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SOCIAL DECISION FUNCTIONS AND STRONGLY DECISIVE SETS

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ABSTRACT

Properties of the strongly decisive sets (some preference for x over y along with no preference for y over x allows coalitional enforcement of x over y) associated with a social decision function are investigated. The collection of such sets does not have the superset preserving property of filters, but is characterized by properties defining a _target_. A 1-1 and onto mapping is exhibited between the class of targets and a certain class of social decision functions, showing that such functions are completely characterized by the structure of their strongly decisive sets. The "ring" structure of targets is shown to be closely related to known results on veto hierarchies.

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1. Introduction

Several recent papers have developed partial or complete characterizations of classes of social decision functions in terms of constructs based upon the associated collections of decisive sets. Hansson (1976) interpreted Arrow's impossibility theorem in terms of the associated ultrafilter of decisive sets. Brown (1973) extended this correspondence to the case of acyclic choice functions and prefilters. To deal with the multiplicity of social decision functions having the same collection of decisive sets, Brown restricted the class of social decision functions while Ferejohn and Fishburn (1979) and Blau and Brown (1980) added structure to the collections of decisive sets, and thereby obtained a characterization of certain social decision functions.

In this paper we investigate properties of what we call the strongly decisive sets associated with a social decision function. Such sets must be able to enforce alternative x over alternative y as long as no one in the set prefers y to x (indifference is allowed) and someone prefers x to y. We show in Section 3, Theorem 1 that the collection of strongly decisive sets form what we call a target, a collection of subsets of the voter set N totally ordered by inclusion (see Figure 1). While targets do not have the superset preserving property exhibited by prefilters, they nonetheless reveal interesting structural information about social decision functions. We show that the rings of the target form a ratification hierarchy. This hierarchy coincides with the notion of a lexicographic dictatorship (Fishburn, 1975) when the social preference is transitive (Theorem 2) and is generally analogous but not identical to the idea of a veto or oligarchy hierarchy (Blau and Deb, 1977).

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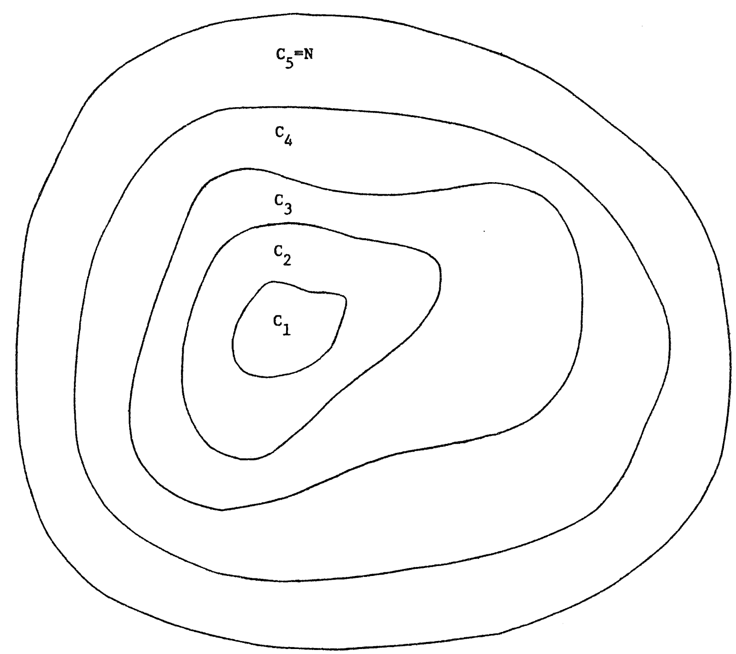
Figure 1 - A TARGET OF VOTER SUBSETS AND ITS RINGS



Target = $\{C_1, C_2, C_3, C_4, N\}$

Center = C_1

ith ring = $C_i \setminus C_{i-1}$ (i=2,3,4,5)

In Section 4 we characterize the set $\mathcal{C}$ of social decision functions that arise naturally from targets. Theorem 4 shows that $\mathcal{C}$ must satisfy the strong Pareto property, a form of neutrality, a new and somewhat technical property which we call unilaterality, and social preference quasitransitivity. Theorem 5 then establishes a 1-1 correspondence between $\mathcal{C}$ and the class of targets over the voter set. The approach of this paper most closely parallels Brown (1973), while treating essentially the opposite extreme case where indifference tends to be treated as support.

