Interior Schwarzschild Solutions and Interpretation of Source Terms

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The solutions of the Einstein field equations, previously used in deriving the self-energy of a point charge, are shown to be nonsingular in a canonical frame, except at the position of the particle. A distribution of "dust" of finite extension is examined as the model whose limit is the point particle. The standard "proper rest-mass density" is related to the bare rest-mass density. The lack of singularity of the initial metric $g_{\mu\nu}$ is in contrast to the Schwarzschild type singularity of standard coordinate systems. Our solutions for the extended source are nonstatic in general, corresponding to the fact that a charged dust is not generally in equilibrium. However, the solutions become static in the point limit for all values of the bare-source parameters. Similarly, the self-stresses vanish for the point particle. Thus, a classical point electron is stable, the gravitational interaction cancelling the electrostatic self-force, without the need for any extraneous "cohesive" forces.

1. INTRODUCTION

THE simplest physical solutions of the Einstein field equations in the presence of matter are the well-known Schwarzschild and Reissner-Nordstrom metrics. In these spherically symmetric cases, the the exterior solutions can be obtained without a detailed description of the mass and charge distributions which produce the gravitational field. However, the interior solutions come into play in relating the source parameters to the observable mass occurring in the exterior solutions. For example, in obtaining the self-energy for static point particles (changed or neutral) one must relate the exterior mass parameter m, to the bare mass m_0 and charge e of the particle, as was discussed in previous work.[1] In this note, we examine in more detail some properties of the inner spherically symmetric solutions in their relations to the sources.

In Sec. 2, we shall concentrate on the initial value problem, which, in the present case, completely determines the spatial part of the metric[2] $(g_{ij})_{t=0}$ independent of $g_{0\mu}$. Our model of the extended particle will consist of a cloud of "dust" (no pressure terms present). In the self-energy analysis, the basic parameter is the bare rest-mass m_0. We will relate it to the more conventional parameter, the "proper rest-mass density" ρ_{00}, usually defined for "dust".[3] The comparison is carried out in isotropic coordinates, where g_{ij} is everywhere regular. (It turns out that the Schwarzschild coordinates are so singular that a relation between ρ_{C0} and m_0 cannot be obtained when the particle's dimensions become sufficiently small.)

In Sec. 3, the rest of the metric $(g_{0\mu})$ is computed at the initial time in the canonical coordinate system[1] whose spatial part is isotropic initially. These canonical coordinate conditions fix the $g_{0\mu}$ uniquely. The analysis is carried out for the simple model of a "shell" distribution of matter and charge of radius ϵ. The $g_{0\mu}$ are shown to be regular everywhere, independent of the size of ϵ. Thus no restrictions on the bare mass due to size arise. This is in contrast to the situation in the usual Schwarzschild solution (even in the isotropic frame) where one regards m as the basic parameter and must require $m\lesssim\epsilon$ in order to avoid singularities. The usual restriction on the mass is due to a singularity in the coordinate system and is not in the geometry. (That the Schwarzschild singularity is spurious has, of course, previously been noted[4] from the exterior solution alone.) A relation does exist between m and the source parameters m_0, e, and ϵ which is manifestly coordinate invariant in the point limit and actually has an invariant significance for extended bodies as well. It is also seen that the solution in the canonical frame is static only in the point limit ($\epsilon=0$), where it represents a stable classical charged particle.

2. ANALYSIS OF SOURCE TERMS

The generally covariant Lagrangian governing the Einstein-Maxwell point charge system has previously been shown (V) to reduce to the form[5]

$$\mathfrak{L}=\pi^{ij}\partial_0 g_{ij}+(-\mathcal{E}^{iT})\partial_0 A_i{}^T +\textstyle\sum_A p_i(t)\{dx^i(t)/dt\}\delta^3(\mathbf{r}-\mathbf{r}(t))-N\bar{R}^0-N_i\bar{R}^i \quad (2.1)$$

where

$$\bar{R}^0\equiv -g^{\frac{1}{2}}\,{}^3R-g^{-\frac{1}{2}}(\tfrac{1}{2}\pi^2-\pi^{ij}\pi_{ij}) +\tfrac{1}{2}g^{-\frac{1}{2}}[\mathcal{E}^i\mathcal{E}_i+\mathcal{B}^i\mathcal{B}_i]+\textstyle\sum_A \delta^3(\mathbf{r}-\mathbf{r}(t)) \times[\bar{m}_0{}^2+(p_i-eA_i{}^T)(p^i-eA^{iT})], \quad (2.2a)$$

