Limiting distributions for a polynuclear growth model with external sources

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Abstract

The purpose of this paper is to investigate the limiting distribution functions for a polynuclear growth model with two external sources, which was considered by Prähofer and Spohn in [13]. Depending on the strength of the sources, the limiting distribution functions are either the Tracy-Widom functions of random matrix theory, or a new explicit function which has the special property that its mean is zero. Moreover, we obtain transition functions between pairs of the above distribution functions in suitably scaled limits. There are also similar results for a discrete totally asymmetric exclusion process.

KEY WORDS: PNG; ASEP; directed polymer; random matrix; limiting distribution.

1 Introduction

Our main object of study is the following combinatorial problem. Fix three real parameters $t > 0$ and $\alpha_\pm \geq 0$. We construct a random set of points in the unit square $[0,1]\times[0,1] \subset \mathbb{R}^2$, as follows. Let $P(\lambda)$ denote a Poisson variable of density λ. We select $P(t^2)$ points at random inside the square $(0,1)\times(0,1)$, $P(\alpha_+ t)$ points at random on the open bottom edge $(0,1)\times\{0\}$, and $P(\alpha_- t)$ points at random on the open left edge $\{0\}\times(0,1)$. Hence no point is selected from the lower left vertex and from the closed top and right edges. For example, in Figure 1, 5 points are selected inside the square, and 2 and 1 points in the bottom and the left edges, respectively. A (weakly) up/right path is given by a sequence of points such that each point is (weakly) above and to the right of its predecessor; thus the solid line in the figure is a (weakly) up/right path from the lower left vertex to the upper right vertex. The length of a (weakly) up/right path is defined by the number of points in the path; thus the up/right path of the example has length 4.

We define $L(t)$ be the length of the longest (weakly) up/right path in this random configuration of points. (This, of course, implicitly depends on the values of $\alpha_\pm$.) E.g., the solid line in Figure 1 is the longest up/right path of the example. In general, there may be more than one longest path, but we are only interested in the

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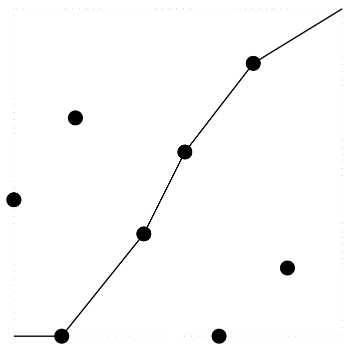

Figure 1: A points selection and the longest up/right path

length $L(t)$, hence the degeneracy is not an issue. We note that the above process can be thought of in an alternative way as a Poisson process of intensity 1 in the open $\mathbb{R}_+^2$, together with a Poisson process of intensity α_+ on the open half-line $\mathbb{R}_+ \times \{0\}$ and a Poisson process of intensity α_- on the open half-line $\{0\} \times \mathbb{R}_-$. Then $L(t)$ is equal to the length of the longest (weakly) up/right path from $(0, 0)$ to (t, t).

The main interest in this paper is the statistics of $L(t)$ as $t \to \infty$, as this problem arose in a polynuclear growth (PNG) model considered by Prähofer and Spohn in [13]. PNG is a simplified model for layer by layer growth in one spatial dimension. At each random nucleation position, an island of height 1 is formed and spreads laterally with speed 1. When two islands meet, they form one island and keep spreading with the same speed. The question is the fluctuations of the height in the large time limit.

As a special case, suppose that a single island starts spreading at the origin and that further nucleation takes place only on top of this ground layer. Furthermore, suppose that there are external nucleation sources at the two ends of the ground layer. Then the height, for example, at the origin at time $\sqrt{2}t$ is equal to $L(t)$ in the above point selection process. We refer the interested readers to the papers [13, 12] for details of the mapping from PNG to $L(t)$ and many other related works.