2. Definitions and Terminology

We assume a finite population N of voters with $|N| = n$. Let A with $|A| \geq 3$ be any set of alternatives under consideration. In discussing the preference relations of voters and of society over the alternatives, we will generally use the (strong) asymmetric preference (P), from which the (weak) preference or indifference (R) and the indifference (I) relations can be deduced in standard fashion. The following classes of relations will be needed:

$\mathcal{B}$ = {asymmetric binary relations on A}.

$\mathcal{R}$ = {P ε $\mathcal{B}$| the R arising from P is transitive}, (transitive preferences).

$\mathcal{Q}$ = {P ε $\mathcal{B}$| P is transitive}, (quasitransitive preferences).

Given a profile $\pi = (P_1, P_2, \ldots, P_n) \varepsilon \mathcal{R}^n$ describing the voter's individual preferences and given x, y ε A, we define

$$[xPy] = \{i \varepsilon N | xP_iy\}$$

and similarly for [xRy] and [xIy]. A social decision function is a function $F : \mathcal{R}^n \to \mathcal{B}$, so for each profile $\pi \varepsilon \mathcal{R}^n$, $F(\pi)$ is the asymmetric preference relation describing society's preference. We occasionally use P in place of $F(\pi)$ with R and I denoting, respectively, society's weak preference and indifference. To simplify the notation, we likewise denote the social preference corresponding to a profile π' by P', with R' and I' playing their expected roles.

We now list some standard conditions which F may be required to satisfy. We streamline these definitions by giving them all the following common quantifiers:

$$\forall x, y, x', y' \varepsilon A, \forall \pi, \pi' \varepsilon \mathcal{R}^n$$

IIA (Independence): $[xRy] = [xR'y]$, $[xPy] = [xP'y]$, and $xF(\pi)y \Rightarrow xF(\pi')y$;

NEU (Neutrality): $[xRy] = [x'Ry']$, $[xPy] = [x'Py']$, and $xF(\pi)y \Rightarrow x'F(\pi)y'$;

MON (Monotonicity): $[xRy] \subseteq [xR'y]$, $[xPy] \subseteq [xP'y]$, and $xF(\pi)y \Rightarrow xF(\pi')y$;

P (Pareto): $[xPy] = N \Rightarrow xF(\pi)y$;

SP (Strong Pareto): $[xRy] = N$ and $[xPy] \neq \emptyset \Rightarrow xF(\pi)y$;

UII (Indifference invariance): $[xIy] = N \Rightarrow xIy$;

URR (Weak preference invariance): $[xRy] = N \Rightarrow xRy$.

A more general form of monotonicity (neutrality) which incorporates neutrality (independence) is often given, but it will be convenient for our purposes to keep these separate. There are obvious connections among the above properties. Thus MON $\Rightarrow$ IIA, NEU $\Rightarrow$ UII, SP $\Rightarrow$ P, and (SP and UII) $\Rightarrow$ URR.

Given $F : \mathcal{R}^n \to \mathcal{B}$, its decisive sets $\mathcal{W}(F)$ are defined by

$$\mathcal{W}(F) = \{C \subseteq N \,|\, [xPy] \supseteq C \Rightarrow xF(\pi)y\}.$$

The focus of what follows will be on the strongly decisive sets $\mathcal{D}(F)$:

$$\mathcal{D}(F) = \{C \subseteq N,\ C \neq \emptyset \,|\, C \subseteq [xRy] \text{ and } C \cap [xPy] \neq \emptyset \Rightarrow xF(\pi)y\}.$$

3. Targets and Voter Hierarchies

Anticipating the structure of the strongly decisive sets $\mathcal{D}(F)$, we call a collection $\mathcal{D} \subseteq 2^N$ a target if

i) $N \in \mathcal{D}$ and $\emptyset \notin \mathcal{D}$

ii) $C_1, C_2 \in \mathcal{D} \Rightarrow C_1 \subseteq C_2$ or $C_2 \subseteq C_1$.

Theorem 1: $F : \mathcal{R}^n \to \mathcal{B}$ and SP $\Rightarrow \mathcal{D}(F)$ is a target.

Proof: i) $N \in \mathcal{D}(F)$ by property SP and $\emptyset \notin \mathcal{D}(F)$ by the definition of $\mathcal{D}(F)$.

ii) Given $C_1, C_2 \in \mathcal{D}(F)$, suppose neither $C_1 \subseteq C_2$ nor $C_2 \subseteq C_1$ holds.