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[1] R. Arnowitt, S. Deser, and C. W. Misner, Nuovo cimento **15**, 487 (1960); Phys. Rev. Letters **4**, 375 (1960), and preceding paper [Phys. Rev. **120**, 313 (1960)]. The last paper is denoted by V in text. The present paper is Va.

[2] Our notation is as in earlier work. Latin indices run from 1 to 3, Greek from 0 to 3, and $x^0=t$. We use units such that $16\pi\gamma c^{-4}=1=c$ where γ is the Newtonian constant.

[3] See, for example, C. Møller, *The Theory of Relativity* (Oxford University Press, New York, 1952).

[4] M. Kruskal (unpublished); and C. Fronsdal, Phys. Rev. **116**, 778 (1959).

[5] All tensors and covariant operations are *three*-dimensional unless specified by a superscript "4", g^{ij} being the matrix inverse of g_{ij} and "$|$" indicating the covariant derivative with respect to

$$\bar{R}^i \equiv -2\pi^{ij}{}_{|j} - g^{ii}\epsilon_{jkl}\mathcal{E}^k\mathcal{B}^l - \textstyle\sum_A (p^i - eA^{iT})\delta^3(\mathbf{r}-\mathbf{r}(t)). \quad (2.2b)$$

The notation is as follows:

$$\pi_{ij} \equiv (-{}^4g)^{\frac{1}{2}}[{}^4\Gamma^0{}_{ij} - g_{ij}\,{}^4\Gamma^0{}_{mn}g^{mn}],$$

$\mathcal{E}^i \equiv \mathfrak{F}^{0i}$ is the electric field *density*, A_i the vector potential, $p_i(t)$ and $x^i(t)$ are the canonical coordinates of a particle, $N \equiv (-{}^4g^{00})^{-\frac{1}{2}}$, $N_i \equiv {}^4g_{0i}$, and $\mathcal{B}^i \equiv \epsilon^{ijk}A_k{}^T{}_{,j}$ is the magnetic field. The superscript "T" appearing in the above means the flat space transverse part of the vector (e.g., $A_i = A_i{}^T + A_i{}^L$, $\boldsymbol{\nabla}\cdot\mathbf{A}^T = 0 = \boldsymbol{\nabla}\times\mathbf{A}^L$). The symbol π stands for $\pi_{ij}g^{ij}$ and ${}^3R$ is the curvature scalar formed from g_{ij}. In Eq. (2.2), $\mathcal{E}^i = \mathcal{E}^{iT} + \mathcal{E}^{iL}$, where the longitudinal part of the electric field, $\boldsymbol{\mathcal{E}}^L$, is the solution of the Maxwell constraint equation

$$\mathcal{E}^{iL}{}_{,i} = \textstyle\sum_A e\delta^3(\mathbf{r}-\mathbf{r}(t)). \quad (2.3)$$

The form of the Lagrangian (2.1) shows immediately that the parameter $\bar{m}_0$ is the bare mass of a particle, i.e., in the flat space limit, $\mathfrak{L}$ reduces to the usual Lagrangian describing a system of charges with bare masses $\bar{m}_0$. The $\sum_A$ in the Lagrangian denotes summation over the point charges. Although we are primarily interested in dealing with only one point particle, the "dust" of many particles is a useful way of representing a single extended particle. In this way the dynamics of the inner structure is correctly treated. The primary quantity that will appear in our later equations is

$$\rho_0(\mathbf{r},t) = \textstyle\sum_A \bar{m}_0\delta^3(\mathbf{r}-\mathbf{r}(t)). \quad (2.4)$$

We will use the bare mass density $\rho_0(\mathbf{r},t)$, in the continuous limit, to describe an extended particle of mass m_0:

$$m_0 \equiv \int \rho_0(\mathbf{r},t)d^3r = \textstyle\sum_A \bar{m}_0. \quad (2.5)$$

It might be noted that the initial shape of ρ_0 may be specified arbitrarily by specifying the initial positions of the particles. Thereafter, the dynamical equations determine ρ_0 uniquely, and in such a way that m_0 remains constant.

