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Representations of Binary Decision Rules by Generalized Decisiveness Structures

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1. INTRODUCTION

This paper is motivated by two apparently dissimilar deficiencies in the theory of social choice and the theory of coöperative games. Both deficiencies stem from what we regard as an inadequate conception of decisiveness or coalitional power. Our main purpose will be to present a more general concept of decisiveness and to show that this notion allows us to characterize broad classes of games and social choice procedures.

Various theorems in social choice theory, beginning with Arrow [1], show that if a binary aggregation procedure satisfies certain axioms, then its underlying power structure must be dictatorial or oligarchical. For this reason among others, the notion of a power structure has come to be identified with the family of decisive coalitions. In many cases, however, the decisiveness structures that result from theorems like Arrow's do not imply all the axioms used to obtain these structures. The reason for this is simply that many decisiveness structures are compatible with a large number of aggregation procedures. The following example illustrates this in connection with Arrow's Theorem.

EXAMPLE 1. Let F be a mapping that assigns an asymmetric binary relation $F(\pi)$ on a set $\{a,b,c\}$ of three alternatives to each n-tuple $\pi = (P_1,\ldots,P_n)$ of individual asymmetric weak orders on $\{a,b,c\}$. Arrow's Theorem says that if each $F(\pi)$ is an asymmetric weak order, and if F satisfies independence and Pareto conditions, then some individual is a dictator. Suppose $n > 1$ and individual 1 is the dictator, so that $xF(\pi)y$ whenever xP_1y. In the usual terminology, every subset of individuals that contains 1 is decisive, and no other coalition is decisive. One specific F that satisfies Arrow's conditions and has 1 as the dictator is F_1 for which $xF_1(\pi)y$ if and only if xP_1y. In this case 1 is an absolute dictator. But consider a different F defined as follows, where I_1 is the indifference relation of individual 1:

$$xF_2(\pi)y \Leftrightarrow xP_1y \text{ or } [xI_1y \text{ and } (x,y) \in \{(a,b),(b,c),(c,a)\}].$$

Here F_2 also satisfies Arrow's independence and Pareto conditions, 1 is the dictator, and the only decisive coalitions are coalitions that contain 1. However, F_2 does not satisfy Arrow's social ordering condition since it yields the social preference cycle $\{aF_2(\pi)b,\ bF_2(\pi)c,\ cF_2(\pi)a\}$ whenever 1 is totally indifferent on $\{a,b,c\}$. Thus F_1 and F_2 have the same ordinary decisiveness structure, but are obviously different mappings.

A common notion of a decisive coalition is a subset of individuals such that when they unanimously prefer one alternative to another then the first is socially preferred to the second. A slightly different definition of decisive coalition is presented later in Section 3, and other definitions exist in the literature. The point we wish to stress, however, is that no conception of decisive coalitions that characterizes decisiveness in terms of single subsets of individuals, even if it is made to depend explicitly on pairs of alternatives, is adequate to characterize certain interesting aggregation procedures. An example of this is the simple majority aggregation procedure, in which x is socially preferred to y if and only if more individuals prefer x to y than prefer y to x. In this case every coalition that contains more than half the individuals is decisive, but what about other coalitions? For example, a nonempty coalition with less than half the individuals is "decisive" when all other individuals are indifferent but is not generally decisive.

In the theory of coöperative games an alternative is said to "dominate" another alternative just when there is a coalition that has the power to enforce this choice and which is unanimous in its preference on the pair of alternatives. This idea of coalitional power, which is usually represented by the game's characteristic function, is also deficient for essentially the same reason as given in the preceding paragraph. It is insufficient to allow the representation of wide classes of objects we might think of as games. For example, the simple majority game that corresponds to the simple majority aggregation procedure of the preceding paragraph cannot be represented as a coöperative game in the usual format. As before, the problem is that the notion of "power", as captured by a characteristic function, is not flexible enough.

Our proposal to remedy the deficiencies noted above is very simple and perhaps obvious by now. It is to characterize decisiveness structures by ordered pairs of disjoint coalitions rather than by single coalitions. Although this approach is not completely new--a similar idea was discussed by Fishburn [6, p. 40] in the context of two-alternative social choice theory, and may well have appeared elsewhere--we are not aware of a general development of it.

Certain structures based on ordered pairs of disjoint coalitions will be referred to as binary constitutions. The next section defines this term precisely and shows that there is a natural bijection between the set of binary constitutions and the set of binary decision rules defined in a traditional manner. Section 3 then shows how a sampling of conditions for binary decision rules in the social choice context maps into equivalent conditions for binary constitutions. Of special interest there is a condition we refer to as "decisiveness" that allows the ordered-pairs approach to be replaced by the traditional notion of decisive coalitions. The final section of the paper analyzes several social ordering axioms from the perspective of binary constitutions and presents special forms of these axioms that tie into recent developments in social choice theory.

2. BINARY DECISION RULES AND CONSTITUTIONS

Throughout this paper, X is the set of alternatives and $N = \{1,2,\ldots,n\}$ is the finite set of individuals. We shall let $\mathcal{A}$ be the collection of all asymmetric binary relations on X. Use will be made later of subsets of $\mathcal{A}$ whose relations are either acyclic, transitive or negatively transitive.

Each $i \in N$ is assumed to have an asymmetric preference relation P_i on X. Indifference I_i and weak preference R_i are defined from P_i by: $xI_iy \Leftrightarrow$ neither xP_iy nor yP_ix; $xR_iy \Leftrightarrow xP_iy$ or xI_iy. All P_i are presumed to lie in a nonempty subset $\mathcal{P}$ of $\mathcal{A}$ which, for the moment, will remain arbitrary. Transitivity conditions for $\mathcal{P}$ will be used later. Within the context of $\mathcal{P}$, a <u>configuration</u> is an ordered n-tuple $\pi = (P_1,P_2,\ldots,P_n)$ in $\mathcal{P}^n$, and a <u>binary decision rule</u> (BDR) is a mapping F from $\mathcal{P}^n$ into $\mathcal{A}$ that satisfies the following binary version of Arrow's independence axiom.

<u>BI</u> (Binary Independence): $(\forall\pi,\pi' \in \mathcal{P}^n)(\forall x,y \in X)$: <u>If</u> $xP_iy \Leftrightarrow xP_i'y$ <u>and</u> $yP_ix \Leftrightarrow yP_i'x$ <u>for all</u> $i \in N$, <u>then</u> $xF(\pi)y \Leftrightarrow xF(\pi')y$.

Within this formulation it is customary to interpret $F(\pi)$ as the social preference relation on X that is assigned by the BDR, F, to the configuration π of individuals' preferences on X. Social indifference would then be

represented by the symmetric complement of $F(\pi)$. For a given $\mathcal{P}$ we shall let $F(\mathcal{P})$ be the set of all BDR's that are defined as above within the context of $\mathcal{P}$ as the set of allowable individual preference relations.

We now present the key concept of this paper, referred to as a binary constitution, that can be used to characterize all BDR's in terms of generalized decisiveness structures. Let $\hat{X} = \{(x,y): \; x,y \in X \;\&\; x \neq y\}$ and let $T = \{(A,B): \; A,B \subseteq N \;\&\; A \cap B = \emptyset\}$ with 2^T the set of all subsets of T. Then a <u>binary constitution</u> is a mapping C from $\hat{X}$ into 2^T that satisfies the following asymmetry axiom.