In the above point selection process, a special case is when $\alpha_\pm = 0$: $P(t^2)$ points are selected inside the square with no points on the edges. (This corresponds to a PNG model without external nucleation sources.) It is interesting that in this case, there is a combinatorial interpretation. Consider a random permutation of S_N. If we take N as Poisson variable $P(t^2)$, the longest increasing subsequence of a random permutation of S_N has the same statistics as $L(t)$ (see [2] and references therein). The limiting fluctuation of $L(t)$ in this case is obtained in [2] : there is a distribution function $\mathrm{F}_{\mathrm{GUE}}(x)$ such that

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{L(t) - 2t}{t^{1/3}} \le x\right) = \mathrm{F}_{\mathrm{GUE}}(x), \qquad \alpha_\pm = 0. \tag{1.1}$$

The convergences of all the moments are also proved in the same paper. Here $\mathrm{F}_{\mathrm{GUE}}$ is the so-called *GUE Tracy-Widom distribution function* of random matrix theory. See Section 2 for the explicit formula of $\mathrm{F}_{\mathrm{GUE}}$, equation (2.9).

The "GUE" in $\mathrm{F}_{\mathrm{GUE}}$ refers to the Gaussian unitary ensemble, the set of all $N \times N$ Hermitian matrices together with the probability measure defined by

$$\frac{1}{Z_N} e^{-trM^2} dM \tag{1.2}$$

where Z_N is a normalization constant (see e.g. [10, 5]). Tracy and Widom in [14] proved that as $N \to \infty$, the properly centered and scaled largest eigenvalue of a random GUE matrix converges in distribution to $\mathrm{F_{GUE}}$. Therefore the above result (1.1) implies that in the limit, $L(t)$ with $\alpha_\pm = 0$ and the largest eigenvalue of a random GUE matrix have the same fluctuations.

In this paper, we obtain similar limiting distributions in the presence of points on the edges. When $\alpha_\pm > 0$, the longest path begins by following one of the edges for a while, then enters the square. Hence when $\alpha_\pm$ are small, the effect coming from edges is small, and we expect $L(t)$ to have the same statistics as when $\alpha_\pm = 0$, i.e. GUE fluctuation. But when $\alpha_\pm$ is large, the longest path lies mostly on one of the edges, hence we expect Gaussian fluctuation. It turns out that the critical case is when $\alpha_\pm = 1$. We need to distinguish four different cases to state the results. In each case, $\alpha_\pm$ are fixed.

- When $\alpha_\pm < 1$, we obtain $\mathrm{F_{GUE}}$ in the limit.
- When either of $\alpha_\pm$ is greater than 1, we obtain Gaussian fluctuation.
- When one of $\alpha_\pm$ is equal to 1 and the other is strictly less than 1, we obtain $\mathrm{F_{GOE}}(x)^2$. In the above definition of the GUE, if we replace Hermitian matrices by real symmetric matrices, we obtain the Gaussian orthogonal ensemble (GOE). The limiting distribution of the (properly centered and scaled) largest eigenvalue of a random GOE matrix is given by $\mathrm{F_{GOE}}$ in [15] (see Section 2 for the explicit formula). The above limit $\mathrm{F_{GOE}}(x)^2$ can be interpreted as follows : take two random GOE matrices and superimpose their eigenvalues. The largest of the superimposition of eigenvalues has limiting fluctuation ${\mathrm{F_{GOE}}}^2$.
- When $\alpha_\pm = 1$, we have a new limiting distribution which we denote $\mathrm{F_0}$, for which we do not yet have a random matrix interpretation. See Section 2 for the definition and discussions.

More explicitly, we have the following theorem.

Theorem 1.1. *For each fixed $\alpha_\pm$, as $t \to \infty$, we have the following results.*

(i). When $0 \le \alpha_\pm \le 1$,

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{L(t) - 2t}{t^{1/3}} \le x \right) = \begin{cases} \mathrm{F_{GUE}}(x), & 0 \le \alpha_\pm < 1, \\ \mathrm{F_{GOE}}(x)^2, & \alpha_+ = 1, 0 \le \alpha_- < 1, \textit{ or } \alpha_- = 1, 0 \le \alpha_+ < 1, \\ \mathrm{F_0}(x), & \alpha_\pm = 1. \end{cases} \tag{1.3}$$