For our initial value problem, the relevant field equation ($\bar{R}^0 = 0$) arises from varying N. Since we are interested in a pure particle initial state with no field excitations or particle momenta present, we shall set p_i, $A_i{}^T$ and $\mathcal{E}^{iT}$ to zero at $t=0$. As was shown in V, the vanishing of the free gravitational modes may now be achieved by choosing a metric of the form $g_{ij} = \chi^4(\mathbf{r})\delta_{ij}$ (isotropic coordinate conditions) and setting $\pi^{ij} = 0$. The Einstein equation, $\bar{R}^0 = 0$, reduces to (in the neutral case)

$$g^{\frac{1}{2}}\,{}^3R \equiv -8\chi\nabla^2\chi = \rho_0(\mathbf{r}). \quad (2.6)$$

To compare with the standard form of the Einstein equations, we note that Eq. (2.6) is $2g^{\frac{1}{2}}G_{**} = g^{\frac{1}{2}}T_{**}$ where $G_{\mu\nu} \equiv {}^4R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\,{}^4R$. The "star" index signifies contraction with the unit normal to the space-like surface, e.g., $T_{**} = n_\mu n^\nu T^\mu{}_\nu$. For the $t=$const space-like surface, $n_\mu = (-N, 0, 0, 0)$. With incoherent matter, the conventional source term used for the interior Schwarzschild solution is

$$T^\mu{}_\nu = \rho_{00}u^\mu u_\nu, \quad (2.7)$$

where u^μ is the velocity field and ρ_{00} is the proper rest-mass density. Hence, for a fluid initially at rest ($u_i = 0$), one has $u_\mu = n_\mu$ and

$$T_{**} = \rho_{00}. \quad (2.8)$$

Comparison with Eq. (2.6) gives

$$\rho_0 = \rho_{00}g^{\frac{1}{2}} = \rho_{00}\chi^6. \quad (2.9)$$

Consequently, the bare mass m_0 is given by

$$m_0 \equiv \int \rho_0(\mathbf{r})d^3r = \int \rho_{00}g^{\frac{1}{2}}d^3r = \int \rho_{00}\chi^6 d^3r. \quad (2.10)$$

We consider the usual case of a matter sphere with constant ρ_{00} and radius ϵ. Hence $-8\chi\nabla^2\chi$ is $\rho_{00}\chi^6$ for $r<\epsilon$ and zero for $r>\epsilon$. The solution of this equation then takes the form[3]

$$\chi = a/(b^2+r^2)^{\frac{1}{2}}, \quad r<\epsilon \quad (2.11a)$$

$$\chi = 1 + m/(32\pi r), \quad r>\epsilon \quad (2.11b)$$

$$\rho_{00} = 24b^2/a^4, \quad (2.11c)$$

where the constant m is the total mass of the system. The mass, along with the constants a and b, is to be determined by the requirement of continuity on χ and $d\chi/dr$ at $r=\epsilon$. One finds

$$a^2 = (1+M/\epsilon)^3\epsilon^3/M, \quad b^2 = \epsilon^3/M, \quad \rho_{00} = 24M\epsilon^{-3}(1+M/\epsilon)^{-6}, \quad (2.12)$$

where $M = m/32\pi$, while Eq. (2.10) relates m to m_0 and ϵ. Since we are ultimately interested in a point particle, we examine these relations for small ϵ. Equations (2.10) and (2.12) yield (to within numerical factors)

$$m \sim (m_0{}^2\epsilon^3)^{1/5}, \quad a \sim (m_0{}^2\epsilon^3)^{1/5}, \quad b \sim (\epsilon^6/m_0)^{1/5}, \quad \rho_{00} \sim m_0{}^{-2}. \quad (2.13)$$

