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METHODS OF ESTIMATION FOR MODELS OF MARKETS WITH BOUNDED PRICE VARIATION

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Abstract

The paper describes methods of estimation for models of markets with controlled prices. Many practical situations involving disequilibrium are due to price ceilings or price floors or both and thus fall in the category of the models described here. The paper also distinguishes between "rationing models" where the short side of the market prevails, and "trading models" where no trading takes place if there is excess demand or excess supply.

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1. INTRODUCTION

There has been a considerable amount of literature on the estimation of models which are in disequilibrium (see Fair and Jaffee (1972), Fair and Kelejian (1974), Amemiya (1974), Maddala and Nelson (1974), Goldfeld and Quandt (1975), Laffont and Garcia (1977), Bowden (1978) etc.). However, there are many cases where some of the observations refer to equilibrium points and some to disequilibrium points. In cases like the market for natural gas with price ceilings, we can have excess demand or equilibrium points. In cases of agricultural commodities with price supports, we can have excess supply or equilibrium points. In the case of commodity futures markets we have exogenously fixed limits within which trading can take place. If the price hits the lower limit we have excess supply, if the price hits the upper limit we have excess demand, otherwise we have an equilibrium. These are all cases of markets with price ceilings, or price floors or both ceilings and floors. Some of the observations refer to equilibrium points and some of the observations refer to disequilibrium points (excess demand or excess supply). The present paper discusses maximum likelihood and two-stage estimation methods for models of such markets.

II. A MODEL WITH PRICE CEILING

Consider the following model with an exogenously set price limit $\overline{P}_t$ (quite a reasonable model for something like natural gas demand):

$$D_t = X_{1t}'\beta_1 + \alpha_1 P_t + u_{1t} \quad (1)$$

$$S_t = X_{2t}'\beta_2 + \alpha_2 P_t + u_{2t} \quad (2)$$

$$Q_t = D_t = S_t \qquad \text{if } P_t < \overline{P}_t \quad (3)$$

$$Q_t = S_t \qquad \text{if } P_t \geq \overline{P}_t \quad (4)$$

Here

D_t is quantity demanded

S_t is quantity supplied

P_t is price

$\overline{P}_t$ is the exogenously set price ceiling

and X_{1t} and X_{2t} are explanatory variables.

Given the observations on P_t and $\overline{P}_t$ we can classify the observations as those belonging to equilibrium and those to excess demand.

Let ψ_1 be the set of observations with $P_t < \overline{P}_t$. For these we have a simultaneous equations model with Q_t and P_t as endogenous variables. Let ψ_2 be the set of observations with $P_t = \overline{P}_t$. For these observations we have $S_t = Q_t$ and $D_t > Q_t$. Note that for this set P_t is an exogenous variable.

Thus, the likelihood function is:

$$L = \prod_{\psi_1} f(Q_t, P_t) \cdot \prod_{\psi_2} \int_{Q_t}^{\infty} g(D_t, Q_t)\, dD_t \quad (5)$$

where $f(Q_t, P_t)$ is the joint density of Q_t and P_t derived from the joint density of (u_1, u_2) as in any simultaneous equations model, and $g(D_t, S_t)$ is the joint density of D_t and S_t derived from the joint density of (u_1, u_2) treating $P_t = \overline{P}_t$ as exogenous. Note that the jacobian of transformation for $f(Q_t, P_t)$ is $|\alpha_1 - \alpha_2|$ which is expected to be nonzero since α_1 and α_2 are of opposite signs and nonzero. The jacobian of transformation for $g(D_t, S_t)$ is, of course, unity.

Here we have a case of a switching simultaneous equation system where P_t is sometimes endogenous and sometimes exogenous. There is a considerable amount of literature on multiple regimes (see e.g. Goldfeld and Quandt (1976)) but not much on problems where the regime shifts involve changes in the number of endogenous variables. Earlier, Barten and Bronsard (1970) derived some two-stage least squares estimators for the case where a regressor may be exogenous or endogenous at different times. Richard (1978) studied some wider aspects of this problem and Davidson (1978) derived the exact maximum likelihood estimators for a fairly general class of models involving shifts between the endogenous and exogenous variables.

