REPORT CTSL– **33**

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*Supported in part by the U. S. Atomic Energy Commission Contract No. AT(11-1)-68.

†Partially supported by the National Science Foundation.

Contents

I. Introduction

The object of particle track analysis is to obtain physical parameters of the particle producing the track from quantities easily measured on the track image. Many of the problems in such analysis are common to bubble chamber, emulsion and cloud chamber work, and, although the following discussion is directed toward bubble chamber track analysis, many of the ideas can be taken over into emulsion or cloud chamber work.

With the advent of more careful temperature and pressure control in bubble chambers, a quantity of interest is the bubble density, which is the average number of bubbles per unit length along the track. For given operating conditions, the bubble density is a function of the particle velocity only; over a wide range it is believed to be approximately proportional to the inverse square of the particle velocity.[1-7] Knowledge of this functional relationship and possession of techniques for calibration and for bubble density measurement permit one to measure to some degree of accuracy the velocity of non-stopping particles in a bubble chamber.

As a matter of convenience, we shall often assume that the relationship between bubble density and particle velocity is the inverse square law valid at lower energies. This particular form of the dependence is not essential to the arguments presented, and the general conclusions should still hold for other dependences, in particular the bubble density rise for highly relativistic particles in some media.[5-7]

II. Theory

Direct counting of the number of bubbles per unit length along a track gives an incorrect result for the bubble density, due primarily to bubble image coalescence. Thus it is necessary to find an estimator for the bubble density. Choice of an estimator is guided by the requirements that it be efficient, that it be conveniently measurable, and that required corrections may be easily applied to it.

We shall now investigate two methods satisfying these criteria. The Gap Length Distribution method is the easier method to apply to measurement by hand, since one must simply decide whether each gap in a track is greater or less than certain fixed distances. The Mean Gap Length method is more amenable to automation since the required quantities are the total number of gaps and the total gap length in a track, with less attention to the details of individual gaps.

III. Gap Length Distribution Method

Since a bubble should be equally likely to occur anywhere along a track, bubble spacings should be describable by a Poisson distribution. Thus, if one plots the logarithm of the number of gaps greater than X in a given track against X, he gets a straight line over some range of X. If the track were infinitely long and he plotted the logarithm of the probability of finding a gap greater than X against X the line would be straight over all positive X. Consideration of finite track length ℓ introduces a term $(1 - \frac{X}{\ell})$ into the exponential distribution so that the line is straight only for X small compared to ℓ: this is discussed in Appendix I.

The probability of finding a gap not small compared to ℓ is so low for ℓ large compared to the average bubble formation length X, that such gaps would not be used in the analysis of a bubble chamber event. Hence we shall neglect the $(1 - \frac{X}{\ell})$ term.

Appendix II demonstrates that the number of bubble gaps greater than a given value varies with the bubble density, the bubble diameter, variations about an average bubble diameter, measurement errors, etc., but that the slope of the straight line on the semilog plot is a function only of the bubble density. Since for a velocity measurement we want to determine the bubble density as accurately as possible without having to worry about a measurement of these other, usually poorly known, quantities, a measurement should be directed toward obtaining the slope of the straight line as accurately as possible.

For a given track, the procedure squeezing the most possible information out of a picture would be to measure each gap between successive bubbles, then do a maximum likelihood fit to these measurements. But for a track containing several hundred bubbles, this is a very cumbersome process without elaborate analysis equipment. A simpler procedure would be to pick several values of X, measure the number of gaps greater than X for each of these values, and fit a straight line to these points on the semilog plot. This is the procedure investigated below.

If we have a track of length L long compared to the average bubble spacing, the mean number of bubbles in the track will be approximately nL, where n is the bubble density. If the effective bubble diameter (the shortest distance between bubble centers such that there is a visible gap between the bubbles) is d, the number of measurable gaps will be approximately nLe^{-nd}. The number of gaps greater than X will be $nLe^{-n(d+x)}$. For a given value of X the number of events should be Poisson distributed, so we may write

$$P(N) = \frac{(K\,e^{-nx})^N}{N!}\,e^{-Ke^{-nx}}$$

where K is nLe^{-nd} with factors taking into account measurement errors, etc.

Let us now consider making r measurements at $X_1, X_2, \ldots, X_r$; $X_i > X_{i+1}$, $X_r = 0$; where a measurement at X_i consists of counting the number of gaps greater than X_i in a given track. This number will be denoted by N_i. The number we would expect is $N_i^* = K\,e^{-nx_i}$. Using the maximum likelihood method[10)] we want to find the value of n that maximizes the joint probability of obtaining N gaps greater than X_1, $N_2 - N_1$ gaps between X_1 and X_2, $N_3 - N_2$ gaps between X_2 and X_3, etc. Assuming that $N_i - N_{i-1}$ will also be Poisson distributed about the expected value, we write the likelihood function

$$\mathcal{L} = \prod_{i=1}^{r} \frac{(K\,e^{-nX_i} - K\,e^{-nX_{i-1}})^{N_i - N_{i-1}}}{(N_i - N_{i-1})!}\,e^{-(K\,e^{-nX_i} - K\,e^{-nX_{i-1}})}$$

where, if $i = 1$, $X_{i-1} = \infty$ and $N_{i-1} = 0$.

$$\mathcal{L} = K^{N_r}\,e^{-K}\prod_{i=1}^{r}\frac{(e^{-nX_i} - e^{-nX_{i-1}})^{N_i - N_{i-1}}}{(N_i - N_{i-1})!}$$

Best Value of n

$$\omega = \ln \mathcal{L} = N_r \ln K - K + \sum_{i=1}^{r} \left[(N_i - N_{i-1}) \ln (e^{-nX_i} - e^{-nX_{i-1}}) - \ln (N_i - N_{i-1})! \right]$$

$$\frac{\partial \omega}{\partial K} = 0 = \frac{N_r}{K} - 1$$

$$K = N_r$$

$$-\frac{\partial \omega}{\partial n} = \sum_{i=1}^{r} \frac{(N_i - N_{i-1})(X_i e^{-nX_i} - X_{i-1} e^{-nX_{i-1}})}{(e^{-nX_i} - e^{-nX_{i-1}})} = 0$$

so we want to find the value of n that satisfies this equation.

Error in n

Let us assume that the statistics are good enough that $\mathcal{L}$ is approximately Gaussian in K and n in the region of the optimum values of K and n. Then the error Δn in n will be given by[10)]

$$\Delta n = \left[(H^{-1})_{22} \right]^{1/2}$$

where $$H_{ij} = -\frac{\partial^2 \omega}{\partial \alpha_i \, \partial \alpha_j} \qquad \alpha_1 = K \qquad \alpha_2 = n \qquad \text{(reference 1)}$$

$$H_{22} = -\frac{\partial^2\omega}{\partial n^2} = \sum_{i=1}^{r} \frac{(N_i - N_{i-1})}{(e^{-nX_i} - e^{-nX_{i-1}})} \left[-X_i^2 e^{-nX_i} + X_{i-1}^2 e^{-nX_{i-1}} + \frac{(X_i e^{-nX_i} - X_{i-1} e^{-nX_{i-1}})^2}{(e^{-nX_i} - e^{-nX_{i-1}})} \right]$$

$$= \sum_{i=1}^{r} \frac{(N_i - N_{i-1})(X_i - X_{i-1})^2 e^{-n(X_i + X_{i-1})}}{(e^{-nX_i} - e^{-nX_{i-1}})^2}$$

$$H_{11} = -\frac{\partial^2\omega}{\partial K^2} = \frac{N_r}{K^2} = \frac{1}{N_r}$$

$$H_{12} = H_{21} = -\frac{\partial^2\omega}{\partial K\,\partial n} = 0$$

$$(H^{-1})_{22} = \frac{H_{11}}{H_{11} H_{22} - H_{12}^2}$$

$$(\Delta n)^{-2} = \sum_{i=1}^{r} \frac{(N_i - N_{i-1})(X_i - X_{i-1})^2 e^{-n(X_i + X_{i-1})}}{(e^{-nX_i} - e^{-nX_{i-1}})^2}$$

