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Robust Adaptive Boundary Control of Semilinear PDE Systems Using a Dyadic Controller

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SUMMARY

In this paper, we describe a dyadic adaptive control (DAC) framework for output tracking in a class of semilinear systems of partial differential equations with boundary actuation and unknown distributed nonlinearities. The DAC framework uses the linear terms in the system to split the plant into two virtual sub-systems, one of which contains the nonlinearities, while the other contains the control input. Full-plant-state feedback is used to estimate the unmeasured, individual states of the two sub-systems as well as the nonlinearities. The control signal is designed to ensure that the controlled sub-system tracks a suitably modified reference signal. We prove well-posedness of the closed-loop system rigorously, and derive conditions for closed-loop stability and robustness using finite-gain $\mathcal{L}$ stability theory. Copyright © 0000 John Wiley & Sons, Ltd.

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KEY WORDS: PDE control; adaptive control; robust control; distributed parameter systems

1. INTRODUCTION

The dyadic adaptive control (DAC) framework was first presented and demonstrated successfully in experiments involving control of beam bending in [1, 2]. Its design was motivated by boundary control of partial differential equations described by

$$\dot{w}(t) \triangleq \frac{\partial w(t)}{\partial t} = \mathfrak{A}w(t) + f(t,w), \ \ \mathfrak{B}w(t) = u(t) \tag{1}$$

where $w(t)$ is the state of the system and $u(t)$ is the boundary control input. The operator $\mathfrak{A}$ is linear, while the nonlinearities are all captured by the function $f(\cdot)$.

A large body of work on the control of PDE systems has focussed on approximating the PDE system by ODEs, and using any of the rich assortment of the ODE control techniques [3, 4, 5, 6]. Control using ODE approximations, however, is known to be vulnerable to the so-called spillover instabilities [7, 8] which arise due to the PDE being approximated with an insufficient number of modes. In order to approximate the PDE accurately, finite order approximations generally require the inclusion of a large number of modes, which complicates the associated control problem, particularly if the PDE system is nonlinear.

In contrast, PDE-based control methods strive to leave the PDE intact for the purpose of designing the controller, and for proving the stability and the performance of the closed-loop systsem. These techniques fall into various classes, some of which we describe briefly here.

Operator-theoretic approaches [9] come closest to those for ordinary differential equations, by the virtue of similar system representation. These include optimal control methods [9, 10] as well as a class of adaptive architectures based on model reference adaptive control [11] and $\mathcal{L}_1$ adaptive control [12]. An alternative lies in employing approaches based around exploiting particular features of a given PDE. These include methods based on Lyapunov functions [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23] and optimal control techniques [24]. While the latter family of techniques typically yields strong theoretical guarantees, they require considerable information about the dynamics of the plant, and tend to lead to PDE-specific architectures.

The development of the DAC method was motivated by practical systems described by a combination of ODEs and PDEs, such as flexible aircraft wings [25] (flexible wing structure combined with the rigid body aircraft dynamics), robotic surgical systems (a multi-segmented flexible robotic arm), temperature control systems (heat diffusion and mass flow of air), etc. In addition to possibly nonlinearly forced PDEs, these systems admit additional finite order dynamics in the form of ODEs. A natural way to deal with such systems is to formulate the control problem in an operator-theoretic framework [9, 12], so that the control architecture is *dimension-independent*; i.e., it can be applied to finite or infinite dimensional systems (or a combination thereof).

The primary novelty of the DAC is the manner in which it uses the linear operator to isolate the control signal and the nonlinearity from each other, by decomposing the original system explicitly into two virtual sub-systems. The control design problem thus simplifies to one of designing observers for the two sub-systems (each of which is referred to as a *half*), and a tracking controller for the linear sub-system together with a suitably chosen reference signal. On the one hand, it opens up the possibility of using well-understood tools from linear control, such as the linear quadratic regulator (LQR), as demonstrated in [26]. This contrasts with the existing approaches to solving optimal control problems for semi-linear systems [24, 10]. On the other hand, it allows us to guarantee robustness rigorously using the small gain theorem, as shown in the present paper.

The DAC architecture is illustrated in Fig. 1. The *particular half* accommodates only the nonlinearity, while the *homogeneous half* accommodates only the control input. The control signal is designed to ensure that the output of the homogeneous half tracks the desired reference signal minus the output of the particular half, thereby ensuring that the output of the two halves put together tracks the reference signal.

The DAC uses full-state feedback to estimate the particular and homogeneous components of the state variable, as well as the nonlinearity. Note that "full-state feedback" refers to the state of the original system, and not the states of the homogeneous and the particular halves. The DAC can accommodate modeling and parametric uncertainties, external disturbances, as well as time delays.

In this paper, we prove the stability of the closed loop using the small gain theorem in the sense of finite-gain $\mathcal{L}$ stability. Bounds for the tracking error are derived, and the application of the DAC framework to systems consisting of a combination of PDEs and/or ODEs is presented. It must be pointed out that the analytical techniques used in the paper closely follow a related, recent paper [12]. However, there are two major differences between the present paper and [12]. The first point of distinction is that the present paper is concerned primarily with control of PDEs under boundary actuation. The second point of distinction is that the DAC is meant primarily for addressing unmatched nonlinearities and disturbances, such as those which arise naturally in boundary control systems with distributed nonlinear forcing terms.

The paper is organized as follows. The necessary preliminaries are recapitulated in Sec. 2, and the problem formulation is given in Sec. 3. The DAC control architecture has been presented in Sec. 4, and the closed-loop stability is proved in Sec. 5. Simulation results are presented in Sec. 6.

A preliminary version of this paper was presented as [27]. The present version is mathematically rigorous and adds a proof of well-posedness of the closed-loop system.

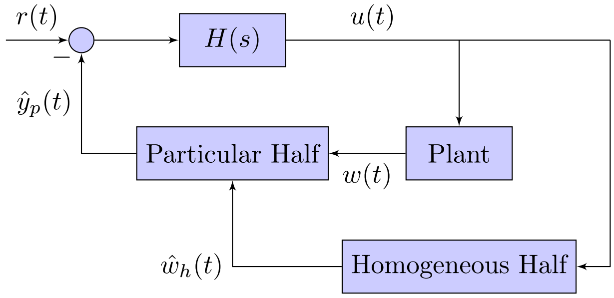

Figure 1. A block diagram of the DAC framework, with the subscripts p and h denoting signals from the particular and homogeneous components. The symbols $w(t)$, $y(t)$, and $r(t)$ denote the system state, output and reference signal, respectively.

2. PRELIMINARIES

Definition 1 ($\mathcal{L}_\infty$ and $\mathcal{L}_1$ norms)
Given $q(t) \in \mathbb{R}^n$ with components $q_i(t)$ $(1 \leq i \leq n)$, we define

$$\|q(t)\|_\infty = \max_{1\leq i\leq n} |q_i(t)|, \ \|q\|_{\mathcal{L}_\infty} = \operatorname*{ess\,sup}_{t\geq 0} \|q(t)\|_\infty$$
$$\|q\|_{\mathcal{L}_\infty,\tau} = \text{ess} \sup_{0\leq t\leq\tau} \|q(t)\|_\infty$$

If $\|q\|_{\mathcal{L}_\infty} < \infty$, then we denote $q \in \mathcal{L}_\infty^n$. The $\mathcal{L}_1$ norm of a linear operator $\mathcal{F} : \mathcal{L}_\infty^m \mapsto \mathcal{L}_\infty^n$ is defined as $\|\mathcal{F}\|_{\mathcal{L}_1} = \sup_{\|q\|_{\mathcal{L}_\infty}=1} \|\mathcal{F}q\|_{\mathcal{L}_\infty}, \ q \in \mathcal{L}_\infty^m$

The spatial domain of interest is the closed, one-dimensional interval $[0, L]$ for some finite $L > 0$. Let $\mathbb{Z}$ be a Hilbert space (in the spatial domain and suitably chosen for the system) of $\mathbb{R}^n$-valued functions, with the inner product $\langle z_1, z_2\rangle_{\mathbb{Z}}$ for all $z_1, z_2 \in \mathbb{Z}$, and with the norm $\|z\|_{\mathbb{Z}} = \sqrt{\int_0^L \langle z, z\rangle_{\mathbb{Z}}\, dx}$ for all $z \in \mathbb{Z}$.

Definition 2
We define the space $\mathbb{W}$ consisting of variables $w(t, x) \in \mathbb{R}^n$ for $x \in [0, L]$ and $t \in \mathbb{R}_{\geq 0}$, satisfying $w(t) \triangleq w(t, \cdot) \in \mathbb{Z}, \ \forall\, t \geq 0$ and ess $\sup_{t\geq 0} \|w(t)\|_{\mathbb{Z}} < \infty$. The space $\mathbb{W}$ is a Banach space with the norm $\|w\|_{\mathbb{W}} = \operatorname{ess\,sup}_{t\geq 0} \|w(t)\|_{\mathbb{Z}}$. We define the corresponding truncated norm as $\|w\|_{\mathbb{W},\tau} = \operatorname{ess\,sup}_{0\leq t\leq\tau} \|w(t)\|_{\mathbb{Z}}$, and the associated Banach space is denoted by $\mathbb{W}_\tau$. Clearly, $\mathbb{W} \subseteq \mathbb{W}_\tau$ for all $\tau \geq 0$.