<u>CO</u>. $(\forall(x,y) \in \hat{X})(\forall(A,B) \in T)$: <u>If</u> $(A,B) \in C(x,y)$ <u>then</u> $(B,A) \notin C(y,x)$.

Despite the fact that this definition makes no reference to individual preferences, it is natural to interpret $C(x,y)$ as the set of ordered pairs of disjoint coalitions of individuals such that x is socially preferred to y if and only if the ordered pair of subsets of N whose members respectively prefer x to y and prefer y to x is in $C(x,y)$. The obvious purpose of axiom CO is to forbid simultaneous social preferences for x over y and for y over x.

The correspondence between BDR's and binary constitutions that is suggested by the preceding description will now be developed. Along with $F(\mathcal{P})$ as the set of BDR's that arise from $\mathcal{P}$, we shall let $\mathcal{C}$ be the set of all binary constitutions. The fact that $\mathcal{C}$ does not depend on $\mathcal{P}$ will have a bearing on our analysis as noted shortly. For each $\pi = (P_1,\ldots,P_n)$ in $\mathcal{P}^n$ and each pair $(x,y) \in \hat{X}$ let

$$\pi(x,y) = (\{i \in N: \; xP_iy\}, \{i \in N: \; yP_ix\}).$$

Thus $\pi(x,y)$ is an ordered pair of disjoint subsets of N that respectively identify the individuals who prefer x to y and those who prefer y to x. Obviously $\pi(x,y) = (A,B) \Leftrightarrow \pi(y,x) = (B,A)$.

For each $F \in F(P)$, the binary constitution induced by F and denoted as C_F is defined on $\hat{X}$ by

$$(\forall(x,y) \in \hat{X}): \quad (A,B) \in C_F(x,y) \Leftrightarrow \exists\pi \in P^n \text{ such that } xF(\pi)y \ \& \ \pi(x,y) = (A,B). \tag{1}$$

The language used in this definition is justified by the following theorem.

THEOREM 1. $C_F \in C$ for each $F \in F(P)$. Moreover, if $F,F' \in F(P)$ and $F \neq F'$ then $C_F \neq C_{F'}$.

Proof. To prove that $C_F \in C$ we need to show that it satisfies CO. If it fails to satisfy CO, with $(A,B) \in C_F(x,y)$ and $(B,A) \in C_F(y,x)$ for some $(x,y) \in \hat{X}$ and $(A,B) \in T$, then (1) implies that there are $\pi, \pi' \in P^n$ such that $xF(\pi)y$, $\pi(x,y) = (A,B)$, $yF(\pi')x$ and $\pi'(y,x) = (B,A)$, in which case $\pi'(x,y) = \pi(x,y)$ in contradiction to BI and the asymmetry of $F(\pi)$. Hence C_F satisfies CO. Next, if $F,F' \in F(P)$ and $F \neq F'$ there exists $\pi \in P^n$ and $(x,y) \in \hat{X}$ such that either $xF(\pi)y$ and not $(xF'(\pi)y)$ or $xF'(\pi)y$ and not $(xF(\pi)y)$. Suppose for definiteness that $xF(\pi)y$ and not $(xF'(\pi)y)$. Then, by (1), $\pi(x,y) \in C_F(x,y)$, but $\pi(x,y) \notin C_{F'}(x,y)$, for otherwise BI is contradicted. Hence $C_F \neq C_{F'}$. Q.E.D.

EXAMPLE 1 (Continued). Suppose F is the BRD, F_2, of Example 1, and let C be the binary constitution induced by F_2. Let $C_1 = \{(A,B) \in T: \ 1 \in A\}$ and $C_2 = \{(A,B) \in T: \ 1 \notin A \cup B\}$. Then $C(a,b) = C(b,c) = C(c,a) = C_1 \cup C_2$ and $C(b,a) = C(c,b) = C(a,c) = C_1$.

In contrast to Theorem 1, it is not necessarily true that only one $C \in C$ is "consistent" with a given $F \in F(P)$. The simplest example arises when $P = \{\emptyset\}$, which asserts that all individuals are always indifferent among all alternatives. In this case P^n contains only one contingency, namely $\pi = (\emptyset,\ldots,\emptyset)$, and the only aspect of C that is relevant to $F \in F(P)$ is whether or not $(\emptyset,\emptyset) \in C(x,y)$ for each $(x,y) \in \hat{X}$. If C and C′ are alike with respect to $(\emptyset,\emptyset)$ but differ in other respects then they will induce the same BDR in $F(P)$.

For each $C \in C$, the BDR induced by C in the context of P and denoted as F_C is defined on P^n by

$$(\forall\pi \in P^n): \quad xF_C(\pi)y \Leftrightarrow (x,y) \in \hat{X} \ \& \ \pi(x,y) \in C(x,y). \tag{2}$$

As shown in the preceding paragraph, if P is sufficiently restricted, then different binary constitutions can induce the same BDR on P^n. Part (c) of the following theorem shows exactly what must be true of P for each C to induce a different F_C.

THEOREM 2. Let P be given.

(a) $F_C \in F(P)$ for each $C \in C$;

(b) For each $F \in F(P)$ let $C(F) = \{C \in C: \ F_C = F\}$ and let $C^*(x,y) = \bigcap_{C \in C(F)} C(x,y)$ for all $(x,y) \in \hat{X}$. Then $C^* = C_F$;

(c) $[(\forall C,C' \in C): \ C \neq C' \Rightarrow F_C \neq F_{C'}] \Leftrightarrow [(\forall(x,y) \in \hat{X}): \exists P,P',P'' \in P$ such that $xPy, yP'x$ and $xI''y]$.

REMARKS. Part (a) shows that each F_C is indeed a BDR. Part (b) says in effect that the intersection of all binary constitutions that induce a given $F \in \mathcal{F}(\mathcal{P})$ is the binary constitution that is induced by F. Hence C* as in (b) is a minimal representative of $\mathcal{C}(F)$, and there will be a one-one induced correspondence between $\mathcal{F}(\mathcal{P})$ and the set of all such minimal representative binary constitutions. Part (c) then says that the set of all such C* will be $\mathcal{C}$ itself if and only if for every $(x,y) \in \hat{X}$ some relation in $\mathcal{P}$ has x preferred to y, another has y preferred to x, and a third has x indifferent to y.

Proof. To prove (a) we need to show that $F_C(\pi)$ is asymmetric and satisfies BI. Asymmetry follows immediately from (2) and CO. If BI were false, then we would have $(x,y) \in \hat{X}$ and $\pi, \pi' \in \mathcal{P}^n$ for which $\pi(x,y) = \pi'(x,y)$, $xF_C(\pi)y$ and not $(xF_C(\pi')y)$. Then (2) would require $\pi(x,y) \in C(x,y)$ and $\pi'(x,y) \notin C(x,y)$, which is impossible since $\pi(x,y) = \pi'(x,y)$.