Pick distinct $x, y \in A$ and any $\pi \in \mathcal{R}^n$ such that $[xPy] = C_1 \setminus C_2$, $[yPx] = C_2 \setminus C_1$, and $[xIy] = (C_1 \cap C_2) \cup (N \setminus (C_1 \cup C_2))$. Then $C_1 \in \mathcal{D}(F) \Rightarrow xF(\pi)y$, and $C_2 \in \mathcal{D}(F) \Rightarrow yF(\pi)x$, contradicting the required asymmetry of $F(\pi)$ and establishing the toally ordered nature of $\mathcal{D}(F)$.

Q.E.D.

We now present several examples for purposes of illustration and subsequent use.

Example 1: Let F be simple majority rule. Then $\mathcal{D}(F) = \{N\}$, while $\mathcal{W}(F) = \{C \subseteq N \,|\, |C| > n/2\}$.

Example 2: Let V be a specific subset of N with $|V| \geq 2$ which acts as an oligarchy in controlling the social decision as follows:

$$xF(\pi)y \Leftrightarrow [xPy] \supseteq V \text{ or } ([xRy] = N \text{ and } [xPy] \neq \emptyset).$$

Again $\mathcal{D}(F) = \{N\}$, while $\mathcal{W}(F) = \{C \subseteq N \,|\, C \supseteq V\}$, the filter generated by V.

Example 3: Let the alternative set X be totally ordered by a total order relation $>$. Pick C_1, C_2, C_3 so that $\emptyset \subsetneqq C_1 \subsetneqq C_2 \subsetneqq C_3 = N$ and define $F : \mathcal{R}^n \to \mathcal{B}$ by

$$xF(\pi)y \Leftrightarrow \begin{cases} [xRy] \supseteq C_i \text{ and } [xPy] \cap C_i \neq \emptyset \ (i = 1,2, \underline{\text{or}}\ 3) \\ [xIy] = N \text{ and } x > y. \end{cases}$$

Here $\mathcal{D}(F) = \{C_1, C_2, N\}$, while $\mathcal{W}(F) = \{C \subseteq N \,|\, C \supseteq C_1\}$.

Example 4: Given $\emptyset \subsetneqq C_1 \subsetneqq C_2 \subsetneqq C_3 = N$, define

$$xF(\pi)y \Leftrightarrow [xRy] \supseteq C_i \text{ and } [xPy] \cap C_i \neq \emptyset \ (i = 1 \ \underline{\text{or}}\ 2).$$

Then $\mathcal{D}(F) = \{C_1, C_2\}$ (not a target since F is not SP) and $\mathcal{W}(F) = \{C \subseteq N \,|\, C \supseteq C_1\}$.

Returning to a general $F : \mathcal{R}^n \to \mathcal{B}$ which is SP, we can order the sets $\{C_i\}_{i=1}^{s}$ in the target $\mathcal{D}(F)$ so that $\emptyset \subsetneq C_1 \subsetneq C_2 \subsetneq \cdots \subsetneq C_s = N$. Now define the collection $\{R_i\}_{i=1}^{s}$ of disjoint sets by

$$R_i = \begin{cases} C_1 & \text{if } i = 1 \\ C_i \setminus C_{i-1} & \text{if } 2 \le i \le s. \end{cases}$$

We refer to the sets R_i as the rings of the target. The ordered partition $\{R_i\}_{i=1}^{s}$ of N can be thought of as a ratification hierarchy in the following sense: as long as some member of R_1 prefers x to y and none are opposed, we have $xF(\pi)y$; if $i > 1$ and members of R_j for all $j < i$ are indifferent, then the members of R_i can force $xF(\pi)y$ in the same fashion as just described for R_1. We now make some observations relating these ideas to established results in social choice theory. The lexicographic dictatorship result of Fishburn (1975) leads directly to:

Theorem 2: Given $F : \mathcal{R}^n \to \mathcal{B}$ satisfying SP, UII, and IIA, then range $(F) \subseteq \mathcal{R}$ $\Leftrightarrow$ $|R_i| = 1$ for each ring R_i of the target $\mathcal{D}(F)$.

Proof: $\Rightarrow$: This implication is proved in Fishburn (1975) and is closely related to Arrow's proof of his impossibility theorem.