Definition 3
The domain of an operator $\mathcal{V}$ is denoted by $\mathcal{D}(\mathcal{V})$. If $\mathcal{V} : X \to Y$ where X and Y are Banach spaces, (obviously, $\mathcal{D}(\mathcal{V}) \subset X$), then we denote the induced norm of $\mathcal{V}$ by $\|\mathcal{V}\|_{(X,Y)}$. If $\mathcal{V} : \mathbb{W} \to \mathbb{W}$, then we use the short-hand notation $\|\mathcal{V}\|_i$ in place of $\|\mathcal{V}\|_{(\mathbb{W},\mathbb{W})}$ for ease of representation.

Definition 4 ([28], Definition 1.1, Ch. 6)
Consider a system $\dot{w} = \mathcal{A}w + f(t, w), \ w(t = 0) = w_0 \in \mathbb{Z}$, where $\mathcal{A}$ is the infinitesimal generator of a C_0 semigroup $\mathcal{T}(t)$. The *mild solution* $w(t)$ is given by

$$w(t) = \mathcal{T}(t)w_0 + \int_0^t \mathcal{T}(t - s)f(s, w(s))\, ds \tag{2}$$

Definition 5 (Convolution)
Given a C_0 semi-group $\mathcal{T}(t)$ and $\tau \geq 0$, we define the convolution operator $\mathcal{T} \star (\tau) : \mathbb{W}_\tau \mapsto \mathbb{W}_\tau$ as $\mathcal{T} \star (\tau)f(\tau) = \int_0^\tau \mathcal{T}(\tau - s)f(s)\, ds$ for all $f \in \mathbb{W}_\tau$. We define the induced norm $\|\mathcal{T} * \|_i \triangleq$ ess $\sup_{(\tau\geq 0)} \|\mathcal{T} \star (\tau)\|_{(\mathbb{W}_\tau, \mathbb{W}_\tau)}$.

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Next, we recall a result from Pazy [28] which postulates the conditions under which the mild solution in Definition 4 is also a classical solution to the initial value problem.

Theorem 1 (Theorems 6.1.4, 6.1.5, [28])
Let $\mathcal{A}$ be the infinitesimal generator of a C_0 semigroup $\mathcal{T}(t)$ on $\mathbb{Z}$, and and let the initial condition $w_0 \in \mathcal{D}(\mathcal{A})$. If $f : [0,\, T] \times \mathbb{Z} \to \mathbb{Z}$ is continuously differentiable with respect to both arguments, for $T > 0$, then the mild solution (2) is a classical solution of the initial value problem (1) for $t \in [0,\, T]$. If the solution exists only up to $T_{\max} < T$, then $\|w(t)\|_{\mathbb{Z}} \to \infty$ as $t \to T_{\max}$.

Finally, we define the projection operator, following [29], which will be used for constructing the adaptive law in the paper. Let $\pi : \mathbb{R}^k \to \mathbb{R}$ be defined by

$$\pi(\alpha) \equiv \pi(\alpha; \kappa, \epsilon) = \frac{\langle \alpha,\, \alpha \rangle - \kappa^2}{\epsilon \kappa^2}, \quad \alpha \in \mathbb{R}^k, \ \kappa \in \mathbb{R}^+,$$

where $\epsilon \in \mathbb{R}^+$ can be arbitrarily small. The derivative of π at $\alpha_1 \in \mathbb{R}^k$, denoted by $\pi'(\alpha_1) \in \mathbb{R}^k$, satisfies $\langle \pi'(\alpha_1),\, \alpha_2 \rangle = \frac{2\langle \alpha_1, \alpha_2 \rangle}{\epsilon \kappa^2} \quad \forall \alpha_2 \in \mathbb{R}^k$.

Definition 6
The projection operator $\mathrm{Proj} :\ \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ is defined as

$$\mathrm{Proj}(\alpha_1, \alpha_2) = \begin{cases} \alpha_2, \ \text{if } \pi(\alpha_1) \leq 0 \text{ or } \langle \pi'(\alpha_1),\, \alpha_2 \rangle \leq 0 \\ \alpha_2 - \frac{\pi'(\alpha_1)}{\|\pi'(\alpha_1)\|_2} \left\langle \frac{\pi'(\alpha_1)}{\|\pi'(\alpha_1)\|_2},\, \alpha_2 \right\rangle \pi(\alpha_1), \ \text{otherwise} \end{cases}$$

The following property of the projection operator will be invoked in the proof of convergence of the observation error. Let Ω_0 and Ω_1 denote the convex sets satisfying

$$\Omega_0 = \{\alpha \mid \pi(\alpha) \leq 0\}, \quad \Omega_1 = \{\alpha \mid \pi(\alpha) \leq 1\}$$

Lemma 1 (Lemma 9, [29])
Suppose that $\alpha_1^* \in \Omega_0$. Then, for all $\alpha_1,\, \alpha_2 \in \mathbb{R}^k$, $(\alpha_1 - \alpha_1^*)\,(\mathrm{Proj}(\alpha_1,\, \alpha_2) - \alpha_2) \leq 0$. Moreover, the solution of the initial value problem $\dot{\alpha}_1 = \mathrm{Proj}\,(\alpha_1,\, \alpha_2)\,,\ \alpha_1(0) = \alpha_{10}$, has the property that if $\alpha_{10} \in \Omega_1$, then $\alpha_1(t) \in \Omega_1$ for all t.

3. PROBLEM FORMULATION

3.1. Problem Statement

We consider a class of systems described by PDEs of the form

$$\begin{aligned} &\dot{w}(t) = \mathfrak{A}w(t) + f(t, w),\ \mathfrak{B}w(t) = u(t) \\ &y(t) = \mathcal{C}w(t), \quad \|w(0)\|_{\mathbb{Z}} \leq \rho_0,\ \rho_0 > 0 \end{aligned} \tag{3}$$

where $\mathfrak{A} : \mathcal{D}(\mathfrak{A}) \subset \mathbb{Z} \mapsto \mathbb{Z}$; $u(t) \in \mathbb{U} = \mathbb{R}^{n_u}$, and the operator $\mathfrak{B} : \mathcal{D}(\mathfrak{B}) \subset \mathbb{Z} \mapsto \mathbb{U}$ captures the boundary actuation and satisfies $\mathcal{D}(\mathfrak{A}) \subset \mathcal{D}(\mathfrak{B})$. The operator $\mathcal{C} : \mathbb{Z} \mapsto \mathbb{R}^{n_y}$ is the output operator. The operators $\mathfrak{A}$, $\mathfrak{B}$ and $\mathcal{C}$ are assumed to be known, and $n_u \geq n_y$. We assume that the possibly nonlinear term on the right-hand side, $f(t, w) : \mathbb{R}_{\geq 0} \times \mathbb{Z} \to \mathbb{Z}$, can be written in the form

$$f(t, w) = \sum_{j=1}^{n_\alpha} \alpha_j(t)\, \phi_j(w), \tag{4}$$

where $\phi_j(w) : \mathbb{Z} \to \mathbb{Z}$ are known C^1 functions of w. The coefficients $\alpha_j(t) \in \mathbb{R}$ are assumed to be continuously differentiable in t and unknown, but with known bounds $\|\alpha_j\|_{\mathcal{L}_\infty} < \nu_\alpha$ and $\|\dot{\alpha}_j\|_{\mathcal{L}_\infty} < \nu_{\dot{\alpha}}$, $\forall\, j$. We have chosen the same bounding value for all j only for brevity.

The control objective is to ensure that the output $y(t)$ tracks a reference signal $r(t) \in C^1([0,\, \infty); \mathbb{R}^{n_y})$, and that $\|w\|_{\mathbb{W}}$ and $u(t)$ are bounded.

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3.2. Formulation and Assumptions

We prescribe that (3) is a boundary control problem; i.e., the operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \mapsto \mathbb{Z}$ with $\mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathfrak{A}) \cap \ker(\mathfrak{B})$ and satisfying $\mathcal{A}z = \mathfrak{A}z$ for all $z \in \mathcal{D}(\mathcal{A})$ is the infinitesimal generator of a C_0 semigroup $\mathcal{T}(t)$ on $\mathbb{Z}$. Furthermore, there exists a bounded operator $\boldsymbol{\beta} : \mathbb{U} \to \mathbb{Z}$ which satisfies $\boldsymbol{\beta}u \in \mathcal{D}(\mathfrak{A})$ for all $u \in \mathbb{U}$; the operator $\mathfrak{A}\boldsymbol{\beta} : \mathbb{U} \to \mathbb{Z}$ is bounded, and $\mathfrak{B}\boldsymbol{\beta}u = u,\ u \in \mathbb{U}$.