Choice of Measurement Spacing

n being the quantity we want to measure, the X_i should be chosen to minimize the error in n, which, from the above equations, means maximizing $-\partial^2\omega/\partial n^2$. Since the N_i are of course unknown, we want to maximize $-\partial^2\omega/\partial n^2$ averaged over the probability distribution of the N_i. But the joint probability of obtaining a particular set of N_i is just the likelihood function. Hence the quantity to maximize is

$$\overline{(\Delta n)^{-2}} = - \sum_{\text{all } N_i} \mathcal{L}(N_i) \frac{\partial^2 \omega}{\partial n^2} = - \sum \mathcal{L}(N_i) \frac{\partial^2 (\ell n\ L)}{\partial n^2}$$

$$= - \sum \mathcal{L} \frac{\partial}{\partial n} \left(\frac{1}{\mathcal{L}} \frac{\partial \mathcal{L}}{\partial n}\right)$$

$$= - \sum \frac{\partial^2 \mathcal{L}}{\partial n^2} + \sum \frac{1}{\mathcal{L}} \left(\frac{\partial \mathcal{L}}{\partial n}\right)^2$$

$$\sum \frac{\partial^2 \mathcal{L}}{\partial n^2} = \frac{\partial^2}{\partial n^2} \sum \mathcal{L} = 0 \qquad \text{since} \qquad \sum \mathcal{L} = 1$$

so
$$\overline{(\Delta n)^{-2}} = \sum_{\text{all } N_i} \frac{1}{\mathcal{L}} \left(\frac{\partial \mathcal{L}}{\partial n}\right)^2$$

Remembering that
$$\mathcal{L} = K^{N_r} e^{-K} \prod_{i=1}^{r} \frac{(e^{-nX_i} - e^{-nX_{i-1}})^{N_i - N_{i-1}}}{(N_i - N_{i-1})!}$$

$$\frac{\partial L}{\partial n} = - \mathcal{L} \sum_{i=1}^{r} \frac{(N_i - N_{i-1})(X_i e^{-nX_i} - X_{i-1} e^{-nX_{i-1}})}{(e^{-nX_i} - e^{-nX_{i-1}})}$$

so that

$$\overline{(\Delta n)^{-2}} = \sum_{\text{all } N_i} \mathcal{L} \left[\sum_{i=1}^{r} \frac{(N_i - N_{i-1})(X_i e^{-nX_i} - X_{i-1} e^{-nX_{i-1}})}{(e^{-nX_i} - e^{-nX_{i-1}})} \right]^2$$

In considering the sum over all N_i, we will think of it as a sum over all pairs $(N_i - N_{i-1})$, which we will denote by n_i. Remembering that we said above that we would let the n_i have Poisson distributions,

$$\overline{(\Delta n)^{-2}} = \sum_{i=1}^{r} \sum_{j=1}^{r} \sum_{\substack{n_k=0 \\ K \neq i \text{ or } j}}^{\infty} \sum_{n_i=0}^{\infty} \sum_{n_j=0}^{\infty} \mathcal{L} \text{ times}$$

$$\frac{n_i\, n_j\, (X_i\, e^{-nX_i} - X_{i-1}\, e^{-nX_{i-1}})(X_j\, e^{-nX_j} - X_{j-1}\, e^{-nX_{j-1}})}{(e^{-nX_i} - e^{-nX_{i-1}})(e^{-nX_j} - e^{-nX_{j-1}})}$$

$$= \sum_{\substack{i=1 \\ i \neq j}}^{r} \sum_{j=1}^{r} \frac{(X_i\, e^{-nX_i} - X_{i-1}\, e^{-nX_{i-1}})(X_j\, e^{-nX_j} - X_{j-1}\, e^{-nX_{j-1}})}{(e^{-nX_i} - e^{-nX_{i-1}})(e^{-nX_j} - e^{-nX_{j-1}})}$$

$$\text{times} \sum_{n_i=0}^{\infty} \sum_{n_j=0}^{\infty} \sum_{\substack{n_k=0 \\ K \neq i \text{ or } j}}^{\infty} \mathcal{L}\, n_i\, n_j$$

$$+ \sum_{i=1}^{r} \frac{(X_i\, e^{-nX_i} - X_{i-1}\, e^{-nX_{i-1}})^2}{(e^{-nX_i} - e^{-nX_{i-1}})^2} \sum_{n_i=0}^{\infty} \sum_{\substack{n_k=0 \\ K \neq i}}^{\infty} \mathcal{L}\, n_i^2$$

Let f_i denote the Poisson distribution $\frac{a_i^n}{n!}\, e^{-a_i}$

where $a_i = K\,(e^{-nX_i} - e^{-nX_{i-1}})$

By factoring $\mathcal{L}$ into its product of f_k and doing the sums over n_k one at a time, they become sums like

$$\sum_{n_k=0}^{\infty} f_k = 1$$

When we get to i and j,

$$\sum_{n_i=0}^{\infty} n_i f_i = \sum_{n=0}^{\infty} \frac{n a_i^n}{n!} e^{-a_i} = \sum_{n=1}^{\infty} \frac{n a_i^n}{n!} e^{-a_i} = a_i \sum_{n=0}^{\infty} \frac{a_i^n}{n!} e^{-a_i} = a_i$$

$$\sum_{n_i=0}^{\infty} n_i^2 f_i = \sum_{n=0}^{\infty} \frac{n(n-1) a_i^n e^{-a_i}}{n!} + \sum_{n=0}^{\infty} \frac{n a_i^n e^{-a_i}}{n!}$$

$$= a_i^2 + a_i$$

so $\Sigma\ \Sigma\ \Sigma\ \mathcal{L}\ n_i\ n_j = K^2 (e^{-nX_i} - e^{-nX_{i-1}})(e^{-nX_j} - e^{-nX_{j-1}})$

$$\Sigma\ \Sigma\ \mathcal{L}\ n_i^2 = K^2 (e^{-nX_i} - e^{-nX_{i-1}})^2 + K (e^{-nX_i} - e^{-nX_{i-1}})$$

Recombining terms,

$$\overline{(\Delta n)^{-2}} = K^2 \sum_{i=1}^{r} \sum_{j=1}^{r} (X_i e^{-nX_i} - X_{i-1} e^{-nX_{i-1}})(X_j e^{-nX_j} - X_{j-1} e^{-nX_{j-1}})$$

$$+ K \sum_{i=1}^{r} \frac{(X_i e^{-nX_i} - X_{i-1} e^{-nX_{i-1}})^2}{(e^{-nX_i} - e^{-nX_{i-1}})}$$

The first term is just $K^2 \left[\sum_{i=1}^{r} (X_i e^{-nX_i} - X_{i-1} e^{-nX_{i-1}}) \right]^2$

But $\sum_{i=1}^{r} X_i e^{-nX_i} - X_{i-1} e^{-nX_{i-1}} = 0$ since $X_r = 0$ and, for $i = 1$, $X_{i-1} = \infty$ (see initial statement of likelihood function, above). Hence the first term vanishes and we have

$$\overline{(\Delta n)^{-2}} = K \sum_{i=1}^{r} \frac{(X_i e^{-nX_i} - X_{i-1} e^{-nX_{i-1}})^2}{(e^{-nX_i} - e^{-nX_{i-1}})}$$