The switch in the regimes considered in this paper is, however, different from the ones discussed in the papers by Barten and Bronsard, Richard, and Davidson. As in their papers, the switch in our model involves changes in the number of endogenous and exogenous variables. But the switching considered in these papers is exogenous. It is a consequence of some abrupt institutional changes or policy changes. (Like shifts from fixed to floating exchange rates, shifts in Federal Reserve policy from manipulation of interest rates to control of money supply, etc.). By contrast, the switch between regimes in our model is endogenous and not exogenous. We thus have an endogenous switching system with changes in the number of endogenous variables (see Maddala and Nelson (1975) for some earlier discussion on exogenous and endogenous switching).

We can write the likelihood function (5) explicitly and obtain the first derivatives. We can then use the Berndt et al. (1974) method to obtain the maximum likelihood estimates. However, even for the computation of the ML estimates, we need some initial consistent estimates and we will now describe some two-stage methods for obtaining them.

III. TWO-STAGE ESTIMATION METHODS

In the set ψ_1, Q_t and P_t are endogenous variables and the reduced form for P_t can be written as

$$P_t = \Pi X_t + v_t \quad \text{with} \quad v_t \sim IID(0, \sigma_v^2) \tag{6}$$

where X_t includes all the exogenous variables in the model. The parameters Π in (6) can be estimated by the tobit method applied to the model

$$\begin{aligned} P_t &= \Pi X_t + v_t \qquad && \text{if } P_t < \overline{P}_t \\ &= \overline{P}_t && \text{if } P_t \geq \overline{P}_t \end{aligned} \tag{7}$$

Once we get estimates $\hat{\Pi}$ of the parameters Π by the tobit method, we can obtain the estimated values $\hat{P}_t$ of P_t. However, the estimated values $\hat{P}_t$ that we have to substitute in the structural equations (1) and (2) are for the subset ψ_1. For this purpose note that

$$\begin{aligned} E(P_t | t \in \psi_1) &= \Pi X_t + E(v_t | v_t < \overline{P}_t - \Pi X_t) \\ &= \Pi X_t + \sigma_v \cdot E\left(\frac{v_t}{\sigma_v} \middle| \frac{v_t}{\sigma_v} < \frac{\overline{P}_t - \Pi X_t}{\sigma_v}\right) \\ &= \Pi X_t - \sigma_v \frac{\phi(Z_t)}{\Phi(Z_t)} \end{aligned} \tag{8}$$

where

$$Z_t = \frac{\overline{P}_t - \Pi X_t}{\sigma_v} \tag{9}$$

and ϕ and Φ are respectively the density function and the distribution function of the standard normal. From the tobit estimates $\hat{\Pi}$ of Π and $\hat{\sigma}_v$ of σ_v, we can get the estimates $\hat{P}_t$ of P_t from equation (8).

Coming next to the structural equations, the demand equation is:

$$Q_t = X_{1t}'\beta_1 + \alpha_1 P_t + u_{1t}$$

and this can be estimated from only the observations in ψ_1. We first have to make adjustment for the fact that $E(u_{1t} | t \in \psi_1) \neq 0$. We have

$$\begin{aligned} E(u_{1t} | t \in \psi_1) &= E\left(u_{1t} \middle| \frac{v_t}{\sigma_v} < Z_t\right) \\ &= -\frac{\sigma_{1v}}{\sigma_v} \frac{\phi(Z_t)}{\Phi(Z_t)} \end{aligned}$$

where $\sigma_{1v} = \text{cov}(u_{1t}, v_t)$ and σ_v and Z_t are as defined earlier. Thus, we can write the demand function as:

$$Q_t = X'_{1t}\beta_1 + \alpha_1 P_t - \frac{\sigma_{1v}}{\sigma_v}\frac{\phi(Z_t)}{\Phi(Z_t)} + \epsilon_t \tag{10}$$

where the new residual ϵ_t has zero mean. The two-stage least squares method now consists of estimating the parameters in (10) by OLS after substituting $\hat{P}_t$ for P_t and $\hat{Z}_t$ for Z_t obtained from the tobit estimates of $\hat{\Pi}$ and $\hat{\sigma}_v$. The tobit two-stage least squares estimates are consistent. Their asymptotic covariance matrix is rather complicated (see Lee, Maddala and Trost (1980).