Consider the abstract differential equation on $\mathbb{Z}$:

$$\dot{v}(t) = \mathcal{A}v(t) - \boldsymbol{\beta}\dot{u}(t) + \mathfrak{A}\boldsymbol{\beta}u(t) + f(t, v + \boldsymbol{\beta}u),\ v(0) = v_0 \tag{5}$$

and suppose that (5) has a unique classical solution for $v_0 \in \mathcal{D}(\mathcal{A})$. Then, we can prove the following result as a direct extension of Theorem 3.3.3 from [9].

Lemma 2
Consider the boundary control problem (3) and the abstract Cauchy equation (5). Suppose that $u \in C^2([0,\,\tau])$ for all $\tau > 0$, and suppose that the classical solution of (5) is unique. Then, if $v(0) = w(0) - \boldsymbol{\beta}u(0) \in \mathcal{D}(\mathcal{A})$, the classical solutions of (3) and (5) are related by $v(t) = w(t) - \boldsymbol{\beta}u(t)$, and the classical solution of (3) is unique.

Proof: Since $u(t)$ and $\dot{u}(t)$ are both continuously differentiable functions of time, it follows that we can find a function $h(t, v)$ which is continuously differential with respect to t and v, depends implicitly on $u(t)$, and satisfies $h(t, v) = \mathfrak{A}\boldsymbol{\beta}u(t) - \boldsymbol{\beta}\dot{u}(t) + f(t, v + \boldsymbol{\beta}u)\ \forall t$. Thus, from Theorem 1, it follows that there exists a classical solution $v(t)$ for (5). The remainder of the proof is identical to that of Theorem 3.3.3 in [9]. ■

We now impose additional structure on the system (3). The first assumption asserts the stability of the semi-group $\mathcal{T}$, as well as the existence of a Lyapunov function corresponding to its infinitesimal generator $\mathcal{A}$.

Assumption 1
The C_0 semigroup $\mathcal{T}(t)$ (whose infinitesimal generator is $\mathcal{A}$) is exponentially stable; i.e., there exist constants $M, \omega \in \mathbb{R}^+$ such that $\|\mathcal{T}(t)\|_{(\mathbb{Z},\mathbb{Z})} \le Me^{-\omega t}\ \forall t \ge 0$. Moreover, $\|\mathcal{T} * \|_i$ is bounded, and there exists a *self-adjoint coercive* operator $\mathcal{P} > 0$ and a constant $\lambda_P > 0$ such that

$$\langle \mathcal{A}z, \mathcal{P}z\rangle_{\mathbb{Z}} + \langle \mathcal{P}z, \mathcal{A}z\rangle_{\mathbb{Z}} \le -\lambda_P \langle z, \mathcal{P}z\rangle_{\mathbb{Z}},\ \forall\ z \in \mathcal{D}(\mathcal{A}) \tag{6}$$

When $\mathcal{T}(t)$ is a *group*, the operator $\mathcal{P}$ of Assumption 1 is bounded on $\mathcal{D}(\mathcal{A})$; for other problems, such as the heat equation, it may be unbounded [12, 30].

Assumption 2
The initial condition $w_0 \in \mathcal{D}(\mathfrak{A})$, and $u(0) = \mathfrak{B}w_0$. This ensures that $v_0 \triangleq w_0 - \boldsymbol{\beta}u(0) \in \mathcal{D}(\mathcal{A})$.

Assumption 3
The output operator $\mathcal{C}$ in (3) is bounded; i.e., $\|y\|_{\mathcal{L}_\infty} = \|\mathcal{C}w\|_{\mathcal{L}_\infty} \le K\|w\|_{\mathbb{W}}$ for some $K > 0$. This is also true if truncated norms are used.

The system dynamics (3) can be viewed as the sum of a linear, exponentially stable, well-posed operator and an external nonlinear forcing term. The control design method can be used for systems of the form $\dot{w} = \mathfrak{A}_g w + f(t, w)$, where $\mathfrak{A}_g$ need not be stable. An extension of the DAC to such systems is examined rigorously in [26], and also illustrated (without a formal proof) via an example in Section VI.

Assumption 4
For every $\rho > 0$, there exist positive constants $\nu_{\phi,1}(\rho)$ and $\nu_{\phi,2}(\rho)$ such that if $\|w(t)\|_{\mathbb{Z}} \le \rho$ for some $t > 0$, then $\|\phi_j(w)\|_{\mathbb{Z}} \le \nu_{\phi,1}(\rho)\|w(t)\|_{\mathbb{Z}} + \nu_{\phi,2}(\rho),\ \forall j$. In general, $\nu_{\phi,1}(\rho)$ and $\nu_{\phi,2}(\rho)$ are class $\mathcal{K}$ functions of ρ.

The following result follows directly from (4) and Assumption 4.

Lemma 3
if $\|w\|_{\mathbb{W},\tau} \le \rho$ for some $\tau > 0$, then $\|f(t, w)\|_{\mathbb{W},\tau} \le \nu_1(\rho)\|w\|_{\mathbb{W},\tau} + \nu_2(\rho)$, where $\nu_1(\rho) = n_\alpha \nu_\alpha \nu_{\phi,1}(\rho)$ and $\nu_2(\rho) = n_\alpha \nu_\alpha \nu_{\phi,2}(\rho)$.

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4. DYADIC ADAPTIVE CONTROLLER DESIGN

4.1. State Observers for the Homogeneous and Particular Sub-Systems

We use the symbol $\hat{(\cdot)}$ to denote observer states, and the subscripts p and h to denote states of the particular and the homogeneous halves, respectively. The dynamics of the two halves (observers for the virtual sub-systems) are given by

$$\dot{\hat{w}}_p = \mathfrak{A}\hat{w}_p + \hat{f}(t,w),\ \ \mathfrak{B}\hat{w}_p = 0,\ \ \hat{y}_p = \mathcal{C}\hat{w}_p \tag{7}$$

$$\dot{\hat{w}}_h = \mathfrak{A}\hat{w}_h,\ \ \mathfrak{B}\hat{w}_h = u(t),\ \ \hat{y}_h = \mathcal{C}\hat{w}_h \tag{8}$$

We choose the initial conditions as follows: $\hat{w}_p(0) = 0$ and $\hat{w}_h(0) = w(0)$. This choice of initial conditions, which ensures that $\hat{w}(0) = w(0)$, will be essential in proving the convergence of the observer error dynamics in Lemma 6.

We write $\hat{f}(\cdot)$ in the form (4), accompanied by the projection operator from Definition 6:

$$\begin{aligned}
\hat{f}(t,w) \quad &= \quad \sum_{j=1}^{n_\alpha} \hat{\alpha}_j(t)\phi_j(w),\ \ 1 \le j \le n_\alpha, && (9)\\
\dot{\hat{\alpha}}_j(t) \quad &= \quad \gamma\,\mathrm{Proj}\left(\hat{\alpha}_j(t),\ -\langle \mathcal{P}\tilde{w}(t), \phi_j(w)\rangle_{\mathbb{Z}}\right), && (10)\\
|\hat{\alpha}_j(t)| \quad &< \quad \nu_\alpha(1+\epsilon),
\end{aligned}$$

where $\epsilon \in \mathbb{R}^+$ is arbitrarily small (see Lemma 1); $\tilde{w} = \hat{w}_p + \hat{w}_h - w$, and $\gamma > 0$ is the adaptation gain.

Remark 1
One may use an *a-priori* estimate for the nonlinearity in place of the projection-based law of (10), as long as the observer states satisfy boundedness properties described in Sec 5.

4.2. Control Signal Design

We design the control signal $u(t)$ to ensure that the output of the homogeneous half, $\hat{y}_h(t)$ in (8), tracks a reference signal $r(t) - \hat{y}_p(t)$, where $r(t)$ is the reference signal for the original system (3).

The input-output dynamics of the linear, exponentially stable homogeneous half (8) can be expressed in the Laplace domain as $\hat{Y}_h(s) = G_c(s)U(s)$, where the transfer function $G_c(s)$ depends on $\mathcal{A}$, as explained in [31]. We choose a control signal of the form

$$U(s) = H(s)(R(s) - \hat{Y}_p(s)) \tag{11}$$

To satisfy the conditions for Lemma 2 in the next section, we require that each element of the transfer function matrix $H(s)$ be stable with a relative degree large enough (≥ 2) to ensure that $u(t) \in C^2([0,\tau])$ for all $\tau > 0$. Moreover, we impose the condition that $G_c(0)H(0) = \mathbb{I}_{n_y}$, the $n_y \times n_y$ identity matrix.

In this sequel, we need a minimal state space realization of (11), such as the observer canonical form,

$$\dot{p}(t) = H_A\,p(t) + H_B\,(r(t) - \hat{y}_p(t)),\quad u(t) = H_C\,p(t),\quad p(0) = 0, \tag{12}$$

where $p(t) \in \mathbb{R}^{n_p}$ $(n_p \ge 2)$ and the matrix H_A is Hurwitz. Since $u(0) = 0$, the Laplace transform of $\dot{u}(t)$ is given by

$$\mathcal{L}(\dot{u}(t)) \quad = \quad sH(s)(R(s) - \hat{Y}_p(s)) \tag{13}$$

Remark 2
The control signal need not be restricted to the form of (11). Rather, it is important that the control signal satisfy the bounds established in the next section and ensure that the output tracking error of the homogeneous half is acceptably small.