Consider for the moment

$$\sum_{i=1}^{r} \frac{(X_i e^{-nX_i} - X_{i-1} e^{-nX_{i-1}})^2}{(e^{-nX_i} - e^{-nX_{i-1}})} - \sum_{i=1}^{r} \frac{(X_{i-1} - X_i)^2 e^{-n(X_{i-1} + X_i)}}{(e^{-nX_i} - e^{-nX_{i-1}})}$$

$$= \sum_{i=1}^{r} \Big\{ X_i^2 e^{-2nX_i} - 2 X_i X_{i-1} e^{-n(X_i + X_{i-1})} + X_{i-1}^2 e^{-2nX_{i-1}}$$

$$- (X_{i-1}^2 - 2 X_i X_{i-1} + X_i^2) e^{-n(X_i + X_{i-1})} \Big\} / (e^{-nX_i} - e^{-nX_{i-1}})$$

$$= \sum_{i=1}^{r} \frac{X_i^2 e^{-nX_i} (e^{-nX_i} - e^{-nX_{i-1}}) - X_{i-1}^2 e^{-nX_{i-1}} (e^{-nX_i} - e^{-nX_{i-1}})}{(e^{-nX_i} - e^{-nX_{i-1}})}$$

$$= \sum_{i=1}^{r} (X_i^2 e^{-nX_i} - X_{i-1}^2 e^{-nX_{i-1}}) = 0 \quad \text{since} \quad X_r = 0,\ X_o = \infty$$

So we may write

$$\overline{(\Delta n)^{-2}} = K \sum_{i=1}^{r} \frac{(X_{i-1} - X_i)^2 e^{-n(X_i + X_{i-1})}}{(e^{-nX_i} - e^{-nX_{i-1}})}$$

The $i = 1$ term vanishes since $X_{i-1} = \infty$, so

$$\overline{(\Delta n)^{-2}} = K \sum_{i=2}^{r} \frac{(X_{i-1} - X_i)^2}{e^{nX_{i-1}} - e^{nX_i}}$$

is the function we will maximize with respect to the X_i.

First consider $1 < i < r$.

$$\frac{\partial}{\partial X_i} \overline{(\Delta n)^{-2}} = K \left[\frac{-2(X_{i-1} - X_i)}{e^{nX_{i-1}} - e^{nX_i}} + \frac{n(X_{i-1} - X_i)^2 e^{nX_i}}{(e^{nX_{i-1}} - e^{nX_i})^2} + \frac{2(X_i - X_{i+1})}{(e^{nX_i} - e^{nX_{i+1}})} - \frac{n(X_i - X_{i+1})^2 e^{nX_i}}{(e^{nX_i} - e^{nX_{i+1}})^2} \right] = 0$$

Defining $\rho_i \equiv n(X_i - X_{i+1})$,

$$\frac{-2\,\rho_{i-1}}{e^{\rho_{i-1}} - 1} + \frac{\rho_{i-1}^2}{(e^{\rho_{i-1}} - 1)^2} = \frac{-2\,\rho_i\, e^{\rho_i}}{e^{\rho_i} - 1} + \frac{\rho_i^2\, e^{2\rho_i}}{(e^{\rho_i} - 1)^2}$$

Add 1 to each side to complete the square. Take the square root, selecting the signs according to the following criteria: If $\rho > 0$,

$$\frac{\rho}{1 - e^{-\rho}} > 1 \quad \text{so} \quad +\sqrt{\left(1 - \frac{\rho}{1-e^{-\rho}}\right)^2} = \frac{\rho}{1 - e^{-\rho}} - 1$$

$$\frac{\rho\, e^{-\rho}}{1 - e^{-\rho}} < 1 \quad \text{so} \quad +\sqrt{\left(1 - \frac{\rho\, e^{-\rho}}{1-e^{-\rho}}\right)^2} = 1 - \frac{\rho\, e^{-\rho}}{1 - e^{-\rho}}$$

The other choice of relative signs in the square root corresponds to $\frac{\rho}{1 - e^{-\rho}} < 1$ or $\rho < 0$, which is physically meaningless. So we obtain

$$\frac{\rho_i}{1 - e^{-\rho_i}} - 1 = 1 - \frac{\rho_{i-1}\, e^{-\rho_{i-1}}}{1 - e^{-\rho_{i-1}}}$$

Consider $i = 1$.

$$\frac{\partial}{\partial X_1} \overline{(\Delta n)^{-2}} = \frac{2(X_1 - X_2)}{e^{nX_1} - e^{nX_2}} - \frac{n(X_1 - X_2)^2 e^{nX_1}}{(e^{nX_1} - e^{nX_2})^2} = 0$$

$$2 = \frac{\rho_i}{1 - e^{-\rho_i}}$$

So we have the recursion relation

$$2(1 - e^{-\rho_1}) = \rho_1$$

$$\frac{\rho_i}{1 - e^{-\rho_i}} = 2 - \frac{\rho_{i-1} e^{-\rho_{i-1}}}{1 - e^{-\rho_{i-1}}} , \qquad \rho_i = n(X_i - X_{i+1})$$

to determine the X_i. Table I shows values for the first few ρ_i.

TABLE I

r	ρ_r	$\sum_{i=1}^{r} \rho_i$	r	ρ_r	$\sum_{i=1}^{r} \rho_i$
1	1.59362	1.59362	9	0.29906	5.89787
2	1.01758	2.61120	10	0.27187	6.16974
3	0.75403	3.36524	11	0.24923	6.41897
4	0.60043	3.96567	12	0.23007	6.64904
5	0.49932	4.46499	13	0.21366	6.86270
6	0.42756	4.89255	14	0.19943	7.06213
7	0.37394	5.26649	15	0.18698	7.24912
8	0.33232	5.59881			

The more measurements we take (the bigger r is), the more accurate the determination of n. If r is large enough, making N_1 small enough that the statistics on N_1 become poor, the Gaussian approximation involved in the error analysis fails, so it is no longer clear that the criteria used above continue to apply. However, making r larger than about 4 does little to improve the accuracy, as can be seen from Table II.

$$\overline{(\Delta n)^{-2}} = N_r \sum_{i=2}^{r} \frac{(X_{i-1} - X_i)^2}{(e^{nX_{i-1}} - e^{nX_i})}$$

We define $\epsilon_r = n\sqrt{\sum_{i=2}^{r} \frac{(X_{i-1} - X_i)^2}{(e^{nX_{i-1}} - e^{nX_i})}}$, X_i determined as above.

TABLE II

r	ϵ_r	r	ϵ_r
2	.8047	10	.9899
3	.9057	11	.9915
4	.9440	12	.9928
5	.9628	13	.9939
6	.9734	14	.9947
7	.9801	15	.9953
8	.9845	16	.9959
9	.9876		

An ideal measurement has $\epsilon = 1$ as an upper limit because there are only N_r events and the statistics on N_r events limits our relative error to being no better than $(N_r)^{-1/2}$. ϵ_4 is already within 6 per cent of this limit, and in particular cases, difficulty of making that many measurements or poor statistics may dictate a smaller number of measurements than this. The recursion formula used to obtain the ϵ_r is given in Appendix III.

The measurement error indicated in Table II is derived under the assumption that the distances chosen for the gap length bins were the optimum ones for the bubble density of that particular track. Of course one does not know the bubble density before he measures it. Figure 1 shows how ϵ varies if one guesses wrong at the bubble density, then optimizes the distances for the gap length bins on the basis of that guess. As above, r is the number of measurements. G is the true bubble density divided by the guessed bubble density. The quantity plotted (F) is the factor by which one must multiply the ϵ_r of Table II to get ϵ such that the relative error in the measurement is $1/\epsilon\sqrt{N_r}$. For a reasonable number of measurements, one can make a rather crude initial estimate of the bubble density to set the spacings and the effect on the error is small. Having this initial guess correct within a factor of two will probably suffice for most applications.

IV. Mean Gap Length Method

An easier and statistically more efficient technique is to measure the total length G of gaps in a track and the number N of the gaps; N/G is then the required bubble density.[3,4,8,9)]

Let us assume that we have a machine capable of scanning a track, deciding whether it sees a region of bubble or a region of gap at each point along the way, and giving us at the end of the scan the total length run, the total gap length, and the number of gaps encountered. We now consider complications of an instrumental nature (the actual bubble chamber, film, and scanning device) and of a physical nature (background bubbles, velocity change due to energy loss along the track, and varying magnification due to track dip in the chamber.)