Coming next to the supply function, now we have observations in both ψ_1 and ψ_2. Again we have to correct for the nonzero means of the residuals in the two regimes. As before, we can obtain

$$E(u_{2t} | t \in \psi_1) = -\frac{\sigma_{2v}}{\sigma_v}\frac{\phi(Z_t)}{\Phi(Z_t)}$$

and

$$E(u_{2t} | t \in \psi_2) = \frac{\sigma_{2v}}{\sigma_v}\frac{\phi(Z_t)}{1-\Phi(Z_t)}$$

where $\sigma_{2v} = \text{cov}(u_{2t}, v_t)$ and σ_v and Z_t are as defined earlier.

Now we can write the supply function as:

$$Q_t = X'_{2t}\beta_2 + \alpha_2 P_t - \frac{\sigma_{2v}}{\sigma_v}\frac{\phi(Z_t)}{\Phi(Z_t)} + \eta_t \quad \text{for} \quad t \in \psi_1 \tag{11}$$

and

$$Q_t = X'_{2t}\beta_2 + \alpha_2 \overline{P}_t + \frac{\sigma_{2v}}{\sigma_v}\frac{\phi(Z_t)}{1-\Phi(Z_t)} + \eta_t \quad \text{for } t \in \psi_2 \tag{12}$$

where the residual η_t now has zero mean. The two-stage least squares method now consists of estimating the parameters β_2, α_2, $\frac{\sigma_{2v}}{\sigma_v}$ by OLS after substituting $\hat{P}_t$ for P_t in equation (11) and $\hat{Z}_t$ for Z_t in equations (11) and (12). The resulting tobit two-stage least squares estimates are consistent. The asymptotic covariance matrix is, however, more complicated than the one for the demand function. It can be obtained by following the methods in Amemiya (1978) and Lee, Maddala and Trost (1980).

One important thing to note is that in the switching simultaneous system that we are considering, we do not need the usual exclusion restrictions to identify the parameters of the supply function so long as there are enough observations in regime ψ_2 (excess demand). However, we need the usual condition that there should be at least one exogenous variable excluded from the demand function in order that the parameters of the demand function be identified. This is also clear by looking at equations (10), (11) and (12). If $X_{1t} = X_t$, the substitution of $\hat{P}_t$ for P_t in (10) introduces perfect multicollinearity. But even if $X_{2t} = X_t$, this does not happen in equations (11) and (12) because $\hat{P}_t$ is substituted for P_t only in equation (11). The switching simultaneous system implied by our model is also different from the switching system considered by Heckman (1978).

This completes the discussion of the two-stage estimation of the structural system given by (1) and (2). These two-stage estimates can be used as starting values in the iterative solution of the likelihood equations. If we use the Berndt et al (1974) method, we need only the first derivatives of the likelihood function (5). These are presented in the Appendix.

IV. PREDICTION OF THE EFFECTS OF REMOVAL OF PRICE CEILING

One interesting question from the policy point of view is what the market equilibrium price would have been in periods the price ceiling is operative. This sort of question is often asked, e.g. in the natural gas market (and other energy markets) the question is raised as to what the market equilbrium price would be if price controls are removed. To answer this question we have to evaluate $E(P_t | t \in \psi_2)$. We have

$$E(P_t | t \in \psi_2) = \Pi X_t + E(v_t | v_t > \overline{P}_t - \Pi X_t)$$

$$= \Pi X_t + \sigma_v \frac{\phi(Z_t)}{1 - \Phi(Z_t)} \tag{13}$$

From the tobit estimates of Π and σ_v we can evaluate (13). A better procedure would be to obtain the solved reduced form estimate of Π.

V. A MODEL WITH BOTH PRICE CEILINGS AND PRICE FLOORS

The preceding analysis can easily be extended to the case of upward and downward limits on prices. Let P_t be controlled to lie between $\overline{P}_{1t}$ and $\overline{P}_{2t}$. There are numerous examples of this. Almost all commodity futures markets have upper and lower limits on price variation. Usually $\overline{P}_{1t} = P_{1,\,t-1} - \gamma$ and $\overline{P}_{2t} = P_{2,\,t-1} + \gamma$ where γ is a given constant. If there are limits on percentage variation from previous price, we will have

$$\overline{P}_{1t} = P_{1,\,t-1}(1-\gamma)$$

and

$$\overline{P}_{2t} = P_{2,\,t-1}(1+\gamma)$$

where γ is, again, a given constant.

We can now classify the observations into three regimes:

<u>Regime 1</u>: $\overline{P}_{1t} < P_t < \overline{P}_{2t}$. Denote this set of points by ψ_1. These are the equilibrium points and Q_t and P_t are both endogenous variables.