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4.3. Generalization to PDE-ODE Systems

The DAC framework described in Sec 4.1 and 4.2 can be applied *mutatis mutandis* to general systems

$$\dot{w} = \mathfrak{A}w + \boldsymbol{\beta}_1 u_1 + f(t, w),\ \ \mathfrak{B}w = u_2 \tag{14}$$

where the underlying Hilbert space is $\mathbb{V} = \mathbb{Z} \times \mathbb{R}^m$, and $m \geq 0$. The "in-line" control operator $\boldsymbol{\beta}_1$ is assumed to be linear and bounded. When we set $\mathfrak{B}w = 0$ for all w, we recover the usual control system with distributed control action and with the additional possibility of including unmatched nonlinearities and disturbances. On the other hand, setting $\boldsymbol{\beta}_1 = 0$ gives us the boundary control formulation of the present paper.

5. CLOSED-LOOP TRACKING AND STABILITY ANALYSIS

The main result of this section determines the conditions under which there exists a constant $\rho > 0$ such that $\|w\|_{\mathbb{W}} < \rho$. We follow the route that is commonly taken while proving results using the small gain theorem; i.e., proof by contradiction. We first prove the boundedness of the observer states and the control signal by assuming that $\|w\|_{\mathbb{W}} \leq \rho$. Thereafter, we show that if a certain small gain condition is satisfied, then $\|w\|_{\mathbb{W}}$ will always be smaller than ρ.

We start by constructing the abstract Cauchy equations for the homogeneous and the particular halves:

$$\dot{\hat{v}}_p = \mathcal{A}\hat{v}_p + \hat{f}(t, v + \boldsymbol{\beta}u) \tag{15}$$

$$\dot{\hat{v}}_h = \mathcal{A}\hat{v}_h + \mathfrak{A}\boldsymbol{\beta}u - \boldsymbol{\beta}\dot{u} \tag{16}$$

Next, we construct two augmented vectors $\bar{v} = [v,\, \hat{v}_p,\, \hat{v}_h, p^\top]^\top$ and $\bar{\bar{v}} = [\bar{v}^\top,\, \tilde{\alpha}_1(t), \ldots, \tilde{\alpha}_{n_\alpha}(t)]^\top$, where $p(t)$ was defined in (12). The vector $\bar{v} \in \mathbb{V} = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{R}^{n_p}$, while $\bar{\bar{v}} \in \mathbb{V}_e = \mathbb{V} \times \mathbb{R}^{n_\alpha}$. It is quite straight-forward to check that $\mathbb{V}$ as well $\mathbb{V}_e$ are both Hilbert spaces. We write the complete closed-loop as a hybrid PDE-ODE system:

$$\frac{\partial \bar{\bar{v}}}{\partial t} = \bar{\bar{\mathcal{A}}}\,\bar{\bar{v}} + \bar{\bar{f}}(t, \bar{\bar{v}}), \tag{17}$$

$$\bar{\bar{\mathcal{A}}} = \begin{bmatrix} \mathcal{A} & \boldsymbol{\beta}H_C H_B \mathcal{C} & 0 & -\boldsymbol{\beta}H_C H_A + \mathfrak{A}\boldsymbol{\beta}H_C & 0 \\ 0 & \mathcal{A} & 0 & 0 & 0 \\ 0 & \boldsymbol{\beta}H_C H_B \mathcal{C} & \mathcal{A} & -\boldsymbol{\beta}H_C H_A + \mathfrak{A}\boldsymbol{\beta}H_C & 0 \\ 0 & -H_B\mathcal{C} & 0 & H_A & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\bar{\bar{f}}(t, \bar{\bar{v}}) = \begin{bmatrix} f(t, v + \boldsymbol{\beta}u) - \boldsymbol{\beta}H_C H_B r(t) \\ \hat{f}(t, v + \boldsymbol{\beta}u) \\ -\boldsymbol{\beta}H_C H_B r(t) \\ H_B r(t) \\ \gamma\,\mathrm{Proj}\,(\hat{\alpha}_1(t),\, -\langle \mathcal{P}\tilde{w}(t), \phi_1(w)\rangle_{\mathbb{Z}}) \\ \vdots \\ \gamma\,\mathrm{Proj}\,(\hat{\alpha}_{n_\alpha}(t),\, -\langle \mathcal{P}\tilde{w}(t), \phi_{n_\alpha}(w)\rangle_{\mathbb{Z}}) \end{bmatrix}$$

Note that the projection operator and the reference signal $r(t)$ have been lumped into the nonlinearity. We also construct a restricted closed-loop system by treating $\hat{\alpha}(t)$ as an exogenous

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signal:

$$\frac{\partial \bar{v}}{\partial t} = \bar{\mathcal{A}}\bar{v} + \bar{f}(t,\bar{v}), \tag{18}$$

$$\bar{\mathcal{A}} = \begin{bmatrix} \mathcal{A} & \boldsymbol{\beta} H_C H_B \mathcal{C} & 0 & -\boldsymbol{\beta} H_C H_A + \mathfrak{A}\boldsymbol{\beta} H_C \\ 0 & \mathcal{A} & 0 & 0 \\ 0 & \boldsymbol{\beta} H_C H_B \mathcal{C} & \mathcal{A} & -\boldsymbol{\beta} H_C H_A + \mathfrak{A}\boldsymbol{\beta} H_C \\ 0 & -H_B\mathcal{C} & 0 & H_A \end{bmatrix}$$

$$\bar{f}(t,\bar{v}) = \begin{bmatrix} f(t, v+\boldsymbol{\beta}u) - \boldsymbol{\beta} H_C H_B r(t) \\ \hat{f}(t, v+\boldsymbol{\beta}u) \\ -\boldsymbol{\beta} H_C H_B r(t) \\ H_B r(t) \end{bmatrix}$$

It is easy to check (e.g., using Lemma 3.2.2 from [9]) that $\bar{\mathcal{A}}$ as well as $\bar{\bar{\mathcal{A}}}$ are both infinitesimal generators of C_0 semigroups. We state the following result as a corollary to (Theorem 1, [12]).

Lemma 4
There exists a time $T_{\max} > 0$ such that the mild solution of (18), $\bar{v}(t)$ found using (2), is unique and continuously differentiable for $t \le T_{\max}$. Moreover, if $T_{\max}$ is finite, then $\lim_{t\to T_{\max}} \|\bar{v}(t)\|_{\mathbb{V}} \to \infty$

Proof: The projection operator in Definition 6 is locally Lipschitz with respect to its arguments (Lemma 1, [12]), and the other terms in $\bar{\bar{f}}(\cdot)$ are C^1 functions of their argument and time. Therefore, the closed-loop system (17) has a unique mild solution (Theorem 6.1.4, [28]). Since the mild solution is a continuous function of time, the projection operator is also a continuous function of time and hence, $\hat{\alpha}_j(t)$ are C^1 functions of time. Furthermore, the projection operator ensures that $\hat{\alpha}_j(t)$ are bounded for all j. Hence, we can treat $\hat{\alpha}_j(t)$ as bounded, C^1 exogenous signals for (18). As a result, $\bar{f}(t,\bar{v})$ is C^1 with respect to its arguments, and from Theorem 1, we deduce that the mild solution of (18) is a classical solution for $t \le T_{\max}$. The final statement of this lemma follows from Theorem 1. ■

The above result ensures that the time-varying state variables of the system and the observer are differentiable with respect to time. In the results that follow, we will prove that $T_{\max}$ cannot be finite, and hence that the states are bounded for all time.

Lemma 5
If $\|w\|_{\mathbb{W},\tau} \le \rho$ for some $\tau > 0$, then there exist constants $\kappa_0 \equiv \kappa_0(\rho)$ and $\kappa_1 \equiv \kappa_1(\rho)$ such that $\|\hat{w}_p\|_{\mathbb{W},\tau} \le \kappa_0 \|w\|_{\mathbb{W},\tau} + \kappa_1$

Proof: From Lemma 4, it follows that the solution $\hat{v}_p(t)$ found using (2) is a classical solution to (15). Thus, the solution $\hat{w}_p = \hat{v}_p$ is a classical solution to the boundary form of the particular half, (8). Using the projection operator for obtaining $\hat{f}(\cdot)$, one can ensure that there exist constants $\kappa_{01}(\rho)$ and $\kappa_{02}(\rho)$ such that $\forall\, t \le \tau,\ \|\hat{f}(t,w)\|_{\mathbb{Z}} \le \kappa_{01}(\rho)\|w(t)\|_{\mathbb{Z}} + \kappa_{02}(\rho)$. Next, using the formula for the mild solution $\hat{v}_p$ (Definition 4), the fact that $\hat{w}_p = \hat{v}_p$, and the definition of the convolution operator (Definition 5), we get $\forall\, t \le \tau$

$$\begin{aligned} & \hat{w}_p(t) = \mathcal{T}(t) \star \hat{f}(t,w), \quad \hat{w}_p(0) = 0 \\ \Longrightarrow \quad & \|\hat{w}_p\|_{\mathbb{W},\tau} \le \|\mathcal{T} * \|_i \big(\kappa_{01}(\rho)\|w\|_{\mathbb{W},\tau} + \kappa_{02}(\rho)\big) \end{aligned}$$

If we define $\kappa_j(\rho) = \|\mathcal{T} * \|_i \kappa_{0j}(\rho)$, for $j = 1, 2$, we get the desired result. ■

Since $\hat{y}_p = \mathcal{C}\hat{w}_p$, it follows as a direct application of Assumption 3 that $\|\hat{y}_p\|_{\mathcal{L}_\infty,\tau}$ is bounded.

