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POWER STRUCTURE AND CARDINALITY RESTRICTIONS FOR PARETIAN SOCIAL CHOICE RULES*

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ABSTRACT

Let f be a multiple-valued Paretian social choice rule for n voters and an outcome set X. The preventing sets for f are shown to form an acyclic majority when $|X| < n$, a prefilter when $|X| \geq n$, and a filter when f also satisfies a binary independence condition. These results are then shown to yield inequalities relating $|X|$, n, and certain preventing sets. In particular, if every coalition of q voters constititues a preventing set, then $|X| \leq [\frac{n-1}{n-q}]$. Other inequalities are obtained if strong equilibria are present for every preference profile.

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I. INTRODUCTION

A fundamental result of Gibbard [4] and Satterthwaite [9] states that a single-valued social choice rule with dominant strategy equilibria for all preference profiles must be dictatorial. As the hypotheses needed for this result are weakened, a variety of positive and negative results have emerged. Thus Peleg [8] and Dutta and Pattanaik [2] have shown that nondictatorial and even anonymous choice functions may be possible when strong equilibria replace dominant strategies. Packel and Saari [7] have shown, on the other hand, that weaker equilibrium conditions still tend to impose either highly indecisive or undemocratic power structures on the decisive sets of a single-valued social choice rule.

In the present paper we consider (multiple-valued) social choice rules satisfying Pareto optimality, which we regard as the least controversial and weakest notion of cooperative equilibrium. We show that significant structural restrictions must exist on the preventing sets of a Paretian social choice rule. These restrictions lead to inequalities relating the cardinalities of the outcome set, the voter set, and certain preventing sets. More specifically, with m possible outcomes and n voters, the preventing sets of a Paretian social choice rule are shown to form either a prefilter (if $m \geq n$) or an acyclic majority (if $m < n$). If every coalition of q voters constitutes a preventing set, it then follows that $m \leq [\frac{n-1}{n-q}]$, where $[\cdot]$ denotes the greatest integer function.

If a Paretian social choice rule is generated in a natural way by a prefilter of preventing sets, then a requirement that true preferences be strong equilibria places cardinality restrictions on the number of outcomes. This parallels, to some extent, ideas considered by Hurwicz and Schmeidler [5]. It also leads to a cardinality result for strong implementation of a class of quota games with a nonempty collegium.

The simple example of the Pareto rule, which chooses for each preference profile the set of Pareto undominated outcomes, shows that dictatorship results analogous to those of Gibbard and Satterthwaite are not to be expected for multiple valued social choice rules. In addition to the cardinality restrictions obtained, the results we develop are noteworthy in that rather weak assumptions in a multiple-valued setting still impose structure on the collection of coalitions that have power.

II. NOTATION AND DEFINITIONS

We consider a finite set N of voters with $|N| = n$. A set X of outcomes is under consideration by the voters, who must choose a nonempty subset from X.

Let $\mathcal{R}$ denote the set of reflexive, transitive, total orderings (weak orders) on X. If $R_i \in \mathcal{R}$ denotes the preference ordering over X for voter i, let

$$\pi = (R_1, R_2, \ldots, R_n) \in \mathcal{R}^n$$

denote the corresponding <u>profile</u> of preferences. Conversely, if $\pi \in \mathcal{R}^n$ is a profile, R_i will always denote the i^{th} component of π and P_i will denote the asymmetric part $(xP_iy \Leftrightarrow \sim yR_ix)$ of R_i. Likewise, $\overline{\pi} \in \mathcal{R}^n$ has i^{th} component $\overline{R}_i$.

A <u>social choice rule</u> is a function $f : \mathcal{R}^n \to 2^X - \{\phi\}$, where 2^X denotes the collection of subsets of X. Thus, for each $\pi \in \mathcal{R}^n$, $f(\pi)$ is a nonempty subset of X representing the social choice of the voters under the preference profile π.

The <u>preventing sets</u> $\mathcal{P}_f$ for a social choice rule $f : \mathcal{R}^n \to 2^X - \{\phi\}$ are defined by

$$\mathcal{P}_f = \{C \subseteq N | \forall\ \pi \in \mathcal{R}^n,\ \forall x,y \in X,\ xP_iy\ \forall i \in C \Rightarrow y \notin f(\pi)\}.$$

It is easy to see that members of a preventing set have power in the sense that any profile π such that $xP_iy\ \forall i \in C,\ \forall y \neq x$ requires that $f(\pi) = \{x\}$.