(ii). When at least one of $\alpha_\pm$ is greater than 1, setting $\alpha = \max\{\alpha_+, \alpha_-\}$,

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{L(t) - (\alpha + \alpha^{-1})t}{\sqrt{\alpha - \alpha^{-1}}\, t^{1/2}} \le x \right) = \begin{cases} \mathrm{erf}(x), & \alpha_+ \ne \alpha_- \\ \mathrm{erf}(x)^2, & \alpha_+ = \alpha_-. \end{cases} \tag{1.4}$$

Remark. As α tends to ∞, $L(t)$ tends to a variable with mean and variance αt; i.e., to $P(\alpha t)$, corresponding to the number of points on the interval $(0, 1)$. For finite α, the longest path leaves the interval near $1 - \alpha^{-2}$, then includes roughly $2\alpha^{-1}t$ points inside the square, thus giving $L(t) \sim P((\alpha - \alpha^{-1})t) + 2\alpha^{-1}t$, agreeing with (1.4).

The functions, $\mathrm{F}_{\mathrm{GUE}}$, $\mathrm{F}_{\mathrm{GOE}}$ and F_0, in the above theorem are defined in Section 2, and we discuss their properties in the same section.

The above theorem shows that there are certain transitions around the points $\alpha_\pm = 1$. It is of interest to investigate these transitions in detail. In Section 3, under proper scalings of $\alpha_\pm \to 1$, we obtain new classes of distribution functions interpolating the functions of the above theorem.

In addition to the above point selection process, there is a closely related lattice directed polymer model in 2-dimensional space, which is a generalization of the model considered by Johansson in [9]. This model and a related exclusion process are discussed in Section 4.

Finally the proofs of the theorems are discussed in Section 5.

Notations. In many papers, the functions $\mathrm{F}_{\mathrm{GUE}}$ and $\mathrm{F}_{\mathrm{GOE}}$ are denoted by F_2 and F_1, respectively.

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2 Limiting distribution functions

Let $u(x)$ be the solution to the Painlevé II (PII) equation,

$$u_{xx} = 2u^3 + xu, \tag{2.1}$$

with the boundary condition

$$u(x) \sim -Ai(x) \quad \text{as} \quad x \to +\infty, \tag{2.2}$$

where Ai is the Airy function. The proofs of the (global) existence and the uniqueness of the solution were established in [8]. The asymptotics as $x \to -\infty$ are given by (see e.g. [8, 7])

$$u(x) = -Ai(x) + O\Big(\frac{e^{-(4/3)x^{3/2}}}{x^{1/4}}\Big), \qquad \text{as } x \to +\infty, \tag{2.3}$$

$$u(x) = -\sqrt{\frac{-x}{2}}\Big(1 + O\big(\frac{1}{x^2}\big)\Big), \qquad \text{as } x \to -\infty. \tag{2.4}$$

Recall that $Ai(x) \sim \frac{e^{-(2/3)x^{3/2}}}{2\sqrt{\pi}x^{1/4}}$ as $x \to +\infty$. Define

$$v(x) := \int_\infty^x (u(s))^2 ds, \tag{2.5}$$

so that $v'(x) = (u(x))^2$. We note another expression

$$v(x) = u(x)^4 + xu(x)^2 - (u'(x))^2 \tag{2.6}$$

which can be obtained by noting that the difference (i) has derivatives equal to zero by the PII equation, and (ii) becomes zero as $x \to \infty$ by (2.3).

The Tracy-Widom distribution functions are defined in terms of u and v.

