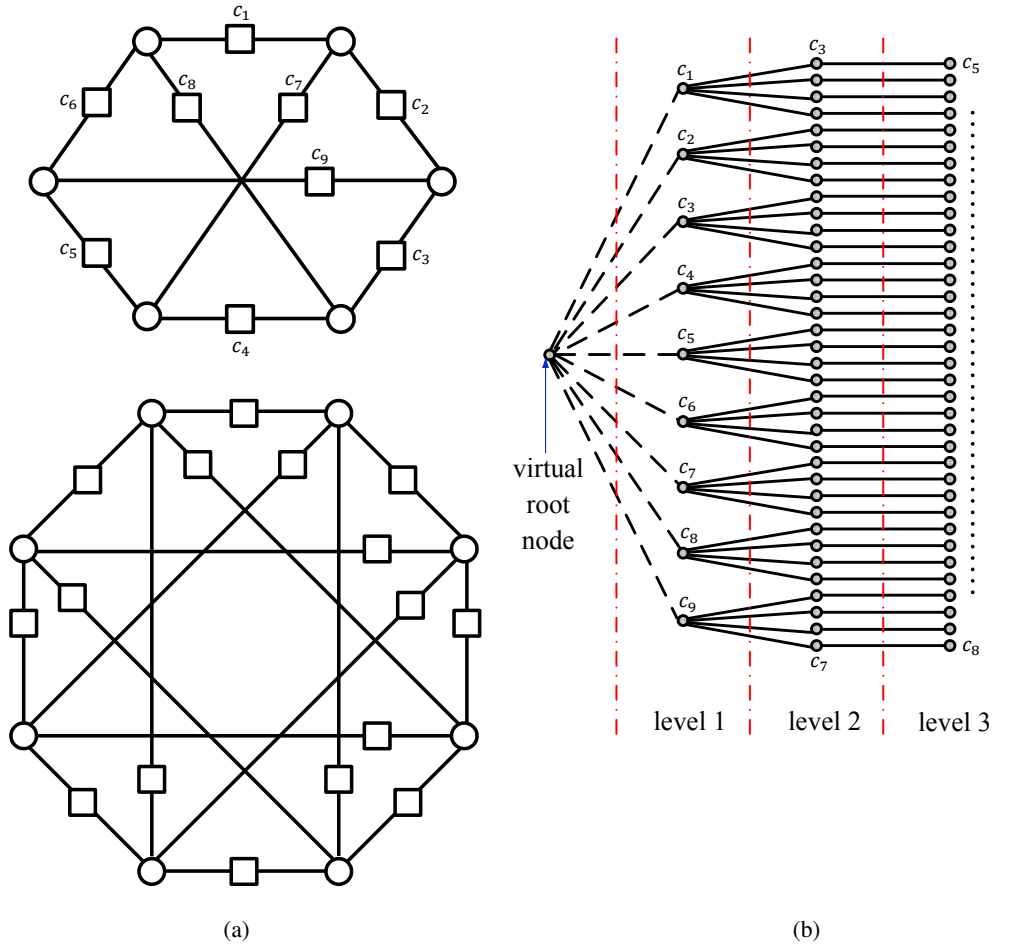


Fig. 7. (a) Upper panel: the u-symmetric $(6, 0, 9, 0)$ unlabeled GAST ($\gamma = 3$). Lower panel: the u-symmetric $(8, 0, 16, 0)$ unlabeled GAST ($\gamma = 4$). (b) The associated unlabeled GAST tree for the $(6, 0, 9, 0)$ unlabeled GAST ($b_{\text{et}} = 3$).

²The reason why it is not $2(\gamma - 1)$ is that two CNs from the group that cannot be selected at level 3 were already accounted for while computing u^1 as they could not be selected at level 2 (recall that the configuration is u-symmetric).

Example 5. Fig. 7(a), upper panel, shows the u -symmetric $(6, 0, 9, 0)$ unlabeled GAST ($\gamma = 3$). Fig. 7(b) confirms that the configuration is u -symmetric with $b_{\text{et}} = b_{\text{ut}} = 3$, $u^0 = 9$, $u^1 = 4$, and $u^2 = 1$. Thus, (13) reduces to (20), implying:

$$t = \frac{1}{3!} \prod_{j=1}^3 u^{j-1} = 3! = 6.$$

The configuration has 6 WCMs (size 6×6), extracted by removing the rows of the following groups of CNs from \mathbf{A} : $\{(c_1, c_3, c_5), (c_1, c_4, c_9), (c_2, c_4, c_6), (c_2, c_5, c_8), (c_3, c_6, c_7), (c_7, c_8, c_9)\}$.

Fig. 7(a), lower panel, shows the u -symmetric $(8, 0, 16, 0)$ unlabeled GAST ($\gamma = 4$). Following the same logic we used for the u -symmetric $(6, 0, 9, 0)$ unlabeled GAST ($\gamma = 3$), we conclude that this configuration has $b_{\text{et}} = 4$, $u^0 = 16$, $u^1 = 9$, $u^2 = 4$, and $u^3 = 1$. Thus, from (20):

$$t = \frac{1}{4!} \prod_{j=1}^4 u^{j-1} = 4! = 24.$$

We conclude Subsection III-B with Table I. Table I lists the number of distinct WCMs for different types of unlabeled GASTs.

TABLE I
NUMBER OF DISTINCT WCMs FOR DIFFERENT TYPES OF UNLABELED GASTs.

Unlabeled GAST type	Number of distinct WCMs (t)
General	$t = \sum_{b_k=b_{\text{st}}}^{b_{\text{et}}} \frac{1}{b_k!} \sum_{i_1=1}^{u^0} \sum_{i_2=1}^{u_{i_1}^1} \sum_{i_3=1}^{u_{i_1, i_2}^2} \cdots \sum_{i_{b_k}=1}^{u_{i_1, i_2, \dots, i_{b_k-1}}^{b_k-1}} T(u_{i_1, i_2, \dots, i_{b_k}}^{b_k})$
Same-size-WCMs	$t = \frac{1}{b_{\text{et}}!} \sum_{i_1=1}^{u^0} \sum_{i_2=1}^{u_{i_1}^1} \sum_{i_3=1}^{u_{i_1, i_2}^2} \cdots \sum_{i_{b_{\text{et}}}=1}^{u_{i_1, i_2, \dots, i_{b_{\text{et}}-1}}^{b_{\text{et}}-1}} (1)$
U-symmetric	$t = \frac{1}{b_{\text{et}}!} \prod_{j=1}^{b_{\text{et}}} u^{j-1}$
U-symmetric, $(2\gamma, 0, \gamma^2, 0)$, with $\gamma \in \{3, 4\}$	$t = \gamma!$

C. Complexity Comparison with an Alternative Idea

We have already proved the optimality of the WCM framework in Subsection III-A. In this subsection, we demonstrate the complexity reduction we gain by focusing only on the set of WCMs, \mathcal{W} , to remove a GAST. We compute the total number of distinct matrices to operate on in an alternative idea (a suboptimal idea), and compare it with the number of distinct WCMs we operate on, which is t derived in Subsection III-B. The suboptimal idea we compare with is operating on the set of all distinct matrices \mathbf{W}^z .

Here, we seek to compare the number of distinct WCMs, which is the size of the set \mathcal{W} , with the number of distinct \mathbf{W}^z matrices, which is the size of the set \mathcal{U} . For convenience, we assume for this comparison that $b_{\text{et}} > 0$ and $u^0 > 0$.

Theorem 3. Given the unlabeled GAST tree, the difference between the cardinalities of the sets \mathcal{U} and \mathcal{W} (the reduction in the number of matrices to operate on) for an (a, d_1, d_2, d_3) unlabeled GAST, with the parameters $b_{\text{st}} > 0$ and $b_{\text{et}} > 0$, is:

$$t' - t = 1 + \sum_{j=1}^{b_{\text{et}}-1} \frac{1}{j!} \sum_{i_1=1}^{u^0} \sum_{i_2=1}^{u_{i_1}^1} \sum_{i_3=1}^{u_{i_1, i_2}^2} \cdots \sum_{i_j=1}^{u_{i_1, i_2, \dots, i_{j-1}}^{j-1}} T_c(u_{i_1, i_2, \dots, i_j}^j), \quad (21)$$

where $t' = |\mathcal{U}|$ and $t = |\mathcal{W}|$. $T_c \left(u_{i_1, i_2, \dots, i_j}^j \right) = 1$ if $u_{i_1, i_2, \dots, i_j}^j \neq 0$, and $T_c \left(u_{i_1, i_2, \dots, i_j}^j \right) = 0$ o/w.

Proof: Given that t (which is $|\mathcal{W}|$) is known from Subsection III-B, we need to derive t' (which is $|\mathcal{U}|$). Since \mathcal{U} is the set of all distinct matrices \mathbf{W}^z , it follows that the cardinality of \mathcal{U} is a function of the total number of nodes in the unlabeled GAST tree. Note that each node at level j in the unlabeled GAST tree is linked to a matrix \mathbf{W}^z (see the proof of Theorem 1). The total number of these tree nodes is given by:

$$\eta_{\text{rep}} = \sum_{j=1}^{b_{\text{et}}} \sum_{i_1=1}^{u^0} \sum_{i_2=1}^{u_{i_1}^1} \sum_{i_3=1}^{u_{i_1, i_2}^2} \cdots \sum_{i_j=1}^{u_{i_1, i_2, \dots, i_{j-1}}^{j-1}} \quad (1). \quad (22)$$

To remove the repeated \mathbf{W}^z matrices from that count, we need to divide by $j!$ the number of tree nodes at each level j . Thus, we reach:

$$\eta = \sum_{j=1}^{b_{\text{et}}} \frac{1}{j!} \sum_{i_1=1}^{u^0} \sum_{i_2=1}^{u_{i_1}^1} \sum_{i_3=1}^{u_{i_1, i_2}^2} \cdots \sum_{i_j=1}^{u_{i_1, i_2, \dots, i_{j-1}}^{j-1}} \quad (1). \quad (23)$$

The cardinality of the set $|\mathcal{U}|$, which is t' , is then:

$$t' = 1 + \eta = 1 + \sum_{j=1}^{b_{\text{et}}} \frac{1}{j!} \sum_{i_1=1}^{u^0} \sum_{i_2=1}^{u_{i_1}^1} \sum_{i_3=1}^{u_{i_1, i_2}^2} \cdots \sum_{i_j=1}^{u_{i_1, i_2, \dots, i_{j-1}}^{j-1}} \quad (1), \quad (24)$$

where the additional 1 is for the particular matrix \mathbf{W}^z extracted by removing d_1 rows from \mathbf{A} corresponding to all degree-1 CNs in the configuration. Note that we can consider the virtual root node as the node linked to this particular \mathbf{W}^z matrix in the tree.

To compute $t' - t$, we subtract (7) from (24). Consequently,

$$\begin{aligned} t' - t &= 1 + \sum_{j=1}^{b_{\text{st}}-1} \frac{1}{j!} \sum_{i_1=1}^{u^0} \sum_{i_2=1}^{u_{i_1}^1} \sum_{i_3=1}^{u_{i_1, i_2}^2} \cdots \sum_{i_j=1}^{u_{i_1, i_2, \dots, i_{j-1}}^{j-1}} \quad (1) \\ &+ \sum_{j=b_{\text{st}}}^{b_{\text{et}}} \frac{1}{j!} \sum_{i_1=1}^{u^0} \sum_{i_2=1}^{u_{i_1}^1} \sum_{i_3=1}^{u_{i_1, i_2}^2} \cdots \sum_{i_j=1}^{u_{i_1, i_2, \dots, i_{j-1}}^{j-1}} \left[1 - T \left(u_{i_1, i_2, \dots, i_j}^j \right) \right]. \end{aligned} \quad (25)$$

Thus, we complete the proof as follows:

$$t' - t = 1 + \sum_{j=1}^{b_{\text{et}}} \frac{1}{j!} \sum_{i_1=1}^{u^0} \sum_{i_2=1}^{u_{i_1}^1} \sum_{i_3=1}^{u_{i_1, i_2}^2} \cdots \sum_{i_j=1}^{u_{i_1, i_2, \dots, i_{j-1}}^{j-1}} \left[1 - T \left(u_{i_1, i_2, \dots, i_j}^j \right) \right] \quad (26.1)$$

$$= 1 + \sum_{j=1}^{b_{\text{et}}-1} \frac{1}{j!} \sum_{i_1=1}^{u^0} \sum_{i_2=1}^{u_{i_1}^1} \sum_{i_3=1}^{u_{i_1, i_2}^2} \cdots \sum_{i_j=1}^{u_{i_1, i_2, \dots, i_{j-1}}^{j-1}} T_c \left(u_{i_1, i_2, \dots, i_j}^j \right). \quad (26.2)$$

Equality (26.1) follows by observing that $\left[1 - T\left(u_{i_1, i_2, \dots, i_j}^j\right)\right] = 1$ for $j \in \{1, 2, \dots, b_{\text{st}} - 1\}$. Equality (26.2) follows from that $T\left(u_{i_1, i_2, \dots, i_j}^j\right) = 1$ for $j = b_{\text{et}}$, and $\left[1 - T\left(u_{i_1, i_2, \dots, i_j}^j\right)\right] = T_{\text{c}}\left(u_{i_1, i_2, \dots, i_j}^j\right)$ (from the definitions of both T and T_{c}). We can simply consider $T_{\text{c}}\left(u_{i_1, i_2, \dots, i_j}^j\right)$ as the complement function (binary inversion) of $T\left(u_{i_1, i_2, \dots, i_j}^j\right)$. ■

Example 6. Consider the $(6, 2, 5, 2)$ unlabeled GAST ($\gamma = 3$) shown in Fig. 4(a). Since $b_{\text{st}} = 1$, $b_{\text{et}} = b_{\text{ut}} = 2$, and $u^0 = 3$, and aided by the unlabeled GAST tree in Fig. 4(b), the complexity reduction (the reduction in the number of matrices to operate on) is (see (21)):

$$t' - t = 1 + \frac{1}{1!} \sum_{i_1=1}^3 T_{\text{c}}\left(u_{i_1}^1\right) = 1 + (1 + 0 + 1) = 3.$$

In other words, the cardinality of the set \mathcal{U} is $t' = 5$, while from Example 2, the cardinality of the set \mathcal{W} (the number of distinct WCMs) is $t = 2$. Thus, the complexity reduction is 60%.

Now, we study the case of same-size-WCMs unlabeled GASTs.

Lemma 3. Given the unlabeled GAST tree, the difference between the cardinalities of the sets \mathcal{U} and \mathcal{W} (the reduction in the number of matrices to operate on) for a same-size-WCMs (a, d_1, d_2, d_3) unlabeled GAST, with the parameter $b_{\text{et}} > 0$, is:

$$t' - t = 1 + \sum_{j=1}^{b_{\text{et}}-1} \frac{1}{j!} \sum_{i_1=1}^{u^0} \sum_{i_2=1}^{u_{i_1}^1} \sum_{i_3=1}^{u_{i_1, i_2}^2} \cdots \sum_{i_j=1}^{u_{i_1, i_2, \dots, i_{j-1}}^{j-1}} (1), \quad (27)$$

where $t' = |\mathcal{U}|$ and $t = |\mathcal{W}|$.