A social choice rule f is <u>Paretian</u> if $x\ P_iy\ \forall i \in N \Rightarrow y \notin f(\pi)$. It is clear that f is Paretian if and only if $N \in \mathcal{P}_f$.

Collections of preventing sets will be shown to satisfy various various set-theoretic properties. We now define the relevant power structures proceeding from most flexible to least democratic.

An <u>acyclic majority</u> is a collection $\mathcal{S}$ of subsets of N satisfying the following properties:

(I) $\phi \notin \mathcal{S}$; $N \in \mathcal{S}$

(II) $C \in \mathcal{S}$ and $C \subseteq D \Rightarrow D \in \mathcal{S}$

(III_{AM}) $C_i \in \mathcal{S}\ \forall i \in I$ and $|I| \leq |X| \Rightarrow \bigcap_{i \in I} C_i \neq \phi$.

The condition III_{AM} becomes increasingly restrictive as the set X of alternatives grows. The cardinality condition $|I| \leq |X|$ will be shown to be relevant only when $|X| < |N|$. With $|X| = 3$ such democratic (anonymous) voting methods as three-fourths rule ($|N| = 4$) and five-sevenths rule ($|N| = 7$) give rise to acyclic majorities.

A prefilter is a collection $\mathcal{S} \subseteq 2^N$ satisfying I, II, and

$$(III_{PF})\ C_i \in \mathcal{S}\ \forall i \in I \Rightarrow \bigcap_{i \in I} C_i \neq \phi.$$

The nonempty set $\bigcap_{C \in \mathcal{S}} C$ in a prefilter is called the collegium. If a collection of preventing sets forms a prefilter, its collegium generally needs outside support (enough to become a preventing set) to effect its collective will. A choice rule with a prefilter structure is called a collegial polity.

A filter is a collection $\mathcal{S} \subseteq 2^N$ satisfying I, II, and

$$(III)\ C, D \in \mathcal{S} \Rightarrow C \cap D \in \mathcal{S}.$$

In a filter on a finite set N, the collegium is the smallest preventing set. This set can be regarded as an oligarchy which controls the choice rule. It is immediate from the definitions that
$\mathcal{S}$ a filter $\Rightarrow \mathcal{S}$ a prefilter $\Rightarrow \mathcal{S}$ an acyclic majority.

III. RESULTS AND PROOFS

The first theorem extends a result from [7] to (multiple-valued) social choice rules.

Theorem 1. Let $f : \mathcal{R}^n \to 2^X - \{\phi\}$ be a Paretian social choice rule. Then

(a) $|X| < n \Rightarrow \mathcal{P}_f$ is an acyclic majority

(b) $|X| \geq n \Rightarrow \mathcal{P}_f$ is a prefilter

Proof. Properties I and II for acyclic majorities and prefilters follow directly from the Paretian assumption and the definition of $\mathcal{P}_f$. Properties III_{AM} and III_{PF} will emerge from the following argument. Given $C_1, C_2, \ldots, C_k \in \mathcal{P}_f$, suppose $\bigcap_{j=1}^{k} C_j = \phi$. For (a) we may also assume $k \leq |X|$. For (b) there is no loss of generality in assuming $k \leq n$ (for each $i \in N$ take one C_j that excludes i and the resultant sets C_j will for a collection of size not exceeding n). Since for (b) $n \leq |X|$, we may again assume $k \leq |X|$, just as for (a). It is then possible to choose distinct $x_1, x_2, \ldots, x_k \in X$ and a profile $\pi \in \mathcal{R}^n$ such that everyone has $x_1, x_2, \ldots, x_k$ among their top k choices and

$$\begin{array}{ll} x_1 P_i x_2 & \forall i \in C_1 \\ x_2 P_i x_3 & \forall i \in C_2 \\ \vdots & \\ x_{j-1} P_i x_j & \forall i \in C_j \\ \vdots & \\ x_k P_i x_1 & \forall i \in C_k. \end{array}$$

Note that transitivity of preferences in π is possible by our supposition that $\bigcap_{j=1}^{k} C_j = \phi$. Since f is Paretian and $f(\pi) \neq \phi$, we must have $x_j \in f(\pi)$ for some $j = 1, 2, \ldots, k$. Since C_{j-1} (or C_k if $j=1$) belongs to $\mathcal{P}_f$, we also have $x_j \notin f(\pi)$. This contradicts the supposition that $\bigcap_{j=1}^{k} C_j = \phi$ and establishes the theorem. ||