Corollary 1
If $\|w\|_{\mathbb{W},\tau} \le \rho$ for some $\tau > 0$, then $\|\hat{y}_p\|_{\mathcal{L}_\infty,\tau} \le K(\kappa_0(\rho)\|w\|_{\mathbb{W},\tau} + \kappa_1(\rho))$, where κ_0 and κ_1 were defined in Lemma 5 and K in Assumption 3.

Theorem 2
Suppose that $\|w\|_{\mathbb{W},\tau} \leq \rho$ and in (12), $\dim(p) \geq 2$. Then, there exist constants $\delta_{iw} \equiv \delta_{iw}(H(s),\rho)$, $\delta_{ir} \equiv \delta_{ir}(H(s),\rho)$ and $\delta_{iu} \equiv \delta_{iu}(H(s),\rho)$ for $i = 0, 1$ such that $\|u\|_{\mathcal{L}_\infty,\tau} \leq \delta_{0w}\|w\|_{\mathbb{W},\tau} + \delta_{0r}\|r\|_{\mathcal{L}_\infty,\tau} + \delta_{0u}$ and $\|\dot{u}\|_{\mathcal{L}_\infty,\tau} \leq \delta_{1w}\|w\|_{\mathbb{W},\tau} + \delta_{1r}\|r\|_{\mathcal{L}_\infty,\tau} + \delta_{1u}$.

Proof : It follows from (11), (13) and Corollary 1 that

$$\begin{aligned} \|u\|_{\mathcal{L}_\infty,\tau} &\leq \|H(s)\|_{\mathcal{L}_1}(\|\hat{y}_p\|_{\mathcal{L}_\infty,\tau} + \|r\|_{\mathcal{L}_\infty,\tau}) \\ \|\dot{u}\|_{\mathcal{L}_\infty,\tau} &\leq \|sH(s)\|_{\mathcal{L}_1}(\|\hat{y}_p\|_{\mathcal{L}_\infty,\tau} + \|r\|_{\mathcal{L}_\infty,\tau}) \end{aligned} \tag{19}$$

Note that, since $H(s)$ is strictly proper, $\|sH(s)\|_{\mathcal{L}_1}$ is finite. The exact expressions for δ_{iu}, δ_{iw} and δ_{ir} can be found readily from the above equations. This completes the proof. ■

Recall that we defined the observation error $\tilde{w} = \hat{w}_p + \hat{w}_h - w$. Likewise, we define the observer output error $\tilde{y} = \hat{y}_p + \hat{y}_h - y$. To prove the boundedness of the observation error, we use (5), (15) and (16) to construct the abstract Cauchy equation

$$\dot{\tilde{v}} = \mathcal{A}\tilde{v} + \hat{f}(t, v + \boldsymbol{\beta}u) - f(t, v + \boldsymbol{\beta}u),\ \tilde{y}(t) = \mathcal{C}\tilde{v}(t) \tag{20}$$

From Lemma 4, it follows that the solution $\tilde{v}(t)$, found by applying the formula (2) to (20), is a classical solution of (20). Moreover, $\tilde{w} = \tilde{v}$ is a classical solution to the observation error equation in the boundary form. We show next that the observation error is uniformly bounded.

Lemma 6
If $\|w\|_{\mathbb{W},\tau} \leq \rho$ for some $\tau > 0$, then the observation errors $\|\tilde{w}(t)\|_{\mathbb{Z}}$ and $\|\tilde{y}(t)\|_\infty$, are bounded $\forall\, t \leq \tau$. Moreover, the bound can be made arbitrarily small by increasing γ.

Proof: Consider the Lyapunov function $V = \langle \tilde{v}(t), \mathcal{P}\tilde{v}(t)\rangle + \frac{1}{\gamma}\sum_{j=1}^{n_\alpha}(\tilde{\alpha}_j(t))^2$ where $\tilde{\alpha}_j(t) = \hat{\alpha}_j(t) - \alpha_j(t)$. Since $\tilde{v}(t)$ is a classical solution to (20), it follows that $\tilde{v}(t)$ and $V(t)$ are C^1 functions of time. It is easy to check that the Lyapunov function is positive definite with respect to $(\tilde{v}, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_{n_\alpha}) \in \mathbb{Z} \times \mathbb{R}^{n_\alpha}$: the first term, $\langle \tilde{v}(t), \mathcal{P}\tilde{v}(t)\rangle$ is positive definite from Assumption 1 and the last term is positive definite by the virtue of being a sum of squares.

Differentiating $V(t)$ with respect to time, we get

$$\begin{aligned} \dot{V}(t) &= \langle \dot{\tilde{v}}(t), \mathcal{P}\tilde{v}(t)\rangle_{\mathbb{Z}} + \langle \tilde{v}(t), \mathcal{P}\dot{\tilde{v}}(t)\rangle_{\mathbb{Z}} + \frac{2}{\gamma}\sum_{j=1}^{n_\alpha}\tilde{\alpha}_j(t)\dot{\tilde{\alpha}}_j(t) \\ &= \langle \mathcal{A}\tilde{v}(t),\, \mathcal{P}\tilde{v}(t)\rangle_{\mathbb{Z}} + \langle \mathcal{P}\tilde{v}(t),\, \mathcal{A}\tilde{v}(t)\rangle_{\mathbb{Z}} \\ &\quad + \langle \sum_{j=1}^{n_\alpha}\tilde{\alpha}_j(t)\phi_j(w),\, \mathcal{P}\tilde{v}(t)\rangle_{\mathbb{Z}} + \langle \mathcal{P}\tilde{v}(t),\, \sum_{j=1}^{n_\alpha}\tilde{\alpha}_j(t)\phi_j(w)\rangle_{\mathbb{Z}} \\ &\quad + \frac{1}{\gamma}\sum_{j=1}^{n_\alpha}\tilde{\alpha}_j(t)(\dot{\hat{\alpha}}_j(t) - \dot{\alpha}_j(t)) + \frac{1}{\gamma}\sum_{j=1}^{n_\alpha}(\dot{\hat{\alpha}}_j(t) - \dot{\alpha}_j(t))\tilde{\alpha}_j(t) \end{aligned} \tag{21}$$

Since $\tilde{v} = \tilde{w}$, we note that

$$\langle \mathcal{P}\tilde{v}(t),\, \sum_{j=1}^{n_\alpha}\tilde{\alpha}_j(t)\phi_j(w)\rangle_{\mathbb{Z}} = \sum_{j=1}^{n_\alpha}\tilde{\alpha}_j(t)\langle \mathcal{P}\tilde{w}(t),\, \phi_j(w)\rangle_{\mathbb{Z}}$$

Using (10) and Lemma 1, we get for all $j \in [1,\, n_\alpha]$ that

$$\begin{aligned} &\frac{1}{\gamma}\tilde{\alpha}_j(t)\dot{\hat{\alpha}}_j(t) + \tilde{\alpha}_j(t)\langle \mathcal{P}\tilde{w}(t),\, \phi_j(w)\rangle_{\mathbb{Z}} \\ &= (\hat{\alpha}_j(t) - \alpha_j(t))\,(\mathrm{Proj}(\hat{\alpha}_j(t),\, -\langle \mathcal{P}\tilde{w}(t),\, \phi_j(w)\rangle_{\mathbb{Z}}) - (-\langle \mathcal{P}\tilde{w}(t),\, \phi_j(w)\rangle_{\mathbb{Z}})) \leq 0 \end{aligned} \tag{22}$$

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Substituting (22) into (21), together with (6), we get

$$\dot{V}(t) \leq -\lambda_P \langle \tilde{v}, \mathcal{P}\tilde{v}\rangle_{\mathbb{Z}} - \frac{2}{\gamma}\left(\sum_{j=1}^{n_\alpha} \tilde{\alpha}_j(t)\dot{\alpha}_j(t)\right)$$

By adding and subtracting $(\lambda_P/\gamma)\sum_{j=1}^{n_\alpha}(\tilde{\alpha}_j(t))^2$, we get $\dot{V} \leq -\lambda_p V + (\lambda_P/\gamma)\sum_{j=1}^{n_\alpha}(\tilde{\alpha}_j(t))^2 - (2/\gamma)\sum_{j=1}^{n_\alpha}\tilde{\alpha}_j(t)\dot{\alpha}_j(t)$. Since $\dot{\alpha}_j$, $\hat{\alpha}_j$ and α_j are bounded for all t with known bounds, it follows that there exists a constant $\rho_1 > 0$, which is independent of γ, such that $\dot{V} \leq -\lambda_P V + \rho_1/\gamma$. Applying the comparison lemma, we get

$$V(t) \leq V(0)e^{-\lambda_P t} + \frac{\rho_1}{\lambda_P \gamma}(1 - e^{-\lambda_P t}) \tag{23}$$