<u>Regime 2</u>: $P_t \geq \overline{P}_{2t}$. Denote this set of points by ψ_2. This set corresponds to excess demand. Here $S_t = Q_t$ and $D_t \geq Q_t$. Also $P_t = \overline{P}_{2t}$ is an exogenous variable.

<u>Regime 3</u>: $P_t \leq \overline{P}_{1t}$. Denote this set by ψ_3. This set corresponds to excess supply. Here $D_t = Q_t$, $S_t \geq Q_t$ and $P_t = \overline{P}_{1t}$ is an exogenous variable.

$$L = \prod_{\psi_1} f(Q_t, P_t) \cdot \prod_{\psi_2} \int_Q^\infty g_2(D_t, Q_t) dD_t \cdot \prod_{\psi_3} \int_Q^\infty g_1(Q_t, S_t) dS_t \tag{14}$$

where g_1 and g_2 are the joint densities of D_t and S_t derived from (1) and (2) after substituting $P_t = \overline{P}_{1t}$ and $P_t = \overline{P}_{2t}$ respectively.

The two-stage estimation method proceeds as before. We first estimate the reduced form equation for P_t by a two-limit tobit method. We have

$$P_t = \overline{P}_{1t} \qquad \text{if } P_t \le \overline{P}_{1t}$$

$$= \Pi X_t + v_t \qquad \text{if } \overline{P}_{1t} < P_t < \overline{P}_{2t}$$

$$= \overline{P}_{2t} \qquad \text{if } P_t \ge \overline{P}_{2t} \tag{15}$$

After we get the tobit estimates of Π and σ_v we next get predicted values of P_t for the observations in regime 1 as done in equation(8). Define, analogous to (9), the following expression.

$$Z_{1t} = \frac{\overline{P}_{1t} - \Pi X_t}{\sigma_v} \quad \text{and} \quad Z_{2t} = \frac{\overline{P}_{2t} - \Pi X_t}{\sigma_v} \; . \tag{16}$$

Then $E(P_t)$ in regime 1 is given by

$$E(P_t|\overline{P}_{1t} < P_t < \overline{P}_{2t}) = \Pi X_t - \sigma_v \frac{\phi(Z_{2t}) - \phi(Z_{1t})}{\Phi(Z_{2t}) - \Phi(Z_{1t})} \tag{17}$$

Our next step would be to adjust for the means of the residuals u_{1t} and u_{2t} in the different regimes as was done in equations (10) to (12). Note that the parameters of the demand function are estimated from observations in regimes 1 and 3, and the parameters of the supply function are estimated from observations in regimes 1 and 2. We have the following expressions for the expectations of u_{1t} and u_{2t} in the different regimes. Note that σ_{1v} and σ_{2v} are, as defined earlier, given by $\sigma_{1v} = \text{cov}(u_{1t}, v_t)$ and $\sigma_{2v} = \text{cov}(u_{2t}, v_t)$. Let us, for compactness define

$$W_{1t} = \frac{-\phi(Z_{2t}) + \phi(Z_{1t})}{\Phi(Z_{2t}) - \Phi(Z_{1t})}$$

$$W_{2t} = \frac{\phi(Z_{2t})}{1 - \Phi(Z_{2t})}$$

and

$$W_{3t} = -\frac{\phi(Z_{1t})}{\Phi(Z_{1t})}$$

Then,

$$E(u_{jt}|t \in \psi_1) = \frac{\sigma_{jv}}{\sigma_v} W_{1t} \qquad j = 1, 2$$

$$E(u_{2t}|t \in \psi_2) = \frac{\sigma_{2v}}{\sigma_v} W_{2t}$$

$$E(u_{1t}|t \in \psi_3) = \frac{\sigma_{1v}}{\sigma_v} W_{3t}$$

Substituting these expressions in the demand and supply functions, we can write them as follows:

<u>Demand Functions</u>

$$Q_t = X'_{1t}\beta_1 + \alpha_1 P_t + \frac{\sigma_{1v}}{\sigma_v} W_{1t} + \epsilon_{1t} \quad \text{for } t \text{ in } \psi_1 \tag{18}$$

$$Q_t = X'_{1t}\beta_1 + \alpha_1 \overline{P}_{1t} + \frac{\sigma_{1v}}{\sigma_v} W_{3t} + \epsilon_{1t} \quad \text{for } t \text{ in } \psi_3 \tag{19}$$