Theorem 1 has direct implications concerning the decisiveness and concentration of power for a Paretian social choice rule. Given an integer q with $n/2 < q < n$, we call f q-preventing if

$$C \subseteq N \text{ and } |C| \geq q \Rightarrow C \in \mathcal{P}_f.$$

If $|X| \geq n$, the prefilter that results from Theorem 1 implies that a Paretian social choice rule f can never be q-preventing. In this case f can only be "democratic" or anonymous when $\mathcal{P}_f = \{N\}$, in which case f may be highly indecisive. If $|X| < n$, anonymous power structures are possible, but only when the size of X is restricted according to the following theorem.

Theorem 2. Given $f : R^n \to 2^X - \{\phi\}$ Paretian and q-preventing. Then $|X| \leq [\frac{n-1}{n-q}]$.

Proof. It is always possible to choose $[\frac{n-1}{n-q}] + 1$ subsets of N of size n-q whose union is all of N. Choose a specific collection of $k = [\frac{n-1}{n-q}] + 1$ such sets and call them $B_1, B_2, \ldots, B_k$. Since f is q-preventing, $C_j = N - B_j \in \mathcal{P}_f$ $\forall j = 1,2,\ldots,k$. Also $\bigcup_{j=1}^{k} B_j = N \Rightarrow \bigcap_{j=1}^{k} C_j = \phi$. Since $\mathcal{P}_f$ is an acyclic majority by Theorem 1, it follows that $|X| < k = [\frac{n-1}{n-q}] + 1$. Thus $|X| \leq [\frac{n-1}{n-q}]$. ||

An inequality equivalent to that of Theorem 2 is obtained by Peleg [8]. His result assumes that f is single-valued and strongly consistent. Since our result merely requires Pareto optimality rather than strong equilibria for each profile, it would appear to be more

general. On the other hand, Peleg assumes q-winning sets rather than the stronger assumption of q-preventing, so the two results are not strictly comparable.

A result giving $\mathcal{P}_f$ a filter structure can also be extended from [7] to our setting. Define a social choice rule $f : R^n \to 2^X - \{\phi\}$ to be binary independent if $\forall \pi, \pi' \in R^n$, $\forall x,y \in X$

$$[x \in f(\pi),\ y \notin f(\pi),\ xP_iy \Leftrightarrow xP_i'y \text{ and } yP_ix \Leftrightarrow yP_i'x\ \forall i \in N] \Rightarrow y \notin f(\pi').$$

Binary independence is a natural version of the well-known independence of irrelevant alternatives applied to our social choice setting. It says that if two profiles are identical in their pairwise rankings of x and y, then one profile cannot result in the selection of y while the other chooses x and excludes y.

Theorem 3. Let $f : R^n \to 2^X - \{\phi\}$ be a Paretian, binary independent social choice rule with $|X| \geq 3$. Then $\mathcal{P}_f$ is a filter.

Proof. Given C, D $\in \mathcal{P}_f$ we must show that $C \cap D \in \mathcal{P}_f$. Suppose some $\pi \in R^n$ has xP_iy $\forall i \in C \cap D$. Choose $\overline{\pi} \in R^n$ with x, y, and z (all distinct) ranked among the top three alternatives for all voters in such a way that:

$x\overline{P}_iz\ \overline{P}_iy$ $\forall i \in C \cap D$

$x\overline{P}_iz$ and $\overline{P}_i$ agrees with P_i on $\{x,y\}$ $\forall i \in C-D$

$z\overline{P}_iy$ and $\overline{P}_i$ agrees with P_i on $\{x,y\}$ $\forall i \in C-D$

$\overline{P}_i$ agrees with P_i on $\{x,y\}$ $\forall i \in N-(C \cup D)$.

Then $C \in \mathcal{P}_f \Rightarrow z \notin f(\overline{\pi})$ and $D \in \mathcal{P}_f \Rightarrow y \notin f(\overline{\pi})$. Since f is Paretian and all other outcomes are ranked below x, y, and z by $\overline{\pi}$, we have $\{x\} = f(\overline{\pi})$. Since π and $\overline{\pi}$ agree on $\{x,y\}$ by construction, binary independence implies that $y \notin f(\pi)$. Hence $C \cap D$ prevents y and is thus a preventing set. ||

It is easy to check that the mapping $f \mapsto \mathcal{P}_f$ from Paretian social choice rules to collections of preventing sets is not one-to-one. Going in the other direction, we can define a social choice rule in a natural way from a prefilter $\mathcal{P}$ of subsets of N. Given $\mathcal{P}$, define $f_{\mathcal{P}} : \mathcal{R}^n \to 2^X - \{\phi\}$ as follows:

$$x \in f_{\mathcal{P}}(\pi) \Leftrightarrow \forall C \in \mathcal{P},\ \forall y \in X,\ \exists i \in C \ni xR_i y.$$

Thus $x \in f_{\mathcal{P}}(\pi)$ precisely when no coalition in $\mathcal{P}$ prevents it.