Definition 1 (TW distribution functions). Set

$$F(x) := \exp\Big(\frac{1}{2}\int_x^\infty v(s)ds\Big) = \exp\Big(-\frac{1}{2}\int_x^\infty (s-x)(u(s))^2 ds\Big), \tag{2.7}$$

$$E(x) := \exp\Big(\frac{1}{2}\int_x^\infty u(s)ds\Big), \tag{2.8}$$

and set

$$\mathrm{F}_{\mathrm{GUE}}(x) := F(x)^2 = \exp\Big(-\int_x^\infty (s-x)(u(s))^2 ds\Big), \tag{2.9}$$

$$\mathrm{F}_{\mathrm{GOE}}(x) := F(x)E(x) = \big(F_2(x)\big)^{1/2} e^{\frac{1}{2}\int_x^\infty u(s)ds}, \tag{2.10}$$

$$\mathrm{F}_{\mathrm{GSE}}(x) := F(x)\big\{E(x)^{-1} + E(x)\big\}/2 = \big(F_2(x)\big)^{1/2}\Big[e^{-\frac{1}{2}\int_x^\infty u(s)ds} + e^{\frac{1}{2}\int_x^\infty u(s)ds}\Big]/2. \tag{2.11}$$

It is proved by Tracy and Widom in [14, 15] that under proper centering and scaling, the distribution of the largest eigenvalue of a random GUE/GOE/GSE matrix converges to $\mathrm{F}_{\mathrm{GUE}}(x)$ / $\mathrm{F}_{\mathrm{GOE}}(x)$ / $\mathrm{F}_{\mathrm{GSE}}(x)$ as the size of the matrix becomes large. The readers are referred to [10, 5] for definitions of various random matrix ensembles and their basic properties. We note that from the asymptotics (2.3) and (2.4), for some positive constant c,

$$F(x) = 1 + O\big(e^{-cx^{3/2}}\big) \qquad \text{as } x \to +\infty, \tag{2.12}$$

$$E(x) = 1 + O\big(e^{-cx^{3/2}}\big) \qquad \text{as } x \to +\infty, \tag{2.13}$$

$$F(x) = O\big(e^{-c|x|^3}\big) \qquad \text{as } x \to -\infty, \tag{2.14}$$

$$E(x) = O\big(e^{-c|x|^{3/2}}\big) \qquad \text{as } x \to -\infty. \tag{2.15}$$

Hence in particular, as $x \to +\infty$, all the above three functions become 1, and as $x \to -\infty$, they become 0. Monotonicity follows from the fact that they are limits of sequences of distribution functions, and therefore (2.9)-(2.11) are indeed distribution functions.

We need a new distribution function for the case when $\alpha_\pm = 1$ in Theorem 1.1.

Definition 2. Set

$$\mathrm{F}_0(x) = \big\{1 - (x + 2u'(x) + 2u(x)^2)v(x)\big\}\big(E(x)\big)^4\,\mathrm{F}_{\mathrm{GUE}}(x). \tag{2.16}$$

The asymptotics (2.3)-(2.5) imply that $\mathrm{F}_0(x)$ has limit 1 as $x \to +\infty$ and 0 as $x \to -\infty$. The monotonicity of F_0 follows from the fact that it is a limit of distribution functions in Theorem 1.1. It would of interest to have random matrix interpretation of the function F_0, but so far we have been unable to identity F_0 as a quantity arising in random matrix theory.

One special property of this distribution function is that it has mean zero.

Proposition 2.1. *We have*

$$\int_{-\infty}^{\infty} x d\,\mathrm{F}_0(x) = 0. \tag{2.17}$$

Proof. We note that the term in front of $v(x)$ in the definition of F_0 has another expression :

$$x + 2u'(x) + 2u(x)^2 = E(x)^{-4} \int_{-\infty}^{x} E(t)^4 dt. \tag{2.18}$$

This follows by noting that $y(x) := x + 2u' + 2u^2$ satisfies

$$y'(x) = 1 + 2u(x)y(x), \qquad y(x) = \frac{1}{\sqrt{-2x}}(1 + o(1)), \quad x \to -\infty. \tag{2.19}$$

Then we have

$$\mathrm{F}_0(x) = \frac{d}{dx}\Big\{ \mathrm{F}_{\mathrm{GUE}}(x) \int_{-\infty}^{x} E(t)^4 dt \Big\}, \tag{2.20}$$

which upon integrating gives

$$\int_{-\infty}^{x} \mathrm{F}_0(t)dt = \mathrm{F}_{\mathrm{GUE}}(x) \int_{-\infty}^{x} E(t)^4 dt = \mathrm{F}_{\mathrm{GUE}}(x)E(x)^4\big(x + 2u'(x) + 2u(x)^2\big). \tag{2.21}$$