Proof: Knowing that the configuration is a same-size-WCMs unlabeled GAST does not simplify the expression of t' in (24). Thus, to compute $(t' - t)$, we subtract (11) from (24). The result of this subtraction is (27). ■

Example 7. Consider the $(7, 9, 13, 0)$ unlabeled GAST ($\gamma = 5$) shown in Fig. 5(a). Since $b_{\text{et}} = b_{\text{ut}} = 2$ and $u^0 = 5$, and aided by the unlabeled GAST tree in Fig. 5(b), the complexity reduction (the reduction in the number of matrices to operate on) is (see (27)):

$$t' - t = 1 + \frac{1}{1!} \sum_{i_1=1}^5 (1) = 1 + 5 = 6.$$

In other words, the cardinality of the set \mathcal{U} is $t' = 11$, while from Example 3, the cardinality of the set \mathcal{W} (the number of distinct WCMs) is $t = 5$. Thus, the complexity reduction is over 50%.

Corollary 2. Given the unlabeled GAST tree, the difference between the cardinalities of the sets \mathcal{U} and \mathcal{W} (the reduction in the number of matrices to operate on) for a u -symmetric (a, d_1, d_2, d_3) unlabeled GAST, with the parameter $b_{\text{et}} > 0$, is:

$$t' - t = 1 + \sum_{j=1}^{b_{\text{et}}-1} \frac{1}{j!} \prod_{i=1}^j u^{i-1}, \quad (28)$$

where $t' = |\mathcal{U}|$ and $t = |\mathcal{W}|$.

Proof: We recall again that the u-symmetric configuration is a special case of the same-size-WCMs configuration. Consequently, we can use the same idea from the proof of Corollary 1 in (27) to reach (28). ■

Example 8. Consider the u-symmetric $(2\gamma, 0, \gamma^2, 0)$ unlabeled GAST. From (19), we know that $w^{j-1} = (\gamma - (j - 1))^2$. Thus, from Corollary 2, the complexity reduction (the reduction in the number of matrices to operate on) is:

$$t' - t = 1 + \sum_{j=1}^{b_{\text{et}}-1} \frac{1}{j!} \prod_{i=1}^j u^{i-1} = 1 + \sum_{j=1}^{\gamma-1} \frac{1}{j!} \prod_{i=1}^j (\gamma - (i - 1))^2. \quad (29)$$

For $\gamma = 3$ (corresponding to the u-symmetric $(6, 0, 9, 0)$ unlabeled GAST), the complexity reduction is $1 + \frac{1}{1!}(9) + \frac{1}{2!}(9)(4) = 28$, which is over 80% (i.e., $t' = 34$ while $t = 6$). For $\gamma = 4$ (corresponding to the u-symmetric $(8, 0, 16, 0)$ unlabeled GAST), the complexity reduction is $1 + \frac{1}{1!}(16) + \frac{1}{2!}(16)(9) + \frac{1}{3!}(16)(9)(4) = 185$, which is about 90% (i.e., $t' = 209$ while $t = 24$).

We conclude Subsection III-C with Table II. Table II lists the reduction, $t' - t$, in the number of matrices to operate on for different types of unlabeled GASTs.

TABLE II
REDUCTION IN THE NUMBER OF MATRICES TO OPERATE ON FOR DIFFERENT TYPES OF UNLABELED GASTS.

Unlabeled GAST type	Reduction in the number of matrices ($t' - t$)
General	$t' - t = 1 + \sum_{j=1}^{b_{\text{et}}-1} \frac{1}{j!} \sum_{i_1=1}^{u^0} \sum_{i_2=1}^{u_{i_1}^1} \sum_{i_3=1}^{u_{i_1, i_2}^2} \cdots \sum_{i_j=1}^{u_{i_1, i_2, \dots, i_{j-1}}^{j-1}} T_c \left(w_{i_1, i_2, \dots, i_j}^j \right)$
Same-size-WCMs	$t' - t = 1 + \sum_{j=1}^{b_{\text{et}}-1} \frac{1}{j!} \sum_{i_1=1}^{u^0} \sum_{i_2=1}^{u_{i_1}^1} \sum_{i_3=1}^{u_{i_1, i_2}^2} \cdots \sum_{i_j=1}^{u_{i_1, i_2, \dots, i_{j-1}}^{j-1}} (1)$
U-symmetric	$t' - t = 1 + \sum_{j=1}^{b_{\text{et}}-1} \frac{1}{j!} \prod_{i=1}^j u^{i-1}$
U-symmetric, $(2\gamma, 0, \gamma^2, 0)$, with $\gamma \in \{3, 4\}$	$t' - t = 1 + \sum_{j=1}^{\gamma-1} \frac{1}{j!} \prod_{i=1}^j (\gamma - (i - 1))^2$

Remark 2. The analysis in Section III is focusing on the case where $b_{\text{et}} > 0$ (thus, $u^0 > 0$) because if $b_{\text{et}} = 0$ (i.e., $u^0 = 0$), $t = 1$ always. In other words, there exists only one matrix \mathbf{W}^z . As a result, there exists only one WCM of size $(\ell - d_1) \times a$, which is the single matrix \mathbf{W}^z itself. Note that if $b_{\text{et}} > 0$, the matrix \mathbf{W}^z of size $(\ell - d_1) \times a$ cannot be a WCM (this is the reason why we do not add 1 in (7) as we do in (24)).

Remark 3. An analysis similar to what we presented in Section III can be done for BASTs.

IV. MORE ON HOW GASTS ARE REMOVED

After demonstrating the complexity reduction achieved by operating only on the set of WCMs to remove a GAST, in this section, we provide more details on the removal of GASTs via their WCMs. We first investigate the dimension of the null space of a WCM. Then, we discuss the best that can be done to break the weight conditions of a short WCM. Finally, we discuss the exact minimum number of edge weight changes needed by the WCM framework to remove a

GAST from the Tanner graph of an NB-LDPC code, and we provide a useful topological upper bound on that minimum. A further discussion about the null spaces of WCMs that belong to GASTs having $b = d_1$ is provided in Appendix C.

A. The Dimension of the Null Space of a WCM

A GAST is removed via breaking the weight conditions of its WCMs. This breaking is performed by forcing the null spaces of these WCMs to have a particular property. Thus, studying the dimension of the null space of a WCM is critical to understand how GASTs are removed.

Consider a WCM \mathbf{W}_h^{cm} , $1 \leq h \leq t$, of a GAST. Recall that $\mathcal{N}(\mathbf{M})$ is the null space of a matrix \mathbf{M} , and let $\dim(\mathcal{N}(\mathbf{M}))$ denote the dimension of the null space of a matrix \mathbf{M} . Moreover, let G_h^{cm} be the subgraph created by removing b_h^{cm} degree ≤ 2 CNs from the GAST subgraph. The b_h^{cm} rows that are removed from \mathbf{A} to reach \mathbf{W}_h^{cm} correspond to these b_h^{cm} CNs. Note that these CNs are the ones on the path from the root node until the tree node linked to \mathbf{W}_h^{cm} in the unlabeled GAST tree (the CNs marked as unsatisfied by [5, Algorithm 1] to extract \mathbf{W}_h^{cm}). Moreover, let $\mathbf{M}(G)$ denote the adjacency matrix of a graph G .

Theorem 4. *The dimension p_h of the null space of a WCM \mathbf{W}_h^{cm} , $1 \leq h \leq t$, of an (a, b, d_1, d_2, d_3) GAST, given that this WCM has unbroken weight conditions, is given by:*

$$p_h = \dim(\mathcal{N}(\mathbf{W}_h^{\text{cm}})) = \sum_{k=1}^{\delta_h} \dim(\mathcal{N}(\mathbf{M}(G_{h,k}^{\text{disc}}))) \geq \delta_h, \quad (30)$$

where δ_h is the number of disconnected components in G_h^{cm} , and $G_{h,k}^{\text{disc}}$ is the k th disconnected component in G_h^{cm} , where $1 \leq k \leq \delta_h$.

Proof: It is known from graph theory that if graph G_h^{cm} has δ_h disconnected components defined as $G_{h,k}^{\text{disc}}$, $1 \leq k \leq \delta_h$, then:

$$p_h = \dim(\mathcal{N}(\mathbf{W}_h^{\text{cm}})) = \dim(\mathcal{N}(\mathbf{M}(G_h^{\text{cm}}))) = \sum_{k=1}^{\delta_h} \dim(\mathcal{N}(\mathbf{M}(G_{h,k}^{\text{disc}}))). \quad (31)$$

Note that by definition of G_h^{cm} , $\mathbf{W}_h^{\text{cm}} = \mathbf{M}(G_h^{\text{cm}})$.

Then, we prove the inequality $p_h \geq \delta_h$. If $\exists G_{h,k}^{\text{disc}}$ s.t. $\dim(\mathcal{N}(\mathbf{M}(G_{h,k}^{\text{disc}}))) = 0$ (which means $\mathcal{N}(\mathbf{W}_h^{\text{cm}}) = \{\mathbf{0}\}$), then it is impossible to have a vector $\mathbf{v} = [v_1 \ v_2 \ \dots \ v_a]^T \in \mathcal{N}(\mathbf{M}(G_h^{\text{cm}})) = \mathcal{N}(\mathbf{W}_h^{\text{cm}})$ s.t. $v_f \neq 0, \forall f \in \{1, 2, \dots, a\}$, where a is the size of the GAST. Thus, in order to have a WCM that has unbroken weight conditions, we must have $\dim(\mathcal{N}(\mathbf{M}(G_{h,k}^{\text{disc}}))) > 0, \forall k \in \{1, 2, \dots, \delta_h\}$. Noting that if $\dim(\mathcal{N}(\mathbf{M}(G_{h,k}^{\text{disc}}))) > 0, \forall k \in \{1, 2, \dots, \delta_h\}$, then $p_h = \sum_{k=1}^{\delta_h} \dim(\mathcal{N}(\mathbf{M}(G_{h,k}^{\text{disc}}))) \geq \delta_h$ completes the proof of Theorem 4. ■

Remark 4. *Consider a WCM \mathbf{W}_h^{cm} that has unbroken weight conditions. In the majority of the GASTs we have studied, if G_h^{cm} (the graph corresponding to \mathbf{W}_h^{cm}) has $\delta_h = 1$ (the graph is fully connected), $\dim(\mathcal{N}(\mathbf{W}_h^{\text{cm}})) = 1$. Similarly, we have typically observed that if $\delta_h > 1$, then $\forall G_{h,k}^{\text{disc}}, 1 \leq k \leq \delta_h, \dim(\mathcal{N}(\mathbf{M}(G_{h,k}^{\text{disc}}))) = 1$. In other words, in most of the cases we have seen, $p_h = \dim(\mathcal{N}(\mathbf{W}_h^{\text{cm}})) = \delta_h$. Having said that, we have already encountered few examples where $p_h = \dim(\mathcal{N}(\mathbf{W}_h^{\text{cm}})) > \delta_h$ (see the next subsection).*

Typically, if $\delta_h = 1$, breaking the weight conditions of a WCM \mathbf{W}_h^{cm} yields $\dim(\mathcal{N}(\mathbf{W}_h^{\text{cm}})) = 0$ (there are few exceptions to that). Contrarily, it is important to note that if $\delta_h > 1$ (which

again means the graph corresponding to \mathbf{W}_h^{cm} has more than one disconnected components), the weight conditions of this WCM can be broken while $\dim(\mathcal{N}(\mathbf{W}_h^{\text{cm}})) > 0$. This situation occurs if $\exists G_{h,k}^{\text{disc}}$ s.t. $\dim(\mathcal{N}(\mathbf{M}(G_{h,k}^{\text{disc}}))) = 0$, and that is not the case for all k . Thus, breaking the weight conditions of such a WCM by making $\dim(\mathcal{N}(\mathbf{W}_h^{\text{cm}})) = 0$ (if possible) is, albeit sufficient, not necessary. A conceptually-similar observation will be presented in the next subsection. We present Example 9 to illustrate Theorem 4 as well as this discussion.

Example 9. Consider the $(6, 0, 0, 9, 0)$ GAST ($\gamma = 3$) over $GF(4)$, where $GF(4) = \{0, 1, \alpha, \alpha^2\}$ and α is a primitive element. This GAST is shown in Fig. 8(a). The matrix \mathbf{A} for this GAST is:

$$\mathbf{A} = \begin{matrix} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ \begin{matrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \\ c_9 \end{matrix} & \begin{bmatrix} w_{1,1} & \alpha & 0 & 0 & 0 & 0 \\ 0 & \alpha^2 & \alpha^2 & 0 & 0 & 0 \\ 0 & 0 & 1 & \alpha^2 & 0 & 0 \\ 0 & 0 & 0 & \alpha^2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ w_{6,1} & 0 & 0 & 0 & 0 & \alpha \\ 0 & \alpha & 0 & 1 & 0 & 0 \\ \alpha & 0 & 0 & 0 & \alpha^2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{matrix}.$$

For the original configuration, we assume that $w_{1,1} = w_{6,1} = 1$. The unlabeled GAST tree of this configuration reveals that it is neither u -symmetric nor same-size-WCMs. The configuration has 10 WCMs (of different sizes), extracted by removing the rows of the following groups of CNs from \mathbf{A} : $\{(c_1, c_3, c_5), (c_1, c_4, c_9), (c_2, c_4, c_6), (c_2, c_5), (c_2, c_8), (c_3, c_6), (c_3, c_8), (c_5, c_7), (c_6, c_7), (c_7, c_8, c_9)\}$. We index these groups of CNs (and consequently, the resulting WCMs) by h , $1 \leq h \leq t = 10$. The WCM of interest in this example is \mathbf{W}_2^{cm} , which is extracted by removing the rows of (c_1, c_4, c_9) from \mathbf{A} . The graph corresponding to \mathbf{W}_2^{cm} , which is G_2^{cm} , is shown in Fig. 8(b). Note that this graph has $\delta_2 = 2$ disconnected components. For the given edge weight assignment, \mathbf{W}_2^{cm} (as well as all the remaining 9 WCMs) has unbroken weight conditions. Thus, according to Theorem 4, $\dim(\mathcal{N}(\mathbf{W}_2^{\text{cm}})) = \sum_{k=1}^2 \dim(\mathcal{N}(\mathbf{M}(G_{2,k}^{\text{disc}}))) \geq 2$ must be satisfied. Solving for the null space of \mathbf{W}_2^{cm} yields:

$$\mathcal{N}(\mathbf{W}_2^{\text{cm}}) = \text{span}\{[\alpha \ 0 \ 0 \ 0 \ 1 \ 1]^T, [0 \ 1 \ 1 \ \alpha \ 0 \ 0]^T\}, \quad (32)$$

which means that $\dim(\mathcal{N}(\mathbf{W}_2^{\text{cm}})) = 2$ because $\dim(\mathcal{N}(\mathbf{M}(G_{2,k}^{\text{disc}}))) = 1, \forall k \in \{1, 2\}$. Note that $\mathcal{N}(\mathbf{M}(G_{2,1}^{\text{disc}})) = \text{span}\{[\alpha \ 1 \ 1]^T\}$, where $G_{2,1}^{\text{disc}}$ is the subgraph grouping $\{v_1, v_5, v_6\}$ in Fig. 8(b), while $\mathcal{N}(\mathbf{M}(G_{2,2}^{\text{disc}})) = \text{span}\{[1 \ 1 \ \alpha]^T\}$, where $G_{2,2}^{\text{disc}}$ is the subgraph grouping $\{v_2, v_3, v_4\}$ in Fig. 8(b). Observe that the existence of the vector:

$$\mathbf{v} = [\alpha \ 0 \ 0 \ 0 \ 1 \ 1]^T + [0 \ 1 \ 1 \ \alpha \ 0 \ 0]^T = [\alpha \ 1 \ 1 \ \alpha \ 1 \ 1]^T \in \mathcal{N}(\mathbf{W}_2^{\text{cm}}) \quad (33)$$

verifies that the weight conditions of \mathbf{W}_2^{cm} are unbroken.