We now wish to assume a form of strong stability on a social choice rule $f : \mathcal{R}^n \to 2^X - \{\phi\}$. To do this in our multiple-valued setting, we need to extend a weak preference order P on X to a weak order $\hat{P}$ on $2^X - \{\phi\}$. There are a variety of ways in which this extension might be made, but the results we obtain require only that the extension procedure satisfy the following reasonable axiom:

$$(\hat{}) \quad \forall x,y \in X,\ xPy \Leftrightarrow \{x,y\}\hat{P}\{y\}.$$

We assume this property of set-preference extensions for the duration.

Given a procedure which extends preferences P on X to preferences $\hat{P}$ on $2^X - \{\phi\}$, we say that $\pi \in \mathcal{R}^n$ is a <u>strong equilibrium</u> for $f : \mathcal{R}^n \to 2^X - \{\phi\}$ if

$$\forall \overline{\pi} \in \mathcal{R}^n,\ \forall \text{ nonempty } C \subseteq N,\ \exists i \in C \ni f(\pi)\hat{R}_i\ f(\overline{\pi}_C,\pi_{N-C})$$

(the profile $(\overline{\pi}_C,\pi_{N-C})$ takes preferences from $\overline{\pi}$ for $i \in C$ and π for $i \in N-C$).

We call $f : \mathcal{R}^n \to 2^X - \{\phi\}$ <u>strongly straightforward</u> if every $\pi \in \mathcal{R}^n$ is a strong equilibrium for f. This equilibrium notion from game theory requires that no coalition can improve the outcome for all its members by misrepresenting their preferences.

The presence of strong equilibria for all profiles is used in Ferejohn and Grether [3] to obtain a result that ties in neatly with Theorem 1. Generalizing preferences to abstract strategies, we let $g : S^n \to X$ be a <u>game form</u> mapping strategy profiles into single outcomes. Let $E_g(\pi)$ denote the set of strong equilibria for profile π, and assume for all $\pi \in \mathcal{R}^n$ that $E_g(\pi) \neq \phi$. Define the <u>decisive sets</u> C_g for g by

$$C_g = \{C \subseteq N \mid \forall x \in X\, \exists s^x \in S^n \text{ s.t. } g(s_c^x,s_{N-C}) = x\ \forall s \in S^n\}.$$

Ferejohn and Grether prove that C_g must be a prefilter when $|X| \geq n$. Call $f : \mathcal{R}^n \to 2^X - \{\phi\}$ <u>strongly and fully implementable</u> by g if

$$f(\pi) = g(E_g(\pi))\ \forall \pi \in \mathcal{R}^n.$$

It is easy to show that f is strongly and fully implemented by g, then $C_g = \mathcal{P}_f$. Thus our Theorem 1 provides an alternative proof that the decisive sets of a strongly and fully implementing game form must be a prefilter when $|X| \geq n$. Theorem 1 also provides a new and analogous result for acyclic majorities when $|X| < n$.

The following theorem and its corollary yield restrictions on the size of the outcome set for strongly straightforward social choice rules arising from proper prefilters (prefilters that are not filters).

Theorem 4. Given a social choice rule f_P defined from a proper prefilter on N. Let $C = \bigcap_{C_i \varepsilon P} C_i$ be the collegium of P and let $\{D_j\}_{j=1}^{k}$ be a nonempty collection of subsets of N-C such that

(i) $C \cup D_j \;\varepsilon\; P \;\forall j = 1,2,\ldots,k$

(ii) $\bigcap_{j=1}^{k} D_j = \phi.$

Then f_P strongly straightforward $\Rightarrow |X| \leq k$.