Defining $n(x)$ to be the true instantaneous "bubble density" at each point along a track and $d(x)$ to be the bubble diameter, we note from the Poisson distribution of the bubble spacings that the probability of having a gap of length at least ℓ between two bubbles is $e^{-n(x)[d(x) + \ell]}$. Therefore, in a track of length L, the expected total gap length is

$$G = \int_0^L e^{-n(x)d(x)} dx$$

and the expected number of gaps is

$$N = \int_0^L n(x) e^{-n(x)d(x)} dx$$

Thus, if n is constant along the track, $n = N/G$ independent of the value or the distribution of the bubble diameters. Even if n varies along the track, the random fluctuations that must occur in d should yield the correct value of N/G on the average.

Unfortunately, there are several instrumental effects that might lead to non-random changes in $d(x)$ and hence to incorrect values of N/G where the bubble density is changing along the track. Variation in lighting intensity in different parts of the chamber could make some bubble images darker and bigger than others. If the size of the film image is primarily due to focus and diffraction effects and not to the actual bubble size, a dipping track will experience a change in bubble diameter due to a change in focus. The combination of projection device and measuring instrument used may have different sensitivities in various parts of the picture; like the lighting variation this could change the bubble diameter. One should minimize these effects as much as possible; if it is impossible to eliminate them, it may be necessary to correct the measured bubble density empirically, depending upon the region where the track occurs (see reference 9).

Gap Filling Errors

Gap filling is an instrumental complication that can be removed electronically. Due to imperfect focus, instrument slit width, and development of film grains adjacent to those illuminated, bubble edges appear fuzzy to the measuring instrument. As an example, suppose that the film density is Gaussian-distributed in the neighborhood of a bubble. The solid line in Figures 2A and 2B shows two such bubbles separated by 1.0 and 1.5 "bubble diameters" (bubble diameter = 2a if the film density is proportional to e^{-x^2/a^2}). Considering the two Gaussian curves to represent illumination and assuming the film darkening to be linear with light intensity (filling is even worse for non-linear darkening, as is the case with ordinary film), the dashed curve shows the film development due to the overlapping tails. Suppose 0.75 to be the discrimination level between gap and bubble. Filling reduces the gap length between the two bubbles by 55 per cent in 2A, but not at all in 2B. Since a large fraction of the total gap length is contributed by the longer gaps which are not changed by gap filling, the gap length distribution is affected more than the total gap length. Much more serious is the loss of counts from the total number of gaps; had the discrimination level in 2A been 0.7 instead of 0.75, the gap would have been missed altogether.

This effect can be eliminated by delaying the bubble-to-gap electronic switchover some distance I from the point where the signal level actually passes the triggering point. Then the electronic circuit ignores all gaps shorter than I and subtracts I from all the other gaps. I is chosen to be the gap size such that the gap filling just becomes negligible, and all gaps considered are essentially free of filling. The effect of this delay circuit is to increase the effective bubble diameter by I, the only disadvantage in this approach being the loss of gap number statistics.

Background Bubbles

When there are many stray bubbles present in the chamber, some of these will lie along the track (i.e., within the acceptance region of the instrument) and be counted. If we assume that such bubbles are equally likely to occur anywhere within the region of interest, the background bubbles will be Poisson-distributed and their "bubble density" adds to the true bubble density along the track. Suppose that a measurement in a region near the track gives n_B bubbles per unit length. Denoting with subscript B the quantities arising from background bubbles and subscript T quantities arising from true track-produced bubbles, measured quantities are:

$$G_{meas} = \int_0^L e^{-(n_B + n_T)d} \, dx$$

$$N_{meas} = \int_0^L (n_B + n_T) \, e^{-(n_B + n_T)d} \, dx$$

Since we assume that d is not a function of x,

$$G_{meas} = e^{-n_B d} \int_0^L e^{-n_T d} \, dx$$

$$N_{meas} = n_B \, e^{-n_B d} \int_0^L e^{-n_T d} \, dx + e^{-n_B d} \int_0^L n_T \, e^{-n_T d} \, dx$$

$$\frac{N_{meas}}{G_{meas}} = n_B + \frac{\int_0^L n_T \, e^{-n_T d} \, dx}{\int_0^L e^{-n_T d} \, dx} = n_B + \left(\frac{N}{G}\right)_T$$

Thus a simple subtraction of the background bubble density from $(N/G)_{meas}$ yields the quantity N/G discussed above.

Change in Particle Velocity: The Residual Range Method

Bubble density is not constant along a track due to energy loss by the particle producing the track. If the change in velocity is not great, this effect may be ignored. But occasionally one would like to bubble count a track whose bubble density changes appreciably. Segmentation of the track into regions of almost constant bubble density may make the statistics quite poor for each segment.

Consider a track segment of length L to be bubble counted. Define R to be the range of the particle in the bubble chamber liquid beyond the segment measured, if the chamber had been big enough to allow the particle to stop. If R were known, β would be known at each point along the segment. Taking $n(x) = n_o/\beta^2(x)$, consider

$$F_1 = \frac{N}{n_o G} = \frac{\int_R^{R+L} \frac{1}{\beta^2} e^{-n_o d/\beta^2} dx}{\int_R^{R+L} e^{-n_o d/\beta^2} dx}$$

$$F_2 = \frac{G}{L} = \frac{1}{L} \int_R^{R+L} e^{-n_o d/\beta^2} dx$$

For a given particle in a given bubble chamber medium, F_1 and F_2 are functions of R, L, and the product $n_o d$ only. Since L is known, simultaneous measurement of F_1 and F_2 will determine R (and incidentally $n_o d$). F_1 and F_2 are written such that dependence on L is slow so that the R determination may be made graphically by making graphs of F_1 vs F_2 showing constant values of R, in steps of L of, say, 0.5 cm. One such graph for L = 8 cm is shown in Fig. 3 (the liquid is CF_3Br at 30°C and 150 psi).

Use of this procedure for determining R requires that track parameters be used with background bubble effects removed. In terms of measured quantities,

$$F_1 = \frac{1}{n_o}\left(\frac{N_{meas}}{G_{meas}} - n_B\right)$$

$$F_2 = \frac{1}{L\,\mathcal{L}_B}\,G_{meas}$$

where $\mathcal{L}_B$ is the measured lacunarity (the average fraction of a track segment consisting of gap) from the background run.

Dipping Tracks

Correction for track dip in emulsion may simply be done by a factor of cos Θ where Θ is the dip angle, as shown by Barkas.[8] Bubble chamber track dip cannot be handled so simply; the requirement that the light rays pass through the camera lens makes the relevant angle the angle between the track and the light ray rather than the dip angle, and the problem is further compounded by the multiple indices of refraction to be taken into account.

An additional complication is introduced by the fact that the chamber-to-film magnification is a function of the depth in the chamber, causing n_o (the bubble density for a particle with $\beta = 1$, assuming a β^{-2} law) to vary in an approximately linear way along a dipping track. We now develop a technique for handling this linear change in n_o.

First, suppose β to be constant along the track segment. By the mean value theorem, there is some point b in the segment such that, if we measure n_o there and calculate assuming n_o constant, we get the same answer for N/G as if we had considered the linear dependence of n_o on position (only N/G is considered since G/L does not explicitly contain n_o). Letting a be the rate of change of n_o with distance and $n_o^* = n_o(b)$, b is determined by

$$\frac{\int_R^{R+L} \frac{n_o^*}{\beta^2}\left[1 + a\,(x - b)\right] e^{-\left[1 + a\,(x - b)\right] n_o^* d/\beta^2}\,dx}{\int_R^{R+L} e^{-\left[1 + a\,(x - b)\right] n_o^* d/\beta^2}\,dx} =$$

$$\frac{\int_R^{R+L} \frac{n_o^*}{\beta^2} e^{-n_o^* d/\beta^2}\,dx}{\int_R^{R+L} e^{-n_o^* d/\beta^2}\,dx}$$

which has the solution

$$b = R + \frac{\beta^2}{a n_o^* d} - \frac{L}{\left(e^{a n_o^* dL/\beta^2} - 1\right)}$$

For $\frac{a n_o^* d}{\beta^2} L = 0$, $b = R + \frac{L}{2}$, as one would expect.