<u>Supply Functions</u>

$$Q_t = X'_{2t}\beta_2 + \alpha_2 P_t + \frac{\sigma_{2v}}{\sigma_v} W_{1t} + \epsilon_{2t} \quad \text{for } t \text{ in } \psi_1 \tag{20}$$

$$Q_t = X'_{2t}\beta_2 + \alpha_2 P_t + \frac{\sigma_{2v}}{\sigma_v} W_{2t} + \epsilon_{2t} \quad \text{for } t \text{ in } \psi_2 \tag{21}$$

The new residuals ϵ_{1t} and ϵ_{2t} have now zero means. The tobit two-stage estimation method now proceeds follows.

We first estimate Π and σ_v from (15). Next we get estimates of Z_{1t} and Z_{2t} from (16), and an estimate of P_t in regime 1 from (17). We then substitute $\hat{P}_t$ for P_t, and $\hat{W}_{1t}$ for W_{1t} in equations (18) and (20), $\hat{W}_{2t}$ for W_{2t} in equation (21), and $\hat{W}_{3t}$ for W_{3t} in equation (19) and estimate equations (18) and (19) together, and equations (20) and (21) together by OLS. The tobit two-stage least squares estimates are consistent. Their asymptotic covariance matrix is, however, complicated. It can be obtained by following the methods in Amemiya (1978) and Lee, Maddala and Trost (1980).

In this model, so long as there are enough observations in ψ_2 and ψ_3, we do not need the usual exclusion restrictions for identification of either the demand function or the supply function.

The two-stage estimates can be used for the iterative computation of the ML estimates. The first derivatives of the likelihood function (14) are presented in the Appendix.

Again, one can estimate what the equilibrium price would be if there were no price floors or price ceilings when these limits are operative. We have

$$E(P_t | P_t \leq P_{1t}) = \Pi X_t + \sigma_v W_{3t} \tag{22}$$

and

$$E(P_t | P_t \geq P_{2t}) = \Pi X_t + \sigma_v W_{2t} \tag{23}$$

where W_{2t} and W_{3t} are defined earlier. For the values of Π we can use the solved reduced form estimates.

VI. CASE WHERE PRICES ARE UNOBSERVED

The above analysis can easily be extended to the case where we have no observations on P_t for the equilibrium points. We do have information on $\bar{P}_{1t}$ and $\bar{P}_{2t}$ and we also know which observations belong to regimes 1, 2 and 3. In this case the likelihood function to be maximized is:

$$L = \prod_{\psi_1} h(Q_t) \prod_{\psi_2} \int_{Q_t}^{\infty} g_2(D_t, Q_t) dD_t \prod_{\psi_3} \int_{Q_t}^{\infty} g_1(Q_t, S_t) dS_t \tag{24}$$

where $h(Q_t)$ is the reduced form equation for Q_t derived from equations (1) and (2). The only difference between the likelihood functions (14) and (24) is in the first expression. Though $h(Q_t)$ involves only the reduced form parameters, the other two expressions in the likelihood function (24) enable us to estimate all the structural parameters. Once the structural parameters have been estimated the prediction of P_t proceeds as before using formulae (22) and (23).

There is a difference, however, in the way the initial estimates of Π and σ_v are computed. In this case we do not observe P_t at all. All we know is the number of observations which are $\leq \bar{P}_{1t}$, between $\bar{P}_{1t}$ and $\bar{P}_{2t}$ and $\geq \bar{P}_{2t}$. We can now use the two-limit probit method described in Rosett and Nelson (1975) to get estimates of Π and σ_v. Once this is done the rest of the two-stage estimation proceeds as in the previous section.

As for maximum likelihood estimation, the first derivatives of the likelihood function (24) are presented in the Appendix.

VII. MODELS WITH PRICE SUPPORTS

Up to this point we have implicitly assumed that the short-side of the market prevails i.e. the quantity transacted is defined by the relation:

$$Q_t = \text{Min}\ (D_t,\ S_t) \tag{25}$$

This is the standard assumption in almost all the econometric literature on disequilibrium (see the different papers listed in the references). This assumption is not a valid one to make except in "rationing models" where, in the case of excess demand, the available supply is rationed out to the demanders by some mechanism, and in the case of excess supply the available demand is 'rationed out' among the suppliers. By contrast, in models with governmental price supports, as for agricultural commodities, as also in the trading models discussed in the next section, we have to replace the condition (25) by something more appropriate for the problem. Consider first the case of governmental price support programs. In this case we have the condition:

$$P_t \geq \overline{P}_t \text{ (a given price).}$$

If $P_t \geq \overline{P}_t$ then we have a simultaneous equations model with $Q_t = D_t$ S_t and Q_t and P_t are the endogenous variables.

