From Power-Localized to Extended States in a Class of One-Dimensional Disordered Systems

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We study a one-dimensional random Kronig-Penney model in the presence of a constant electric field. We rigorously prove for the first time the existence of a transition between a regime of extended states for large field and a regime of power-localized states for small field. There the large-distance behavior of the states is $|x|^{-\alpha(F)}$ with $\alpha(F) \sim C/F$ for small field F, confirming a numerical computation of Soukoulis *et al.*

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Disordered systems have been and are being extensively studied because of their relevance to a large variety of physical situations.[1] It has been proven recently that, as predicted by Mott and Twose,[2] one-dimensional (1D) Schrödinger Hamiltonians with disorder possess a complete set of exponentially localized states.[3,4] Much less is known about such systems in electric fields.[5-7] Recently, Soukoulis *et al.*[7] studied the Kronig-Penney model

$$H = -\frac{d^2}{dx^2} + \sum_{n=-\infty}^{\infty} V_n \delta(x-n) - Fx, \tag{1}$$

where we shall take here, for simplicity, the V_n to be independent, identically distributed random variables with a distribution $B(E)dE$, with all moments finite and $\langle V_n \rangle = 0$. They performed numerical calculations of a transmission coefficient on an approximate[8] version of the model; it presents a power-law decay, suggesting (but this is known to be sometimes a dangerous extrapolation) that H had localized states, but ones which are only power-law localized (i.e., with eigenfunctions decaying as $|x|^{-\alpha(F)}$ at $+\infty$) rather than exponentially localized. Moreover, their calculation suggested that $\alpha(F)$ diverged as F^{-1} for small F. Such behavior had also been theoretically predicted by Prigodin[5] in the case of an electron in a white-noise potential and an electric field.

These results are somewhat surprising since Bentosela *et al.*[6] had studied the model where $\sum_{n=-\infty}^{+\infty} V_n \delta(x-n)$ is replaced by an arbitrary (disordered) potential V, which is sufficiently smooth[9]; they proved that such a model has no localized staes—in fact, the initial wave packets would be uniformly accelerated. Given this fact, the naturalness of model (1) for calculations, and the approximate nature of the arguments in Ref. 7, it seems useful to confirm or refute the conclusions of Ref. 7, with the tools of rigorous mathematical physics.

When $F = 0$, the model (1) has a complete set of exponentially localized states (nevertheless the localization length diverges at all energies of the form $E = k^2\pi^2$, k an integer). We have proven that for small F, (1) has a complete set of localized states, decaying of course as $\exp(-cF^{1/2}|x|^{3/2})$ for $x \to -\infty$, but more interestingly decaying as a power for $x \to +\infty$, and one has, for example,

$$c_1|x|^{-a/F} \leq ||\psi_n||(x) \leq c_2|x|^{-a/F}, \tag{2}$$

where[10]

$$||\psi_n||(x) = [\int_{x-1}^{x+1} |\psi(y)|^2 dy]^{1/2},$$

and the diffusion constant vanishes. For large F in contrast, all states are extended.

Let us now give an idea of how one can get (2). Let ψ be a solution of the equation $H\psi = E\psi$, H given by (1). Since we can integrate the equation between the δ functions, we can get a relation equivalent to $H\psi = E\psi$ but involving ψ only at points $x = n$. One then gets the following recursion relation on the $\psi(n)$:

$$\begin{pmatrix} \psi(n+1) \\ \psi(n) \end{pmatrix} = \begin{pmatrix} \alpha_n + V_n\gamma_n & \beta_n \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi(n) \\ \psi(n-1) \end{pmatrix}, \tag{3}$$

where the α_n, β_n, and γ_n will be functions of Airy functions. The important point is that γ_n will be mainly a product of two Airy functions, each of them decaying as $n^{-1/4}$ at $+\infty$. Hence γ_n will behave at $+\infty$ as $n^{-1/2}$, and more precisely one finds for n large and modulo oscillations that

$$\gamma_n \sim F^{-1/2} n^{-1/2}. \tag{4}$$

By studying α_n and β_n, one can show that the qualitative behavior is the same as if Eq. (3) were replaced by

$$\begin{pmatrix} \psi(n+1) \\ \psi(n) \end{pmatrix} = \begin{pmatrix} V_n\gamma_n & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi(n) \\ \psi(n-1) \end{pmatrix}, \tag{5}$$

which is the equation associated with an Anderson

model with potential $n^{-1/2}V_n$ at site n and coupling constant $F^{-1/2}$. From this we can understand why (1) has power-localized states for F small. Here is a first heuristic argument: Since wave functions represent standing waves, we can imagine the wave being set up by transmissions from a central peak. In the approximation of ignoring multiple reflections and approximating the randomness of potentials at distinct sites by independent reflections, we see that

$$\psi(n) \sim \prod_{j=1}^{n}(1-r_j), \tag{6}$$

where r_j is the reflection probability at site j. For weak coupling, the Born approximation says that r is proportional to the square of the potential strength, so that

$$\psi(n) \sim \prod_{j=1}^{n}[1-c(Fn)^{-1}] \sim n^{-c/F}. \tag{7}$$

Having shown this heuristic argument, let us just say that the upper bound of this kind, stated in Eq. (2) for small field, can then be proven rigorously[11] with use of techniques analogous to those of Delyon, Kunz, and Souillard[4] and Simon.[12]