Proof. First note that since P is a proper prefilter, nonempty collections $\{D_j\}$ with the desired properties must exist. We can assume, without loss of generality, that $\{D_j\}_{j=1}^{k}$ has no proper subcollections satisfying (ii) and that, for each D_j, no proper subset of D_j satisfies (i). Suppose that $|X| > k$. Then we can choose k + 1 distinct outcomes $x_1, x_2, x_3, \ldots, x_k$, and $z \;\varepsilon\; X$ and a profile $\pi \;\varepsilon\; R^n$ ranking these outcomes above all others and satisfying:

$$\forall_i \;\varepsilon\; C, \quad zP_i x_1 P_i x_2 \ldots P_i x_k$$

$$\forall_i \;\varepsilon\; D_1, \quad zP_i x_1,$$

$$\forall_i \;\varepsilon\; D_2, \quad x_1 P_i x_2$$

$$\forall_i \;\varepsilon\; D_3, \quad x_2 P_i x_3$$

$$\vdots$$

$$\forall_i \;\varepsilon\; D_{k-1}, \quad x_{k-2} P_i x_{k-1}$$

$$\forall_i \;\varepsilon\; D_k, \quad x_{k-1} P_i x_k P_i z.$$

(Failure to specify a preference between outcomes on a set D_j means there is no unanimity on D_j or on any other set whose union with C belongs to P). Since $C \cup D_j \;\varepsilon\; P$ it follows that $x_j \notin f_P(\pi)\; \forall j = 1,2,\ldots,k$ and hence that $f_P(\pi) = \{\phi\}$. Consider the profile $\overline{\pi}$ which is identical to π except that

$$\forall_i \;\varepsilon\; D_k, \quad x_k \overline{P}_i x_{k-1} \overline{P}_i z.$$

The definition of f_P now requires that $f(\overline{\pi}) = \{x_k, z\}$. Since $x_k P_i z\; \forall i \;\varepsilon\; D_k$, it follows that $\{x_k, z\}\; \hat{P}_i\; \{z\}\; \forall i \;\varepsilon\; D_k$. Thus all members of D_k are better off misrepresenting π with $\overline{\pi}$ and strong straightfowardness is violated. It follows that we cannot have $|X| > k$. ||

We now apply Theorem 4 to a prefilter $P_{C,q,n}$ defined by a collegium C with $|C| = c < n-1$ and a quota q $(n>q>c)$ so that

$$B \;\varepsilon\; P \Leftrightarrow C \subseteq B \text{ and } |B| \geq q.$$

We denote the social choice rule arising from such a prefilter by $f_{C,q,n}$.

Corollary. If the social choice rule $f_{C,q,n} : R^n \to 2^X - \{\phi\}$ is strongly straightforward, then $|X| \leq [\frac{n-c-1}{n-q}] + 1$.

Proof. It is always possible to choose $[\frac{n-c-1}{n-q}] + 1$ subsets of N-C of size n-q whose union is all of N-C. Choose a specific collection of $k = [\frac{n-c-1}{n-q}] + 1$ such sets and call them $A_1, A_2, \ldots, A_k$. Then the collection $\{D_j\}_{j=1}^{k}$ defined by $D_j = N - (C \cup A_j)$ satisfies conditions (i) and (ii) of Theorem 4 for the proper prefilter $P_{C,q,n}$. If $f_{C,q,n}$ is strongly straightforward, it follows that $|X| \leq k = [\frac{n-c-1}{n-q}] + 1$. ||

Theorem 4 and its corollary show that strong straightforwardness imposes severe restrictions on the size of the outcome set for social choice rules defined from prefilters. It should be noted that any social choice rule defined from a filter must satisfy strong straightforwardness for reasonable set-preference extensions. A converse to the corollary along these lines would be an interesting result.

CONCLUDING REMARKS

Since our concern here has been with social choice rules which are multiple-valued, the issue of implementation via a mechanism or game form (see [1,6]) might seem relevant. The unanimity aspect of Pareto optimality as an equilibrium concept, however, makes questions of implementation uninteresting in this setting. Our use of strong equilibria in Theorem 4 does admit the possibility of meaningful

consideration of implementation via indirect mechanisms as opposed to the true-preference setting we have employed. The restrictions we have obtained on $|X|$ may not exist with the use of strongly stable indirect mechanisms.

While there exist difficulties with interpreting Pareto optimality as an equilibrium concept, especially in noncooperative game theory, it does provide parallels with results for other equilibrium concepts (see [7]). Whether or not this interpretation is employed, the normative acceptability and widespread existence of Paretian social choice rules give our results a degree of generality not present with more restrictive equilibrium conditions.

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