$\Leftarrow$: Given $\pi \in \mathcal{R}^n$, xRy, and yRz (societal weak preference), we must show that xRz. If xRy is, more specifically, xIy, then we must have $[xIy] = N$ since otherwise the first i ($1 \le i \le n$) such that $R_i = \{j\}$ and $\sim xI_jy$ would determine a strict social preference (this is a consequence of $|R_i| = 1 \; \forall i$). In this xIy case, $[yRz] = [xRz]$ and $[yPz] = [xPz]$, from which xRz follows (UII is needed here in some cases). A similar argument holds if yRz results from yIz. We now consider the final case where xPy and yPz (strict societal preferences). Let

q be the first ring subscript such that $R_q = \{r\}$ and xP_ry and let k be the first ring subscript such that $R_k = \{m\}$ and yP_mz (such subscripts exist by UII). Let $i = \min\{q,k\}$ and $R_i = \{j\}$. Since individual preferences are in $\mathcal{R}$, we have $xI_pz \;\; \forall p \in R_t (t < i)$ and xP_jz. If follows from the definition of $\mathcal{D}(F)$ and $|R| = 1$ for all rings that xPz. Hence xRz and $F(\pi) \in \mathcal{R}$. Since $\pi \in \mathcal{R}^n$ was arbitrary, range $(F) \subseteq \mathcal{R}$.

Q.E.D.

If we are given range $(F) \subseteq \mathcal{Q}$, it is well known (assuming IIA and P) that there will be an oligarchy V which can force a social decision by unanimous strong preference of its members. Furthermore, each member of V has a veto ($i \in V$ and $xP_iy \Rightarrow \sim yF(\pi)x$) and $V = \bigcap_{C \in \mathcal{W}(F)} C$. Blau and Deb (1977) develop the idea of a veto hierarchy, and Theorem 2 gives a rather restrictive set of conditions (range $(F) \subseteq \mathcal{R}$, SP, UII, and IIA) under which the veto and ratification hierarchies coincide. Example 2 shows that these hierarchies cannot generally be expected to coincide. The most that we can expect is that the first round vetoers V_1 are a subset of the first ring R_1 (i.e. the center of the target) of the ratification hierarchy. The characterization of the next section will provide another special setting in which the two hierarchies coincide.

4. Characterization Theorem

Given a target $\mathcal{D} \subseteq 2^N$, it induces for any $\pi \in \mathcal{R}^n$ a natural social preference $F(\mathcal{D})(\pi)$ by means of the following definition:

$$xF(\mathcal{D})(\pi)y \Leftrightarrow \exists C \in \mathcal{D} \ni C \subseteq [xRy] \text{ and } C \cap [xPy] \ne \emptyset.$$

We would like to characterize those social decision functions which arise from targets in the manner defined above. Since different functions can have identical targets (compare Examples 1 and 2) we will need to restrict the class of functions considered. Our final social choice function condition requires the following definitions.

Given $F : \mathcal{R}^n \to \mathcal{B}$, a subset M of voters is called minimal winning for x over y (written $M \in \mathcal{M}_{x,y}(F)$) if $\exists \bar{\pi} \in \mathcal{R}^n$ such that

i) $[x\bar{R}y] = M$ and $xF(\bar{\pi})y$;

ii) $(\forall \pi' \in \mathcal{R}^n)[xR'y] \subsetneq M$, $[xP'y] \subseteq [x\bar{P}y]$, and $[xI'y] \subseteq [x\bar{I}y] \Rightarrow \sim xF(\pi')y$.

Any $\bar{\pi}$ which "works" in the above definition is called a minimal winning profile for M relative to x and y. The natural interpretation of the definition should clarify it. We will have $M \in \mathcal{M}_{x,y}(F)$ precisely when the cast of supporting and indifferent voters equals M and can effect a social preference for x over y in such a manner that any defection (yP_ix) by members of M will destroy the societal preference for x over y. In our Example 1, $\mathcal{M}_{x,y}(F) = \mathcal{W}(F)\ \forall\ x \neq y$; while in Example 4, $\mathcal{M}_{x,y}(F) = \mathcal{D}(F)\ \forall\ x \neq y$.