To prove (b) note first that for each $F \in \mathcal{F}(\mathcal{P})$, $C_F \in \mathcal{C}$ by Theorem 1, and it follows from (2) that F is the BDR induced by C_F. Hence $\mathcal{C}(F)$ is not empty. By (2), if $xF(\pi)y$, then $\pi(x,y) \in C(x,y)$ for every $C \in \mathcal{C}(F)$, and if not $(xF(\pi)y)$ then $\pi(x,y) \notin C(x,y)$ for all $C \in \mathcal{C}(F)$. Hence C* as defined in (b) must be in $\mathcal{C}(F)$, and it is readily seen that C* is identical to C_F.

For (c) suppose first that $C \neq C'$ and that the given condition on $\mathcal{P}$ holds. Assume without loss in generality that $(A,B) \in C(x,y)$ and $(A,B) \notin C'(x,y)$ for some $(A,B) \in T$ and $(x,y) \in \hat{X}$. Then, by the condition on $\mathcal{P}$, there is a $\pi \in \mathcal{P}^n$ that has $\pi(x,y) = (A,B)$, and, by (2), $xF_C(\pi)y$ and not $(xF_{C'}(\pi)y)$ so that $F_C \neq F_{C'}$. Conversely, suppose there is $(x,y) \in \hat{X}$ such that no $P \in \mathcal{P}$ has xPy or that no $P \in \mathcal{P}$ has xIy. If no $P \in \mathcal{P}$ has xPy let C and C′ be binary constitutions that are alike in all respects except

that C(x,y) contains only $(\{1\},\emptyset)$ and $C'(x,y)$, C(y,x) and $C'(y,x)$ are all empty. Then $C \neq C'$ but $F_C = F_{C'}$. If no $P \in \mathcal{P}$ has xIy let C and C′ be alike in all respects except that C(x,y) contains only $(\emptyset,\emptyset)$ and $C'(x,y)$, C(y,x) and $C'(y,x)$ are all empty. Then $C \neq C'$ but $F_C = F_{C'}$. Q.E.D.

Theorems 1 and 2 show that there is a natural bijection between $\mathcal{C}$ and $\mathcal{F}(\mathcal{P})$ provided that, for each distinct pair of alternatives, an individual can either prefer either one to the other or be indifferent between them. If $\mathcal{P}$ happens to be restricted in some way that violates this provision, then as noted in Theorem 2(b) there is a natural way to identify a subset of $\mathcal{C}$ that has a natural bijection with $\mathcal{F}(\mathcal{P})$. This latter situation will not be explored further in the present paper. In other words, we assume henceforth that $\mathcal{P}$ satisfies the condition in the second half of Theorem 2(c).

3. PROPERTIES OF BDR's AND CONSTITUTIONS

The natural bijection between binary decision rules and binary constitutions shows that BDR's can be studied either from the traditional perspective of F functions or from the perspective of generalized decisiveness structures as characterized by binary constitutions. Although binary constitutions seem quite attractive from a conceptual viewpoint and may be more manageable than BDR's in certain types of investigations, the need to manipulate individual preferences in specific ways in certain derivations (e.g. in a proof of Arrow's impossibility theorem) may favor the use of BDR's in some cases.

Our main purpose in the rest of this paper will be to identify and illustrate equivalences between selected special properties for BDR's and binary constitutions. We consider first several types of dictators and then look at various conditions such as unanimity and monotonicity. The next section will examine social "rationality" conditions.

Dictators and Oligarchies

The following definitions for a BDR, F, identify certain potential dictatorial features that F might possess.

D1. Individual i is a weak dictator (or vetoer) if and only if $(\forall \pi \in P^n)(\forall (x,y) \in \hat{X})$: $xP_iy \Rightarrow$ not $(yF(\pi)x)$.

D2. Individual i is a dictator if and only if $(\forall \pi \in P^n)(\forall (x,y) \in \hat{X})$: $xP_iy \Rightarrow xF(\pi)y$.

D3. Individual i is an absolute dictator if and only if $(\forall \pi \in P^n)$ $(\forall (x,y) \in \hat{X})$: $xP_iy \Leftrightarrow xF(\pi)y$.

D4. A nonempty subset $A^* \subseteq N$ is an oligarchy if and only if $(\forall \pi \in P^n)$ $(\forall (x,y) \in \hat{X})$: xP_iy for all $i \in A^* \Rightarrow xF(\pi)y$, and xP_iy for some $i \in A^* \Rightarrow$ not $(yF(\pi)x)$.

Thus an absolute dictator determines social indifference by his indifference as well as determining social preferences according to his preferences. Although a BDR could have a number of weak dictators, which is true if it has an oligarchy with $|A^*| \geq 2$, there can be at most one dictator or absolute dictator for a given F. In addition, F can have at most one oligarchy. An oligarchy consists of a single individual if and only if this oligarchy contains a dictator.

The appropriate definitions for a binary constitution C that correspond to the foregoing are as follows.

D1*. Individual i is a weak dictator if and only if $(\forall (A,B) \in T)$ $(\forall (x,y) \in \hat{X})$: $i \in B \Rightarrow (A,B) \notin C(x,y)$.

D2*. Individual i is a dictator if and only if $(\forall (A,B) \in T)$ $(\forall (x,y) \in \hat{X})$: $i \in A \Rightarrow (A,B) \in C(x,y)$.

D3*. Individual i is an absolute dictator if, and only if, $(\forall (A,B) \in T)(\forall (x,y) \in \hat{X})$: $i \in A \Leftrightarrow (A,B) \in C(x,y)$.

D4*. A nonempty subset $A^* \subseteq N$ is an oligarchy if and only if $(\forall (A,B) \in T)$ $(\forall (x,y) \in \hat{X})$: $A^* \subseteq A \Rightarrow (A,B) \in C(x,y)$, and $A^* \cap B \neq \emptyset \Rightarrow (A,B) \notin C(x,y)$.

The equivalences between the F definitions and the C definitions are easily proved using (1) and (2). The reader may find it instructive to prove one or more of these equivalences.

In addition to the above definitions, various related concepts are found in the literature. For example, each of the definitions can be specialized to subsets of $\hat{X}$, in which case different dictators or different oligarchies may reign in different regions of $\hat{X}$. Another example is a hierarchy of dictators, in which the secondary dictator determines social preference only if the primary dictator is indifferent, the tertiary dictator does likewise only if the primary and secondary dictators are indifferent, and so forth.

Unanimity and Two-Configuration Conditions

We now present a sample of common binary-based conditions for BDR's and specify the forms that these conditions take for binary constitutions. The proof of equivalence for each pair is straightforward using (1) and (2) and will be omitted.

<u>B1</u> (Pareto). $(\forall \pi \in P^n)(\forall (x,y) \in \hat{X})$: If $xP_i y$ for all $i \in N$ then $xF(\pi)y$.

<u>B1</u>*. $(\forall (x,y) \in \hat{X})$: $(N,\emptyset) \in C(x,y)$.