Note also that $m \sim (\epsilon^3/\rho_{00})^{1/5}$. Thus to obtain a finite total mass, ρ_{00} must vanish as ϵ^3. Physically, however, the bare mass m_0 is to be taken independent of ϵ, showing again[1] that m vanishes for the point particle. The relation between m, m_0 and ϵ is coordinate independent since ϵ may be expressed invariantly in terms of the proper length of the particle's circumference. The spatial metric is everywhere nonsingular and no limitation on m of the form $m \lesssim \epsilon$ is therefore encountered. With the model employed here, that of a uniform spherical distribution of bare mass, the total mass vanishes more rapidly ($\sim\epsilon^{3/5}$) than with a shell distribution,[1] where $m \sim \epsilon^{\frac{1}{2}}$. It is physically clear that the shell should provide an upper limit in this respect, since in this case the extent of cancellation of the bare mass by gravitational self-interaction is least.

g_{ij} (not ${}^4g_{\mu\nu}$). The totally antisymmetric symbols $\epsilon_{ijk} = \epsilon^{ijk} = 0, \pm 1$ are three-tensor densities of weight -1 and $+1$, respectively.

Our analysis relating the standard "proper mass density" to the bare mass was performed in isotropic coordinates. A corresponding attempt in Schwarzschild coordinates cannot be carried out, due to the singular nature of this frame. Here Eq. (2.6) reads

$$a'/a^2r+r^{-2}(1-a^{-1})=\rho_{00}, \quad r<\epsilon \tag{2.14}$$

where $a\equiv g_{rr}$ and $a'\equiv da/dr$. The interior solution is (again with a uniform distribution)

$$a(r)=(1-r^2/R^2)^{-1}, \tag{2.15}$$

where $R^2=6/\rho_{00}$. The relation of ρ_{00} to the bare mass is obtained from Eq. (2.10):

$$m_0=12\pi R\{-(\epsilon/R)[1-\epsilon^2/R^2]^{\frac{1}{2}}+\sin^{-1}(\epsilon/R)\}, \tag{2.16}$$

while the total mass[3] is $m=(4\pi/3)\rho_{00}\epsilon^3$. For a fixed value of m_0, there is *no* solution for $\rho_{00}=\rho_{00}(\epsilon,m_0)$ possible if $\epsilon<m_0/6\pi^2$. This difficulty expresses, in terms of the bare parameters, the unsuitability of the Schwarzschild coordinate system for describing point particles.

3. REGULARITY OF THE CANONICAL FRAME

We turn now to the determination of the remaining components, $g_{0\mu}$, of the metric in the canonical frame. For this purpose we employ the appropriate field equations to determine N and N_i. The canonical coordinate conditions read, in the notation of V:

$$x^0=-\tfrac{1}{2}(1/\nabla^2)(\pi^T+\nabla^2\pi^L), \tag{3.1a}$$

$$x^i=g_i-\tfrac{1}{4}(1/\nabla^2)g^T{}_{,i}. \tag{3.1b}$$

For our purposes, a differentiated form of Eqs. (3.1) is more relevant. If one expresses the right-hand sides in terms of π^{ij} and g_{ij}, one finds (retaining the summation convention for *all* repeated indices)

$$\pi^{ii}=0, \tag{3.2a}$$

$$g_{ij,jkk}-\tfrac{1}{4}g_{kj,kji}-\tfrac{1}{4}g_{jj,ikk}=0. \tag{3.2b}$$

In order to maintain these conditions in time, N and N_i must be chosen appropriately. Thus, taking the time derivatives of Eqs. (3.2) and using the field equations to eliminate $\partial_0\pi^{ij}$ and $\partial_0 g_{ij}$, one obtains four equations independent of time derivatives to determine N and N_i.