Now, assume that in the process of removing the GAST, we break the weight conditions of \mathbf{W}_2^{cm} via the following set of a single edge weight change: $\{w_{6,1} : 1 \rightarrow \alpha^2\}$. This change results in breaking the weight conditions of \mathbf{W}_2^{cm} , i.e., $\nexists \mathbf{v} = [v_1 \ v_2 \ \dots \ v_6] \in \mathcal{N}(\mathbf{W}_2^{\text{cm}})$ s.t. $v_f \neq 0, \forall f \in \{1, 2, \dots, 6\}$. However, $\mathcal{N}(\mathbf{W}_2^{\text{cm}}) = \text{span}\{[0 \ 1 \ 1 \ \alpha \ 0 \ 0]^T\} \neq \{\mathbf{0}\}$, i.e., $\dim(\mathcal{N}(\mathbf{W}_2^{\text{cm}})) = 1$ (it was originally 2). This is an example of how the weight conditions of a WCM that has a corresponding graph with $\delta_h > 1$ can be broken while $\dim(\mathcal{N}(\mathbf{W}_h^{\text{cm}})) > 0$ (for $h = 2$ here).

Obviously, it is possible to make another edge weight change for an edge in $G_{2,2}^{\text{disc}}$ to make $\mathcal{N}(\mathbf{W}_2^{\text{cm}}) = \{\mathbf{0}\}$. However, this is by no means necessary for the GAST removal process.

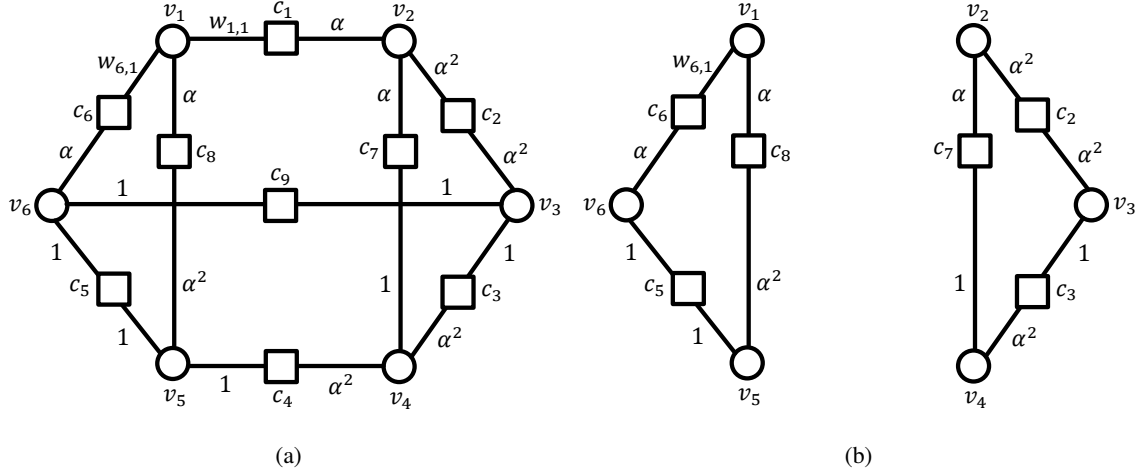


Fig. 8. (a) A $(6, 0, 0, 9, 0)$ GAST ($\gamma = 3$). (b) The graph created by removing (c_1, c_4, c_9) from the GAST graph. $\text{GF}(4)$ is assumed.

B. Breaking the Weight Conditions of Short WCMs

In this subsection, we discuss the best that can be done to break the weight conditions of a short WCM. The following lemma states this result.

Lemma 4. *The null space of a short WCM \mathbf{W}_h^{cm} (a WCM that has fewer rows than columns) after a successful removal process for the (a, b, d_1, d_2, d_3) GAST to which this WCM belongs satisfies the following two conditions³:*

- 1) *Its dimension is strictly more than 0, i.e., $p_h = \dim(\mathcal{N}(\mathbf{W}_h^{\text{cm}})) > 0$.*
- 2) *For any $\mathbf{v} = [v_1 \ v_2 \ \dots \ v_a]^T \in \mathcal{N}(\mathbf{W}_h^{\text{cm}})$, where a is the size of the GAST, the number of non-zero elements in \mathbf{v} is strictly less than a , i.e., $\|\mathbf{v}\|_0 < a$.*

Proof: Since the number of rows is less than the number of columns in this WCM, the WCM cannot have a full column rank. Thus, $\mathcal{N}(\mathbf{W}_h^{\text{cm}}) \neq \{\mathbf{0}\}$, which means $\dim(\mathcal{N}(\mathbf{W}_h^{\text{cm}})) > 0$, and proves the first condition in the lemma. Moreover, if a GAST is removed successfully, this implies that each WCM in the set \mathcal{W} associated with that GAST has broken weight conditions. Thus, the second condition in the lemma is automatically satisfied for the short WCM. ■

Lemma 4 further emphasizes on the fact that the weight conditions of a WCM \mathbf{W}_h^{cm} can be broken while $\dim(\mathcal{N}(\mathbf{W}_h^{\text{cm}})) > 0$ (i.e., $\mathcal{N}(\mathbf{W}_h^{\text{cm}}) \neq \{\mathbf{0}\}$). One way this case can happen is if $\delta_h > 1$ (which is discussed in the previous subsection). Another way is if the WCM is short, even with $\delta_h = 1$. For many short WCMs, the reason why this case occurs is that before breaking the weight conditions of a short WCM, it typically has $p_h = \dim(\mathcal{N}(\mathbf{W}_h^{\text{cm}})) > \delta_h$. Here, we are more interested in short WCMs with $\delta_h = 1$. The difference between the two ways is that if $\delta_h > 1$ and \nexists any disconnected component having a short adjacency matrix, we can still break the weight conditions of the WCM by making $\dim(\mathcal{N}(\mathbf{W}_h^{\text{cm}})) = 0$. However, such processing

³Note that Lemma 4 also applies to any WCM \mathbf{W}_h^{cm} that has G_h^{cm} with $\delta_h > 1$ and at least one of the disconnected components having a short adjacency matrix (a very rare case).

is not necessary, and it would require more edge weight changes than the minimum needed. Contrarily, if the WCM is short, it is impossible to break the weight conditions by making $\dim(\mathcal{N}(\mathbf{W}_h^{\text{cm}})) = 0$, and the best we can do is what is described in Lemma 4. The following example demonstrates Lemma 4.

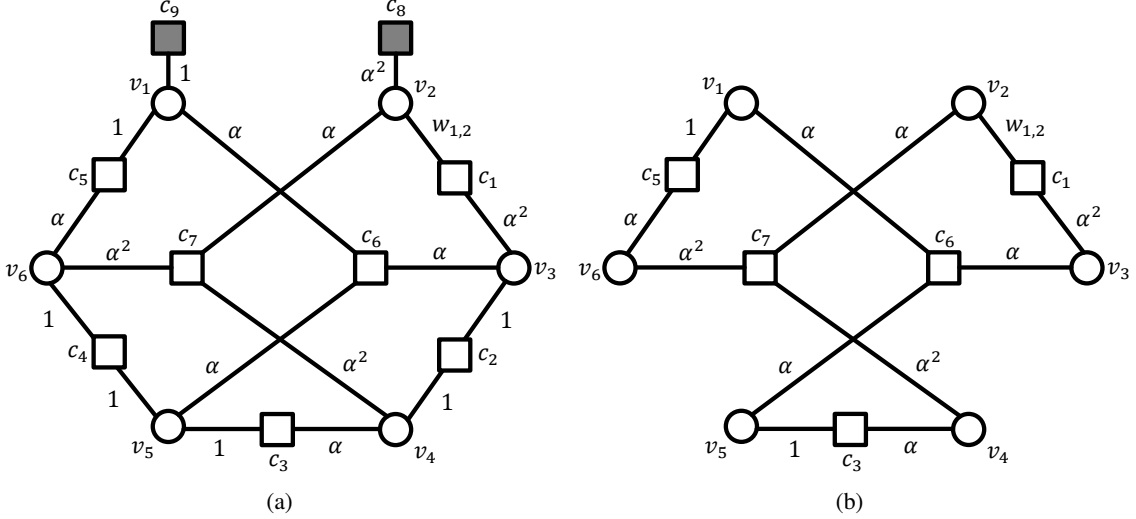


Fig. 9. (a) A $(6, 2, 2, 5, 2)$ GAST ($\gamma = 3$). (b) The subgraph created by removing (c_2, c_4, c_8, c_9) from the GAST subgraph. $\text{GF}(4)$ is assumed.

Example 10. Consider the $(6, 2, 2, 5, 2)$ GAST ($\gamma = 3$) over $\text{GF}(4)$, where $\text{GF}(4) = \{0, 1, \alpha, \alpha^2\}$, that is shown in Fig. 9(a). The matrix \mathbf{A} for this configuration is:

$$\mathbf{A} = \begin{matrix} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ \begin{matrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \\ c_9 \end{matrix} & \begin{bmatrix} 0 & w_{1,2} & \alpha^2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & \alpha \\ \alpha & 0 & \alpha & 0 & \alpha & 0 \\ 0 & \alpha & 0 & \alpha^2 & 0 & \alpha^2 \\ 0 & \alpha^2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix}.$$

For the original configuration, we assume that $w_{1,2} = \alpha^2$. From Example 2, this configuration has 2 WCMs, extracted by removing the rows of the following groups of CNs from \mathbf{A} : $\{(c_3, \mathcal{O}), (c_2, c_4, \mathcal{O})\}$, where $\mathcal{O} = \{c_8, c_9\}$. We index these groups of CNs (and consequently, the resulting WCMs) by h , $1 \leq h \leq t = 2$. The WCM of interest in this example is \mathbf{W}_2^{cm} , which is extracted by removing the rows of (c_2, c_4, c_8, c_9) from \mathbf{A} . The graph corresponding to \mathbf{W}_2^{cm} , which is G_2^{cm} , is shown in Fig. 9(b). Note that \mathbf{W}_2^{cm} is of size 5×6 (a short matrix). For the given edge weight assignment, \mathbf{W}_2^{cm} (as well as \mathbf{W}_1^{cm}) has unbroken weight conditions. Solving for the null space of \mathbf{W}_2^{cm} yields the following:

$$\mathcal{N}(\mathbf{W}_2^{\text{cm}}) = \text{span}\{[0 \ 1 \ 1 \ \alpha^2 \ 1 \ 0]^T, [1 \ 1 \ 1 \ 0 \ 0 \ \alpha^2]^T\}, \quad (34)$$

This is one of the cases where we have $\dim(\mathcal{N}(\mathbf{W}_h^{\text{cm}})) > 1$ (for $h = 2$) with $\delta_h = 1$ (the corresponding graph to \mathbf{W}_2^{cm} is fully connected). Observe also that the existence of the vector:

$$\mathbf{v} = [0 \ 1 \ 1 \ \alpha^2 \ 1 \ 0]^T + \alpha[1 \ 1 \ 1 \ 0 \ 0 \ \alpha^2]^T = [\alpha \ \alpha^2 \ \alpha^2 \ \alpha^2 \ 1 \ 1]^T \in \mathcal{N}(\mathbf{W}_2^{\text{cm}}) \quad (35)$$

verifies that the weight conditions of \mathbf{W}_2^{cm} are unbroken.

Now, assume that in the process of removing the GAST, we break the weight conditions of \mathbf{W}_2^{cm} via the following set of a single edge weight change: $\{w_{1,2} : \alpha^2 \rightarrow \alpha\}$. This change results in breaking the weight conditions of \mathbf{W}_2^{cm} , i.e., $\nexists \mathbf{v} = [v_1 \ v_2 \ \dots \ v_6] \in \mathcal{N}(\mathbf{W}_2^{\text{cm}})$ s.t. $v_f \neq 0$, $\forall f \in \{1, 2, \dots, 6\}$. However, $\mathcal{N}(\mathbf{W}_2^{\text{cm}}) = \text{span}\{[1 \ 0 \ 0 \ \alpha^2 \ 1 \ \alpha^2]^T\} \neq \{\mathbf{0}\}$, i.e., $\dim(\mathcal{N}(\mathbf{W}_2^{\text{cm}})) = 1$ (it was originally 2). This example illustrates that the weight conditions of the short WCM can only be broken with $\dim(\mathcal{N}(\mathbf{W}_2^{\text{cm}})) > 0$ regardless of the edge weight change(s) we perform.

C. The Number of Edge Weight Changes Needed

In this subsection, we discuss the minimum number of edge weight changes needed, in addition to how to select these edge weight changes in order to have a successful removal of the problematic object. Recall that we need to break the weight conditions of all the WCMs of a GAST in order to remove the GAST.

Lemma 5. *The minimum number of edge weight changes (with respect to the original configuration) needed to remove an $(a, b, b_2, d_1, d_2, d_3)$ GAS (convert it into a non-AS) is given by:*

$$E_{\text{GAS},\min} = g - b_{\text{vn},\max} + 1, \quad (36)$$

where $g = \lfloor \frac{\gamma-1}{2} \rfloor$, and $b_{\text{vn},\max}$ is the maximum number of existing unsatisfied CNs per VN in the GAS. A topological upper bound on that minimum is given by:

$$E_{\text{GAS},\min} \leq g - d_{1,\text{vn},\max} + 1, \quad (37)$$

where $d_{1,\text{vn},\max}$ is the maximum number of existing degree-1 CNs per VN in the GAS.

Proof: The set of GASs is simply the set of absorbing sets (ASs). Thus, the proof of (36) is exactly the same as the proof of $E_{\text{AS},\min}$ in [10, Lemma 2].

Now, we prove the upper bound in (37). Recall that degree-1 CNs are always unsatisfied. Thus, irrespective of whether the VN that has the maximum number of existing unsatisfied CNs is the same as the VN that has the maximum number of existing degree-1 CNs or not, the following inequality is always satisfied:

$$b_{\text{vn},\max} \geq d_{1,\text{vn},\max}. \quad (38)$$

Substituting (38) in (36) gives (37), and completes the proof. \blacksquare

While theoretically a GAST can be removed by forcing a single degree > 2 CN to be unsatisfied via a single edge weight change, this is not the strategy we follow. The reason is that the described process can result in another GAS with a degree > 2 unsatisfied CN ($b > d_1 + b_2$). Despite that GASs with $b > d_1 + b_2$ are generally less harmful than GASTs, it is not preferred to remove GASTs by converting some of them into other types of GASs. Thus, we remove a GAST by performing $E_{\text{GAST},\min} = E_{\text{GAS},\min}$ edge weight changes for edges connected to only degree-2 CNs (all the weights of edges connected to degree > 2 CNs remain untouched). In other words, (36) and (37) are applicable to both GASTs and GASs. Furthermore, $E_{\text{GAST},\min}$ we use in [5, Algorithm 2] is given by (36) and bounded by (37).

The bound in (37) is purely topological (determined from the unlabeled GAST), i.e., it does not require any knowledge of b of the GAST being processed (nor $b_{\text{vn,max}}$, consequently). The importance of this topological bound will be illustrated shortly.