For $\frac{a n_o^* d}{\beta^2} L = \pm \infty$, $b = (R + \frac{L}{2}) \pm \frac{L}{2}$. For many bubble chamber operating conditions, b will not vary greatly from $R + \frac{L}{2}$. For example, in the Caltech 12-inch heavy liquid chamber, b cannot vary from $R + \frac{L}{2}$ by more than 0.16 L in the worst case.

In this formula, a and L are known. Remembering that F_2 is the track lacunarity corrected for background, we use

$$\frac{n_o^* d}{\beta^2} = - \ln (F_2)$$

This same approach may be followed when β changes appreciably in the track segment, as $-\ell n\ F_2$ is still a measure of some average $n_o d/\beta^2$.

Summary of Procedure

Incorporating the various corrections above, bubble density analysis proceeds as follows:

1) Measure N, G, L for the track and N_B, G_B, L_B on a background run near the track.

2) $F_2 = \frac{L_B}{G_B} \frac{G}{L}$.

3) For n_o, use the value for the depth in the chamber corresponding to the depth of the point b on the track where

$$b = R - \frac{1}{a\ \ell n\ F_2} - \frac{L}{\left(e^{-a\ L\ \ell n\ F_2} - 1\right)} .$$

4) $F_1 = \frac{1}{n_o}\left(\frac{N}{G} - \frac{N_B}{G_B}\right)$.

5) Use curve of F_1 vs F_2 for appropriate value of L to determine "residual range", R.

6) Determine initial particle velocity from range-energy relationship in the bubble chamber liquid.

In all of the above, it is assumed that the particle type is known, and it is the energy that is to be measured. This is the way in which we are using the gap counting technique. Of course, if gap counting is to be used in conjunction with another parameter such as track curvature in a magnetic field for particle identification, then the above procedure can be used as well. In this latter case one tries several particle masses to find the best agreement with the other measured parameter.

Chamber Calibration

In order to use this particle velocity analysis, the chamber is calibrated using stopping pions, or, if long tracks are available, stopping protons. This measurement of n_o should be redone each time the chamber temperature or pressure might have changed.

To measure n_o from a stopping particle, one first computes a graph of F_1 vs F_2 for various values of L (R is of course zero). Measuring F_2 and L on the track then graphically yields F_1, which, divided into the measured value of N/G, gives n_o. Since statistics on stopping tracks may be poor, several measurements may have to be averaged to give a value of n_o. Thus, unless there are many stopping particles in each picture, it is important for the chamber to be stable from picture to picture.

Information from Blob Length

The residual range method of bubble density analysis as discussed so far is analogous to the usual mean gap length analysis. Barkas[9)] has shown that additional information is contained in the mean blob length if one knows the effective bubble diameter. It is on just those tracks requiring the residual range method of analysis that blob length information would be most useful, since a particle with a large velocity change was probably close to stopping and hence would have low lacunarity (see reference 9). Therefore let us now consider how this added information may be obtained using the Residual Range method.

Again looking at Fig. 3, one sees that, for the larger values of R (implying more constant velocity along the segment being considered), the lines of constant R approach vertical straight lines. Here the Residual Range method becomes the familiar constant velocity mean gap length method, where a measurement of N/G (or F_1) suffices to determine the bubble density and hence the velocity. For lacunarity (F_2) near unity, lines of constant $n_o d$ are horizontal; knowledge of the lacunarity combined with knowledge of the bubble diameter does not add information to the bubble density measurement.

As the lacunarity (F_2) decreases, lines of constant $n_o d$ attain more slope, so that a second measurement of R (and hence the velocity) may come from combining the known bubble diameter (or $n_o d$) with the measured lacunarity (F_2).

Since mean gap length and mean blob length (with known bubble diameter) afford two independent measurements of particle velocity, the same approach should be followed here. In the present notation, mean blob length is $\frac{L - G}{N}$, or

$$\frac{L - \int_R^{R+L} e^{-nd}\, dx}{\int_R^{R+L} n\, e^{-nd}\, dx} .$$

Denoting the mean blob length times n_o by F_3, we may write

$$F_3 = n_o \frac{\dfrac{L}{\int_R^{R+L} e^{-nd}\,dx} - 1}{\dfrac{\int_R^{R+L} n\, e^{-nd}\,dx}{\int_R^{R+L} e^{-nd}\,dx}}$$

Or, in terms of the measured quantities F_1 and F_2,

$$F_3 = \frac{\dfrac{1}{F_2} - 1}{F_1} \quad .$$

Thus one might include on his graphs of F_1 vs F_2 lines of constant F_3. Then the gap length information would be obtained exactly as described above for the Residual Range method. To obtain the blob length information, one computes F_3 from F_1 and F_2, then finds the intersection of that value of F_3 with the known value of n_od, giving a second value for R.

In order to avoid complicating the graphs further by including the F_3 plot, one may calculate where the F_3 and n_od intersection will lie. With F_1 plotted on a linear scale and F_2 plotted on a logarithmic scale, lines of constant n_od and lines of constant F_3 are nearly straight. Let F_1^* and F_2^* represent the measured values of F_1 and F_2; let F_1^{**} represent the intersection of the line for the known n_od with the line $F_2 = F_2^*$, and F_2^{**} represent the n_od intersection with the line $F_1 = F_1^*$. The approximation to the n_od line becomes

$$\ell n\, F_2 = \frac{\ell n(F_2^{**}/F_2^{*})}{F_1^{*} - F_1^{**}} F_1 + \frac{F_1^{*}}{F_1^{*} - F_1^{**}} \ell n\, F_2^{*} - \frac{F_1^{**}}{F_1^{*} - F_1^{**}} \ell n\, F_2^{**} \quad .$$

The equation for the F_3 line is

$$\frac{\frac{1}{F_2} - 1}{F_1} = \frac{\frac{1}{F_2^*} - 1}{F_1^*} \;, \quad \text{or} \quad \ln F_2 = -\ln\left[1 + \frac{F_1}{F_1^*}\left(\frac{1}{F_2^*} - 1\right)\right] \;.$$

In linear approximation (Taylor expansion about $\frac{F_1^* + F_1^{**}}{2}$),

$$\ln F_2 = -\ln\left[1 + \frac{(F_1^* + F_1^{**})}{2F_1^*}\left(\frac{1}{F_2^*} - 1\right)\right]$$

$$- \frac{\frac{1}{F_1^*}\left(\frac{1}{F_2^*} - 1\right)}{1 + \frac{(F_1^* + F_1^{**})}{2F_1^*}\left(\frac{1}{F_2^*} - 1\right)} \left[F_1 - \frac{(F_1^* + F_1^{**})}{2}\right] \;.$$

Combining the two equations yields for F_1:

$$\frac{F_1^{**} \ln(F_2^{**}/F_2^*)\left[F_2^*(F_1^* - F_1^{**}) + F_1^* + F_1^{**}\right] + 2F_1^*(1-F_2^*)(F_1^* - F_1^{**})}{\ln(F_2^{**}/F_2^*)\left[F_2^*(F_1^* - F_1^{**}) + F_1^* + F_1^{**}\right] + 2(1-F_2^*)(F_1^* - F_1^{**})}$$

The intersection of this value of F_1 with the known $n_o d$ line will yield the mean blob length estimate of R.