The field equations obtained from varying the Lagrangian of Eq. (2.1) with respect to π^{ij} and g_{ij} read

$$\partial_0 g_{ij}=2Ng^{-\frac{1}{2}}(\pi_{ij}-\tfrac{1}{2}g_{ij}\pi)+N_{i|j}+N_{j|i}, \tag{3.3a}$$

$$\begin{aligned}\partial_0\pi^{ij}=&-Ng^{\frac{1}{2}}({}^3R^{ij}-\tfrac{1}{2}g^{ij}\,{}^3R)+\tfrac{1}{2}Ng^{-\frac{1}{2}}g^{ij}(\pi^{mn}\pi_{mn}-\tfrac{1}{2}\pi^2)\\&-2Ng^{-\frac{1}{2}}(\pi^{im}\pi_m{}^j-\tfrac{1}{2}\pi\pi^{ij})+g^{\frac{1}{2}}(N^{|ij}-g^{ij}N^{|m}{}_{|m})\\&+[(\pi^{ij}N^m)_{|m}-N^i{}_{|m}\pi^{mj}-N^j{}_{|m}\pi^{mi}]+\tfrac{1}{2}N\mathcal{T}_m{}^{ij}\end{aligned} \tag{3.3b}$$

where the matter stress-tensor $\mathcal{T}_m{}^{ij}$ is

$$\begin{aligned}\mathcal{T}_m{}^{ij}\equiv&\,\delta^3(\mathbf{r}-\mathbf{r}(t))(p^i-eA^{iT})(p^j-eA^{jT})\\&\times[m_0{}^2+(p_m-eA_m{}^T)(p^m-eA^{mT})]^{-\frac{1}{2}}\\&+g^{-\frac{1}{2}}[\tfrac{1}{2}g^{ij}(\mathcal{E}^m\mathcal{E}_m+\mathcal{B}^m\mathcal{B}_m)\\&-(\mathcal{E}^i\mathcal{E}^j+\mathcal{B}^i\mathcal{B}^j)].\end{aligned} \tag{3.3c}$$

Fortunately, the initial conditions and coordinate conditions for our problem reduce these equations to quite simple form. These are $\pi^{ij}=\mathcal{E}^{iT}=A^{iT}=p_i=0$ and $g_{ij}=\chi^4\delta_{ij}$. At the initial time, Eq. (3.3a) then reduces to

$$\partial_0 g_{ij}=N_{i|j}+N_{j|i}. \tag{3.4a}$$

Taking the time derivative of Eq. (3.2b), one obtains a homogeneous equation for N_i which has the solution

$$N_i(\mathbf{r},0)=0. \tag{3.5}$$

The relevant component of Eq. (3.3b) is now just

$$\begin{aligned}\partial_0\pi^{ii}=&-Ng^{\frac{1}{2}}({}^3R^{ii}-\tfrac{1}{2}g^{ii}\,{}^3R)+g^{\frac{1}{2}}(N^{|ii}-g^{ii}N^{|m}{}_{|m})\\&+\tfrac{1}{2}Ng^{-\frac{1}{2}}(g^{ii}\mathcal{E}^{mL}\mathcal{E}^{nL}g_{mn}-\mathcal{E}^{iL}\mathcal{E}^{iL}).\end{aligned} \tag{3.4b}$$

The coordinate condition (3.2a), the isotropic form of g_{ij} and the constraint equation

$$g^{\frac{1}{2}}\,{}^3R=\rho_0(r)+\tfrac{1}{2}\chi^{-2}\mathcal{E}^{mL}\mathcal{E}^{mL} \tag{3.6}$$

reduce Eq. (3.4b) further to

$$\partial_m(\chi^2\partial_m N)=\tfrac{1}{4}N\{\chi^{-2}\mathcal{E}^{iL}\mathcal{E}^{iL}+\rho_0\}, \tag{3.7}$$

where $\mathcal{E}^{iL}$ is determined by $\nabla\cdot\boldsymbol{\varepsilon}^L=(e/m_0)\rho_0$. With the initial mass and charge distribution chosen to be a "shell" of radius ϵ:

$$\rho_0(r)=m_0\delta(r-\epsilon)/(4\pi r^2), \tag{3.8}$$

$\boldsymbol{\varepsilon}^L$ is zero for $r<\epsilon$ and equals $-(e/4\pi)\nabla(1/r)$ for $r>\epsilon$. Also, since the source terms vanish for $r<\epsilon$, both χ and N are constant in the interior (i.e., $N(r)=N(\epsilon)$ for $r<\epsilon$). The exterior equation for N becomes