A useful definition and a corollary, which simplify the process of selecting $E_{\text{GAST,min}}$ edge weights to change, are proposed below:

Definition 10. A *borderline VN* in an (a, b, d_1, d_2, d_3) GAST in a code with column weight γ is a VN that is connected to exactly $g = \lfloor \frac{\gamma-1}{2} \rfloor$ degree-1 CNs.

Corollary 3. An (a, b, d_1, d_2, d_3) GAST that has at least one borderline VN has $E_{\text{GAST,min}} = 1$, and the upper bound on $E_{\text{GAST,min}}$ is also 1.

Proof: A borderline VN already has the maximum number of unsatisfied CNs a VN can have in a GAST (or in an AS in general), which is $g = \lfloor \frac{\gamma-1}{2} \rfloor$. Consequently, a GAST with at least one borderline VN has:

$$b_{\text{vn,max}} = d_{1,\text{vn,max}} = g = \left\lfloor \frac{\gamma-1}{2} \right\rfloor. \quad (39)$$

Substituting (39) in (36) gives $E_{\text{GAST,min}} = 1$. Noting that $b_{\text{vn,max}} = d_{1,\text{vn,max}}$ proves that the upper bound is also 1 (see (37)). ■

Remark 5. Lemma 5, Corollary 3, and the discussion below give the minimum number of edge weight changes in addition to the specification of which edge weights need to be changed in order to remove a GAST. However, they do not determine what these particular changes should be, i.e., they do not specify the new values of the edge weights to be changed. Specifying the new values of the edge weights is performed by the WCM framework via checking the null spaces of all WCMs of the GAST being processed, and making sure that all the WCMs have broken weight conditions after the edge weight changes (see also [5, Theorem 3], [5, Algorithm 2], and Example 11). This justification is the reason why the word “**properly**” is used to describe the edge weight changes in this subsection.

It can be concluded from Corollary 3 that any degree-2 unsatisfied CN connected to a borderline VN results in an object that is not a GAST. Thus, assuming that the object being processed is detected to be a GAST via the WCM framework (at least one of its WCMs has unbroken weight conditions), we select the edge weights to be changed based on the following two cases⁴:

- 1) If the GAST has at least one borderline VN ($E_{\text{GAST,min}} = 1$), then we **properly** change the weight of an edge connected to a degree-2 CN connected to any of the borderline VNs. If every VN in the GAST is borderline, then we change the weight of an edge connected to any degree-2 CN.
- 2) If the GAST does not have any borderline VNs ($E_{\text{GAST,min}} \geq 1$), then we determine the VN(s) that has (have) the maximum number, $d_{1,\text{vn,max}}$, of degree-1 CNs connected to it (them). Then we **properly** change the weights of a maximum of $(g - d_{1,\text{vn,max}} + 1)$ edges

⁴Note that these two items are for a stand-alone GAST. It happens in few cases that we need more than the minimum number of edge weight changes to remove a GAST because of previously removed GASTs that share edges with the GAST being processed (or other reasons).

connected to different degree-2 CNs connected to a particular VN of those having $d_{1,\text{vn,max}}$ neighboring degree-1 CNs.

To relate the above analysis to the WCMs, recall that every CN in the GAST has a corresponding row in the matrix \mathbf{A} of this GAST. The GAST is removed by breaking the weight conditions of all its WCMs. To achieve this, we operate on a set of rows in \mathbf{A} , that has the minimum cardinality and corresponds to degree-2 CNs, with the property that every WCM has at least one row in that set. Any $(g - d_{1,f} + 1)$ rows in \mathbf{A} satisfy the stated property if they correspond to degree-2 CNs connected to the same VN, v_f , $f \in \{1, 2, \dots, a\}$, where $d_{1,f}$ is the number of degree-1 CNs connected to VN v_f . The reason is that we cannot together remove $(g - d_{1,f} + 1)$ rows of degree-2 CNs connected to VN v_f from \mathbf{A} to extract a WCM because then the resulting matrix will not be a valid \mathbf{W}^z . Thus, a set of $(g - d_{1,\text{vn,max}} + 1)$ rows of degree-2 CNs connected to the same VN that achieves $d_{1,\text{vn,max}}$ is indeed a set of minimum cardinality with the property that every WCM has at least one row in that set. Consequently, the topological upper bound in (37) provides the cardinality of that set of rows satisfying the stated property. Properly operating on a maximum of $(g - d_{1,\text{vn,max}} + 1)$ weights (only one weight per row) in these rows is what is needed to remove the GAST. Examples 11 and 12 illustrate the process performed by the WCM framework to remove a stand-alone GAST.

Remark 6. *Typically, we only need to perform $(g - b_{\text{vn,max}} + 1) \leq (g - d_{1,\text{vn,max}} + 1)$ edge weight changes to remove the GAST. When $b_{\text{vn,max}} \neq d_{1,\text{vn,max}}$, the number of WCMs with unbroken weight conditions becomes strictly less than t , and only $(g - b_{\text{vn,max}} + 1)$ rows are enough to establish a set of minimum cardinality with the property that every WCM with unbroken weight conditions has at least one row in that set.*

Example 11. *We again discuss the $(6, 0, 0, 9, 0)$ GAST ($\gamma = 3$) in Fig. 8(a), with $w_{1,1} = w_{6,1} = 1$. The null spaces of the 10 WCMs of that GAST are given in (51), Example 14 (see Example 9 for how the WCMs are extracted). Since $b_{\text{vn,max}} = d_{1,\text{vn,max}} = 0$, (36) gives $E_{\text{GAST,min}} = \lfloor \frac{3-1}{2} \rfloor - 0 + 1 = 2$ (same as the upper bound). Given that all VNs have 0 degree-1 neighboring CNs, either VN can be selected. Suppose that v_1 is selected. Each WCM of the 10, has at least one of the two rows corresponding to c_1 and c_6 (both are connected to v_1) in \mathbf{A} . Thus, we consider the following two sets of edge weight changes for $w_{1,1}$ and $w_{6,1}$ (cardinality 2). The first set is $\{w_{1,1} : 1 \rightarrow \alpha, w_{6,1} : 1 \rightarrow \alpha\}$. Using this set of edge weight changes, the null spaces of the 10 WCMs become:*

$$\begin{aligned} \mathcal{N}(\mathbf{W}_1^{\text{cm}}) &= \mathcal{N}(\mathbf{W}_3^{\text{cm}}) = \mathcal{N}(\mathbf{W}_4^{\text{cm}}) = \mathcal{N}(\mathbf{W}_6^{\text{cm}}) = \mathcal{N}(\mathbf{W}_8^{\text{cm}}) = \mathcal{N}(\mathbf{W}_9^{\text{cm}}) = \{\mathbf{0}\}, \\ \mathcal{N}(\mathbf{W}_2^{\text{cm}}) &= \text{span}\{[0 \ 1 \ 1 \ \alpha \ 0 \ 0]^T\}, \text{ and} \\ \mathcal{N}(\mathbf{W}_5^{\text{cm}}) &= \mathcal{N}(\mathbf{W}_7^{\text{cm}}) = \mathcal{N}(\mathbf{W}_{10}^{\text{cm}}) = \text{span}\{[1 \ 1 \ 1 \ \alpha \ 1 \ 1]^T\}. \end{aligned} \quad (40)$$

Clearly, the GAST is not removed as there are 3 WCMs with unbroken weight conditions: \mathbf{W}_5^{cm} , \mathbf{W}_7^{cm} , and $\mathbf{W}_{10}^{\text{cm}}$. The second set is $\{w_{1,1} : 1 \rightarrow \alpha, w_{6,1} : 1 \rightarrow \alpha^2\}$. Using this set of edge weight changes, the null spaces of the 10 WCMs become:

$$\begin{aligned} \mathcal{N}(\mathbf{W}_1^{\text{cm}}) &= \mathcal{N}(\mathbf{W}_3^{\text{cm}}) = \mathcal{N}(\mathbf{W}_4^{\text{cm}}) = \mathcal{N}(\mathbf{W}_5^{\text{cm}}) = \mathcal{N}(\mathbf{W}_6^{\text{cm}}) \\ &= \mathcal{N}(\mathbf{W}_7^{\text{cm}}) = \mathcal{N}(\mathbf{W}_8^{\text{cm}}) = \mathcal{N}(\mathbf{W}_9^{\text{cm}}) = \mathcal{N}(\mathbf{W}_{10}^{\text{cm}}) = \{\mathbf{0}\} \text{ and} \\ \mathcal{N}(\mathbf{W}_2^{\text{cm}}) &= \text{span}\{[0 \ 1 \ 1 \ \alpha \ 0 \ 0]^T\}, \end{aligned} \quad (41)$$

which means that the GAST is successfully removed as the 10 WCMs have broken weight conditions. As a result, it can be concluded that properly identifying which edge weights to change is not enough. Checking the null spaces of all WCMs is what determines which set of edge weight changes is sufficient for a successful GAST removal.

Now, consider the case of $w_{1,1} = \alpha$ and $w_{6,1} = 1$ for the original configuration (i.e., before any removal attempt). The configuration in this case is a $(6, 1, 0, 9, 0)$ GAST with $b_{\text{vn,max}} = 1$ and $d_{1,\text{vn,max}} = 0$. Thus, (36) gives $E_{\text{GAST,min}} = 1$, while the upper bound is 2 from (37). The null spaces of the 10 WCMs are:

$$\begin{aligned} \mathcal{N}(\mathbf{W}_1^{\text{cm}}) &= \text{span}\{[\alpha \ 1 \ 1 \ \alpha \ 1 \ 1]^T\}, \quad \mathcal{N}(\mathbf{W}_2^{\text{cm}}) = \text{span}\{[\alpha \ 0 \ 0 \ 0 \ 1 \ 1]^T, [0 \ 1 \ 1 \ \alpha \ 0 \ 0]^T\}, \quad \text{and} \\ \mathcal{N}(\mathbf{W}_3^{\text{cm}}) &= \mathcal{N}(\mathbf{W}_4^{\text{cm}}) = \mathcal{N}(\mathbf{W}_5^{\text{cm}}) = \mathcal{N}(\mathbf{W}_6^{\text{cm}}) = \mathcal{N}(\mathbf{W}_7^{\text{cm}}) = \mathcal{N}(\mathbf{W}_8^{\text{cm}}) \\ &= \mathcal{N}(\mathbf{W}_9^{\text{cm}}) = \mathcal{N}(\mathbf{W}_{10}^{\text{cm}}) = \{\mathbf{0}\}. \end{aligned} \quad (42)$$

Only 2 WCMs, \mathbf{W}_1^{cm} and \mathbf{W}_2^{cm} , have unbroken weight conditions. Both of them share the row corresponding to c_6 . Consequently, only one edge weight change is needed to break the weight conditions of the 2 WCMs and remove the object ($E_{\text{GAST,min}}$ is achieved). A set of a single change, e.g., $\{w_{6,1} : 1 \rightarrow \alpha^2\}$, is sufficient to perform the removal (see also Remark 6).

Example 12. We discuss the $(6, 2, 2, 5, 2)$ GAST ($\gamma = 3$) in Fig. 9(a), with $w_{1,2} = \alpha^2$. The null spaces of the 2 WCMs of that GAST are given in (52), Example 14 (see Example 10 for how the WCMs are extracted). Since $b_{\text{vn,max}} = d_{1,\text{vn,max}} = 1$, (36) gives $E_{\text{GAST,min}} = 1$ (same as the upper bound). Either v_1 or v_2 can be selected as both are borderline VNs. Suppose that v_2 is selected. Each WCM of the 2, has the row of c_1 (see Example 10). Applying the set of a single edge weight change, $\{w_{1,2} : \alpha^2 \rightarrow \alpha\}$, yields the following null spaces:

$$\mathcal{N}(\mathbf{W}_1^{\text{cm}}) = \{\mathbf{0}\} \quad \text{and} \quad \mathcal{N}(\mathbf{W}_2^{\text{cm}}) = \text{span}\{[1 \ 0 \ 0 \ \alpha^2 \ 1 \ \alpha^2]^T\}, \quad (43)$$

which means that the GAST is successfully removed.

V. REMOVING OSCILLATING SETS TO ACHIEVE MORE GAIN

Now that we have presented the in-depth analysis of the baseline WCM framework, we now introduce an extension to the framework. In particular, in this section, we discuss a new set of detrimental objects, namely the oscillating sets of type two (OSTs), that are the second-order cause of the error floor of NB-LDPC codes with even column weights over asymmetric channels. We show how to remove OSTs using the WCM framework. In the simulation results section, we will show that performing another optimization phase that addresses OSTs, after the GASTs removal phase, secures up to nearly 2.5 orders of magnitude overall performance gain in the error floor region over practical (asymmetric) Flash channels.

A. Defining OSs and OSTs

Before we introduce an oscillating set (OS), we define an oscillating VN.

Definition 11. Consider a subgraph induced by a subset \mathcal{V} of VNs in the Tanner graph of an NB-LDPC code. Set all the VNs in \mathcal{V} to values $\in GF(q) \setminus \{0\}$ and set all other VNs to 0. A VN in \mathcal{V} is said to be an **oscillating VN** if the number of its neighboring satisfied CNs equals

the number of its neighboring unsatisfied CNs (for some set of given VN values). The set of all oscillating VNs in \mathcal{V} is referred to as \mathcal{S} .

It is clear that for codes with fixed column weights (fixed VN degrees), there can exist an oscillating VN only under the condition that the column weight γ is even. Based on Definition 11, we define the oscillating set.

Definition 12. Consider a subgraph induced by a subset \mathcal{V} of VNs in the Tanner graph of an NB-LDPC code. Set all the VNs in \mathcal{V} to values $\in GF(q) \setminus \{0\}$ and set all other VNs to 0. The set \mathcal{V} is said to be an $(a, b, b_2, d_1, d_2, d_3)$ **oscillating set (OS)** over $GF(q)$ if and only if the size of \mathcal{V} is a , the number of unsatisfied (resp., degree-2 unsatisfied) CNs connected to \mathcal{V} is b (resp., b_2), the number of degree-1 (resp., 2 and > 2) CNs connected to \mathcal{V} is d_1 (resp., d_2 and d_3), the set of oscillating VNs $\mathcal{S} \subseteq \mathcal{V}$ is not empty, and each VN (if any) in $\mathcal{V} \setminus \mathcal{S}$ is connected to strictly more satisfied than unsatisfied neighboring CNs (for some set of given VN values).

The unlabeled OS is defined similar to the unlabeled GAS except for that the word ‘‘more’’ is replaced by the word ‘‘at least’’ in Definition 2. Moreover, [5, Lemma 1] can be changed to suit OSs by referring to the unlabeled OS instead of the unlabeled GAS in the topological conditions, and by using the following equation instead of (2) in the weight conditions:

$$\left(\sum_{e=1}^{\ell-b} F(\psi_{e,f}) \right) \geq \left(\sum_{k=1}^b F(\theta_{k,f}) \right). \quad (44)$$

Note that the equality in (44) must hold for at least one VN in \mathcal{V} . We also define an oscillating set of type two (OST) as follows.

Definition 13. An OS that has $d_2 > d_3$ and all the unsatisfied CNs connected to it $\in \{\mathcal{O} \cup \mathcal{T}\}$ (having either degree 1 or degree 2), is defined as an (a, b, d_1, d_2, d_3) **oscillating set of type two (OST)**. Similar to the unlabeled OS definition, we also define the (a, d_1, d_2, d_3) **unlabeled OST**.