The choice of the initial conditions in Sec. 4.1 ensures that $\tilde{w}(0) = 0$, and hence, $V(0) = \sum_{j=1}^{n_\alpha}(\tilde{\alpha}_j(0))^2/\gamma$. Furthermore, since $\mathcal{P}$ is assumed to be coercive, it follows that there exists a constant ρ_2 satisfying $\rho_2\|\tilde{v}(t)\|_{\mathbb{Z}}^2 \leq \langle\tilde{v}(t), \mathcal{P}\tilde{v}(t)\rangle_{\mathbb{Z}} \leq V(t)$. Substituting into (23), we deduce that there exists a constant $c > 0$ independent of γ, such that $\|\tilde{w}(t)\|_{\mathbb{Z}} = \|\tilde{v}(t)\|_{\mathbb{Z}} \leq c/\sqrt{\gamma}$. This proves that $\tilde{w}(t)$ is uniformly bounded, and the bound can be made arbitrarily small by increasing γ. Since $\tilde{y}(t) = \mathcal{C}\tilde{w}(t)$, and $\mathcal{C}$ is bounded (Assumption 3), it follows that $\|\tilde{y}(t)\|_\infty \leq Kc/\sqrt{\gamma}$. ■

Remark 3
If the semi-group generated by $\mathcal{A}$ is a *group* (e.g., for the wave equation), the requirement that $\tilde{w}(0) = 0$ can be relaxed because it is possible to find a Lyapunov operator $\mathcal{P}$ that, in addition to satisfying the terms of Assumption 1, is bounded on $\mathcal{D}(\mathcal{A})$.

Assumption 5 (Small-gain condition)
We assume that there exist constants ρ and ρ_0, an arbitrarily small $\epsilon_s > 0$, and a stable strictly proper $H(s)$, with relative degree ≥ 2, such that:

$$\frac{\Delta_1\|r\|_{\mathcal{L}_\infty} + \Delta_2}{1 - \Delta_0} \leq \rho - \epsilon_s \tag{24}$$

where $\Delta_0 = \|\mathcal{T} * \|_i\delta_0 + \|\boldsymbol{\beta}\|_{(\mathbb{R}^{n_u},\mathbb{Z})}\delta_{0w}$, $\Delta_1 = \|\mathcal{T} * \|_i\delta_1 + \|\boldsymbol{\beta}\|_{(\mathbb{R}^{n_u},\mathbb{Z})}\delta_{0r}$, $\Delta_2 = \|\mathcal{T} * \|_i\delta_2 + \|\boldsymbol{\beta}\|_{(\mathbb{R}^{n_u},\mathbb{Z})}\delta_{0u} + M\rho_0$, and $\delta_0 = \nu_1(\rho) + \|\mathfrak{A}\boldsymbol{\beta}\|_{(\mathbb{R}^{n_u},\mathbb{Z})}\delta_{0w} + \|\boldsymbol{\beta}\|_{(\mathbb{R}^{n_u},\mathbb{Z})}\delta_{1w}$, $\delta_1 = \|\mathfrak{A}\boldsymbol{\beta}\|_{(\mathbb{R}^{n_u},\mathbb{Z})}\delta_{0r} + \|\boldsymbol{\beta}\|_{(\mathbb{R}^{n_u},\mathbb{Z})}\delta_{1r}$, and $\delta_2 = \nu_2(\rho) + \|\mathfrak{A}\boldsymbol{\beta}\|_{(\mathbb{R}^{n_u},\mathbb{Z})}\delta_{0u} + \|\boldsymbol{\beta}\|_{(\mathbb{R}^{n_u},\mathbb{Z})}\delta_{1u}$. The constants $\delta_{i(\cdot)}$ were derived in Theorem 2, while $\nu_i(\rho)$ were defined in Lemma 3.

We are now ready to state and prove the main result of this section.

Theorem 3
Consider The closed-loop system (3), (7), (8), (10), and (11). Then, the state w is bounded in the sense of $\mathbb{W}$ for all times if Assumption 5 is satisfied.

Proof: We will prove this result by contradiction. Suppose that $\|w(\tau)\|_{\mathbb{Z}} = \rho$ for some time $\tau > 0$ and that $\|w(t)\|_{\mathbb{Z}} < \rho$, $\forall\, t < \tau$. Recall the abstract Cauchy differential equation (5) in Lemma 2:

$$\dot{v} = \mathcal{A}v + \mathfrak{A}\boldsymbol{\beta}u - \boldsymbol{\beta}\dot{u} + f(t, v + \boldsymbol{\beta}u)$$

From Lemma 4, the classical solution, $v(t)$, is found by applying (2) to the abstract Cauchy equation. It follows from Lemma 2 that $w(t) = v(t) + \boldsymbol{\beta}u(t)$ is a classical solution to (3). This gives $w(t) = \mathcal{T}(t)w(0) + \mathcal{T}(t) \star (\mathfrak{A}\boldsymbol{\beta}u(t) - \boldsymbol{\beta}\dot{u}(t) + f(t, w)) + \boldsymbol{\beta}u(t)$. Using Assumption 1,

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Lemma 3, and Theorem 2, we get

$$
\begin{aligned}
\|w(\tau)\|_{\mathbb{Z}} \;\; & \leq \;\; M\|w(0)\|_{\mathbb{Z}} + \|\mathcal{T}*\|_i\big(\|\mathfrak{A}\boldsymbol{\beta}\|_{(\mathbb{R}^{n_u},\mathbb{Z})}\|u\|_{\mathcal{L}_\infty,\tau} + \|\boldsymbol{\beta}\|_{(\mathbb{R}^{n_u},\mathbb{Z})}\|\dot{u}\|_{\mathcal{L}_\infty,\tau} \\
& \qquad +\nu_1(\rho)\|w\|_{\mathbb{W},\tau} + \nu_2(\rho)\big) + \;\|\boldsymbol{\beta}\|_{(\mathbb{R}^{n_u},\mathbb{Z})}\|u\|_{\mathcal{L}_\infty,\tau} \\
& \leq \;\; M\rho_0 + \|\boldsymbol{\beta}\|_{(\mathbb{R}^{n_u},\mathbb{Z})}(\delta_{0w}\rho + \delta_{0r}\|r\|_{\mathcal{L}_\infty} + \delta_{0u}) + \\
& \qquad \|\mathcal{T}*\|_i\Bigg(\|\mathfrak{A}\boldsymbol{\beta}\|_{(\mathbb{R}^{n_u},\mathbb{Z})}(\delta_{0w}\rho + \delta_{0r}\|r\|_{\mathcal{L}_\infty} + \delta_{0u}) \\
& \qquad +\|\boldsymbol{\beta}\|_{(\mathbb{R}^{n_u},\mathbb{Z})}(\delta_{1w}\rho + \delta_{1r}\|r\|_{\mathcal{L}_\infty} + \delta_{1u}) + \nu_1(\rho)\rho + \nu_2(\rho)\Bigg)
\end{aligned} \tag{25}
$$

Grouping the terms together and noting that $\|w(\tau)\|_{\mathbb{Z}} = \rho$, the above inequality implies that $\rho \leq \frac{\Delta_1\|r\|_{\mathcal{L}_\infty}+\Delta_2}{1-\Delta_0} \leq \rho - \epsilon_s$, the latter inequality being the small-gain condition in (24). We thus have a contradiction to our initial assumption that $\|w(\tau)\|_{\mathbb{Z}} = \rho$, and $\|w\|_{\mathbb{W},\tau} < \rho$. From Lemma 4, it follows that $w(t)$ exists and is unique for all t, else it should have diverged to infinity for some $T_{\max} > 0$. Therefore, it follows that $\|w\|_{\mathbb{W}} < \rho$. This completes the proof. ■

Remark 4
Although Theorem 3 only posits a bound on $\|w\|_{\mathbb{W}}$, it must be noted that this bound automatically ensures that the control input is bounded, due to Theorem 2. Furthermore, Lemma 4 ensures that the remaining states (i.e., $\hat{w}_p$, $\hat{w}_h$, p, and $\hat{\alpha}$) are also bounded.

Theorem 4
If $\lim_{t\to\infty}\{\hat{y}_p(t), r(t), y(t)\}$ exist, then the control law in (11) ensures that $\lim_{t\to\infty}\|y(t) - r(t)\|_\infty \leq \dfrac{Kc}{\sqrt{\gamma}}$, where $c \in \mathbb{R}^+$, defined in the proof of Lemma 6, is a constant.