$$(r^2\chi^2)(r^2\chi^2N')'=4\bar{e}^2N, \quad r>\epsilon \tag{3.9}$$

where $N'\equiv dN/dr$ and the spherical symmetry of the problem has been utilized. Here χ^2 takes on the known exterior form[1] $\chi^2=(1+M/r)^2-(\bar{e}/r)^2$, ($M\equiv m/32\pi$ and $\bar{e}\equiv|e|/16\pi$), with $M^2+\epsilon M-(M_0\epsilon+\bar{e}^2)=0$. In terms of the radial coordinate σ defined according to

$$dr/d\sigma=(r\chi)^2/(2\bar{e})\equiv[(r+M)^2-\bar{e}^2]/2\bar{e}, \tag{3.10}$$

Eq. (3.9) becomes $d^2N/d\sigma^2=N$. With the usual boundary conditions that $N\to1$ as $r\to\infty$, one finds

$$N=Ae^{\sigma}+(1-A)e^{-\sigma}, \tag{3.11a}$$

or

$$N=A\frac{r+M-\bar{e}}{r+M+\bar{e}}+(1-A)\frac{r+M+\bar{e}}{r+M-\bar{e}}. \tag{3.11b}$$

The constant A is determined by the matching condition at $r=\epsilon$. Thus, the jump discontinuity in $r^2\chi^2N'\equiv dN/d\sigma$ gives rise to the $\delta(r-\epsilon)$ term in Eq. (3.7):

$$[d(\ln N)/d\sigma]_{r=\epsilon}=2M_0, \quad M_0\equiv m_0/32\pi. \tag{3.12}$$

This gives

$$A=[1+(1-M_0/\bar{e})(1+M_0/\bar{e})^{-1}\exp 2\sigma_\epsilon]^{-1}, \tag{3.13}$$

where $\sigma_\epsilon\equiv(\sigma)_{r=\epsilon}$.

Our main interest in the solution (3.11), (3.13) for N resides in the question of whether it introduces a singularity into the metric (as does the N for the usual isotropic Reissner-Nordstrom case[6]). We therefore check whether $N=0$ or $N=\infty$ can occur. The constant interior solution is, of course, well behaved. The condition that $N=0$ for $r>\epsilon$ would imply that

$$\exp 2(\sigma-\sigma_\epsilon)=(M_0/\bar{e}-1)(M_0/\bar{e}+1)^{-1}. \qquad (3.14)$$

Since $dr/d\sigma>0$ [from Eq. (3.10)], it follows that $\sigma(r)\geqslant\sigma(\epsilon)$ for $r\geqslant\epsilon$. One can easily see, therefore, that no solution of Eq. (3.14) can exist for any values of M_0 and $\bar{e}$. From Eq. (3.11b), N can equal infinity only when $r=\bar{e}-M\geqslant\epsilon$. However,

$$\bar{e}-M=\tfrac{1}{2}\epsilon+\bar{e}-(\bar{e}^2+M_0\epsilon+\tfrac{1}{4}\epsilon^2)^{\frac{1}{2}}<\tfrac{1}{2}\epsilon. \qquad (3.15)$$

Thus the *full metric*, g_{ij} and $g_{0\mu}$, is everywhere *nonsingular* initially. The dynamical equations (3.3) guarantee that this property is maintained for a finite time.

For finite ϵ, our solution is not static. Physically, this corresponds to the fact that an initial distribution of electrically charged dust is not stable. (There is a special case, $M_0=\bar{e}$, which implies $M=\bar{e}$, and consequent cancellation of electrostatic and gravitational forces, where a static solution for arbitrary ϵ exists.) In the limit of a point particle, one has $M=\bar{e}$, independent of M_0 [by Eq. (3.15)] and therefore a static situation. Thus, the inclusion of gravitation leads to the existence of a stable classical point charge without the introduction of any external cohesive forces. In the point limit, N becomes

$$N=(1+2M/r)^{-1}, \qquad (3.16a)$$

and

$$\chi^2=1+2M/r, \qquad (3.16b)$$

which are the form of the standard static Reissner-Nordstrom solution with $M=\bar{e}$.