For combining the gap length and blob length estimates of bubble density, Barkas[9] assumes that the two estimates are sufficiently close that a linear interpolation will suffice. He obtains

$$n_{ave.} = \omega n_{gap} + (1 - \omega)\, n_{blob}$$

where $$\frac{\omega}{1-\omega} = \frac{\mathcal{L}^2 (\ln \mathcal{L})^2}{(1 - \mathcal{L} + \ln \mathcal{L})^2} \left(\frac{\sigma_b}{d}\right)^2 ,$$

$\mathcal{L}$ being the measured lacunarity, d the effective bubble diameter, and $\sigma_b^{\,2}$ the variance of the mean blob length.

Making the same assumption for the Residual Range method, we may write

$$R_{ave.} = \omega R_{gap} + (1 - \omega) R_{blob} ,$$

using the above expression for $\frac{\omega}{1-\omega}$.

ACKNOWLEDGMENTS

The authors would like to express their appreciation to E. D. Alyea, Jr., Daniel D. Sell, J. Howard Marshall III, Arpad Barna, and John Walsh for their help in the theoretical and instrumental aspects of this investigation.

Appendix I

The probability of finding a distance between two bubbles equal to or greater than X is e^{-nX}, where n is the bubble density, in an infinitely long track. Bubble diameter for the moment is assumed zero. If the track is of length L, this probability of gap length X will equal the probability of finding a gap of length X times the number of places the gap could begin and remain within the interval, divided by the number of places the gap could begin without having to end in the interval. For a gap between X and X + dx,

$$dP(x) = \frac{\int_0^{L-X} da \, e^{-nX} \, n \, dx}{\int_0^{L} da}$$

$$= e^{-nX} \, n \, dx \, (1 - \frac{X}{L})$$

$$P(x) = \int_X^L e^{-nX} \, n \, dx \, (1 - \frac{X}{L})$$

$$= e^{-nX} - e^{-nL} + e^{-nL} (1 + \frac{1}{nL}) - e^{-nL} (\frac{X}{L} + \frac{1}{nL})$$

$$= e^{-nX} (1 - \frac{(X + n^{-1})}{L}) \quad \text{if} \quad L \gg X, \; L \gg \frac{1}{n} .$$

Appendix II

Consider an interval long enough that its finite size may be neglected. For zero bubble size, as we saw in Appendix I, the probability of a gap occurring of length between X and X + dX is

$$dP(X) = e^{-nX}\, n\, dX$$

If the bubble diameter is d, for a gap X, the centers of the bubbles must be separated by X + d; therefore,

$$dP(X) = e^{-n(X + d)}\, n\, dX$$

If the bubble diameter is Gaussian distributed with radius r, variance σ, the probability that a bubble has radius Y is

$$\phi(Y) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-1/2\,(y-r)^2/\sigma^2}\, dy$$

$$dP(X) = \frac{1}{2\pi\sigma^2} \int_{y_1=0}^{\infty} \int_{y_2=0}^{\infty} e^{-1/2\,(y_1-r)^2/\sigma^2}\, e^{-1/2\,(y_2-r)^2/\sigma^2} \text{ times}$$

$$e^{-n(X+y_1+y_2)}\, n\, dX\, dy_1\, dy_2$$

$$= \frac{e^{-nX - r^2/\sigma^2}}{2\pi\sigma^2} \int_{y_1=0}^{\infty} e^{-1/2\, y_1^2/\sigma^2 + (r/\sigma^2 - n)y_1}\, dy_1 \text{ times}$$

$$\int_{y_2=0}^{\infty} e^{-1/2\, y_2^2/\sigma_2^2 + (r/\sigma^2 - n)y_2}\, dy_2$$

If σ is small compared to r,

$$dP(X) = \frac{e^{-nX - r^2/\sigma^2 + (r/\sigma - n\sigma)^2}}{2\pi\sigma^2} \left[\int_{y=-\infty}^{\infty} e^{-1/2\,(y - [r/\sigma - n\sigma])^2}\, dy \right]^2$$

times $n\,dX$

$$= e^{-nX}\,(e^{-2nr}\; e^{n^2\sigma^2})\; n\,dX \approx e^{-n(X + d)}\; n\,dX$$

By a similar treatment, if there is a measurement error with variation $\sigma \ll d$, Gaussian distributed,

$$dP(X) = e^{-nX}\,(e^{-nd}\; e^{n^2\sigma^2/2})\; n\,dX \approx e^{-n(X + d)}\; n\,dX$$

In each case, the value of dP(X) depends rather critically on d, less severely on σ, but its X dependence is a function only of n.

Appendix III

Recursion formula for ϵ_r :

$$\epsilon_r^2 = \sum_{i=2}^{r} \frac{(X_{i-1} - X_i)^2}{(e^{nX_{i-1}} - e^{nX_i})}$$

$$= \sum_{i=2}^{r} \frac{\rho_{i-1}^2 \, e^{-nX_i}}{e^{n(X_{i-1} - X_i)} - 1} = \sum_{i=2}^{r-1} \frac{\rho_{i-1}^2 \, e^{-nX_i^{(r)}}}{e^{\rho_{i-1}} - 1} + \frac{\rho_r^2}{e^{\rho_r} - 1}$$

where $X_i^{(r)}$ is X_i for $i = 1, \ldots, r$. $X_i^{(r)} = X_i^{(r-1)} + \frac{\rho_r}{n}$.

$$\epsilon_r^2 = \left(\sum_{i=2}^{r-1} \frac{\rho_{i-1} \, e^{-nX_i^{(r-1)}}}{e^{\rho_{i-1}} - 1} \right) e^{-\rho_r} + \frac{\rho_r^2}{e^{\rho_r} - 1}$$

$$\epsilon_r^2 = \left(\epsilon_{r-1}^2 + \frac{\rho_r^2}{1 - e^{-\rho_r}} \right) e^{-\rho_r}$$

To show that $\epsilon \to 1$ as $r \to \infty$, we will first show that $\rho \to 0$ as $r \to \infty$. From the definition $\rho_i > 0$, for all i.

Define $X_i = \dfrac{\rho_i}{1 - e^{-\rho_i}}$.

$$\frac{\partial X_i}{\partial \rho_i} = \frac{1}{1 - e^{-\rho_i}} - \frac{\rho_i \, e^{-\rho_i}}{(1 - e^{-\rho_i})^2} = \frac{1 - e^{-\rho_i}(1 + \rho_i)}{(1 - e^{-\rho_i})^2} .$$

But $1 + \rho < e^{\rho}$ if $\rho > 0$, so $e^{-\rho_i}(1 + \rho_i) < 1$

$$\therefore \quad \frac{\partial X_i}{\partial \rho_i} > 0 \qquad \text{for} \qquad \rho_i > 0 \; .$$

From the recursion relation, $X_i = 2 - X_{i-1}\, e^{-\rho_{i-1}}$,

$$X_i - X_{i-1} = 2 - X_{i-1}\,(1 + e^{-\rho_{i-1}}) = 2\,(1 - \frac{\rho_{i-1}}{2} \operatorname{ctnh} \frac{\rho_{i-1}}{2})$$

$$1 - X \operatorname{ctnh} X < 0 \qquad \text{for} \qquad 0 < X < \pi$$

$$\therefore \; X_i < X_{i-1} \qquad \text{since} \qquad 0 < \rho_{i-1} < \pi \qquad \text{for} \qquad i > 1$$

$$\therefore \; \rho_i < \rho_{i-1} \qquad \text{since} \qquad \frac{\partial X}{\partial \rho} > 0 \qquad \text{if} \qquad \rho > 0$$

Thus the ρ_i form a decreasing, bounded sequence. By the Bolzano-Weierstrass theorem, the ρ_i must have at least one limit point. It is easily shown that the decreasing nature of the sequence requires that there can be only one limit point. Hence the ρ_i converge to some value we will call ρ_∞. Applying the same arguments to the X_i, they also converge.