Let us now turn to the lower bound stated in Eq. (2) and to the absence of localized states for large enough field. Let ϕ^+ and ϕ^- be two solutions of $-\phi''-Fx\phi=E\phi$ with Wronskian equal to 1. We can write any solution ψ of (1) on an interval $(n,n+1)$ as $\psi(x)=a_n\phi^+(x)+b_n\phi^-(x)$ and one deduces readily the recurrence relation

$$\begin{pmatrix}a_{n+1}\\ b_{n+1}\end{pmatrix}=\left[1+V_n\begin{pmatrix}\phi_+(n)\phi_-(n) & \phi_-^2(n)\\ -\phi_+^2(n) & -\phi_+(n)\phi_-(n)\end{pmatrix}\right]\begin{pmatrix}a_n\\ b_n\end{pmatrix}=(1+V_nM_n)\begin{pmatrix}a_n\\ b_n\end{pmatrix}. \tag{8}$$

Since the $\phi_\pm(n)$ are of magnitude $F^{-1/4}n^{-1/4}$, the matrix M_n is of magnitude $(Fn)^{-1/2}$. Furthermore, since $\mathrm{Tr}M=\det M=0$, we have $\det(1+V_nM_N)=1$ for all n. In order to get a *lower* bound on the solution of (8), we use the fact that it is sufficient[13] to get an *upper* bound on the growth of product of transfer matrices: Indeed we have

$$\begin{pmatrix}a_n\\ b_n\end{pmatrix}=A_n\begin{pmatrix}a_1\\ b_1\end{pmatrix}$$

so that

$$\left\|\begin{pmatrix}a_n\\ b_n\end{pmatrix}\right\| \geq \|A_n^{-1}\|^{-1}\left\|\begin{pmatrix}a_1\\ b_1\end{pmatrix}\right\|,$$

but A_n is a 2×2 matrix with determinant 1 and thus $\|A_n^{-1}\|^{-1}=\|A_n\|^{-1}$, which implies

$$\left\|\begin{pmatrix}a_n\\ b_n\end{pmatrix}\right\| \geq \|\prod_{i=1}^{n}(1+V_iM_i)\|^{-1}\left\|\begin{pmatrix}a_1\\ b_1\end{pmatrix}\right\|. \tag{9}$$

Fixing now the field F, and making the dependence on the energy E explicit, we set

$$g_n(E)=\mathrm{Tr}(P_n^*P_n),\quad P_n=\prod_{i=1}^{n}(1+V_iM_i). \tag{10}$$

If angular brackets denote averaging with respect to the disorder, because of the independence of the V's an easy calculation shows that $\langle g_n(E)\rangle \leq c_1 n^{c_2/F}$ with c_2 independent of E and c_1 independent of E on bounded sets. A slightly more involved calcuation shows that for any integer p

$$\langle g_n(E)^p\rangle \leq c_1 n^{c_2(p)/F},$$

$$\langle |g_n(E)-g_n(E')|^p\rangle \leq c_3|E-E'|^2 n^{2+c_4(p)/F}.$$

By using an idea similar to the one used by Kolmogorov in the study of stocastic processes,[14] one can deduce[11] from these inequalities that for almost all choices of V (i.e., with probability one),

$$g_n(E) \leq C_5 n^{4/p+C_6(p)/F}$$

uniformly for E in bounded sets. This proves power lower bounds on

$$\begin{pmatrix}a_n\\ b_n\end{pmatrix}$$

and thus on ψ:

$$|\psi(n)| \geq \tilde{C}_5^{-1} n^{-[1/4+4/p+C_6(p)/F]}.$$

By large enough choice of p, this also implies the absence of localized states for large F, and in particular shows that the spectrum is then purely continuous. This, in turn, implies the vanishing of the inverse participation ratio by the general arguments of Kunz and Souillard.[3]

In the same way as we studied model (1) here, we can also study a tight-binding model

$$(H\psi)(n)=\psi(n+1)+\psi(n-1)+(\lambda/n^\alpha)V_n\psi(n) \tag{11}$$

and show that, in the unperturbed spectrum (i.e., the interval $[-2, +2]$), it undergoes, when $\alpha = \frac{1}{2}$, a transition from a pure point spectrum with power-localized states to a continuous spectrum with nonlocalized states under variation of the coupling constant λ or the energy. We do not know if, when continuous, the spectrum is absolutely continuous or singular continuous, or if particles have diffusive behavior or not. The same question is pending for the case of the Kronig-Penney model with large electric field. For $\alpha < \frac{1}{2}$, (11) has pure point spectrum,[12] and for $\alpha > \frac{1}{2}$ it follows from the techniques of the present paper that the spectrum within $[-2, 2]$ is continuous.

As a conclusion, we have proven a transition from a regime of extended states to a regime of power-localized states in model (1) and model (11) with $\alpha = \frac{1}{2}$. We mentioned above that in the electric field case, if we have a smooth potential, the results are very different[6]: One may ask which case is relevant, and whether the Kronig-Penney potential is only of academic interest, introduced for its analytic simplicity. This question is of interest since there are nowadays experiments on thin wires, e.g., made of GaAs, which are thin enough to present 1D behavior at low temperature and one may see there, for example, the crossover from 2D to 1D behavior.[15] In some of these wires, the variations of potential at the impurities is much larger than the energies involved by the weak electric field used to measure the conductivity; the Kronig-Penney model is then the natural relevant model in these cases.

We finally mention that, while writing this Letter, we learned that Bentosela, Grecchi, and Zironi have developed numerical studies of model (1) and obtained results[16] beyond those of Ref. 6; they no longer compute a transmission coefficient but compute directly the behavior of eigenfunctions, and their numerical studies agree with our analytical results.

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