<u>B2</u> (Anonymity). $(\forall \pi,\pi' \in P^n)(\forall (x,y) \in \hat{X})(\forall$ permutations σ on $N)$: If $xP_i y \Leftrightarrow xP'_{\sigma(i)}y$ and $yP_i x \Leftrightarrow yP_{\sigma(i)}x$ for all $i \in N$, then $xF(\pi)y \Leftrightarrow xF(\pi')y$.

<u>B2</u>*. $(\forall (x,y) \in \hat{X})(\forall (A,B),(A',B') \in T)$: If $|A| = |A'|$ and $|B| = |B'|$ then $(A,B) \in C(x,y) \Leftrightarrow (A',B') \in C(x,y)$.

<u>B3</u> (Monotonicity). $(\forall \pi,\pi' \in P^n)(\forall (x,y) \in \hat{X})$: If $xP_i y \Rightarrow xP'_i y$ and $xI_i y \Rightarrow xR'_i y$ for all $i \in N$, and if $xF(\pi)y$, then $xF(\pi')y$.

<u>B3</u>*. $(\forall (x,y) \in \hat{X})(\forall (A,B),(A',B') \in T)$: If $A \subseteq A', B' \subseteq B$ and $(A,B) \in C(x,y)$ then $(A',B') \in C(x,y)$.

<u>B4</u> (Semineutrality). $(\forall \pi,\pi' \in P^n)(\forall (x,y),(x,z) \in \hat{X})$: If $xP_i y \Leftrightarrow xP_i z$ and $yP_i x \Leftrightarrow zP_i x$ for all $i \in N$, then $xF(\pi)y \Leftrightarrow xF(\pi')z$.

<u>B4</u>*. $(\forall (x,y),(x,z) \in \hat{X})$: $C(x,y) = C(x,z)$.

<u>B5</u> (Neutrality). $(\forall \pi,\pi' \in P^n)(\forall (x,y),(z,w) \in \hat{X})$: If $xP_i y \Leftrightarrow zP'_i w$ and $yP_i x \Leftrightarrow wP'_i z$ for all $i \in N$, then $xF(\pi)y \Leftrightarrow zF(\pi')w$.

<u>B5</u>*. $(\forall (x,y),(z,w) \in \hat{X})$: $C(x,y) = C(z,w)$.

<u>B6</u> (Decisiveness). $(\forall \pi,\pi' \in P^n)(\forall (x,y) \in \hat{X})$: If $xP_i y \Leftrightarrow xP'_i y$ for all $i \in N$, and if $xF(\pi)y$, then $xF(\pi')y$.

<u>B6</u>*. $(\forall (x,y) \in \hat{X})(\forall (A,B),(A,B') \in T)$: If $(A,B) \in C(x,y)$ then $(A,B') \in C(x,y)$.

The effects of the final three conditions on a binary constitution are especially noteworthy. Condition B4* says that C is constant on each subset

$\{(x,y): y \in X\backslash\{x\}\}$ of $\hat{X}$ whose ordered pairs have the same first alternative. Hence $C(x,y)$ in this case can be abbreviated as $C(x)$, with $C(x,y) = C(x)$ for all $y \neq x$.

Condition B5* asserts much more, namely that C is constant on $\hat{X}$ with the same image for every $(x,y) \in \hat{X}$. When B5* holds, we shall let $C_0 = C(x,y)$ for all $(x,y) \in \hat{X}$. It follows from this that many aspects of neutral binary constitutions can be examined from the perspective of neutral (or dual) decision rules on two alternatives. A detailed discussion of such rules is given in Chapters 3 through 5 in Fishburn [6].

Two familiar examples of C_0 type rules are the Pareto rule $C_0 = \{(N,\emptyset)\}$, and the simple majority rule $C_0 = \{(A,B) \in T: |A| > |B|\}$. Ferejohn and Grether [5] examine majority rules for which $C_0 = \{(A,B) \in T: |A| > \alpha n\}$ and $C_0 = \{(A,B) \in T: |A| > \alpha(|A| + |B|)\}$ with $1/2 \leq \alpha < 1$. Other C_0 rules include various forms of weighted majorities (different weights perhaps for different individuals) and representative systems, which are essentially hierarchical structures based on weighted majorities [6, 10].

Although we hesitate to refer to B6 as <u>decisiveness</u> since this term is used for several other concepts, it seemed more appropriate than other designations in the context of this paper. Condition B6 says that if x is socially preferred to y for some configuration in which $\{i: xP_i y\} = A$, then x will be socially preferred to y for all other configurations that have $\{i: xP_i y\} = A$. In other words, if A is decisive for x over y in one situation in which nobody else prefers x to y, then A is decisive for x over y in all situations in which nobody else prefers x to y. Defining

$$D(x,y) = \{A: \ (A,B) \in C(x,y) \text{ for some } (A,B) \in T\},$$

B6 implies that x is socially preferred to y if, and only if, $\{i: \ xP_iy\} \in D(x,y)$. As just suggested, sets in D(x,y) are often referred to as <u>decisive coalitions</u> for x over y.

When B6 or B6* holds, other conditions can be written in terms of D rather than C as follows:

C0: $A \in D(x,y)$ and $A \cap B = \emptyset \Rightarrow B \notin D(y,x)$,

B1*: $N \in D(x,y)$,

B3*: $A \in D(x,y)$ and $A \subseteq B \Rightarrow B \in D(x,y)$,

B5*: $D(x,y) = D(z,w)$ for all $(x,y), (z,w) \in \hat{X}$.

When <u>both</u> B5* and B6* hold, binary constitutions are completely characterized by a set D_0 of coalitions such that $D_0 = D(x,y)$ for all $(x,y) \in \hat{X}$. In terms of C_0 as defined earlier under B5*, $D_0 = \{A: \ (A,B) \in C_0 \text{ for some } (A,B) \in T\}$.

Notions of decisiveness for single coalitions that do not presume B6 can of course be given. In particular, regardless of whether B6 or B5 and B6 hold for an arbitrary binary decision rule F, let

$$D_F(x,y) = \{A \subsetneq N: \ (\forall \pi \in P^n)(\forall B \subsetneq N \setminus A), \pi(x,y) = (A,B) \Rightarrow xF(\pi)y\},$$

$$W_F = \cap_{\hat{X}} D_F(x,y).$$

Coalitions in $D_F(x,y)$ can be thought of as coalitions that are decisive for x over y whenever everyone in the coalition prefers x to y and no other individual prefers x to y. Coalitions in W_F are decisive in this sense for all $(x,y) \in \hat{X}$. As suggested earlier, D_F and W_F are incomplete descriptors of F unless F satisfies

B6 or B5 and B6 respectively. In other words, when $\mathcal{D}$ is the set of all functions D from $\hat{X}$ into the set of subsets of 2^N that satisfy

$$\underline{\text{D0}}: \ (\forall (x,y) \in \hat{X})(\forall (A,B) \in T): A \in D(x,y) \Rightarrow B \notin D(y,x),$$

then there is a natural bijection between $\mathcal{D}$ and the set of all BDR's that satisfy B6. And, when $\mathcal{W}$ is the set of all subsets of 2^N that satisfy

$$\underline{\text{W0}}: \ (\forall (A,B) \in T): \ A \notin W \text{ or } B \notin W,$$

there is a natural bijection between $\mathcal{W}$ and the set of all BDR's that satisfy B5 and B6.