If hard decision, majority rule decoding, e.g., Gallager B decoding [1], is assumed, an oscillating VN in error receives the exact same number of ‘‘stay’’ and ‘‘flip’’ messages. This observation makes it harder for the decoder to correct it compared with a VN with more neighboring unsatisfied than satisfied CNs. Over aggressively asymmetric channels, oscillating VNs in error are even less likely to be corrected in many cases under soft decision decoding because of the high error magnitudes. Consequently, OSTs typically contribute to between 5% and 10% of the errors of NB-LDPC codes with even γ in the error floor region over practical (asymmetric) Flash channels, making OSTs the second-order cause, after GASTs, of the error floor in such channels. As we shall see in Section VI, removing OSTs from the Tanner graphs of NB-LDPC codes guarantees about 0.5 of an order of magnitude or more additional performance gain.

Fig. 10(a) shows an $(8, 4, 3, 13, 1)$ OST that has $\mathcal{S} = \{v_1\}$. Fig. 10(b) shows a $(6, 6, 2, 11, 0)$ OST that has $\mathcal{S} = \{v_2, v_3, v_4, v_5\}$. Some OSTs have underlying GASTs as subgraphs, while others do not. For example, if the VN v_1 is eliminated from the $(8, 4, 3, 13, 1)$ OST in Fig. 10(a), the underlying object is a $(7, 4, 3, 11, 1)$ GAST (the two CNs shaded in red will be degree-1 unsatisfied CNs as a result of the elimination of v_1). Contrarily, the $(6, 6, 2, 11, 0)$ OST in Fig. 10(b) does not have an underlying GAST.

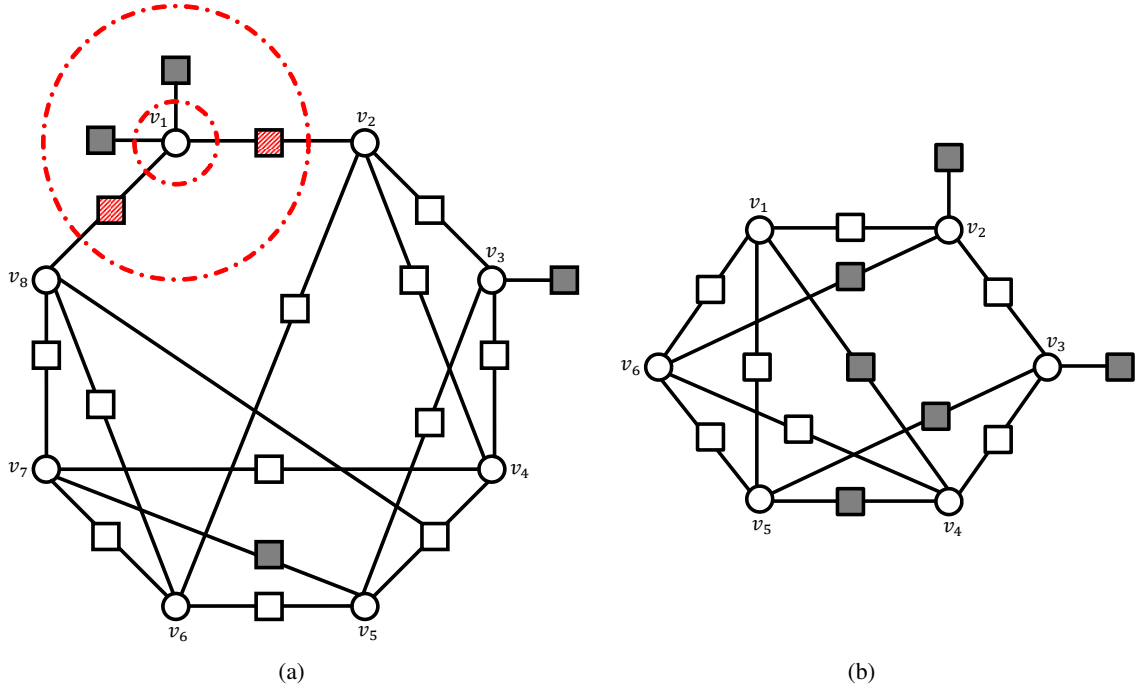


Fig. 10. (a) An $(8, 4, 3, 13, 1)$ OST ($\gamma = 4$). (b) A $(6, 6, 2, 11, 0)$ OST ($\gamma = 4$). Appropriate non-binary edge weights are assumed. Unlabeled OSTs are reached by setting all the weights in the configurations to 1.

B. How to Remove OSTs Using WCMs

Before we propose the lemma that discusses the removal of OSTs, we need to state several auxiliary results.

Lemma 6. Consider an (a, d_1, d_2, d_3) unlabeled OST with its sets \mathcal{T} and \mathcal{H} . A CN $c \in \mathcal{T}$ can be unsatisfied in the resulting OST (with proper edge labeling) resulting in $b > d_1$ if and only if the two neighboring VNs of c (with respect to this unlabeled OST) each has the property that strictly more than $\frac{\gamma}{2}$ of its neighboring CNs belong to $\{\mathcal{T} \cup \mathcal{H}\}$.

Proof: The proof follows the same logic as the proof of [5, Theorem 1]. ■

Lemma 7. Given an (a, d_1, d_2, d_3) unlabeled OST, the maximum number of unsatisfied CNs, b_{o_max} , in the resulting OST after edge labeling is upper bounded by:

$$b_{o_max} \leq d_1 + b_{o_ut}, \text{ where} \quad (45)$$

$$b_{o_ut} = \left\lfloor \frac{1}{2} \left(a \left(\frac{\gamma}{2} \right) - d_1 \right) \right\rfloor. \quad (46)$$

Proof: The proof follows the same logic as the proof of [5, Theorem 2]. The main equation in the proof is:

$$b_{o_ut} = \sum_{f=1}^a \left[\frac{\gamma}{2} - b_{o_up,f} \right] = a \left(\frac{\gamma}{2} \right) - (d_1 + b_{o_ut}), \quad (47)$$

where b_{o_ut} is the upper bound on the maximum number of degree-2 unsatisfied CNs the resulting OST can have after labeling, and $b_{o_up,f}$ is the number of the already-unsatisfied CNs connected to VN v_f , $f \in \{1, 2, \dots, a\}$, updated by what has been done for all the VNs processed prior to VN v_f . ■

The following example illustrates Lemmas 6 and 7.

Example 13. Both configurations in Fig. 10 can have degree-2 unsatisfied CNs in the resulting OSTs. For the $(8, 3, 13, 1)$ unlabeled OST, $b_{o_ut} = 6$ (1 of these 6 CNs is unsatisfied in the $(8, 4, 3, 13, 1)$ OST in Fig. 10(a)), while for the $(6, 2, 11, 0)$ unlabeled OST, $b_{o_ut} = 5$ (4 of these 5 CNs are unsatisfied in the $(6, 6, 2, 11, 0)$ OST in Fig. 10(b)). The upper bound of b_{o_max} is achieved for both.

For a given (a, b, d_1, d_2, d_3) OST, let \mathcal{Z}_o be the set of all (a, b'_o, d_1, d_2, d_3) GASTs/OSTs with $d_1 \leq b'_o \leq b_{o_max}$, which have the same unlabeled GAST/OST as the original OST. Here, b_{o_max} is the largest allowable number of unsatisfied CNs for these configurations.

Definition 14. An (a, b, d_1, d_2, d_3) OST is said to be **removed** from the Tanner graph of an NB-LDPC code if and only if the resulting object (after edge weight processing) $\notin \mathcal{Z}_o$.

We thus augment the code optimization process for asymmetric channels to consist of two phases. The first phase, as before, focuses on the removal of GASTs, and the second phase focuses on the removal of OSTs. The ordering of the phases is critical because of the following. While it is allowed to remove a GAST by converting it to an OST during the first phase because GASTs, not OSTs, are the root cause of the error floor, it is not allowed to remove an OST by converting it into a GAST during the second phase because GASTs are generally more harmful compared with OSTs. This is the reason why the set \mathcal{Z}_o is the set of all (a, b'_o, d_1, d_2, d_3) GASTs/OSTs with $d_1 \leq b'_o \leq b_{o_max}$. For example, to remove the $(6, 6, 2, 11, 0)$ OST in Fig. 10(b), the configuration needs to be converted into an object $\notin \{(6, 2, 2, 11, 0)$ GAST, $(6, 3, 2, 11, 0)$ GAST/OST, $(6, 4, 2, 11, 0)$ GAST/OST, $(6, 5, 2, 11, 0)$ OST, $(6, 6, 2, 11, 0)$ OST, $(6, 7, 2, 11, 0)$ OST $\}$ ($d_1 = 2$ and $b_{o_max} = d_1 + b_{o_ut} = 7$).

For a given OST, define a matrix \mathbf{W}_o^z to be the matrix obtained by removing b'_o , $d_1 \leq b'_o \leq b_{o_max}$, rows corresponding to CNs $\in \{\mathcal{O} \cup \mathcal{T}\}$ from the matrix \mathbf{A} , which is the OST adjacency matrix. These b'_o CNs can simultaneously be unsatisfied under some edge labeling that produces a GAST/an OST which has the same unlabeled GAST/OST as the given OST. Let \mathcal{U}_o be the set of all matrices \mathbf{W}_o^z . Each element $\in \mathcal{Z}_o$ has one or more matrices $\in \mathcal{U}_o$.

Definition 15. For a given (a, b, d_1, d_2, d_3) OST and its associated adjacency matrix \mathbf{A} and its associated set \mathcal{Z}_o , we construct a set of t_o matrices as follows:

- 1) Each matrix $\mathbf{W}_h^{o_cm}$, $1 \leq h \leq t_o$, in this set is an $(\ell - b_h^{o_cm}) \times a$ submatrix, $d_1 \leq b_h^{o_cm} \leq b_{o_max}$, formed by removing **different** $b_h^{o_cm}$ rows from the $\ell \times a$ matrix \mathbf{A} of the OST. These $b_h^{o_cm}$ rows to be removed correspond to CNs $\in \{\mathcal{O} \cup \mathcal{T}\}$ that can simultaneously be unsatisfied under some edge labeling that produces a GAST/an OST which has the same unlabeled GAST/OST as the given OST.
- 2) Each matrix $\mathbf{W}_o^z \in \mathcal{U}_o$, for every element $\in \mathcal{Z}_o$, contains at least one element of the resultant set as its submatrix.
- 3) This resultant set has the **smallest cardinality**, which is t_o , among all the sets which satisfy conditions 1 and 2 stated above.

We refer to the matrices in this set as **oscillating weight consistency matrices (OWCMs)**, and to this set itself as \mathcal{W}_o .

Similar to b_{et} in GASTs, we also define $b_{o_et} \leq b_{o_ut}$ for OSTs such that $b_{o_max} = d_1 + b_{o_et}$.

The following lemma addresses the removal of OSTs via their OWCMs. In other words, the lemma shows how the WCM framework can be customized to remove OSTs.

Lemma 8. *The necessary and sufficient processing needed to remove an (a, b, d_1, d_2, d_3) OST, according to Definition 14, is to change the edge weights such that for every OWCM $\mathbf{W}_h^{\text{o-cm}} \in \mathcal{W}_o$, there does not exist any vector with all its entries $\neq 0$ in the null space of that OWCM. Mathematically, $\forall h$:*

$$\text{If } \mathcal{N}(\mathbf{W}_h^{\text{o-cm}}) = \text{span}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{p_h}\}, \text{ then } \nexists \mathbf{r} = [r_1 \ r_2 \ \dots \ r_{p_h}]^T$$

$$\text{for } \mathbf{v} = r_1 \mathbf{x}_1 + r_2 \mathbf{x}_2 + \dots + r_{p_h} \mathbf{x}_{p_h} = [v_1 \ v_2 \ \dots \ v_a]^T \text{ s.t. } v_j \neq 0, \forall j \in \{1, 2, \dots, a\}, \quad (48)$$

where p_h is the dimension of $\mathcal{N}(\mathbf{W}_h^{\text{o-cm}})$. Computations are performed over $GF(q)$.

Proof: The proof follows the same logic as the proof of [5, Theorem 3]. ■

A similar analysis to the one in Section III can be performed to compute the number of OWCMs in the set \mathcal{W}_o , with few changes (e.g., b_{o_et} should be used instead of b_{et}). Now, we propose the minimum number of edge weight changes needed to remove an OST from the Tanner graph of an NB-LDPC code with even column weight.

Corollary 4. *The minimum number of edge weight changes (with respect to the original configuration) needed to remove an $(a, b, b_2, d_1, d_2, d_3)$ OS (convert it into a non-OS/non-AS) is given by:*

$$E_{\text{OS}, \min} = \frac{\gamma}{2} - b_{\text{vn}, \max} + 1 = 1 \leq \frac{\gamma}{2} - d_{1, \text{vn}, \max} + 1, \quad (49)$$

where $d_{1, \text{vn}, \max}$ is the maximum number of existing degree-1 CNs per VN in the OS.

Proof: The proof follows the same logic as the proof of Lemma 5 (see also [10]). Note that by definition of an OS, at least one of its VNs has exactly $\frac{\gamma}{2}$ neighboring unsatisfied CNs. Thus,

$$E_{\text{OS}, \min} = \frac{\gamma}{2} - b_{\text{vn}, \max} + 1 = \frac{\gamma}{2} - \frac{\gamma}{2} = 1, \quad (50)$$

where $b_{\text{vn}, \max}$ is the maximum number of existing unsatisfied CNs per VN in the OS, which equals $\frac{\gamma}{2}$ for any OS. ■

Similar to GASTs, since we only change the weights of edges connected to degree-2 CNs to remove OSTs, (49) also holds for $E_{\text{OST}, \min}$. Moreover, a **topologically-oscillating VN** in an OST in a code with column weight γ is connected to exactly $\frac{\gamma}{2}$ degree-1 CNs. An OST with such a VN has the upper bound on $E_{\text{OST}, \min}$ equal to 1.

The following simple algorithm illustrates the procedure we follow to optimize NB-LDPC codes with even column weights for usage over asymmetric channels.

Algorithm 1 Optimizing NB-LDPC Codes with Even Column Weights

- 1: Apply [5, Algorithm 2] to optimize the NB-LDPC code by removing the detrimental GASTs.
 - 2: Using initial simulations and combinatorial techniques (e.g., [42]) for the output code of step 1, determine the set of OSTs to be removed.
 - 3: Apply a customized version of [5, Algorithm 2] to further optimize the NB-LDPC code generated in step 1 by removing the detrimental OSTs.
-

A crucial check to make while removing a certain OST is that the edge weight changes to be performed do not undo the removal of any of the already removed GASTs nor OSTs.

VI. APPLICATIONS OF THE WCM FRAMEWORK

In this section, we apply the WCM framework to optimize NB-LDPC codes with different structures and for various applications, demonstrating significant performance gains in the error floor region. We used a finite-precision, fast Fourier transform based q -ary sum-product algorithm (FFT-QSPA) LDPC decoder [43], which performs a maximum of 50 iterations (except for PR channel simulations), and it stops if a codeword is reached sooner.