We say that $F : \mathcal{R}^n \to \mathcal{B}$ satisfies unilaterality (UL) if $(\forall x,y\in A)(\forall M \in \mathcal{M}_{x,y}(F))$, $[xRy] \supseteq M$ and $[xPy] \cap M \neq \emptyset \Rightarrow F(\pi)y$. This condition thus says that on a set in $\mathcal{M}_{x,y}(F)$, any voter can unilaterally force $xF(\pi)y$ as long as no other voters in the set prefer y to x (they may all be indifferent).

The class of social decision functions we now consider is defined by

$$\mathcal{C} = \{F : \mathcal{R}^n \to \mathcal{B} \mid F \text{ is SP, NEU, and UL}\}.$$

Theorem 3: Let $\mathcal{D}$ be a target and $F(\mathcal{D})$ its induced social decision function. Then

i) $F(\mathcal{D}) \in \mathcal{C}$

ii) Range $(F(\mathcal{D})) \subseteq \mathcal{Q}$

Proof: i) The fact that $F(\mathcal{D})(\pi)$ is asymmetric follows readily from the definition of $F(\mathcal{D})$ and property ii) of targets. Since $N \in \mathcal{D}$, SP is immediate. Condition NEU results from the uniform way that any pair $x, y \in A$ is treated in the definition of $xF(\mathcal{D})(\pi)y$. Finally, given $M \in \mathcal{M}_{x,y}(F(\mathcal{D}))$ by virtue of a minimal winning profile $\bar{\pi}$, we then have $[x\bar{R}y] = M$ and $xF(\mathcal{D})(\bar{\pi})y$. By definition of $xF(\mathcal{D})(\bar{\pi})y$, there must be some $C \in \mathcal{D}$ such that $[x\bar{R}y] = M \supseteq C$ and $[x\bar{P}y] \cap C \neq \emptyset$. By minimality of M, we have $M \subseteq C$ and we conclude that $M = C \in \mathcal{D}$ and, in particular, the conclusion of UL is satisfied.

ii) Suppose $xF(\mathcal{D})(\pi)y$ and $yF(\mathcal{D})(\pi)z$. Then $\exists C, C' \in \mathcal{D}$ with $C \subseteq [xRy]$, $C \cap [xPy] \neq \emptyset$ and $C' \subseteq [yRz]$, $C' \cap [yPz] \neq \emptyset$. Assume without loss of generality that $C \subseteq C'$. Then $C \subseteq [xRz]$ since each P_i making up π has transitive R_i. Also, xP_iy and yR_iz for some $i \in C$ from which it follows by transitivity of R_i that xP_iz. Hence $C \cap [xPz] \neq \emptyset$ and $xF(\mathcal{D})(\pi)z$ follows. Thus $F(\mathcal{D})(\pi) \in \mathcal{Q}$.

Q.E.D.

The properties SP, NEU, and UL making up the definition of $\mathcal{C}$ can be seen to be independent (no two imply the third) by looking at Examples 2, 3, 4, where, respectively, only UL, NEU, and SP are lacking. In each case range $(F) \subseteq \mathcal{Q}$. The following lemma will be useful in examining some consequences of UL and in our final characterization.

Minimality Lemma. Given $F : \mathcal{R}^n \to \mathcal{B}$ and $\pi \in \mathcal{R}^n$ with $xF(\pi)y$, $\exists\, M \subseteq [xRy] \ni M \in \mathcal{M}_{x,y}(F)$. Furthermore, any $\bar{\pi} \in \mathcal{R}^n$ with $[x\bar{R}y] = M$ and $[x\bar{P}y] = [xPy] \cap M$ is a minimal winning profile for M relative to x and y.

Proof: Define $\mathcal{M} = \{C \subseteq [xRy] \mid \exists \pi' \in \mathcal{R}^n \ni [xR'y] = C, [xP'y] = [xPy] \cap C$, and $xF(\pi')y\}$. Since $\mathcal{M}$ is nonempty ($[xRy] \in \mathcal{M}$) and finite, $\mathcal{M}$ has a minimal element with respect to the partial order $\subseteq$. Let $M \in \mathcal{M}$ be such a minimal element with $\bar{\pi}$ any associated profile as prescribed in the definition of $\mathcal{M}$. Then $M \in \mathcal{M}_{x,y}(F)$ and $\bar{\pi}$ is a minimal winning profile for M relative to x and y.

Q.E.D.

The next result establishes some connections among UL and some of the other conditions that $F : \mathcal{R}^n \to \mathcal{B}$ may satisfy. It also shows that functions in $\mathcal{C}$ must have quasitransitive range values.