$$\therefore \quad \lim_{i \to \infty} X_i - X_{i-1} = 0$$

so

$$\lim_{i \to \infty} \frac{\rho_{i-1}}{2} \operatorname{ctnh} \frac{\rho_{i-1}}{2} = 1$$

$$\therefore \quad \lim_{i \to \infty} \rho_i = 0$$

For physical reasons ϵ_r must be a bounded increasing function; each additional measurement must improve the accuracy of the result, but with a finite N_r the error cannot be zero (which corresponds to ϵ being infinite). Therefore, by arguments similar to the above, ϵ_r converges to a limit, ϵ. Therefore, ϵ must satisfy

$$\epsilon^2 = \left(\epsilon^2 + \frac{\rho_r^{\;2}}{1 - e^{-\rho_r}}\right) e^{-\rho_r}$$

in the limit $r \to \infty$.

$$\lim_{r \to \infty} \epsilon^2 (1 - e^{-\rho_r}) = \lim_{r \to \infty} \frac{\rho_r^{\;2} \; e^{-\rho_r}}{1 - e^{-\rho_r}}$$

$$\epsilon^2 = \lim_{r \to \infty} \frac{\rho_r^{\;2} \; e^{-\rho_r}}{(1 - e^{-\rho_r})^2}$$

$$= \lim_{\rho \to 0} \frac{\rho^2 \; e^{-\rho}}{(1 - e^{-\rho})^2}$$

$$= 1$$

so

$$\epsilon = 1$$

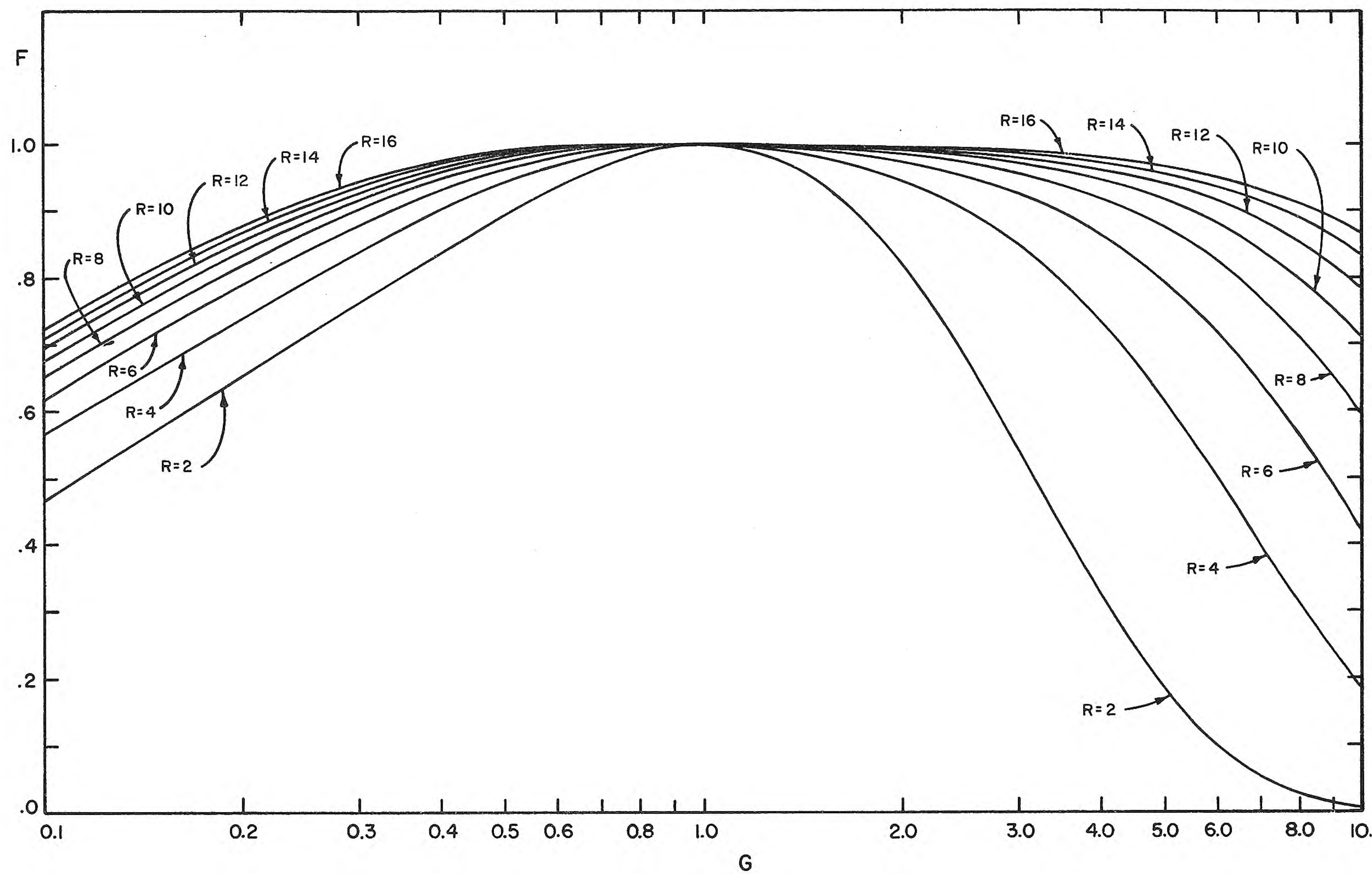

F
1.0
.8
.6
.4
.2
.0
R=2
R=4
R=6
R=8
R=10
R=12
R=14
R=16
0.1
0.2
0.3
0.4
0.5
0.6
0.8
1.0
2.0
3.0
4.0
5.0
6.0
8.0
10.
G

FIGURE 1

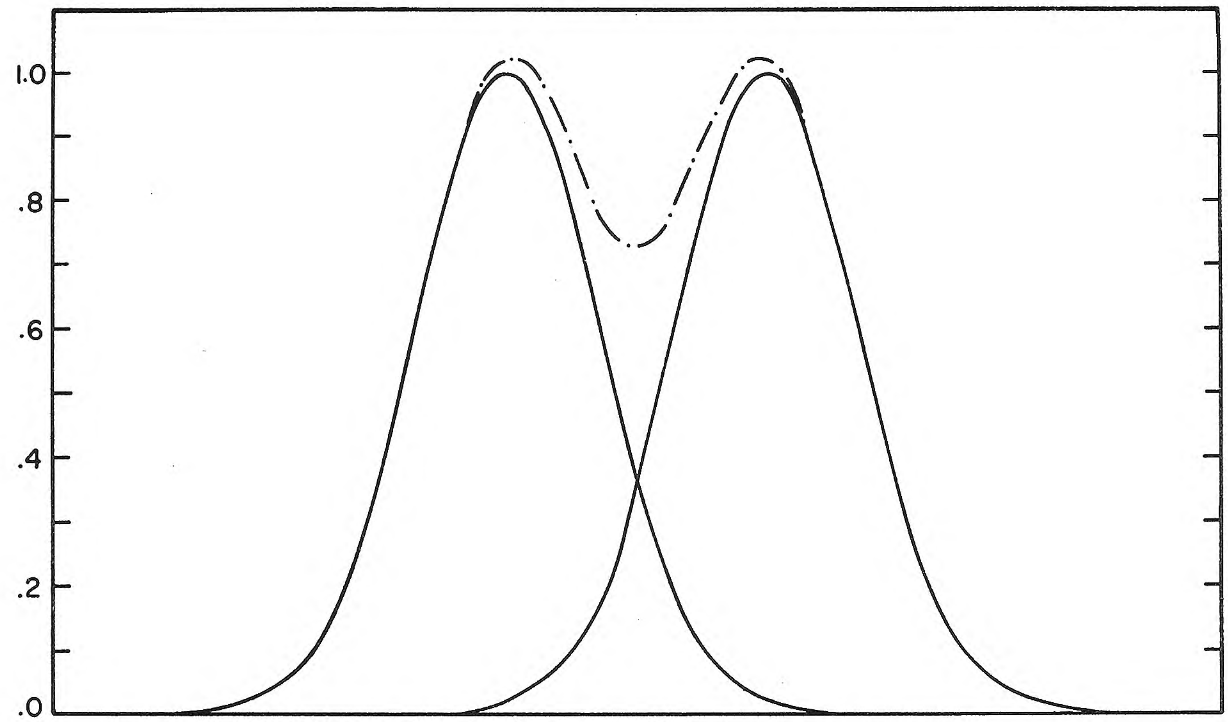


FIGURE 2a

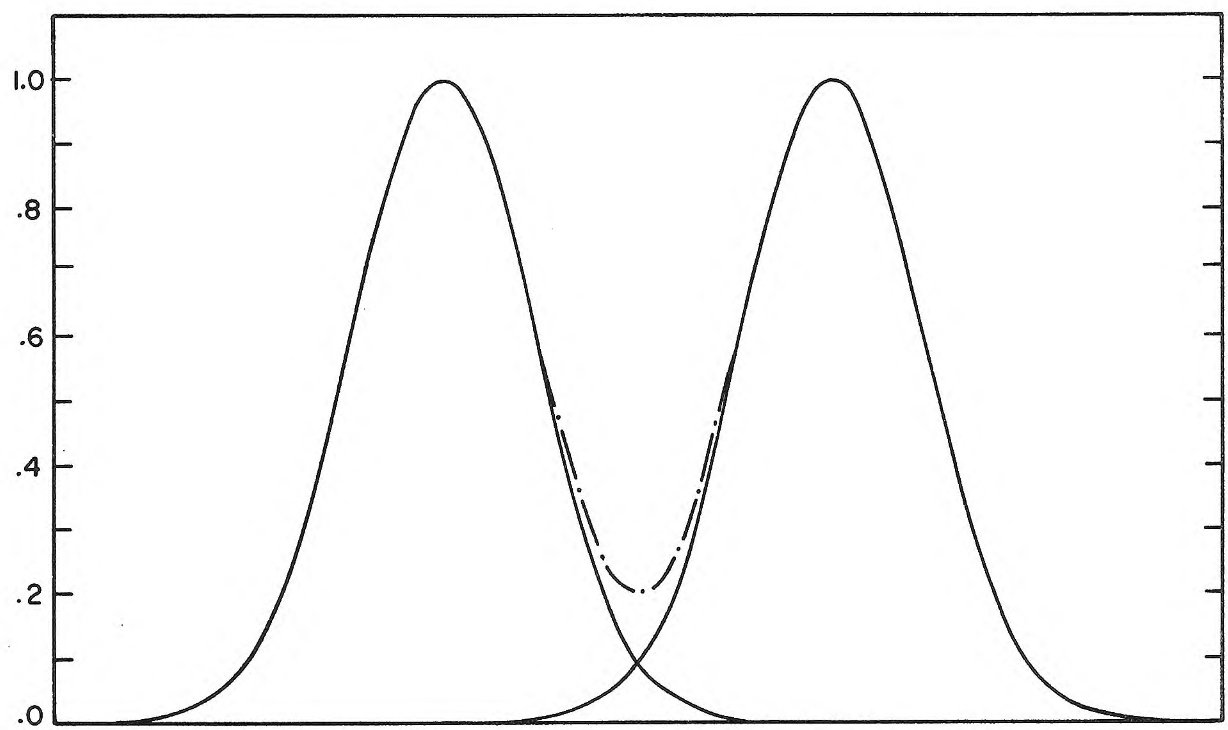
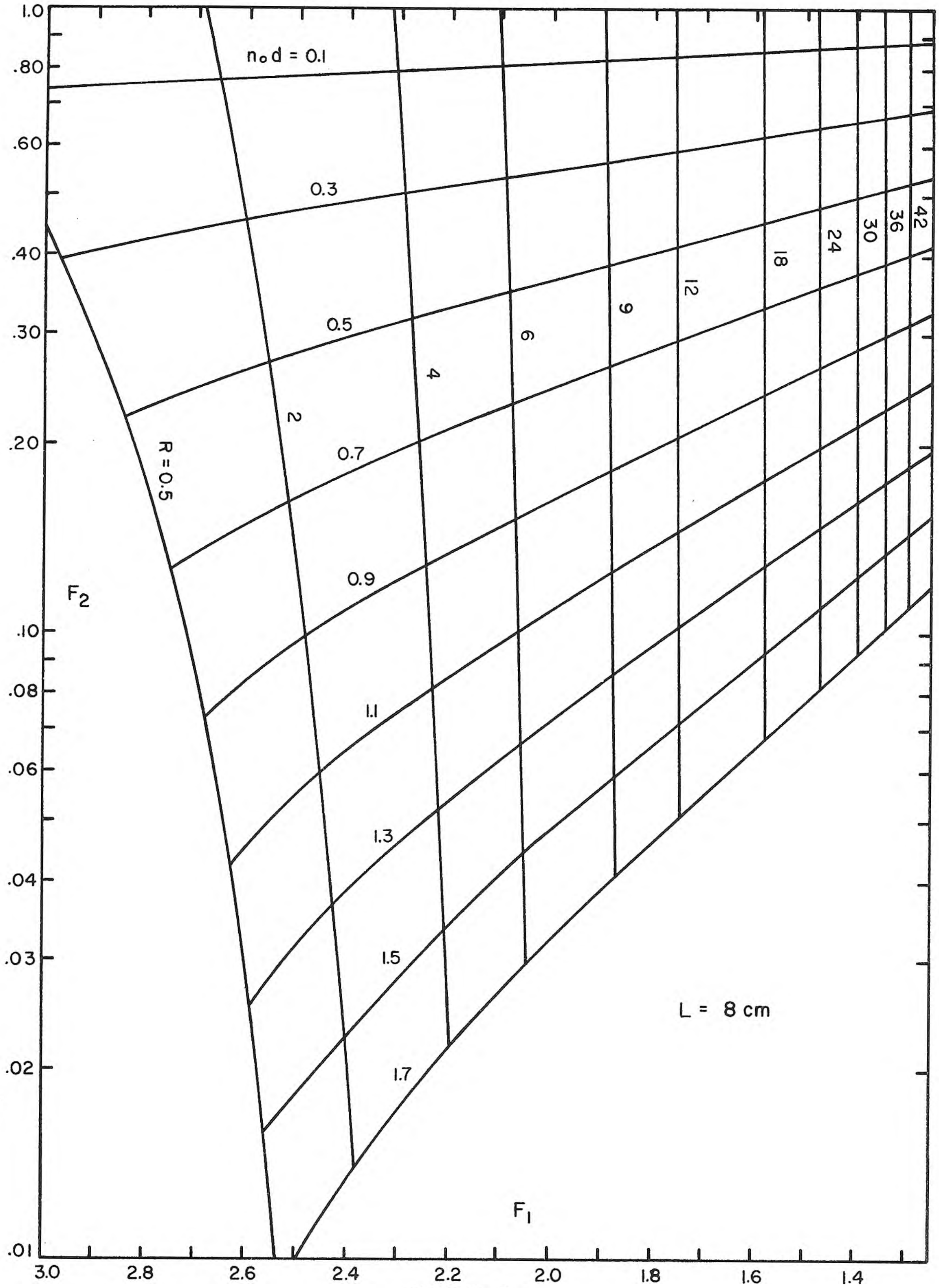


FIGURE 2b



$n_o d = 0.1$
0.3
0.5
0.7
0.9
1.1
1.3
1.5
1.7
R = 0.5
2
4
6
9
12
18
24
30
36
42
F_2
F_1
L = 8 cm
1.0
.80
.60
.40
.30
.20
.10
.08
.06
.04
.03
.02
.01
3.0
2.8
2.6
2.4
2.2
2.0
1.8
1.6
1.4

FIGURE 3

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