Special subsets of $\mathcal{W}$ that have been discussed by Hansson [8], Kirman and Sondermann [9], and Brown [3,4], will be examined in the next section. These are based on the following types of W (or D_0) sets. A family W of subsets of N is a <u>prefilter</u> if and only if

(i) $N \in W$,

(ii) $(\forall A,B \subsetneq N)$: $A \in W$ and $A \subseteq B \Rightarrow B \in W$,

(iii) $\cap_W A \neq \emptyset$;

a <u>filter</u> if and only if it is a prefilter that satisfies

(iv) $(\forall A,B \subsetneq N)$: $A,B \in W \Rightarrow A \cap B \in W$;

and an <u>ultrafilter</u> if and only if it is a filter that satisfies

(v) $(\forall A \subsetneq N)$: $A \in W$ or $(N \setminus A) \in W$.

4. SOCIAL ORDERING CONDITIONS

Because social ordering conditions are of interest only if individual preferences are fairly well structured, it will be assumed throughout this section that $\mathcal{P}$ is the set of all negatively transitive relations in A. In other words, $\mathcal{P}$ is the set of all asymmetric weak orders on X, with $xPz \Rightarrow xPy$ or yPz, for all $x,y,z \in X$. It should be noted that the proofs of equivalence given below depend on this assumption. The interested reader is invited to construct alternative conditions for binary constitutions that are equivalent to the social ordering conditions under other forms for $\mathcal{P}$.

Let Π be the set of all n-tuples of asymmetric weak orders on X. The main purpose of this section is to establish conditions on binary constitutions that are equivalent to the following social ordering conditions for a BDR, F.

R1 (Acyclicity). $(\forall \pi \in \Pi)$: $F(\pi)$ is acyclic.

R2 (Partial Order). $(\forall \pi \in \Pi)$: $F(\pi)$ is transitive.

R3 (Weak Order). $(\forall \pi \in \Pi)$: $F(\pi)$ is negatively transitive.

Given asymmetry, $R3 \Rightarrow R2 \Rightarrow R1$. Recall that a binary relation R on X is acyclic if and only if the transitive closure R^t of R is irreflexive (it is never true that $x_1Rx_2, x_2Rx_3, \ldots, x_{m-1}Rx_m$ and x_mRx_1), and that R is negatively transitive if and only if $(\forall x,y,z \in X)$: $xRz \Rightarrow xRy$ or yRz, or equivalently, $(\forall x,y,z \in X)$: [not (xRy) & not (yRz)] $\Rightarrow$ not (xRz).

Acyclicity

The acyclicity condition that we shall use for a binary constitution C takes the following form.

R1*. $(\forall m > 1)(\forall$ distinct $x_1,x_2,\ldots,x_m \in X)(\forall (A_1,B_1),\ldots,(A_m,B_m) \in T)$: If $(A_k,B_k) \in C(x_k,x_{k+1})$ for all $k < m$, and if $(A_m,B_m) \in C(x_m,x_1)$, then either

$$A_k \not\subseteq \bigcup_{\substack{j=1 \\ j\neq k}}^{m} B_j \quad \text{for some } k \in \{1,\ldots,m\} \tag{3}$$

or

$$B_k \not\subseteq \bigcup_{\substack{j=1 \\ j\neq k}}^{m} A_j \quad \text{for some } k \in \{1,\ldots,m\}. \tag{4}$$

Alternatively, R1* says that if $A_k \subseteq \cup_{j\neq k} B_j$ and $B_k \subseteq \cup_{j\neq k} A_j$ for every k, then either $(A_k,B_k) \notin C(x_k,x_{k+1})$ for some $k < m$ or else $(A_m,B_m) \notin C(x_m,x_1)$.

THEOREM 3. Suppose F and C are related as in (1) and (2). Then R1 holds for F if and only if R1* holds for C.

Two preliminary lemmas will be proved before we complete the proof of Theorem 3.

LEMMA 1. Suppose (3) and (4) are false. Then, for all $k \in \{1,\ldots,m\}$, $i \in B_k \Rightarrow \exists j \neq k$ with $i \in A_j$, and $i \in A_k \Rightarrow \exists j \neq k$ with $i \in B_j$.

Proof. If $i \in B_k$ and $i \notin A_j$ for all $j \neq k$, then (4) holds. If $i \in A_k$ and $i \notin B_j$ for all $j \neq k$, then (3) holds. Q.E.D.

LEMMA 2. Suppose $x_1,x_2,\ldots,x_m$ are distinct alternatives in X and that, with $x_{m+1} \equiv x_1$, P' is an asymmetric binary relation that is included in $\cup_{j=1}^{m}\{(x_j,x_{j+1}),(x_{j+1},x_j)\}$ and satisfies the following for all $k \in \{1,\ldots,m\}$: (i) $x_{k+1}P'x_k \Rightarrow \exists j \neq k$ with $x_jP'x_{j+1}$; (ii) $x_kP'x_{k+1} \Rightarrow \exists j \neq k$ with $x_{j+1}P'x_j$. Then there is an asymmetric weak order P on $\{x_1,\ldots,x_m\}$ such that

$$P' = P \cap [\bigcup_{j=1}^{m} \{(x_j, x_{j+1}), (x_{j+1}, x_j)\}]. \quad (5)$$

Proof. Given the lemma's hypotheses let

$$K_1 = \{k \in \{1,\ldots,m\}: x_k P' x_{k+1}\}$$
$$K_2 = \{k \in \{1,\ldots,m\}: x_{k+1} P' x_k\}$$
$$K_3 = \{1,\ldots,m\} \setminus (K_1 \cup K_2).$$

Then $K_1 \cap K_2 = \emptyset$ by asymmetry, and (i) and (ii) imply that $K_1 = \emptyset \Leftrightarrow K_2 = \emptyset$. [$P = \emptyset$ satisfies (5) if $K_1 = K_2 = \emptyset$.] With I' the symmetric complement of P' on $\{x_1,\ldots,x_m\}$, partition $\{x_1,\ldots,x_m\}$ into subsets $Y_1, Y_2, \ldots, Y_M$ such that x_k and x_{k+1} are in the same Y_j if, and only if, $x_k I' x_{k+1}$, and such that $x_1 \in Y_1$ with $x_{k+1} \in Y_{j+1}$ if $\{x_k \in Y_j$, not $(x_{k+1} I' x_k)$, $j < M$ and $k < m\}$. [$M = 1 \Leftrightarrow K_1 = \emptyset$. Y_1 might contain $x_m, x_{m-1}, \ldots$ as well as $x_2, x_3, \ldots$. If $m = 3$, $x_1 I' x_2$, $x_2 I' x_3$ and not $(x_3 I' x_1)$ then (i) or (ii) will be violated.] Next, define Q' on $\{Y_1,\ldots,Y_m\}$ by

$$Y_j Q' Y_k \Leftrightarrow x_\alpha P' x_\beta \text{ for some } x_\alpha \in Y_j \text{ and } x_\beta \in Y_k.$$