If $P_t < \overline{P}_t$ then $D_t < S_t$ but we do observe <u>both</u> D_t and S_t. The observed price is, of course, $\overline{P}_t$ which is now exogenous. We have observations on both the market demand and the amount the government buys under the price support program.

Thus, if we denote by ψ_1 the set of observations for which $D_t = S_t$ and ψ_2 the set of observations for which $D_t < S_t$ then the appropriate likelihood function for this model is:

$$L = \prod_{\psi_1} f(Q_t,\ P_t) \prod_{\psi_2} g(D_t,\ S_t) \tag{26}$$

where $f(Q_t, P_t)$ and $g(D_t, S_t)$ are as defined in (5).

Note that even if $\overline{P}_t$ is an exogenous variable for the subset of observations ψ_2, we cannot estimate the demand function and the supply function by OLS using just the observations in this subset.

The two-stage procedure for this model follows the same lines as that described in Section III, except that since both the quantity demanded and quantity supplied are observed in both the regimes, all observations can be used to estimate both the demand and supply functions (and not just the demand function if we were to be considering a "rationing" model). The maximization of the likelihood function (26) is easily accomplished since there are no integral signs involved as in the previous models.

VIII. TRADING MODELS

In Section V we considered models with both price ceilings and price floors and it was mentioned that an example of this is the case of futures markets where there are limits to the price variation. However, this example is not, strictly speaking, appropriate for the model analyzed there because we considered a "rationing model" where the implicit assumption was that the quantity transacted under excess demand or excess supply is given by (25). By contrast, in a "trading model," as in the case of the futures markets, if P_t is within the specified limits, trading takes place and Q_t and P_t are endogenous, but if P_t is outside the specified limits, no trading takes place. Thus, in the notation of Section V, if $P_t < \overline{P}_{1t}$ or $P_t > \overline{P}_{2t}$ we have $Q_t = 0$, since no trading takes place.

Define the partitioning of the data points into three sets as before:

$$\Psi_1: \quad \overline{P}_{1t} < P_t < \overline{P}_{2t}$$

$$\psi_2: \quad P_t \geq \overline{P}_{2t}$$

$$\psi_3: \quad P_t \leq \overline{P}_{1t}$$

Then the likelihood function for this model is:

$$L = \prod_{\psi_1} f(Q_t, P_t) \prod_{\psi_2} \int_{\overline{P}_{2t}}^{\infty} g(P_t) dP_t \prod_{\psi_3} \int_{-\infty}^{\overline{P}_{1t}} g(P_t) dP_t \tag{27}$$

where $g(P_t)$ is the distribution of the equilibrium price P_t i.e. the distriburion of P_t derived from the reduced form equation for P_t implied by the structural equations (1) and (2).

The two-stage estimation of this model proceeds as before except that the parameters of the demand and supply functions are estimated respectively from equations (18) and (20) only.

The prediction of the equilibrating prices in the absence

of the controls on prices proceeds as described before in Section V.

IX. SOME CONCLUDING REMARKS

The present paper describes methods of estimation for models of markets with controlled price variation. Many practical situations involving disequilibrium are due to price ceilings or price floors or both and thus fall in the category of the models described here. We distinguish between "rationing models" of disequilibrium where the short side of the markets prevails, and "trading models" of disequilibrium where no trading takes place if there is excess demand or excess supply. The econometric literature on disequilibrium has considered only "rationing models" but not trading models. We have presented the likelihood functions for the different models and described two-stage estimation methods that will give initial consistent estimates that can be used as starting values in the iterative solution of the likelihood equations. We have also described how one can estimate the market equilibrium price that would prevail if the price controls are removed.

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APPENDIX

If we use the Berndt et. al. (1974) method or the Davidson-Fletcher-Powell method, we need only the first derivatives of the likelihood function to obtain the ML estimates by an iterative method. In many problems similar to the ones described in this paper, it has been found that the Berndt et. al. method has better convergence properties than the Newton-Raphson or other methods depending on the evaluation of the second derivatives of the likelihood function. Hence, in this Appendix we will outline some formulae that will be useful in obtaining the first derivatives of the different likelihood functions mentioned in the paper.