Theorem 4: Given $F : \mathcal{R}^n \to \mathcal{B}$,

i) SP, UII, and UL $\Rightarrow$ MON and IIA

ii) SP, NEU, and UL $\Rightarrow$ Range (F) $\subseteq \mathcal{Q}$, MON and IIA

Proof: i) Given $[xRy] \subseteq [xR'y]$, $[xPy] \subseteq [xP'y]$, and $xF(\pi)y$, apply the Minimality Lemma to obtain $M \subseteq [xRy]$ with $M \in \mathcal{M}_{x,y}(F)$ and a $\bar{\pi} \in \mathcal{R}^n$ with $[x\bar{R}y] = M$ and $[x\bar{P}y] = [xPy] \cap M$, a minimal winning profile. By SP and UII, $[xPy] \cap M \neq \emptyset$ (otherwise we violate SP when $M \subsetneq N$ and UII when $M = N$). Noting that $[xR'y] \supseteq M$ and $[xP'y] \cap M \neq \emptyset$ and invoking UL, we conclude that F satisfies MON and hence IIA.

ii) We first note that the hypotheses of i) above hold since NEU $\Rightarrow$ UII. Thus we immediately have that F satisfies both MON and IIA. By using IIA and then NEU repeatedly it can be shown that for all $x \neq y$ and $z \neq w$, $\mathcal{M}_{x,y}(F) = \mathcal{M}_{z,w}(F)$ (we omit the details). Now given $xF(\pi)y$ and $yF(\pi)z$, use the Minimality Lemma to obtain $M_1 \in \mathcal{M}_{x,y}(F)$ and $M_2 \in \mathcal{M}_{y,z}(F)$. From the remarks above and UL we have $\mathcal{M}_{x,y}(F) = \mathcal{M}_{y,z}(F) = \mathcal{D}(F)$. Thus M_1 and M_2 are in $\mathcal{D}(F)$ and we can assume by Theorem 1, without loss of generality, that $M_1 \subseteq M_2$. Now $xF(\pi)z$ follows precisely as it did in Theorem 3,ii). Thus range (F) $\subseteq \mathcal{Q}$.

Q.E.D.

It is seen from these results that $\mathcal{C}$ implicitly requires much more than its definition states. We have attempted to define $\mathcal{C}$ as "weakly" as possible for the characterization which follows. Letting $\mathcal{T}$ denote the class of all targets from N, we have a map from social decision functions in $\mathcal{C}$ to targets in $\mathcal{T}$ (Theorem 1) and a return path from $\mathcal{T}$ to $\mathcal{C}$ (Theorem 3,i).

Theorem 5: The mapping $\mathcal{D} : \mathcal{C} \to \mathcal{T}$ is 1-1 and onto with inverse $F : \mathcal{T} \to \mathcal{C}$.

Proof: We will show that $\forall\, D \in \mathcal{T}$, $\mathcal{D}(F(D)) = D$, and that $\forall f \in \mathcal{C}$, $F(\mathcal{D}(f)) = f$. Given a target D with $C \in D$, we have $C \in \mathcal{D}(F(D))$ since $[xRy] \supseteq C$ and $[xPy] \cap C \neq \emptyset \Rightarrow xF(D)(\pi)y$. Thus $\mathcal{D}(F(D)) \supseteq D$. Conversely, if $C \in \mathcal{D}(F(D))$, we know that C is strongly decisive for F(D), from which it follows that

$$[xRy] = C \text{ and } [xPy] \cap C \neq \emptyset \Rightarrow xF(D)(\pi)y. \qquad (1)$$