Since $\int_{-\infty}^{\infty} x d\,\mathrm{F}_0(x) = \lim_{x\to\infty}\big[x\,\mathrm{F}_0(x) - \int_{-\infty}^{x} \mathrm{F}_0(y)dy\big]$ from integration by parts, subtracting (2.21) from $x\,\mathrm{F}_0(x)$ and taking the limit $x \to \infty$, we find that F_0 has mean 0, as required. □

Remark. The mean zero property of $\mathrm{F}_0(x)$ was suggested in [13] by numerical computation. Moreover, by an indirect argument for PNG with $\alpha_\pm = 1$, it is shown that the average of $L(t)$ is exactly $2t$ (see [11]), which implies that F_0 has mean zero. We note that the means of $\mathrm{F}_{\mathrm{GUE}}$ and $\mathrm{F}_{\mathrm{GOE}}$ are $-1.77109\cdots$ and $-0.76007\cdots$, respectively.

3 Around the transition : $\alpha_\pm \to 1$

In this section, we investigate the transition around $\alpha_\pm = 1$ in detail.

To state the results, Theorem 3.3, we need some preliminary definitions. Let Γ be the real line $\mathbb{R}$, oriented from $+\infty$ to $-\infty$. Let $m(\cdot; x)$ be the solution to the Painlevé II Riemann-Hilbert problem :

$$\begin{cases} m(z;x) \qquad \text{is analytic in } z \in \mathbb{C} \setminus \Gamma, \\ m_+(z;x) = m_-(z;x) \begin{pmatrix} 1 & -e^{-2i(\frac{4}{3}z^3 + xz)} \\ e^{2i(\frac{4}{3}z^3 + xz)} & 0 \end{pmatrix} \quad \text{for } z \in \Gamma, \\ m(z;x) = I + O\big(\frac{1}{z}\big) \qquad \text{as } z \to \infty. \end{cases} \tag{3.1}$$

Here $m_+(z;x)$ (resp., m_-) is the limit of $m(z';x)$ as $z' \to z$ from the left (resp., right) of the contour Γ : $m_\pm(z;x) = \lim_{\epsilon\downarrow 0} m(z \mp i\epsilon; x)$. The relation between the above Riemann-Hilbert problem and the Painlevé II equation is the following (see, e.g., [7]). If we expand

$$m(z;x) = I + \frac{m_1(x)}{z} + O\big(\frac{1}{z^2}\big), \qquad \text{as } z \to \infty, \tag{3.2}$$

we have

$$
\begin{aligned}
2i(m_1(x))_{12} = -2i(m_1(x))_{21} &= u(x), &(3.3)\\
2i(m_1(x))_{22} = -2i(m_1(x))_{11} &= v(x), &(3.4)
\end{aligned}
$$

where $u(x)$ and $v(x)$ are defined in (2.1)-(2.5). The above Riemann-Hilbert problem is the special case of monodromy data $p = -q = 1$, $r = 0$ in the standard family of Painlevé II Riemann-Hilbert problems.

Define

$$
a(x,w) = \begin{cases} m_{22}(-iw;x) & w > 0, \\ -m_{21}(-iw;x)e^{\frac{8}{3}w^3-2xw} & w < 0, \end{cases} \tag{3.5}
$$

$$
b(x,w) = \begin{cases} m_{12}(-iw;x) & w > 0, \\ -m_{11}(-iw;x)e^{\frac{8}{3}w^3-2xw} & w < 0. \end{cases} \tag{3.6}
$$

From the jump condition of the Riemann-Hilbert problem (3.1), $a(x,w)$ and $b(x,w)$ are continuous at $w = 0$. Indeed since m_-v in the upper half plane of $\mathbb{C}$ is an analytic continuation of m_+ in the lower half plane of $\mathbb{C}$, $a(x,w)$ and $b(x,w)$ are analytic in w. We have the following properties of a, b.