Proof: The input-output dynamics of (8) are written in the Laplace domain as $\hat{Y}_h(s) = G_c(s)U(s)$, and from (11), it follows that $\hat{Y}_h(s) = G_c(s)H(s)(R(s) - \hat{Y}_p(s))$. Thus, $\hat{Y}(s) = \hat{Y}_p(s) + \hat{Y}_h(s) = G_c(s)H(s)R(s) + (\mathbb{I}_{n_y} - G_c(s)H(s))\hat{Y}_p(s)$. The Laplace transform of the tracking error, $e(t) = y(t) - r(t)$, is then given by

$$
E(s) = Y(s) - R(s) = (\mathbb{I}_{n_y} - G_c(s)H(s))\hat{Y}_p(s) + (G_c(s)H(s) - \mathbb{I}_{n_y})R(s) - \tilde{Y}(s) \tag{26}
$$

Since $G_c(0)H(0) = \mathbb{I}_{n_y}$, and $\lim_{t\to\infty}\{\hat{y}_p(t),\, r(t)\}$ are assumed to exist, we can use the final value theorem to deduce that $\lim_{t\to\infty}\|e(t)\|_\infty = \lim_{t\to\infty}\|\tilde{y}(t)\|_\infty \leq \frac{Kc}{\sqrt{\gamma}}$. This completes the proof. ■

If the limiting values do not exist, which would be the result of $\lim_{t\to\infty} r(t)$ not existing, then Eq. (26) can be used to derive a bound on $\|e\|_{\mathcal{L}_\infty}$.

6. SIMULATION

Consider the unstable forced wave equation

$$
\begin{aligned}
& \ddot{\theta}(t,x) - 0.1\dot{\theta}_{xx}(t,x) - 2\theta_{xx}(t,x) = 1000\,\theta(t,x) + 1000\,\sin(t) \\
& \theta_x(t,0.1) = 0,\;\; \theta(t,0) = u(t), y(t) = \int_0^{0.1} \theta(t,x)\,dx
\end{aligned} \tag{27}
$$

where the value of 1000 (multiplying $\theta(t,x)$) and the signal $\sigma(t) = 1000\,\sin(t)$ are assumed to be unknown to the controller. The underlying (spatial) Hilbert space is $\mathbb{Z} = H_1([0,\,L];\mathbb{R}) \times \mathcal{L}_2([0,\,L];\mathbb{R})$. It is easy to check, using the Cauchy-Schwarz inequality, that $y(t) = \int_0^L \theta(t,x)dx$ is bounded on $\mathbb{Z}$.

The observers of the DAC are designed as follows:

$$\begin{aligned}
\ddot{\hat{\theta}}_p(t,x) - 0.1\dot{\hat{\theta}}_{p,xx}(t,x) - 2\hat{\theta}_{p,xx}(t,x) &= \hat{\alpha}(t)\theta(t,x) + \hat{\sigma}(t) - \upsilon\big(0.1(\dot{\hat{\theta}}_p(t,x) - \dot{\theta}(t,x)) \\
&\quad + 2(\hat{\theta}_p(t,x) - \theta(t,x))\big) \\
\ddot{\hat{\theta}}_h(t,x) - 0.1\dot{\hat{\theta}}_{h,xx}(t,x) - 2\hat{\theta}_{h,xx}(t,x) &= -\upsilon(0.1\dot{\hat{\theta}}_h(t,x) + 2\hat{\theta}_h(t,x)) \qquad (28) \\
\hat{\theta}_p(t,0) = \hat{\theta}_{p,x}(t,0.1) = \hat{\theta}_{h,x}(t,0.1) = 0,\ \hat{\theta}_h(t,0) = u(t)
\end{aligned}$$

The additional linear terms on the right hand side of (28) are added in order to inject stability into the unstable plant. While they resemble the usual error-based terms in a Luenberger observer, their specific form is chosen based on PDE backstepping [2]. In particular, as part of backstepping, we are able to derive a control law which maps an unstable wave equation of the form (27) into a stable wave equation with the additional stabilizing terms in (28). It would be an interesting open problem to determine whether our control law is implicitly equal to a backstepping controller.

We note that the initial condition for $\hat{\theta}_h$ need not satisfy $\hat{\theta}_h(0,x) = \theta(0,x)$, as explained in Remark 3. In fact, it is convenient to set the initial conditions to zero in practice [1]. The estimates $\hat{\alpha}(t)$ and $\hat{\sigma}(t)$ are found using the projection operator:

$$\begin{aligned}
\dot{\hat{\alpha}}(t) &= \gamma\,\mathrm{Proj}\left(\hat{\alpha}(t), -\int_0^{0.1} (\dot{\tilde{\theta}} + \delta\tilde{\theta})\theta\, dx\right),\ \hat{\alpha}(0) = 0 \\
\dot{\hat{\sigma}}(t) &= \gamma\,\mathrm{Proj}\left(\hat{\sigma}(t), -\int_0^{0.1} (\dot{\tilde{\theta}} + \delta\tilde{\theta})\, dx\right),\ \hat{\sigma}(0) = 0 \qquad (29)
\end{aligned}$$

$|\hat{\alpha}(t)| \le 1200$, $|\hat{\sigma}| < 1200$, and we set $\delta = 10$ (see Appendix 1 for stability analysis of the observation error dynamics). The adaptive gain $\gamma = 10000$, and $\upsilon = 1000$. The low-pass filter in (11) is chosen as $H(s) = p_h/(s^2 + 120s + 3600)$, where $p_h = 3600/G_c(0)$ ensures that $G_c(0)H(0) = 1$. The closed-loop is observed in simulations to be stable for $\upsilon \in (500, 16000)$.

Simulation results in Figs. 2(a) and (c) demonstrate that the steady-state tracking error is negligible. Fig. 2(c) shows that transient response characteristics are uniform with respect to the initial condition and the amplitude of the reference input. Figures (b) and (d) show the time histories of $\hat{\alpha}$ and $\hat{\sigma}$. Interestingly, both parameters converge, albeit slowly, to the corresponding true values (1000 in both cases). This behavior is not necessarily guaranteed as part of the DAC architecture, and this point is demonstrated in Fig. 3.

Figure 3 shows the time histories of the outputs and the adapted parameters when the adaptive laws are designed using finite dimensional approximations of the system and the observers, rather than the PDE-based (29). The values of ν and γ are left unchanged. It is evident that $\hat{\sigma}$ does not converge to the true value at least during the duration of the simulations, although the time history of the output is identical to that in Fig. 2 (c).

7. CONCLUDING REMARKS

We presented a novel output tracking method, featuring a two-stage (dyadic) adaptive controller, for semi-linear infinite dimensional systems with boundary actuation and state feedback. The architecture breaks the system down into two virtual sub-systems, one of which contains the control signal while the other contains the nonlinearties and the disturbances. The individual states of the two sub-systems are estimated by observers which are referred to as the homogeneous and particular halves. The control signal is designed for the homogeneous half of the observer, while disturbances and the nonlinearities are estimated by putting together both halves. The robustness of the controller was proved using the small gain theorem. The DAC architecture presented here is applicable to general semi-linear systems (14), where control

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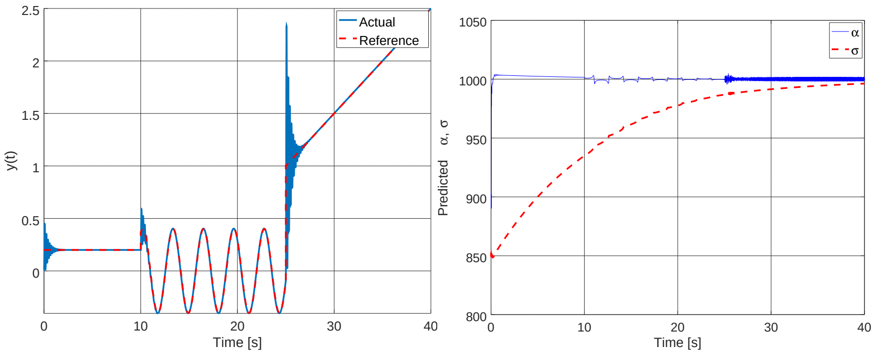
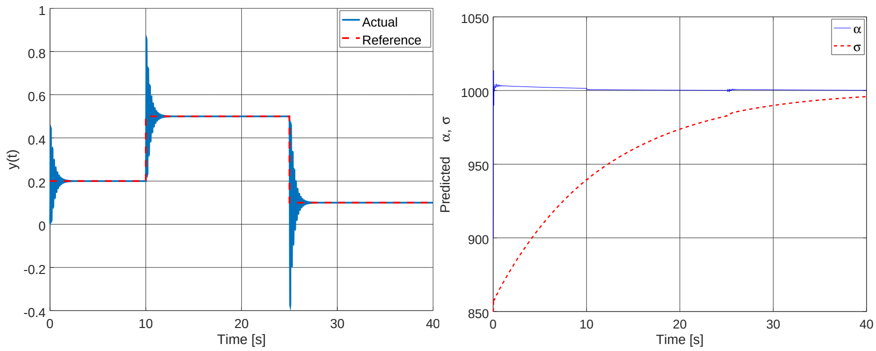
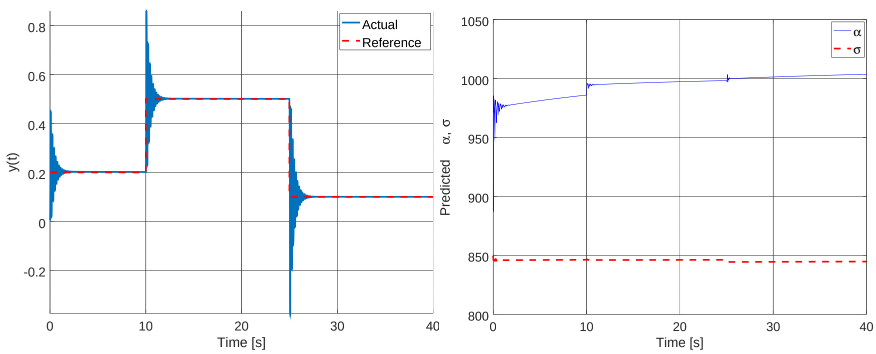


(a) Tracking (b) Adapted parameters $\hat{\alpha}(t)$, $\hat{\sigma}(t)$



(c) Uniform transients (d) $\hat{\alpha}(t)$, $\hat{\sigma}(t)$

Figure 2. DAC illustrated for two sample reference inputs, together with the adapted parameters.