Let C' be the largest proper subset of C such that C' ε D (if no such C' exists, take C' = $\emptyset$). Choose distinct x', y' ε A and a $\pi' \varepsilon \mathcal{R}^n$ such that [x'R'y'] = C and [x'P'y'] $\cap$ C $\neq \emptyset$ (hence x'F(D)(π')y' by (1)) and [x'P'y'] $\cap$ C' = $\emptyset$. Then we must have C ε D since no other C' ε D will give us x'F(D)(π')y'. Thus we have shown that $\mathcal{D}$(F(D)) = D $\forall$ D ε $\mathcal{T}$. For the reverse composition, consider any f ε $\mathcal{C}$ and suppose xF($\mathcal{D}$(f))(π)y. Then $\exists$ C ε $\mathcal{D}$(f) $\ni$ C $\subseteq$ [xRy] and C $\cap$ [xPy] $\neq \emptyset$. It follows from the definition of C ε $\mathcal{D}$(f) that xf(π)y, and we have shown that xF($\mathcal{D}$(f))(π)y $\Rightarrow$ xf(π)y. Conversely, given xf(π)y, apply the Minimality Lemma to obtain M $\subseteq$ [xRy] with M ε $\mathcal{M}_{x,y}$(f) and a $\bar{\pi} \varepsilon \mathcal{R}^n$ with $[x\bar{R}y]$ = M and $[x\bar{P}y]$ = [xPy] $\cap$ M. As in the proof of Theorem 4,i, we must have [xPy] $\cap$ M $\neq \emptyset$ and M ε $\mathcal{D}$(F). Since M $\subseteq$ [xRy] we have xF($\mathcal{D}$(f))(π)y. Thus xF($\mathcal{D}$(f))(π)y $\Leftrightarrow$ xf(π)y, showing that F($\mathcal{D}$(f)) = f $\forall$ f ε $\mathcal{C}$.

Q.E.D.

We have now characterized through social decision function axioms on $\mathcal{C}$ the ratification hierarchy process discussed earlier. The highly powerful central committee R_1 considers pairs of alternatives on a consensus basis. We interpret this to mean that consensus of x over y occurs as long as no one in R_1 prefers y to x and someone prefers x to y. If such a consensus occurs, then the central committee forces the whole population to choose x over y. If everyone in R_1 is indifferent between x and y, the second ring R_2 may pick up the ball, operating as does R_1, on a consensus basis. The process continues and can, in the unlikely case of massive indifference, filter all the way

down to the outer ring R_s. If we add that societal preferences can _only_ arise in the above consensus fashion, Theorem 5 shows that the above general procedure is the _only_ example of a social decision function in $\mathcal{C}$. Thus the strongly decisive sets of an f ε $\mathcal{C}$ completely determine f and the associated veto and ratification hierarchies coincide.

5. Concluding Comments

One major consequence of considering strongly decisive sets rather than decisive ones (i.e. unanimity is required on the set) is that supersets of _strongly_ decisive sets need not remain so. This fact allows for more variety in the class of possible targets and makes possible our rather unstructured characterization.

Some, but not all, of what we have done carries over to cases where N is infinite. One obstacle occurs in the proof of the Minimality Lemma where we call for a minimal element of a finite set. If the set becomes infinite, some added structure may be needed for a full characterization and Zorn's lemma might have to be invoked.

Note that omission or weakening of the rather severe UL condition would broaden the class of social decision functions from $\mathcal{C}$ to a considerably larger class $\mathcal{C}'$ in which many functions would correspond to each possible target. A less concrete approach to characterization would be to define an equivalence relation on $\mathcal{C}'$ by F ~ F' $\Leftrightarrow$ $\mathcal{D}$(F) = $\mathcal{D}$(F'). Theorem 5 then shows that each equivalence class in $\mathcal{C}'$ contains precisely one member of $\mathcal{C}$. Thus there is a natural social decision function satisfying UL associated with any function in $\mathcal{C}'$.

Recalling that the proof of Theorem 1 only used asymmetry and condition SP on $F \in \mathcal{C}$ to show that $\mathcal{D}(F)$ is a target, we see that the procedure of the last paragraph can be carried out for even larger classes of social decision functions. The ultimate claim is then that there is a natural function in $\mathcal{C}$ associated with any asymmetric (not necessarily transitive) preference function satisfying SP.

The connections among IIA, NEU, MON, SP, UII, and range (F) $\subseteq \mathcal{Q}$ have less impact because of the use of the restrictive UL property. While there are neater results such as (range (F) $\subseteq \mathcal{Q}$, SP, and IIA) $\Rightarrow$ MON and NEU (see Guha (1972) and Blau (1976)), it is of some interest that implications from SP and UII to MON and from SP and NEU to range (F) $\subseteq \mathcal{Q}$ are provided here.

We conclude by observing that our approach essentially treats indifference as support (as long as some other member in the ring provides support). This is at the extreme from the "decisive set" approach, where indifference essentially becomes opposition. This observation suggests a variety of possible intermediate notions obtainable by placing cardinality conditions on the set [xPy]. The social decision procedures so considered would appear to be meaningful and worthwhile for the added decisiveness they provide.

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