The solution corresponding to a neutral particle may be obtained from the above results by carefully limiting $\bar{e}$ to zero [or by direct solution of Eq. (3.7)]. One finds

$$N=1-2M_0[1+2M_0/(\epsilon+M)]^{-1}/(r+M), \quad r\geqslant\epsilon \qquad (3.17)$$

and $N(r)=N(\epsilon)$ for $r<\epsilon$. The rest of the metric is given from $\chi=1+M/r$, where $M^2+\epsilon M-\epsilon M_0=0$. Again there are no singularities in $g_{\mu\nu}$ for any ϵ and M_0, in contrast to the standard isotropic form of the Schwarzschild solution. For a finite ϵ, the solution is again not static. In the limit $\epsilon\to 0$, one finds $N\to 1$ and $\chi\to 1$ since the mass M of the neutral point particle vanishes.

4. DISCUSSION

We have examined here the relation between matter sources and the interior solution to which they give rise for spherically symmetric cases. The sources were considered to be built up from a "dust" of particles. For both neutral and charged sources, the full initial metric $g_{\mu\nu}$ was found, in the canonical coordinate frame, to be everywhere nonsingular, independent of the size of the system and its bare mass and charge. This is in contrast with the Schwarzschild type of singularity appearing in the usual coordinate systems. In the course of the derivation, the relation between the standard "proper rest-mass density" and the bare mass (i.e., the mass the particle possesses when all its couplings are removed), was brought out. In our discussion, no external pressure terms were considered in the source stress-tensor. Such terms (for a perfect fluid) do not affect the calculation of the spatial metric g_{ij} and hence the relation between m_0 and the total mass. However, since a pressure term summarizes the presence of other forces (not being treated dynamically), these forces would contribute to the clothing of the original mechanical mass and hence the parameter m_0 would now represent this original mechanical mass plus the clothing due to the forces giving rise to the pressure term.

Since no pressure has been introduced, our initial distributions of matter and charge are not, in general, stable, and hence our solution are not static for arbitrary size of the distributions. However, for the point charge (where $m=2|e|$ independent of m_0) our solution is indeed static, the gravitational forces having counteracted the repulsive electrostatic self-forces. The conservation requirement on the total stress tensor $T^{\mu\nu}$ implies that $T^{ij}{}_{,j}=-T^{0i}{}_{,0}=0$ for the static case, where T^{ij} are the total system's spatial stresses. Thus, in the notation of the orthogonal decomposition,[1] T^{ij} reduces to $T^{ij}=T^{ijTT}+T^{ijT}$. Spherical symmetry means that T^{ijTT} vanishes, since no preferred transverse direction can be distinguished. Hence, T^{ij} has at most one independent component which may be taken as T^{ii}. For our static point solutions, T^{ii} (as calculated from either the Landau-Lifshitz or Einstein pseudotensors) vanishes everywhere for arbitrary m_0 and e. In the rest frame, then, $T^{\mu\nu}=\rho\delta^{\mu 0}\delta^{\nu 0}$ where $\int d^3r\,\rho=m$, the clothed mass. In a moving system, therefore, $T^{\mu\nu}=\rho u^\mu u^\nu$. The vanishing of T^{ij} in the rest frame, therefore, is necessary so that the structure of the total stress tensor be that of a particle of renormalized mass m. [This requirement is stronger than the usual one that $P^\mu=\int d^3r\,T^{0\mu}$ transform like the energy-momentum of a particle.[7]] Thus the point charge is a completely stable object, without any ad hoc pressure terms required, and its mass is completely determined by its field interactions.

[6] See, for example, C. W. Misner and J. A. Wheeler, Ann. Phys. 2, 592 (1957).

[7] The standard static Reissner-Nordstrom solution has $T^{ii}\neq 0$ but $\int d^3r\,T^{ii}=0$. Thus it obeys the weaker condition on P^μ but its total stress-tensor is not that of a particle. This is due to the presence of the *phenomenological* pressure terms which were needed there to stabilize the particle.