(a) Tracking (b) $\hat{\alpha}(t)$, $\hat{\sigma}(t)$

Figure 3. DAC with the adaptive law designed using a finite dimensional approximation, together with the adapted parameters. This figure should be compared with Fig. 2.

inputs can enter through the boundary or "in-line." Aside from projection-based adaptation and linear control signals, the architecture can accommodate other control signals and observers, provided they yield bounds similar to Theorem 2 and Lemma 5. A limitation of the DAC architecture presented here is the lack of a specific procedure for designing the controller, and the conservatism of the small gain condition. In particular, the choice of the filter $H(s)$ is arbitrary within the bounds of the small gain condition. Some initial results on using LQR to design the control law have been presented by the authors in [26], but further development is needed along those lines. Other avenues for extending this work include further experimental validation and demonstration, understanding how to use output feedback in place of full-state feedback, and determining fundamental restrictions, if any, on the ability of this technique to accommodate unstable linear dynamics.

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APPENDIX 1: PROOF OF THE ADAPTIVE LAW IN SECTION 6

The observer error dynamics is given by

$$\ddot{\tilde{\theta}} - b\dot{\tilde{\theta}}_{xx} - a\tilde{\theta}_{xx} = \tilde{\alpha}\theta + \tilde{\sigma} - bv\dot{\tilde{\theta}} - av\tilde{\theta}$$

where we have omitted the arguments of the variables for brevity, but with the understanding that they are well-known to the reader. Consider the Lyapunov function

$$V = \int_0^{0.1} \left(\dot{\tilde{\theta}}^2 + (a+b\delta)\tilde{\theta}_x^2 + v(a+b\delta)\tilde{\theta}^2 + 2\delta\tilde{\theta}\dot{\tilde{\theta}}\right) dx + \frac{\tilde{\alpha}^2}{\gamma} + \frac{\tilde{\sigma}^2}{\gamma}$$

We write its derivative $\dot{V}$ as

$$\begin{aligned} \dot{V} = & 2\dot{V}_1 + 2(a+b\delta)\int_0^{0.1} \tilde{\theta}_x\dot{\tilde{\theta}}_x\, dx + 2v(a+b\delta)\int_0^{0.1} \tilde{\theta}\dot{\tilde{\theta}}\, dx \\ & + 2\dot{V}_2 + \frac{2}{\gamma}\tilde{\alpha}\dot{\hat{\alpha}} + \frac{2}{\gamma}\tilde{\sigma}(\dot{\hat{\sigma}} - \dot{\sigma}) \end{aligned} \tag{30}$$

where each $\dot{V}_i$ corresponds to one of the remaining terms in the integral. We first expand $\dot{V}_1$:

$$\begin{aligned} \dot{V}_1 &= \int_0^{0.1} \dot{\tilde{\theta}}\ddot{\tilde{\theta}}\, dx = \int_0^{0.1} \dot{\tilde{\theta}}(b\dot{\tilde{\theta}}_{xx} + a\tilde{\theta}_{xx} - bv\dot{\tilde{\theta}} - av\tilde{\theta})\, dx + \tilde{\alpha}\int_0^L \dot{\tilde{\theta}}\theta\, dx + \tilde{\sigma}\int_0^L \dot{\tilde{\theta}}\, dx \\ &= -b\int_0^{0.1} (\dot{\tilde{\theta}}_x^2 + v\dot{\tilde{\theta}}^2)\, dx - av\int_0^{0.1} \tilde{\theta}\dot{\tilde{\theta}}\, dx - a\int_0^L \dot{\tilde{\theta}}_x\tilde{\theta}_x\, dx + \tilde{\alpha}\int_0^{0.1} \dot{\tilde{\theta}}\theta\, dx + \tilde{\sigma}\int_0^L \dot{\tilde{\theta}}\, dx \end{aligned} \tag{31}$$

Next,

$$\begin{aligned} \dot{V}_2 &= \delta\int_0^{0.1} \dot{\tilde{\theta}}^2\, dx + \delta\int_0^{0.1} \tilde{\theta}\ddot{\tilde{\theta}}\, dx \\ &= \delta\int_0^{0.1} \dot{\tilde{\theta}}^2\, dx + \delta\int_0^{0.1} \tilde{\theta}(b\dot{\tilde{\theta}}_{xx} + a\tilde{\theta}_{xx} - bv\dot{\tilde{\theta}} - av\tilde{\theta})\, dx + \tilde{\alpha}\int_0^{0.1} \delta\tilde{\theta}\theta\, dx + \tilde{\sigma}\int_0^{0.1} \delta\tilde{\theta}\, dx \\ &= \delta\int_0^{0.1} \dot{\tilde{\theta}}^2\, dx - b\delta\int_0^{0.1} \tilde{\theta}_x\dot{\tilde{\theta}}_x dx - bv\delta\int_0^{0.1} \tilde{\theta}\dot{\tilde{\theta}}\, dx - a\delta\int_0^{0.1} (\tilde{\theta}_x^2 + v\tilde{\theta}^2) dx \\ &\quad + \tilde{\alpha}\delta\int_0^{0.1} \tilde{\theta}\theta\, dx + \tilde{\sigma}\delta\int_0^{0.1} \tilde{\theta}\, dx \end{aligned} \tag{32}$$

Furthermore, the projection operator (29) ensures that

$$\tilde{\alpha}\left(\dot{\hat{\alpha}} + \gamma\int_0^{0.1} (\dot{\tilde{\theta}} + \delta\tilde{\theta})\theta\, dx\right) \leq 0, \quad \tilde{\sigma}\left(\dot{\hat{\sigma}} + \gamma\int_0^{0.1} (\dot{\tilde{\theta}} + \delta\tilde{\theta})\, dx\right) \leq 0 \tag{33}$$

Substituting (31), (32) and (33) into (30) gives

$$\begin{aligned}\dot{V} &\leq -2b\int_0^{0.1}\dot{\tilde{\theta}}_x^2 dx - 2\int_0^{0.1}\left((bv-\delta)\dot{\tilde{\theta}}^2 + a\delta\tilde{\theta}_x^2 + a\delta v\tilde{\theta}^2\right)dx - \frac{2}{\gamma}\tilde{\sigma}\dot{\sigma} \\ &\leq -2\int_0^{0.1}\left((bv-\delta)\dot{\tilde{\theta}}^2 + 2a\tilde{\theta}_x^2 + 2av\tilde{\theta}^2\right)dx - \frac{2}{\gamma}\tilde{\sigma}\dot{\sigma} \end{aligned} \tag{34}$$

Completing the squares in the Lyapunov equation allows us to write

$$V \leq \int_0^{0.1}\left((1+\delta)\dot{\tilde{\theta}}^2 + (a+b\delta)\tilde{\theta}_x^2 + (\delta + v(a+b\delta))\tilde{\theta}^2\right)dx + \frac{\tilde{\alpha}^2}{\gamma} + \frac{\tilde{\sigma}^2}{\gamma} \tag{35}$$

Define $s_1 = \min((bv-\delta),\, a\delta,\, a\delta v)$ and $s_2 = \max((1+\delta),\, (a+b\delta),\, (\delta + v(a+b\delta)))$. It can be checked readily that

$$\dot{V} \leq -\frac{2s_1}{s_2}V + \frac{2}{\gamma}\left(\frac{s_1}{s_2}\left(\tilde{\alpha}^2 + \tilde{\sigma}^2\right) + |\dot{\sigma}||\tilde{\sigma}|\right)$$

Note that $|\tilde{\alpha}|$ is bounded because α is constant and $|\hat{\alpha}|$ is bounded by the projection law. Likewise, $|\dot{\sigma}|$ is bounded, and projection ensures that $|\hat{\sigma}|$ and $|\tilde{\sigma}|$ are bounded. It follows that there exists a constant $\nu_w > 0$ independent of γ such that $V(t) \leq V(0)e^{(-2s_1/s_2)t} + \nu_w/\gamma$. It follows that $\|\tilde{\theta}(t)\|_{\mathcal{L}_2}$, $\|\dot{\tilde{\theta}}(t)\|_{\mathcal{L}_2}$ and $\|\tilde{\theta}_x(t)\|_{\mathcal{L}_2}$ are bounded for all time, and the asymptotic value of the bound can be made arbitrarily small by increasing γ.