Consider first the likelihood function given by (26). We can write

$$\text{Log } L = \sum_{\psi_1} \text{Log } f + \sum_{\psi_2} \text{Log } g$$

Since the Jacobian of the transformation from (u_1, u_2) to (Q,P) is $(\alpha_2 - \alpha_1)$ we have

$$\text{Log } f = \text{Log } (\alpha_2 - \alpha_1) - \frac{1}{2} \text{Log } (y_1' \Lambda^{-1} y_1)$$

where $$y_1 = \begin{pmatrix} Q_t - X_{1t}'\beta_1 - \alpha_1 P_t \\ Q_t - X_{2t}'\beta_2 - \alpha_2 P_t \end{pmatrix}$$

and

$$\Lambda = \text{Cov } (u_1, u_2) = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}$$

similarly $$\text{Log } g = -\frac{1}{2} \text{Log } (Y_2' \Lambda^{-1} Y_2)$$

where $$Y_2 = \begin{pmatrix} D_t - X_{1t}'\beta_1 - \alpha_1 \bar{P}_t \\ S_t - X_{2t}'\beta_2 - \alpha_2 \bar{P}_t \end{pmatrix}$$

We first obtain derivatives of $Y_1' \Lambda Y_1$ with respect to Y_1 and Λ and then the derivatives of Y_1 with respect to $\alpha_1, \alpha_2, \beta_1, \beta_2$. These formulae are well-known. e.g. if A is symmetric, $\frac{\partial(y'Ay)}{\partial y} = 2Ay$

$$\frac{\partial(y'A^{-1}y)}{\partial \chi} = -y'A^{-1} \frac{\partial A}{\partial \chi} A^{-1} y$$

where χ is an element of A.

Coming next to likelihood function (27), it can be written as:

$$\text{Log } L = \sum_{\psi_1} \text{Log } f + \sum_{\psi_2} \text{Log } [1-\Phi(Z_{2t})] + \sum_{\psi_3} \text{Log } \Phi(Z_{1t})$$

where Z_{1t}, Z_{2t} are defined in (16) and $\Phi(\cdot)$ is the cumulative distribution function of the standard normal. The derivatives of Log f are obtained as before. As for the others, we make use of the fact that

$$\frac{\partial}{\partial Z} \Phi(Z) = \phi(Z)$$

where $\phi(Z)$ is the density function of the standard normal evaluated at Z.

Next we evaluate the derivatives of Z_1 and Z_2 with respect to the structural parameters. To do this we have to obtain the relationship between (Π, σ_v) and the structural parameters. Since this will depend on the specification of the structural equations

(1) and (2) and can be easily worked out in individual cases, details are omitted here.

Coming next to the likelihood function (5), we first write $g(D_t, S_t)$ as the product of the marginal density of S_t and the conditional density of D_t given S_t.

i.e. $$g(D_t, S_t) = g_1(S_t)\, g_2(D_t|S_t)$$

$g_1(S_t)$ being the usual normal density is not a problem. $g_2(D_t|S_t)$ is normal with mean

$$M = X'_{1t}\beta_1 + \alpha_1 P_t - \frac{\sigma_{12}}{\sigma_2^2}(Q_t - X'_{2t}\beta_2 - \alpha_2 P_t)$$

and variance $$\sigma_{1.2}^2 = \sigma_1^2 - \frac{\sigma_{12}^2}{\sigma_2^2}$$

hence $$\int_{Q_t}^{\infty} g_2(D_t|S_t)dD_t = 1 - \Phi\left(\frac{Q_t - M}{\sigma_{1.2}}\right)$$

Now we proceed as done before with the likelihood function (27).

The procedure for the likelihood function (14) is similar. For the set ψ_2, we consider the factorization:

$$g(D_t, S_t) = g_1(D_t)\, g_2(S_t|D_t)$$

and we proceed as before.

Finally, for the likelihood function (24), the terms in ψ_2 and ψ_3 are handled as in the case of the likelihood function (14). As for the first term, we have merely to obtain the reduced form equation for Q_t from the structural equations (1) and (2). Since it depends on the detailed specifications of the equations (1) and (2) and can be easily worked out in individual cases, it is omitted here.