Then Q' is asymmetric and, with $Y_{M+1} \equiv Y_1$, we have $Y_j Q' Y_{j+1}$ for some $j \in \{1,\ldots,M\}$ if and only if $Y_{k+1} Q' Y_k$ for some $k \in \{1,\ldots,M\}$. Moreover, for each $j \in \{1,\ldots,M\}$, either $Y_j Q' Y_{j+1}$ or $Y_{j+1} Q' Y_j$, and if $|k - j| > 1$ and $\{j,k\} \neq \{1,M\}$ then neither $Y_j Q' Y_k$ nor $Y_k Q' Y_j$. Next, let R' be the transitive closure of Q'. Then R' is transitive by definition and, because of the properties for Q' that were noted above, R' is asymmetric. Hence R' is an asymmetric partial order. It then follows from Szpilrajn's extension theorem [11] that there is a linear order R on $\{Y_1, Y_2, \ldots, Y_M\}$ that includes R'. [That is, R is irreflexive, transitive and complete, with $R' \subseteq R$.] Given such an R, define P on $\{x_1, x_2, \ldots, x_m\}$ by

$$x_\alpha P x_\beta \Leftrightarrow Y_j R Y_k \text{ when } x_\alpha \in Y_j \text{ and } x_\beta \in Y_k.$$

Then P is an asymmetric weak order on $\{x_1,\ldots,x_m\}$. Moreover, if $x_k P' x_{k+1}$ and $x_k \in Y_j$, then $x_{k+1} \in Y_{j+1}$ [recall that $Y_{M+1} \equiv Y_1$], hence $Y_j Q' Y_{j+1}$, hence $Y_j R' Y_{j+1}$, hence $Y_j R Y_{j+1}$, hence $x_k P x_{k+1}$; similarly, if $x_{k+1} P' x_k$ then $x_{k+1} P x_k$; and, if $x_k I' x_{k+1}$, then x_k and x_{k+1} are in the same Y_j and therefore $x_k I x_{k+1}$. Therefore P satisfies (5). Q.E.D.

Proof of Theorem 3. With F and C induced by each other by (1) and (2), suppose first that R1 is false. Then there is a $\pi \in \Pi$ and distinct $x_1,\ldots,x_m \in X$ (with $m > 2$ since $F(\pi)$ is asymmetric) such that $x_1 F(\pi) x_2, x_2 F(\pi) x_3, \ldots, x_m F(\pi) x_1$. Then, with $(A_k, B_k) = \pi(x_k, x_{k+1})$ for all $k < m$, and with $(A_m, B_m) = \pi(x_m, x_1)$, R1*, if true, requires that either $(\exists i \in N)(\exists k \in \{1,\ldots,m\})$ with $i \in A_k$ and $i \notin \bigcup_{j \neq k} B_j$--in which case $x_1 R_i x_2, x_2 R_i x_3, \ldots, x_m R_i x_1$ and $x_k P_i x_{k+1}$, which contradicts the assumption that P_i is an asymmetric weak order--or else $(\exists i \in N)(\exists k \in \{1,\ldots,m\})$ with $i \in B_k$ and $i \notin \bigcup_{j \neq k} A_j$--which similarly contradicts weak order for P_i. Therefore not (R1) $\Rightarrow$ not (R1*).

Conversely, suppose R1* is false so that its hypotheses hold for a situation in which both (3) and (4) are false. Then, for this situation, let $\pi' = (P'_1,\ldots,P'_n)$ be such that $P'_i \subseteq \bigcup_{j=1}^{m} \{(x_j, x_{j+1}), (x_{j+1}, x_j)\}$ for each

For example, if B5* and B6* hold so that the D_0 form is applicable, and if $|X| = 3$, then D_0 never gives rise to cyclic social preferences if $A_1 \cap A_2 \cap A_3 \neq \emptyset$ whenever $A_1, A_2, A_3 \in D_0$.

We conclude this subsection with a corollary that connects with Brown's work on prefilters as defined at the conclusion of the preceding section. See also Ferejohn and Grether [5] for related results.

COROLLARY 1. For a given set X of alternatives and a given set $N = \{1,2,\dots,n\}$ of individuals, let $\mathcal{D}_1$ denote the set of all D_0 that characterize all binary constitutions that satisfy B1*, B3*, B5*, B6* and R1*. Then every $D_0 \in \mathcal{D}_1$ is a prefilter on N if, and only if, $|X| \geq n$.

Proof. If $|X| \geq n$ suppose to the contrary of the corollary that $D_0 \in \mathcal{D}_1$ and $\cap_{D_0} A = \emptyset$. Then there must be a subset of n or fewer coalitions in D_0 such that each $i \in N$ is not in at least one coalition in this subset. Since the intersection of these coalitions is empty, $R1^*_6$ is violated and hence $D_0 \in \mathcal{D}_1$ is contradicted. Conversely, if $|X| < n$, let D_0 consist of all $n - 1$ member coalitions plus N. Then D_0 is not a prefilter since $\cap_{D_0} A = \emptyset$. But R1* holds since the intersection of any $m \leq |X|$ sets in D_0 is not empty. Since D_0 satisfies the other conditions in the corollary by its construction, $D_0 \in \mathcal{D}_1$. Q.E.D.

Partial Orders

The following transitivity condition for a binary constitution C is an appropriate counterpart to the transitivity condition R2 for BDR's.

R2*. ($\forall$ distinct $x,y,z \in X$)($\forall (A,B),(A',B') \in T$): If $(A,B) \in C(x,y)$, $(A',B') \in C(y,z)$, and if A^* and B^* are disjoint subsets of $(A\cap B')\cup(A'\cap B)$, then $(A^*\cup[(A\cup A')\setminus(B\cup B')], B^*\cup[(B\cup B')\setminus(A\cup A')]) \in C(x,z)$.

THEOREM 4. Suppose F and C are related as in (1) and (2). Then R2 holds for F if and only if R2* holds for C.

Proof. Given $(A,B) \in C(x,y)$ and $(A',B') \in C(y,z)$ let Π' be the subset of all configurations in Π for which $\pi(x,y) = (A,B)$ and $\pi(y,z) = (A',B')$. In other words, $\pi = (P_1,\dots,P_n)$ is in Π' if, and only if, $\{i\colon xP_iy\} = A$, $\{i\colon yP_ix\} = B$, $\{i\colon yP_iz\} = A'$ and $\{i\colon zP_iy\} = B'$. Then, because each P_i is an asymmetric weak order, $\pi \in \Pi'$ implies that $(A\cup A')\setminus(B\cup B') \subseteq \{i\colon xP_iz\}$, $(B\cup B')\setminus(A\cup A') \subseteq \{i\colon zP_ix\}$, $\{[N\setminus(A\cup B)]\cap[N\setminus(A'\cup B')]\} = \{i\colon xI_iy \;\&\; yI_iz\}$ is disjoint from $\{i\colon xP_iz \text{ or } zP_ix\}$, and any one of xP_iz, zP_ix and xI_iz can hold for each $i \in (A\cap B')\cup(A'\cap B)$. Consequently, $\pi \in \Pi'$, given $\pi(x,y) = (A,B)$ and $\pi(y,z) = (A',B')$, if and only if there are disjoint $A^*,B^* \subseteq (A\cap B')\cup(A'\cap B)$ such that $\pi(x,z) = (A^*\cup[(A\cup A')\setminus(B\cup B')], B^*\cup[(B\cup B')\setminus(A\cup A')])$. Since R2 implies that $xF(\pi)z$ for each such π, it follows that, given R2, $\pi(x,z) \in C(x,z)$ for each $\pi \in \Pi'$. Therefore, if R2 holds for F then R2* must hold for C. Conversely, if R2* holds for C, then, whenever $xF(\pi)y$ and $yF(\pi)z$, it must be true that $xF(\pi)z$ since $\pi(x,z) \in C(x,z)$. Hence R2* implies R2. Q.E.D.