All the unoptimized NB-LDPC codes we are using in Subsections VI-A, VI-B, and VI-C are regular non-binary protograph-based LDPC (NB-PB-LDPC) codes. These codes are constructed as follows. First, a binary protograph matrix \mathbf{H}_p is designed. Then, \mathbf{H}_p is lifted via a lifting parameter ζ to create the binary image of \mathbf{H} , which is \mathbf{H}_b . The lifting process means that every 1 in \mathbf{H}_p is replaced by a $\zeta \times \zeta$ circulant matrix, while every 0 (if any) in \mathbf{H}_p is replaced by a $\zeta \times \zeta$ all-zero matrix. The circulant powers are adjusted such that the unlabeled Tanner graph of the resulting code does not have cycles of length 4. Then, the 1's in \mathbf{H}_b are replaced by non-zero values $\in \text{GF}(q)$ to generate \mathbf{H} . These unoptimized codes are high performance NB-PB-LDPC codes (see also [44] and [45], in addition to [5]). Note that the WCM framework works for any regular, or even irregular with fixed column weight, NB-LDPC codes. Moreover, the WCM framework also works for any GF size q and for any code rate.

In this paper, RBER is the raw bit error rate, which is the number of raw (uncoded) data bits in error divided by the total number of raw (uncoded) data bits read [46]. UBER is the uncorrectable bit error rate, which is a metric for the fraction of bits in error out of all bits read after the error correction is applied via encoding/decoding [46]. One formulation of UBER, as recommended by industry, is the frame error rate (FER) divided by the sector size in bits.

A. Optimizing Column Weight 5 Codes

In this subsection, we use the WCM framework to optimize NB-LDPC codes with column weight 5 for the first time. Column weight 5 codes generally guarantee better performance compared with column weight 3 and 4 codes in the error floor region. We show in this subsection, that more than 1 order of magnitude performance gain is still achievable via the WCM framework for such codes despite their improved error floor performance. The channel used in this subsection is a practical Flash channel: the normal-Laplace mixture (NLM) Flash channel [34]. Here, we use 3 reads, and the sector size is 512 bytes.

In the NLM channel, the threshold voltage distribution of sub-20nm multi-level cell (MLC) Flash memories is carefully modeled. The four levels are modeled as different NLM distributions, incorporating several sources of error due to wear-out effects, e.g., programming errors, thereby resulting in significant asymmetry [34]. Furthermore, the authors provided accurate fitting results of their model for program/erase (P/E) cycles up to 10 times the manufacturer's endurance specification. We implemented the NLM channel based on the parameters described in [34].

In this subsection, Code 1 is an NB-PB-LDPC code defined over $\text{GF}(4)$, with block length = 6724 bits, rate ≈ 0.88 , and $\gamma = 5$. Code 2 is the result of optimizing Code 1 for the asymmetric NLM channel by attempting to remove the GASTs in Table III using the WCM framework.

Fig. 11 shows that more than 1 order of magnitude performance gain is achieved via optimizing Code 1 to arrive at Code 2 using the WCM framework. The figure also shows that by deploying the WCM framework, an UBER value of approximately 4.53×10^{-15} is achievable at RBER of approximately 4.69×10^{-3} on the NLM Flash channel (an aggressively asymmetric channel) with only 3 reads.

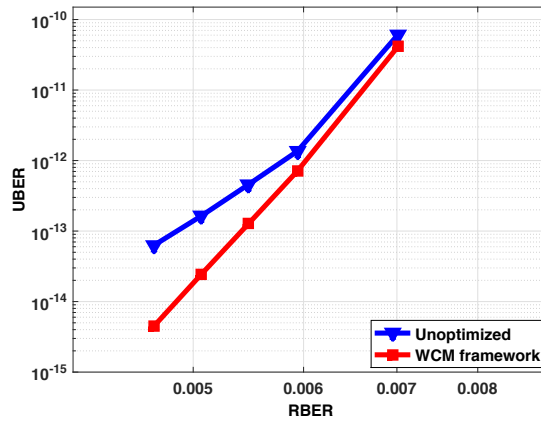


Fig. 11. Simulation results over the NLM channel with 3 reads for Code 1 (unoptimized) and Code 2 (WCM framework). The two codes have $\gamma = 5$.

TABLE III
ERROR PROFILE OF CODES 1 AND 2 OVER THE NLM CHANNEL WITH 3 READS, $RBER \approx 4.69 \times 10^{-3}$, $UBER$ (UNOPTIMIZED) $\approx 6.31 \times 10^{-14}$, AND $UBER$ (WCM FRAMEWORK) $\approx 4.53 \times 10^{-15}$ (SEE FIG. 11).

Error type	Count	
	Code 1	Code 2
(4, 8, 8, 6, 0)	18	0
(6, 8, 8, 11, 0)	9	0
(6, 10, 8, 11, 0)	11	0
(7, 5, 5, 15, 0)	4	0
(7, 9, 9, 13, 0)	4	0
(7, 10, 10, 9, 2)	7	1
(8, 6, 6, 17, 0)	23	0
(8, 8, 6, 17, 0)	15	0
Other	9	7

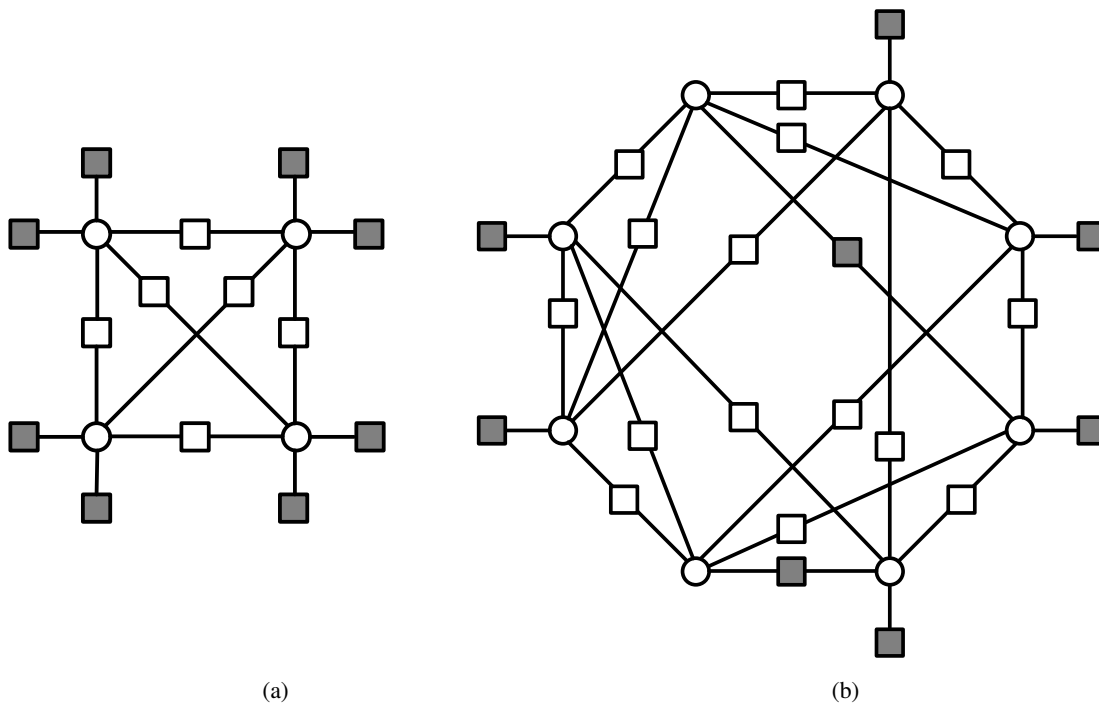


Fig. 12. (a) A (4, 8, 8, 6, 0) GAST ($\gamma = 5$). (b) An (8, 8, 6, 17, 0) GAST ($\gamma = 5$). Appropriate non-binary edge weights are assumed.

Table III shows the error profiles of Codes 1 and 2 over the NLM channel with 3 reads. The table reveals that 33% of the errors in the error profile of Code 1 are non-elementary GASTs. The table also demonstrates the effectiveness of the WCM framework in removing the detrimental objects. Two of the GASTs that strongly contribute to the error profile of Code 1 are $(4, 8, 8, 6, 0)$ and $(8, 8, 6, 17, 0)$ GASTs, which are shown in Fig. 12. The key difference between GASTs in codes with $\gamma = 5$ (or 6) and GASTs in codes with $\gamma \in \{3, 4\}$ is that for the former GASTs, $g = 2$, while for the latter GASTs, $g = 1$. In other words, a VN in an object in a code with $\gamma = 5$ (or 6) can be connected to a maximum of 2 unsatisfied CNs while the object is classified as a GAST (see also Fig. 12 and Example 3); for $\gamma \in \{3, 4\}$, this maximum is 1.

B. Achieving More Gain by Removing Oscillating Sets

In this subsection, we demonstrate the additional gains that can be achieved for NB-LDPC codes with even column weights (particularly, $\gamma = 4$) over practical asymmetric channels by removing OSTs as described in Section V.

First, we present results for the NLM channel described in the previous subsection (still with 3 reads). Code 3 is an NB-PB-LDPC code defined over $\text{GF}(4)$, with block length = 8480 bits, rate ≈ 0.90 , and $\gamma = 4$. Code 4 is the result of optimizing Code 3 by attempting to remove the dominant GASTs $(4, 4, 4, 6, 0)$, $(6, 4, 4, 10, 0)$, $(6, 5, 5, 8, 1)$, and $(8, 4, 2, 15, 0)$ using the WCM framework (see [5]). Code 5 is the result of optimizing Code 4 for the asymmetric NLM channel by attempting to remove the OSTs in Table IV, left panel, using the WCM framework. The performance curves of Code 3 (unoptimized) and Code 4 (WCM framework, no OSTs removal) in Fig. 13(a) were introduced in [5].

It is demonstrated by Fig. 13(a) that removing the dominant OSTs to generate Code 5 results in nearly 0.5 of an order of magnitude gain in performance over Code 4 (for which only the dominant GASTs are removed) even though Code 4 is highly optimized (it outperforms Code 3 by about 2 orders of magnitude). Thus, applying Algorithm 1 to remove OSTs after removing GASTs raises the gain to almost 2.5 orders of magnitude for Code 5 compared with the unoptimized code (Code 3) over the NLM channel. Table IV, left panel, shows the significant reduction in the number of OSTs in the error profile of Code 5 compared with Code 3.

Second, we present results for another asymmetric Flash channel: the Cai-Haratsch-Mutlu-Mai (CHMM) Flash channel [35]. The authors developed a model in [35] for the threshold voltage distribution that is suitable for 20nm and 24nm MLC Flash memories. The four levels are modeled as different Gaussian distributions that are shifted and broadened with the increase in P/E cycles, resulting in limited asymmetry relative to the NLM channel. We implemented the CHMM channel based on the data and the model provided in [35]. In this subsection, we use 3 reads, and the sector size is 512 bytes.

Here, Code 6 is an NB-PB-LDPC code defined over $\text{GF}(4)$, with block length = 1840 bits, rate ≈ 0.80 , and $\gamma = 4$. Code 7 is the result of optimizing Code 6 by attempting to remove the dominant GASTs $(4, 4, 4, 6, 0)$, $(6, 4, 2, 11, 0)$, $(6, 4, 4, 10, 0)$, $(7, 4, 3, 11, 1)$, $(8, 5, 5, 12, 1)$, and $(9, 5, 5, 14, 1)$ using the WCM framework (see also [5]). Code 8 is the result of optimizing Code 7 for the asymmetric CHMM channel by attempting to remove the OSTs in Table IV, right panel, using the WCM framework. The performance curves of Code 6 (unoptimized) and Code 7 (WCM framework, no OSTs removal) in Fig. 13(b) were introduced in [5].

Fig. 13(b) reveals that removing the dominant OSTs to design Code 8 results in more than 0.5 of an order of magnitude performance gain over Code 7 (for which only the dominant GASTs

are removed). Consequently, applying Algorithm 1 to remove OSTs (after removing GASTs) raises the performance gain to more than 1.5 orders of magnitude for Code 8 compared with the unoptimized code (Code 6) over the CHMM channel. Table IV, right panel, clarifies the significant reduction in the number of OSTs in the error profile of Code 8 compared with Code 6.

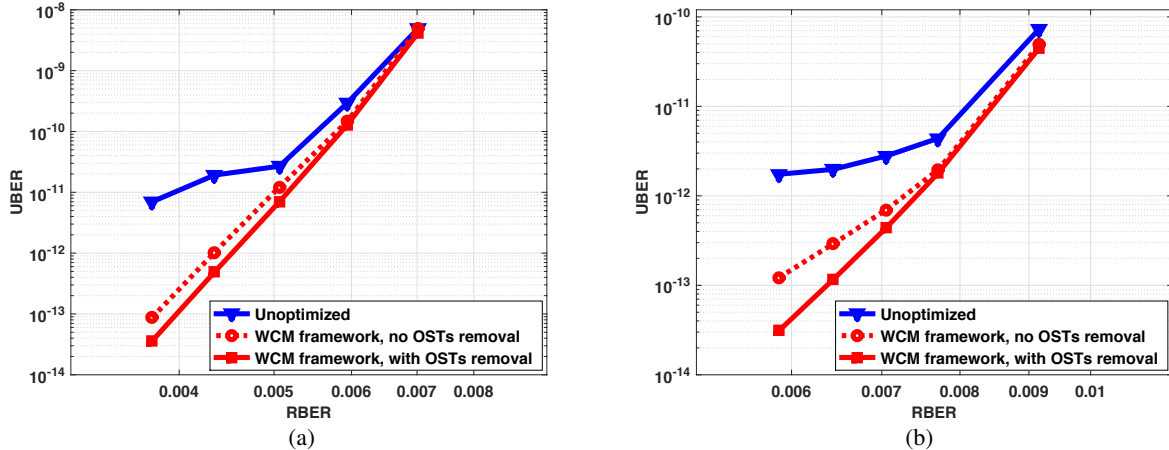


Fig. 13. (a) Simulation results over the NLM channel with 3 reads for Code 3 (unoptimized), Code 4 (WCM framework, no OSTs removal), and Code 5 (WCM framework, with OSTs removal). (b) Simulation results over the CHMM channel with 3 reads for Code 6 (unoptimized), Code 7 (WCM framework, no OSTs removal), and Code 8 (WCM framework, with OSTs removal). The six codes have $\gamma = 4$.

TABLE IV

LEFT PANEL: OSTs ERROR PROFILE OF CODES 3 AND 5 OVER THE NLM CHANNEL WITH 3 READS, $RBER \approx 3.75 \times 10^{-3}$, $UBER$ (UNOPTIMIZED) $\approx 6.98 \times 10^{-12}$, AND $UBER$ (WCM FRAMEWORK, WITH OSTs REMOVAL) $\approx 3.58 \times 10^{-14}$ (SEE FIG. 13(A)). RIGHT PANEL: OSTs ERROR PROFILE OF CODES 6 AND 8 OVER THE CHMM CHANNEL WITH 3 READS, $RBER \approx 5.87 \times 10^{-3}$, $UBER$ (UNOPTIMIZED) $\approx 1.74 \times 10^{-12}$, AND $UBER$ (WCM FRAMEWORK, WITH OSTs REMOVAL) $\approx 3.11 \times 10^{-14}$ (SEE FIG. 13(B)).