Lemma 3.1. *For all $x, w \in \mathbb{R}$, we have :*

(i). $a(x,w), b(x,w)$ are real.

(ii). For each fixed $w \in \mathbb{R}$,

$$
\begin{aligned}
a(x,w) &= I + O(e^{-cx^{3/2}}), & x \to +\infty, &\quad (3.7)\\
b(x,w) &= -e^{\frac{8}{3}w^3-2xw}\big(I + O(e^{-cx^{3/2}})\big), & x \to +\infty, &\quad (3.8)\\
a(x,w) &\sim \frac{1}{\sqrt{2}}e^{\frac{4}{3}w^3-\frac{\sqrt{2}}{3}|x|^{3/2}+|x|w-\sqrt{2}w^2|x|^{1/2}} & x \to -\infty, &\quad (3.9)\\
b(x,w) &\sim -\frac{1}{\sqrt{2}}e^{\frac{4}{3}w^3-\frac{\sqrt{2}}{3}|x|^{3/2}+|x|w-\sqrt{2}w^2|x|^{1/2}} & x \to -\infty. &\quad (3.10)
\end{aligned}
$$

(iii).

$$
\begin{aligned}
\lim_{w\to+\infty} a(x,w) &= 1, & \lim_{w\to+\infty} b(x,w) &= 0, &(3.11)\\
\lim_{w\to-\infty} a(x,w) &= 0, & \lim_{w\to-\infty} b(x,w) &= 0, &(3.12)\\
a(x,0) &= E(x)^2, & b(x,0) &= -E(x)^2. &(3.13)
\end{aligned}
$$

(iv).

$$
\begin{aligned}
\frac{\partial}{\partial x}a(x,w) &= u(x)b(x,w), &(3.14)\\
\frac{\partial}{\partial x}b(x,w) &= u(x)a(x,w) - 2wb(x,w), &(3.15)\\
\frac{\partial}{\partial w}a(x,w) &= 2(u(x))^2a(x,w) - \big(4wu(x) + 2u'(x)\big)b(x,w), &(3.16)\\
\frac{\partial}{\partial w}b(x,w) &= \big(-4wu(x) + 2u'(x)\big)a(x,w) + \big(8w^2 - 2x - 2(u(x))^2\big)b(x,w). &(3.17)
\end{aligned}
$$

(v).

$$a(x,w) = -b(x,-w)e^{\frac{8}{3}w^3-2xw}, \tag{3.18}$$
$$b(x,w) = -a(x,-w)e^{\frac{8}{3}w^3-2xw}. \tag{3.19}$$

(vi). For each fixed $y \in \mathbb{R}$, as $w \to -\infty$,

$$a(2y\sqrt{|w|}+4w^2, w) \to \operatorname{erf}(y), \tag{3.20}$$
$$b(2y\sqrt{|w|}+4w^2, w) \sim -e^{\frac{16}{3}|w|^3+4y|w|^{3/2}}. \tag{3.21}$$

Proof. The properties (i)-(v) are consequences of (3.1) and Lemma 2.1 in [4]. The result (vi) is obtained by applying the Deift-Zhou method to the Riemann-Hilbert problem (3.1). The specialty of the scaling $x = 2y\sqrt{|w|}+4w^2$ is related to the fact that the exponent term $\frac{4}{3}z^3 + xz$ of the anti-diagonal entry in the jump matrix of the Riemann-Hilbert problem (3.1) has the critical points at $z = \pm i\frac{\sqrt{x}}{2}$, which are the stationary phase points in the asymptotic analysis (see [7]). Recalling $z = -iw$, we see that $x = 4w^2$ corresponds to one of these stationary phase points. By analyzing $m(-iw;x)$ around this stationary phase point, we obtain $m_{12}(-iw;x)e^{\frac{8}{3}w^3-2xw} \sim -\operatorname{erf}(y)$ and $m_{11}(-iw;x) \sim 1$ as $w \to -\infty$ with the above x. Similar computation appeared in Section 10.3 of [4], and we omit the detailed computations here. □

We now define the following functions with parameters w_+, w_-, w.