The specializations of R2* in the context of B5* [$C(x,y) = C_0$ for all $(x,y) \in \hat{X}$] and in the context of B5* and B6* [$D_0 = \{A\colon (A,B) \in C_0$ for some $B\}$] will now be stated. It is to be understood that these conditions apply only if X has at least three alternatives.

$R2^*_5$. $(\forall (A,B),(A',B') \in T)$: If $(A,B),(A',B') \in C_0$ and A^*,B^* are disjoint subsets of $(A\cap B')\cup(A'\cap B)$, then $(A^*\cup[(A\cup A')\setminus(B\cup B')], B^*\cup[(B\cup B')\setminus(A\cup A')]) \in C_0$.

$i \in N$ and $(A_k,B_k) = \pi'(x_k,x_{k+1})$ for $k = 1,\ldots,m$, with $x_{m+1} \equiv x_1$. Then, by Lemma 1, for all $i \in N$ and for all $k \in \{1,\ldots,m\}$, $x_{k+1}P_i'x_k \Rightarrow \exists j \neq k$ with $x_jP_i'x_{j+1}$, and $x_kP_i'x_{k+1} \Rightarrow \exists j \neq k$ with $x_{j+1}P_i'x_j$. Hence, by Lemma 2, for each $i \in N$ there is an asymmetric weak order P_i on $\{x_1,\ldots,x_m\}$ that satisfies $P_i' = P_i \cap [\cup\{(x_j,x_{j+1}), (x_{j+1},x_j)\}]$. Extend P_i to all of X by making all alternatives in $X\setminus\{x_1,\ldots,x_m\}$ indifferent to one another and less preferred than everything in $\{x_1,\ldots,x_m\}$. Each P_i thus extended is therefore an asymmetric weak order on X so that $\pi = (P_1,\ldots,P_n)$ is in Π. [The initial configuration π' need not be in Π, which is why the lemmas were needed in this proof.] In addition, property (5) for each i implies that $\pi(x_k,x_{k+1}) = (A_k,B_k)$ for $k = 1,\ldots,m$. Hence, by (2) and the hypotheses of R1* for the situation at hand, we get $x_1F(\pi)x_2, x_2F(\pi)x_3,\ldots,x_mF(\pi)x_1$, which contradicts R1 for F. Therefore not (R1*) $\Rightarrow$ not (R1). Q.E.D.

Condition R1* can be simplified when certain other conditions are presupposed to hold for the binary constitution C. We illustrate this for two cases, first when neutrality (B5*) holds, in which case we let $C_0 = C(x,y)$ for all $(x,y) \in \hat{X}$, and second when neutrality and decisiveness (B5* and B6*) hold, in which case we let $D_0 = \{A\colon (A,B) \in C_0$ for some $(A,B) \in T\}$. The following versions of R1* apply to these cases.

<u>R1$^*_5$</u>. ($\forall$ finite integers m with $1 < m \leq |X|$)($\forall (A_1,B_1),\ldots,(A_m,B_m) \in T$): If $A_k \subseteq \cup_{j\neq k}B_j$ and $B_k \subseteq \cup_{j\neq k}A_j$ for $k = 1,\ldots,m$, then $(A_k,B_k) \notin C_0$ for some $k \in \{1,\ldots,m\}$.

<u>R1$^*_6$</u>. ($\forall$ finite integers m with $1 < m \leq |X|$)($\forall A_1,\ldots,A_m \subseteq N$): If $\cap_{k=1}^m A_k = \emptyset$, then $A_k \notin D_0$ for some $k \in \{1,\ldots,m\}$.

LEMMA 3. <u>Given</u> B5*: R1* <u>holds for</u> C <u>if and only if</u> R1$^*_5$ <u>holds for</u> C_0. <u>Given</u> B5* <u>and</u> B6*: R1* <u>holds for</u> C <u>if and only if</u> R1$^*_6$ <u>holds for</u> D_0.

<u>Proof</u>. The C_0 proof is immediate from R1* and the observation made just before Theorem 3. To prove the D_0 result we show first that, given $A_1,\ldots,A_m \subseteq N$, there exist $B_1,\ldots,B_m \subseteq N$ for which

$$B_k \cap A_k = \emptyset \text{ for all } k \text{ [so that } (A_k,B_k) \in T],$$
$$B_k \subseteq \cup_{j\neq k}A_j \text{ for all } k,$$
$$A_k \subseteq \cup_{j\neq k}B_j \text{ for all } k,$$

if and only if $A_k \subseteq N\setminus(\cap_{j\neq k}A_j)$ for all k. Since the three given conditions on the B_k will hold if and only if the third holds when each B_k is made as large as possible subject to the first two conditions, take $B_k = (\cup_{j\neq k}A_j) \cap (N\setminus A_k)$ for all k. Then it is easily verified that

$$\bigcup_{j\neq k} B_j = (\bigcup_{j=1}^m A_j) \cap [N \setminus \bigcap_{j\neq k} A_j)].$$

It follows that $A_k \subseteq \cup_{j\neq k}B_j$ for all k if, and only if, $A_k \subseteq N\setminus\cap_{j\neq k}A_j$ for all k. Since the latter condition holds if and only if $\cap A_k = \emptyset$, it then follows from the C_0 result and the hypotheses of the D_0 part of Lemma 3 that C which satisfies B5* and B6* also satisfies R1* if and only if $A_k \notin D_0$ for some k whenever $1 < m \leq |X|$ and $\cap A_k = \emptyset$. Q.E.D.

Conditions R1$^*_5$ and R1$^*_6$ can of course be used to investigate families of C_0 and D_0 constitutions that never give rise to cyclic social preferences.

$R2^*_6$. $(\forall A,A'\subseteq N)$: If $A,A' \in D_0$ then $\{E\colon A\cap A'\subseteq E\subseteq A\cup A'\}\subseteq D_0$.

LEMMA 4. Given B5*: R2* holds for C if and only if $R2^*_5$ holds for C_0. Given B5* and B6*: R2* holds for C if and only if $R2^*_6$ holds for D_0.