Error type	Count	
	Code 3	Code 5
(5, 5, 5, 6, 1)	22	0
(6, 5, 4, 10, 0)	29	0
(8, 4, 2, 12, 2)	24	0
(8, 5, 3, 13, 1)	25	0

Error type	Count	
	Code 6	Code 8
(6, 5, 2, 11, 0)	29	0
(6, 6, 2, 11, 0)	11	0
(7, 5, 3, 11, 1)	34	0
(8, 4, 3, 13, 1)	15	0
(9, 4, 2, 14, 2)	11	1

C. Effect of Soft Information in Flash Channels

In this subsection, we show the performance of NB-LDPC codes optimized by the WCM framework over practical Flash channels with additional soft information. The NLM and CHMM Flash channels used here are as described in the previous two subsections, except that we now consider 6 threshold voltage reads instead of 3. The additional reads increase the amount of soft information provided to the decoder from the Flash channel.

In the simulations of this subsection, Code 9 is an NB-PB-LDPC code defined over $GF(4)$, with block length = 3996 bits, rate ≈ 0.89 , and $\gamma = 3$. Code 10 is the result of optimizing Code 9 for the asymmetric NLM channel (with 6 reads this time) by attempting to remove the dominant GASTs (4, 2, 2, 5, 0), (4, 3, 2, 5, 0), (5, 2, 2, 5, 1), (6, 0, 0, 9, 0), (6, 1, 0, 9, 0), (6, 1, 1, 7, 1), (6, 2, 2, 5, 2), and (6, 2, 2, 8, 0) using the WCM framework.

Furthermore, Code 11 is another NB-PB-LDPC code defined over GF(4), with block length = 3280 bits, rate ≈ 0.80 , and $\gamma = 4$. Code 12 is the result of optimizing Code 11 for the asymmetric NLM channel (with 6 reads) by attempting to remove the dominant GASTs (4, 4, 4, 6, 0), (6, 2, 2, 11, 0), (8, 4, 3, 13, 1), and (8, 5, 2, 15, 0) in addition to the dominant OSTs (6, 5, 4, 10, 0), (7, 6, 4, 12, 0), (8, 4, 2, 12, 2), and (9, 4, 2, 14, 2) using the WCM framework. We also reuse Code 6 in this subsection (its parameters are stated in the previous subsection). Code 13 is the result of optimizing Code 6 for the asymmetric CHMM channel (with 6 reads) by attempting to remove the dominant GASTs (4, 4, 4, 6, 0), (6, 4, 4, 11, 0), and (7, 4, 3, 11, 1) in addition to the dominant OSTs (6, 5, 2, 11, 0), (7, 5, 3, 11, 1), (7, 5, 4, 9, 2), (7, 6, 6, 8, 2), (8, 6, 2, 15, 0), and (10, 7, 5, 11, 4) using the WCM framework.

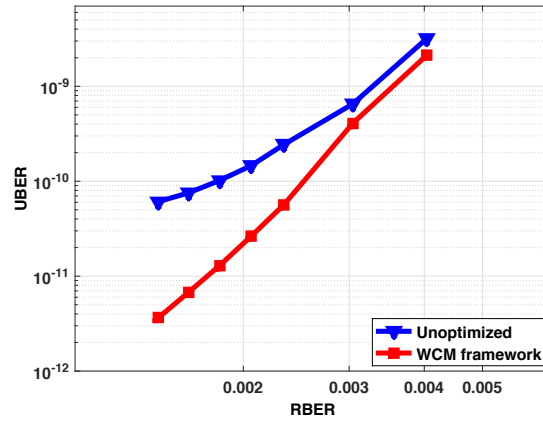


Fig. 14. Simulation results over the NLM channel with 6 reads for Code 9 (unoptimized) and Code 10 (WCM framework). The two codes have $\gamma = 3$.

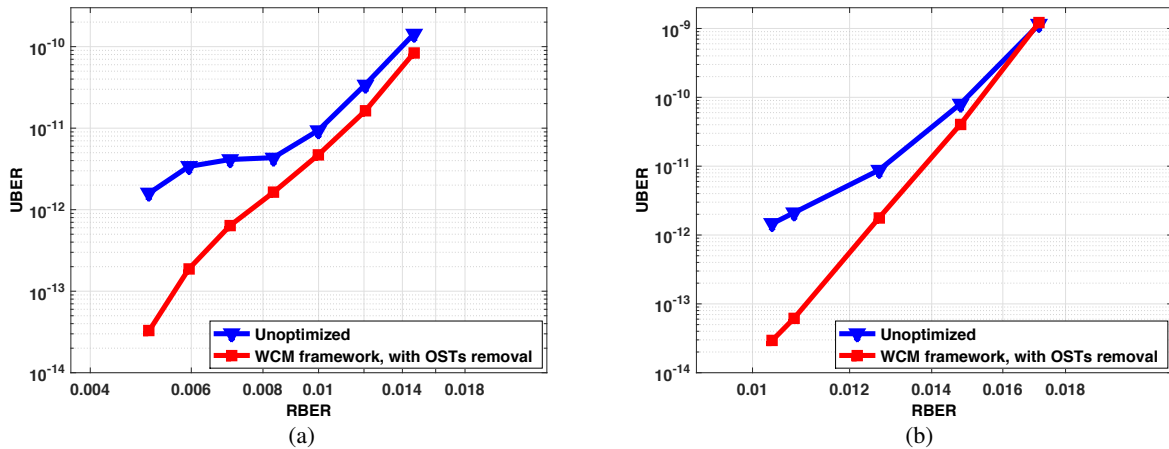


Fig. 15. (a) Simulation results over the NLM channel with 6 reads for Code 11 (unoptimized) and Code 12 (WCM framework, with OSTs removal). (b) Simulation results over the CHMM channel with 6 reads for Code 6 (unoptimized) and Code 13 (WCM framework, with OSTs removal). The four codes have $\gamma = 4$.

According to our simulations, the most dominant GASTs in the error floor of the unoptimized codes (Codes 9, 11, and 6) are hardly affected by the additional soft information (compare the dominant GASTs listed above for Codes 9, 11, and 6 with the dominant GASTs in [5, Table I], [5, Table II], and [5, Table IV], respectively). Moreover, Figures 14, 15(a), and 15(b) show that the performance gains achieved by applying the WCM framework over practical Flash channels

with 6 reads are in the same range as the gains achieved over the same channels with 3 reads. In particular, more than 1 order of magnitude gain is achieved in Fig. 14, and more than 1.5 orders of magnitude (> 0.5 of an order of magnitude is due to OSTs removal) gain is achieved in both Fig. 15(a) and 15(b). Furthermore, similar to the case of 3 reads demonstrated in [5], the more asymmetric the Flash channel is, the higher the percentage of relevant non-elementary GASTs ($b > d_1$ or/and $d_3 > 0$) that appear in the error profile of the NB-LDPC code.

The major difference between the results over practical Flash channels with 3 and 6 reads is the gain achieved in RBER. Consider the $\gamma = 4$ codes simulated over the CHMM channel, and assume that the target UBER is 10^{-13} . In Fig. 13(b), Code 8 achieves the target UBER at RBER $\approx 6.5 \times 10^{-3}$. On the contrary, Code 13 achieves the target UBER at RBER $\approx 1.1 \times 10^{-2}$, as revealed by Fig. 15(b). Thus, using 6 reads achieves in this case about 70% RBER gain compared with using only 3 reads. This RBER gain is directly translated into P/E cycles gain, which means an extension in the lifetime of the Flash device. Similar gains are also observed for codes with different column weights over both the NLM and CHMM channels.

D. Optimizing Spatially-Coupled Codes

In this subsection, we extend the scope of the WCM framework to irregular codes with fixed column weights (fixed VN degrees). In particular, we use the WCM framework to optimize non-binary spatially-coupled (NB-SC) codes with $\gamma \in \{3, 4\}$ for PR and AWGN channels, showing more than 1 order of magnitude performance gain.

SC codes are a class of LDPC codes that have capacity-approaching asymptotic performance, and very good finite-length performance. Literature works studying the asymptotic performance of SC codes include [36], [37], [47] for the binary case, and [38], [48] for the non-binary case. Recent results on the finite-length constructions of SC codes include [39]–[41], [49]–[51] for the binary case, and [52]–[54] for the non-binary case. Most of these finite-length constructions are based on protographs. SC codes are constructed by partitioning an underlying block LDPC code, and then rewiring the partitioned components together multiple times [51], [54]. We demonstrate the effectiveness of the WCM framework by optimizing the edge weights of NB-SC codes designed using two different finite-length construction techniques (both are also based on protographs). First, we show results for NB-SC codes partitioned using single cutting vectors (CVs) [53], and the underlying block LDPC codes used are array-based LDPC (AB-LDPC) codes [19]. More details about the CV technique can be found in [53]. Second, we show results for better NB-SC codes, designed using the optimal overlap, circulant power optimizer (OO-CPO) technique [51], [54]. The partitioning here is derived through solving an optimization problem aiming at minimizing the total number of detrimental objects in the protograph of the SC code. Next, circulant powers of the underlying block code are optimized to further reduce the number of detrimental objects in the final unlabeled graph of the SC code. More details about the OO-CPO technique can be found in [51] and [54]. In this subsection, we focus on the case of partitioning the underlying block code into only two component matrices (memory $m = 1$), and all the SC codes do not have cycles of length 4 in their unlabeled graphs.

The WCM framework requires the initial unoptimized code to have a fixed column weight (fixed VN degree) but not necessarily a fixed row weight (fixed CN degree). NB-SC codes that are based on the underlying structured and regular block codes incorporate irregularities in their CN degrees (different row weights), while having fixed VN degrees [51], [53], making them suitable for optimization using the WCM framework for various applications.

We use the PR channel described in [10]. This PR channel incorporates inter-symbol interference (intrinsic memory), jitter, and electronic noise. The normalized channel density [11], [55], [56] is 1.4, and the PR equalization target is [8 14 2]. The receiver consists of filtering units followed by a Bahl Cocke Jelinek Raviv (BCJR) detector [57], which is based on pattern-dependent noise prediction (PDNP) [58], and an FFT-QSPA LDPC decoder [43]. The number of global (detector) iterations is 10, and the number of local (decoder) iterations is 20. Unless a codeword is reached, the decoder performs its prescribed number of local iterations for each global iteration. More details can be found in [10].

Code 14 is an NB-SC code designed using the CV technique, and defined over $\text{GF}(4)$, with block length = 8464 bits, rate ≈ 0.85 , and $\gamma = 3$. The underlying block code is a non-binary AB-LDPC code defined over $\text{GF}(4)$, with circulant size = 23 and $\gamma = 3$. The coupling length $L = 8$ [53], [54], and the underlying block code is partitioned using the optimal CV [5 11 18] (see also [53] for more details about determining the optimal CV). Code 15 is the result of optimizing Code 14 for the PR channel by attempting to remove the dominant BASTs $(6, 0, 0, 9, 0)$, $(6, 1, 0, 9, 0)$, $(6, 2, 0, 9, 0)$, $(8, 0, 0, 10, 1)$, and $(8, 0, 0, 12, 0)$ using the WCM framework.

Fig. 17(a) shows that the SC code optimized using the WCM framework (Code 15) outperforms the unoptimized SC code (Code 14) by more than 1.5 orders of magnitude over the PR channel. Note that this significant performance gain is achieved despite the unlabeled Tanner graphs of Codes 14 and 15 both being designed using the optimal CV. In the caption of Fig. 17(a) we precede the names of Codes 14 and 15 with “SC” for clarity.

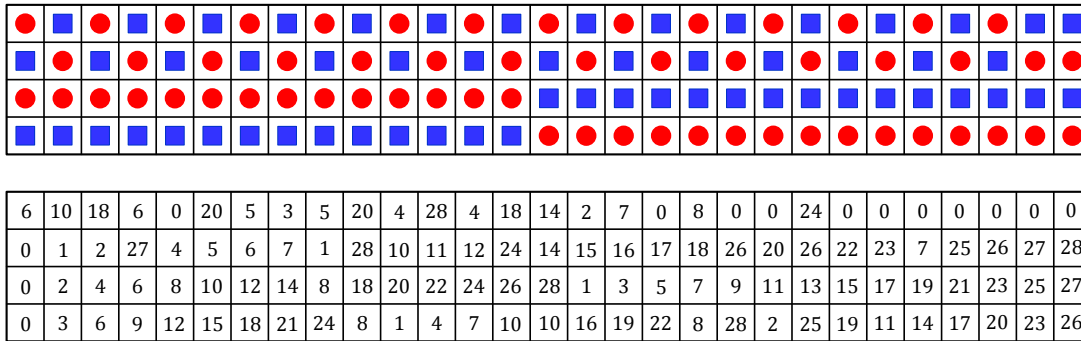


Fig. 16. Upper panel: the OO partitioning of the underlying block code of Code 17 (and Code 19). Entries with circles (resp., squares) are assigned to the first (resp., second) component matrix. Lower panel: the circulant power arrangement for the circulants in the underlying block code of Code 17 (and Code 19) after applying the CPO.

In the AWGN simulations, Code 16 (resp., Code 17) is an NB-SC code designed using the CV (resp., OO-CPO) technique, and defined over $\text{GF}(8)$, with block length = 12615 bits, rate ≈ 0.83 , and $\gamma = 4$. The underlying block code is defined over $\text{GF}(8)$, with circulant size = 29, $\gamma = 4$, and row weight = 29. The coupling length $L = 5$ [54]. The underlying block code of Code 16 is partitioned using the CV [5 11 18 24], and it is a non-binary AB-LDPC code. The underlying block code of Code 17 is partitioned according to Fig. 16, upper panel, and the circulant power arrangement is given in Fig. 16, lower panel (see also [51] and [54]). Code 18 (resp., Code 19) is the result of optimizing Code 16 (resp., Code 17) for the AWGN channel by attempting to remove the dominant EASs $(4, 4, 4, 6, 0)$ (only from Code 17), $(6, 4, 4, 10, 0)$, $(6, 6, 6, 9, 0)$, and $(8, 2, 2, 15, 0)$ using the WCM framework.

Fig. 17(b) shows that the SC codes optimized using the WCM framework outperform the unoptimized SC codes by more than 1 order of magnitude over the AWGN channel. Again, note that this significant performance gain is achieved despite the unlabeled Tanner graphs of Codes 16 and 18 (resp., Codes 17 and 19) both being designed using the same technique. An important observation is that despite the very good performance of Code 17 (the unoptimized code designed using the OO-CPO technique), optimizing Code 17 using the WCM framework to reach Code 19 still achieves over 1 order of magnitude performance gain. In the caption of Fig. 17(b) we precede the names of Codes 16, 17, 18, and 19 with “SC” for clarity.

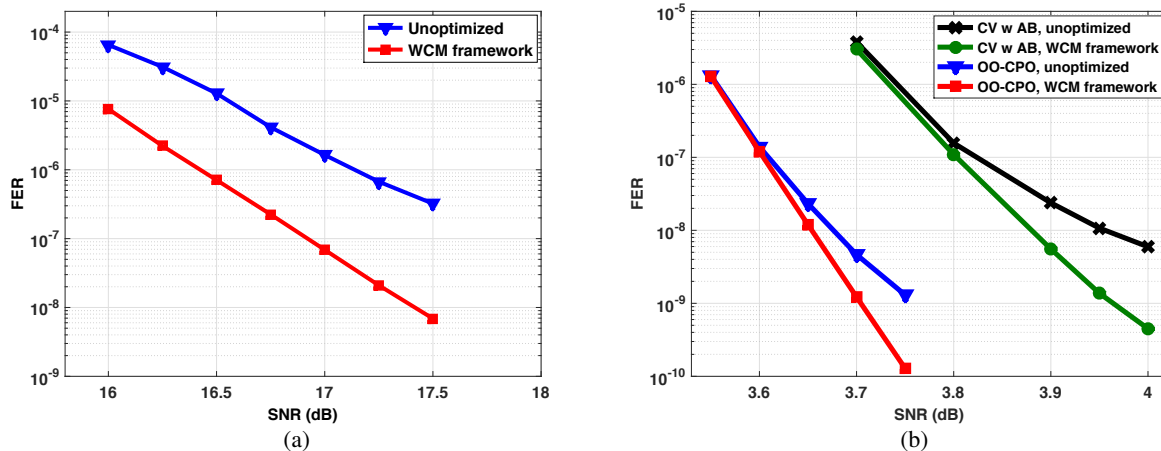


Fig. 17. (a) Simulation results over the PR channel for SC Code 14 (unoptimized) and SC Code 15 (WCM framework); both codes have $\gamma = 3$. (b) Simulation results over the AWGN channel for SC Codes 16 (CV, unoptimized), 17 (OO-CPO, unoptimized), 18 (CV, WCM framework), and 19 (OO-CPO, WCM framework); all codes have $\gamma = 4$.

VII. CONCLUSION

In this paper, we have provided a theoretical analysis of a general combinatorial framework for optimizing non-binary graph-based codes. In particular, we proved the optimality of the WCM framework, and we demonstrated its efficiency by comparing the number of matrices it operates on with that number in an alternative idea. We have also detailed the theory behind the removal of a GAST; we discussed the dimension of the null space of a WCM and the minimum number of edge weight changes needed to remove a GAST. Furthermore, we proposed new combinatorial objects, OSTs, and showed how to extend the WCM framework to remove them and achieve additional performance gains for NB-LDPC codes with even column weights. On the applications side, the WCM framework was applied to different codes over a variety of channels with different characteristics, where performance gains of at least 1 order, and up to nearly 2.5 orders, of magnitude were achieved. A notable extension of the WCM framework was to use it for optimizing spatially-coupled codes over multiple channels. We believe that this framework will serve as an effective code optimization tool for emerging multi-dimensional storage devices, e.g., 3-D Flash and two-dimensional magnetic recording (TDMR) devices.

APPENDIX A FINDING THE WCMs OF A GIVEN GAST

The steps of [5, Algorithm 1] are:

- 1) **Input:** Tanner graph G_s of the GAST s , with edge weights over $\text{GF}(q)$, from which the matrix \mathbf{A} is formed.

- 2) Set the maximum number of nested **for** loops, `loop_max`.
- 3) Mark all the CNs $\in \{\mathcal{T} \cup \mathcal{H}\}$ as satisfied. (CNs $\in \mathcal{O}$ are always unsatisfied.)
- 4) Check if \exists in G_s at least one degree-2 CN connecting two VNs, each is connected to $> \lceil \frac{\gamma+1}{2} \rceil$ CNs that are marked as satisfied.
- 5) **if** \nexists any of them **then**
- 6) \exists only one $(\ell - d_1) \times a$ WCM. Extract it by removing all the rows corresponding to degree-1 CNs from the matrix **A**.
- 7) Go to 26.
- 8) **else**
- 9) Count such CNs (that satisfy the condition in 4), save the number in u^0 , and save their indices (the indices of their rows in **A**) in $\mathbf{y}^0 = [y^0(1) \ y^0(2) \ \dots \ y^0(u^0)]^T$.
- 10) **end if**
- 11) Compute b_{ut} from (4). If $b_{\text{ut}} = 1$, go to 25.
- 12) **for** $i_1 \in \{1, 2, \dots, u^0\}$ **do** (Level 1)
- 13) Remove the marking performed in levels ≥ 1 , and mark the selected CN $c_{y^0(i_1)}$ as unsatisfied.
- 14) Redo the counting in 9, but save in $u_{i_1}^1$ ($< u^0$) and $\mathbf{y}_{i_1}^1$ (instead of u^0 and \mathbf{y}^0 , resp.).
- 15) If $b_{\text{ut}} = 2 \parallel u_{i_1}^1 = 0$, go to 12.
- 16) **for** $i_2 \in \{1, 2, \dots, u_{i_1}^1\}$ **do** (Level 2)
- 17) Remove the marking performed in levels ≥ 2 , and mark the selected CN $c_{y_{i_1}^1(i_2)}$ as unsatisfied.
- 18) Redo the counting in 9, but save in u_{i_1, i_2}^2 ($< u_{i_1}^1$) and \mathbf{y}_{i_1, i_2}^2 .
- 19) If $b_{\text{ut}} = 3 \parallel u_{i_1, i_2}^2 = 0$, go to 16.
- 20) ...
- 21) The lines from 16 to 19 are repeated (`loop_max-2`) times, with the nested (`loop_max-2`) **for** loops executed over the running indices $i_3, i_4, \dots, i_{\text{loop_max}}$.
- 22) ...
- 23) **end for**
- 24) **end for**
- 25) Obtain the WCMs via the indices in the **y** arrays. In particular, by removing permutations of the rows corresponding to $c_{y^0(i_1)}, c_{y_{i_1}^1(i_2)}, \dots, c_{y_{i_1, i_2, \dots, i_{b_{\text{ut}}-1}}^{b_{\text{ut}}-1}(i_{b_{\text{ut}}})}$, and the degree-1 CNs from **A**, all the WCMs are reached.
- 26) Eliminate all the repeated WCMs to reach the final set of WCMs, \mathcal{W} , where $t = |\mathcal{W}|$.
- 27) **Output:** The set \mathcal{W} of all WCMs of the GAST.

APPENDIX B

OPTIMIZING NB-LDPC CODES BY REDUCING THE NUMBER OF GASTS

The steps of [5, Algorithm 2] are:

- 1) **Input:** Tanner graph G_T of the NB-LDPC code with edge weights over $\text{GF}(q)$.
- 2) Using initial simulations and combinatorial techniques (e.g., [42]), determine \mathcal{G} , the set of GASTs to be removed.
- 3) Let \mathcal{X} be the set of GASTs in \mathcal{G} that cannot be removed, and initialize it with \emptyset .
- 4) Let \mathcal{P} be the set of GASTs in \mathcal{G} that have been processed, and initialize it with \emptyset .
- 5) Sort the GASTs in \mathcal{G} according to their sizes (parameter a) from the smallest to the largest.
- 6) Start from the smallest GAST (smallest index).

- 7) **for** every GAST $s \in \mathcal{G} \setminus \mathcal{P}$ **do**
- 8) If the unlabeled configuration of s does not satisfy the unlabeled GAST conditions in Definitions 2 and 3, skip s and go to 7.
- 9) Determine the minimum number of edge weight changes needed to remove the GAST s , $E_{\text{GAST},\min}$, by using [10, Lemma 2] (see also Lemma 5 in this paper).
- 10) Extract the subgraph G_s of the GAST s , from G_T .
- 11) Use [5, Algorithm 1] to determine the set \mathcal{W} of all WCMs of s ($|\mathcal{W}| = t$).
- 12) **for** $h \in \{1, 2, \dots, t\}$ **do**
- 13) Find the null space $\mathcal{N}(\mathbf{W}_h^{\text{cm}})$ of the h th WCM.
- 14) **if** (6) is satisfied (i.e., the WCM already has broken weight conditions) **then**
- 15) Go to 12.
- 16) **else**
- 17) Keep track of the changes already performed in G_s . (*The total number of changes to remove the GAST should be as close as possible to $E_{\text{GAST},\min}$.*)
- 18) Determine the smallest set of edge weight changes in G_s needed to achieve (6) for the h th WCM, without violating (6) for WCMs prior to h .
- 19) If this set of edge weight changes does not undo the removal of any GAST $\in \mathcal{P} \setminus \mathcal{X}$, perform these changes in G_s and go to 12.
- 20) **if** \nexists more edge weights to execute 18 and 19 **then**
- 21) Add GAST s to the set \mathcal{X} and go to 27.
- 22) **else** Go to 18 to determine a new set of changes.
- 23) **end if**.
- 24) **end if**
- 25) **end for**
- 26) Update G_T by the changes performed in G_s .
- 27) Add GAST s to the set \mathcal{P} .
- 28) If $\mathcal{P} \neq \mathcal{G}$, go to 7 to pick the next smallest GAST.
- 29) **end for**
- 30) If $\mathcal{X} = \emptyset$, then all the GASTs have been removed. Else, only the remaining GASTs in \mathcal{X} cannot be removed.
- 31) **Output:** Updated Tanner graph G_T of the optimized NB-LDPC code with edge weights over $\text{GF}(q)$.

APPENDIX C

NULL SPACES OF WCMs OF GASTs WITH $b = d_1$

In this appendix, we investigate the null spaces, along with their dimensions, of WCMs that belong to GASTs with $b = d_1$.

Remark 7. *There are few configurations that can be categorized as (a, b_g, d_1, d_2, d_3) GASTs, with $b_g \in \{b_i, b_{ii}, \dots\}$ (b_g is not unique). In other words, it is possible to have a configuration which is an (a, b_i, d_1, d_2, d_3) GAST for some set of VN vectors, and it is an $(a, b_{ii}, d_1, d_2, d_3)$ GAST for another set of VN vectors, where $b_i \neq b_{ii}$. For example, the configuration in Fig. 8(a), with $w_{1,1} = w_{6,1} = 1$, is a $(6, 0, 0, 9, 0)$ GAST for the vector $[\alpha \ 1 \ 1 \ \alpha \ 1 \ 1]^T$ (along with others), while the same configuration is a $(6, 3, 0, 9, 0)$ GAST for the vector $[\alpha^2 \ 1 \ 1 \ \alpha \ \alpha \ \alpha]^T$ (along with others). In cases like these, we identify the configuration with its **smallest** b_g in*

the set $\{b_i, b_{ii}, \dots\}$. Thus, we identify the configuration in Fig. 8(a) as a $(6, 0, 0, 9, 0)$ GAST as mentioned in Example 9. Note that this situation is not problematic for our GAST removal process, as our goal is to convert the GAST into another object $\notin \mathcal{Z}$, where \mathcal{Z} is the set of all (a, b', d_1, d_2, d_3) , $d_1 \leq b' \leq b_{\max}$.

Corollary 5. An (a, d_1, d_1, d_2, d_3) GAST, which is a GAST with $b = d_1$, has unbroken weight conditions for all its WCMs.

Proof: From [5, Lemma 1] (see Section II), a GAST that has $b = d_1$ must have the particular \mathbf{W}^z matrix of size $(\ell - d_1) \times a$ (extracted by removing the rows of all degree-1 CNs from \mathbf{A}) with unbroken weight conditions, i.e., $\exists \mathbf{v} = [v_1 \ v_2 \ \dots \ v_a]^T \in \mathcal{N}(\mathbf{W}^z)$ s.t. $v_f \neq 0, \forall f \in \{1, 2, \dots, a\}$. Since by definition of WCMs, each $\mathbf{W}_h^{\text{cm}}, \forall h \in \{1, 2, \dots, t\}$, is a submatrix of this particular \mathbf{W}^z , it follows that $\mathcal{N}(\mathbf{W}^z) \subseteq \mathcal{N}(\mathbf{W}_h^{\text{cm}}), \forall h$. In other words, $\exists \mathbf{v} = [v_1 \ v_2 \ \dots \ v_a]^T \in \mathcal{N}(\mathbf{W}_h^{\text{cm}}), \forall h$, s.t. $v_f \neq 0, \forall f \in \{1, 2, \dots, a\}$. ■

Corollary 5 highlights that each WCM of an (a, d_1, d_1, d_2, d_3) GAST has $\dim(\mathcal{N}(\mathbf{W}_h^{\text{cm}})) > 0, \forall h$, which is a consequence of all of them having unbroken weight conditions. The following example further discusses the null spaces of WCMs belonging to GASTs with $b = d_1$.

Example 14. We once more return to the $(6, 0, 0, 9, 0)$ GAST in Fig. 8(a), with $w_{1,1} = w_{6,1} = 1$. The configuration is a $(6, 0, 0, 9, 0)$ GAST because the vector $\mathbf{v} = [\alpha \ 1 \ 1 \ \alpha \ 1 \ 1]^T$, for example, is in the null space of the 9×6 matrix $\mathbf{W}^z = \mathbf{A}$ (note that there are no degree-1 CNs in this configuration). The null spaces of the 10 WCMs, extracted according to Example 9, of that GAST are detailed below:

$$\begin{aligned} \mathcal{N}(\mathbf{W}_1^{\text{cm}}) &= \mathcal{N}(\mathbf{W}_3^{\text{cm}}) = \mathcal{N}(\mathbf{W}_4^{\text{cm}}) = \mathcal{N}(\mathbf{W}_5^{\text{cm}}) = \mathcal{N}(\mathbf{W}_6^{\text{cm}}) = \mathcal{N}(\mathbf{W}_7^{\text{cm}}) \\ &= \mathcal{N}(\mathbf{W}_8^{\text{cm}}) = \mathcal{N}(\mathbf{W}_9^{\text{cm}}) = \mathcal{N}(\mathbf{W}_{10}^{\text{cm}}) = \text{span}\{[\alpha \ 1 \ 1 \ \alpha \ 1 \ 1]^T\} \text{ and} \\ \mathcal{N}(\mathbf{W}_2^{\text{cm}}) &= \text{span}\{[\alpha \ 0 \ 0 \ 0 \ 1 \ 1]^T, [0 \ 1 \ 1 \ \alpha \ 0 \ 0]^T\}. \end{aligned} \quad (51)$$

We now turn our attention to the $(6, 2, 2, 5, 2)$ GAST in Fig. 9(a), with $w_{1,2} = \alpha^2$. The configuration is a $(6, 2, 2, 5, 2)$ GAST because the vector $\mathbf{v} = [\alpha^2 \ 1 \ 1 \ 1 \ \alpha \ \alpha]^T$, for example, is in the null space of the 7×6 matrix \mathbf{W}^z , extracted by removing the rows of the 2 degree-1 CNs (that are c_8 and c_9) from \mathbf{A} . The null spaces of the 2 WCMs, extracted according to Example 10, of that GAST are:

$$\begin{aligned} \mathcal{N}(\mathbf{W}_1^{\text{cm}}) &= \text{span}\{[\alpha^2 \ 1 \ 1 \ 1 \ \alpha \ \alpha]^T\} \text{ and} \\ \mathcal{N}(\mathbf{W}_2^{\text{cm}}) &= \text{span}\{[0 \ 1 \ 1 \ \alpha^2 \ 1 \ 0]^T, [1 \ 1 \ 1 \ 0 \ 0 \ \alpha^2]^T\}. \end{aligned} \quad (52)$$

It is clear that, all the WCMs for both GASTs have unbroken weight conditions, which is expected according to Corollary 5 (both GASTs have $b = d_1$).

Note that in Example 14, all the WCMs have $p_h = \dim(\mathcal{N}(\mathbf{W}_h^{\text{cm}})) = \delta_h$, except for one WCMs; \mathbf{W}_2^{cm} of the $(6, 2, 2, 5, 2)$ GAST has $\dim(\mathcal{N}(\mathbf{W}_2^{\text{cm}})) = 2 > \delta_2 = 1$. As mentioned before, it is typically the case that $p_h = \dim(\mathcal{N}(\mathbf{W}_h^{\text{cm}})) = \delta_h$.

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