Proof. Assume throughout that $|X| \geq 3$, for otherwise there is nothing to prove. The validity of the first part of Lemma 4 is obvious from the statements of R2* and $R2^*_5$. The second part then follows from the first part if, given $A,A' \in D_0$, $\{E\colon\ E = A^*\cup[(A\cup A')\setminus(B\cup B')]$ for some B,B' such that $A\cap B = A'\cap B' = \emptyset$ and some $A^*\subseteq(A\cap B')\cup(A'\cap B)\} = \{E\colon\ A\cap A'\subseteq E\subseteq A\cup A'\}$. Taking $B = N\setminus A$ and $B' = N\setminus A'$, it follows easily that the second E set is included in the first E set. Moreover, since $A\cap A'\subseteq[(A\cup A')\setminus(B\cup B')]\subseteq A\cup A'$ and since $(A\cap B')\cup(A'\cap B)\subseteq A\cup A'$ for all B and B' that are respectively disjoint from A and A', the two E sets must be identical. Q.E.D.

The presence of axiom R2* allows certain implications among other conditions that are not available otherwise. Two of these are noted in the following lemmas. The first lemma uses the so-called strong Pareto condition, which in the C context can be expressed as follows.

B1**. $(\forall(x,y) \in \hat{X})(\forall A\subseteq N)$: $A \neq \emptyset \Leftrightarrow (A,\emptyset) \in C(x,y)$.

LEMMA 5. $(|X| \geq 3$, B1**,R2*$)\Rightarrow$ B5*.

LEMMA 6. $(|X| \geq 3$,B1*,B6*,R2*$)\Rightarrow$ B5*.

These lemmas show that neutrality follows from partial order and other conditions. The first lemma was proved by Blau [2] as a correction to an incorrect "theorem" in Guha [7]. A proof of the second lemma is similar in form to part of the usual proof of Arrow's Theorem and will be omitted.

The following corollary of Lemmas 4 and 6 is similar to a result in Hansson [8]. Again we omit its proof since the proof follows easily from the lemmas and the observation that (B1*,B5*,B6*,$R2^*_6$) $\Rightarrow$ B3*. The corollary is not quite correctly stated since the neutrality implication of Lemma 6 is presupposed by the D_0 specification, but that should cause no problems.

COROLLARY 2. Given X with $|X| \geq 3$ and given N, let $\mathcal{D}_2$ denote the set of all D_0 that characterize all binary constitutions that satisfy B1*,B6* and R2*. Then $\mathcal{D}_2$ is the set of all filters on N.

Two further remarks are in order here. First, there is a natural one-one correspondence between $\mathcal{D}_2$ and the set of oligarchies in N. The oligarchy that corresponds to D_0 is $\bigcap_{D_0} A$. The filter that corresponds to oligarchy A is $\{B\colon A\subseteq B\}$. Second, we note the importance of the decisiveness condition B6* by two examples. Let $X = \{a,b,c\}$ with $N = \{1,2,3\}$. Suppose first that C is as follows: $C(x,y) = \{(N,\emptyset)\}$ for all $(x,y) \in \hat{X}\setminus\{(a,b)\}$, and $C(a,b) = \{(\{1\},\emptyset),(N,\emptyset)\}$. Then C satisfies B1* and R2* but does not satisfy either B5* or B6*. Suppose next that $C(x,y) = \{(N,\emptyset),(\{1\},\emptyset)\}$ for all $(x,y) \in \hat{X}$. This C then satisfies B1*,B5* and R2* but it does not satisfy B6*.

Weak Orders

The following condition on C corresponds to the condition of negative transitivity for each $F(\pi)$.

R3*. $(\forall$ distinct $x,y,z \in X)(\forall(A,B),(A',B') \in T)$: If $(A,B) \notin C(x,y)$, $(A',B') \notin C(y,z)$, and if A^* and B^* are disjoint subsets of $(A\cap B')\cup(A'\cap B)$, then $(A^*\cup[(A\cup A')\setminus(B\cup B')],\ B^*\cup[(B\cup B')\setminus(A\cup A')]) \notin C(x,z)$.

THEOREM 5. Suppose F and C are related as in (1) and (2). Then R3 holds for F if and only if R3* holds for C.

We omit the proof of Theorem 5 since it is very similar to the proof of Theorem 4. The specializations of R3* for the C_0 and D_0 contexts are as follows. Again it is to be understood that these conditions apply only if $|X| \geq 3$.

$R3^*_5$. $(\forall (A,B),(A',B') \in T)$: If $(A,B),(A',B') \notin C_0$ and A^*,B^* are disjoint subsets of $(A \cap B') \cup (A' \cap B)$, then $(A^* \cup [(A \cup A') \setminus (B \cup B')], B^* \cup [(B \cup B') \setminus (A \cup A')]) \notin C_0$.

$R3^*_6$. $(\forall A,A' \subseteq N)$: If $A,A' \notin D_0$ then $\{E: A \cap A' \subseteq E \subseteq A \cup A'\} \cap D_0 = \emptyset$.

LEMMA 7. Given B5*: R3* holds for C if and only if $R3^*_5$ holds for C_0. Given B5* and B6*: R3* holds for C if and only if $R3^*_6$ holds for D_0.

The proof of Lemma 7 is similar to the proof of Lemma 4 and will be omitted. The following corollary of the second part of Lemma 7 is again similar to a result in Hansson [8].

COROLLARY 3. Given X with $|X| \geq 3$ and given N, let $\mathcal{D}_3$ be the set of all D_0 that characterize all binary constitutions that satisfy B1*,B6* and R3*. Then $\mathcal{D}_3$ is the set of all ultrafilters on N.

Proof. Since R3* ⇒ R2*, Corollary 2 shows that all $D_0 \in \mathcal{D}_3$ are filters. If $A, N \setminus A \in D_0$ and $R3^*_6$ holds, then D_0 must be empty, and this contradicts B1*. Hence, using the latter part of Lemma 7, every $D_0 \in \mathcal{D}_3$ is an ultra-filter. Moreover, every D_0 that is an ultrafilter on N clearly satisfies the conditions of Corollary 3. Q.E.D.

It may be noted in addition that every $D_0 \in \mathcal{D}_3$ corresponds to an absolute dictator in N. That is, for each D_0 in $\mathcal{D}_3$ there is an $i \in N$ such that $D_0 = \{A \subseteq N: i \in A\}$ with x socially preferred to y if and only if xP_iy.

The principal results in this section are Theorems 3 through 5, which identify conditions for binary constitutions that are equivalent to familiar social ordering conditions for binary decision rules under the assumption that $P^n = \Pi$. The specializations of the ordering conditions for binary constitutions under neutrality or neutrality and decisiveness provide examples of what the conditions look like in some special cases and, in addition, provide connections to prior work of Hansson, Brown, and others. Although a great deal more could be said about special decisiveness structures and about conditions on BRD's that correspond to these structures, we shall refrain from doing so at the present time.

5. DISCUSSION

Our principal objective in this paper was to develop a notion of power or decisiveness that is sufficiently rich to characterize the class of binary decision rules. Secondly, we wished to be able to rewrite the axioms on the BDRs as axioms on the decisiveness structures (which could be interpreted as explicit restrictions on the distribution of power in society). Finally, we have also been able to determine conditions under which the traditional concept of decisiveness (found in Arrow [1]) is useful in characterizing various binary procedures.

The results obtained here indicate exactly how the theory of binary social choice is parallel to what we might call the theory of binary decisiveness structures developed by various authors including Brown [3, 4] and Hansson [8]. Evidently only a beginning has been made in this area and additional work remains to be done.

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