Definition 3. For each $w_+, w_- \in \mathbb{R}$, when $w_+ + w_- \neq 0$, set

$$H(x;w_+,w_-) = \left\{a(x,w_+)a(x,w_-) - \frac{a(x,w_+)a(x,w_-) - b(x,w_+)b(x,w_-)}{2(w_+ + w_-)}v(x)\right\}\mathrm{F}_{\mathrm{GUE}}(x). \tag{3.22}$$

When $w_+ + w_- = 0$, we use the l'Hopital's rule (note Lemma 3.1 (v)). Also using Lemma 3.1 (iii), set

$$G(x;w) = \lim_{w_- \to +\infty} H(x;w,w_-) = a(x,w)\,\mathrm{F}_{\mathrm{GUE}}(x). \tag{3.23}$$

From Lemma 3.1 (ii), H and G have the limit 1 as $x \to \infty$ and has the limit 0 as $x \to -\infty$ for each fixed w_+, w_-, w. Also theorem 3.3 below shows that they are limits of distribution functions. Therefore H and G are distribution functions. These distribution functions interpolate between the functions $\mathrm{F}_{\mathrm{GUE}}$, ${\mathrm{F}_{\mathrm{GOE}}}^2$ and F_0 of theorem 1.1. The following results follow from Lemma 3.1 (iii), (vi).

Proposition 3.2. *For fixed $x \in \mathbb{R}$, we have*

$$H(x;w_+,w_-) \to \begin{cases} \mathrm{F}_{\mathrm{GUE}}(x), & w_+, w_- \to +\infty \\ \mathrm{F}_{\mathrm{GOE}}(x)^2, & w_+ = 0, w_- \to +\infty \text{ or } w_- = 0, w_+ \to +\infty \\ \mathrm{F}_0, & w_+ = w_- = 0 \\ 0, & w_+ \text{ or } w_- \to -\infty, \end{cases} \tag{3.24}$$

and

$$G(x;w)\to\begin{cases}\mathrm{F}_{\mathrm{GUE}}(x), & w\to+\infty\\ \mathrm{F}_{\mathrm{GOE}}(x)^2, & w=0\\ 0, & w\to-\infty.\end{cases}\tag{3.25}$$

Also we have for fixed $x\in\mathbb{R}$, with $w=-\max\{-w_+,-w_-\}$, as $w\to-\infty$,

$$H(2x\sqrt{|w|}+4w^2;w_+,w_-)\quad\to\quad\begin{cases}\mathrm{erf}(x), & w_+\neq w_-\\ \mathrm{erf}^2(x), & w_+=w_-,\end{cases}\tag{3.26}$$

$$G(2x\sqrt{|w|}+4w^2;w)\quad\to\quad\mathrm{erf}(x).\tag{3.27}$$

Now the main theorem in this section is that if we take proper scaling of $\alpha_\pm\to 1$ in t, we obtain the above functions in the limit.

Theorem 3.3. *Set $w_\pm$ by*

$$\alpha_\pm=1-\frac{2w_\pm}{t^{1/3}}.\tag{3.28}$$

We have the followings.

(i). When $0\le\alpha_+<1$ and $w_-\in\mathbb{R}$ are fixed,

$$\lim_{t\to\infty}\mathbb{P}\Big(\frac{L(t)-2t}{t^{1/3}}\le x\Big)=G(x;w_-).\tag{3.29}$$

When $w_+\in\mathbb{R}$ and $0\le\alpha_-<1$ are fixed, the limit is $G(x;w_+)$.

(ii). When $w_\pm\in\mathbb{R}$ are fixed, as $t\to\infty$,

$$\lim_{t\to\infty}\mathbb{P}\Big(\frac{L(t)-2t}{t^{1/3}}\le x\Big)=H(x;w_+,w_-).\tag{3.30}$$

A special case of (ii) is when $w_+=-w_-$, which corresponds to $\alpha_+\alpha_-=1$. In this case, the limiting shape of PNG has curvature 0 (see [13]). Hence we obtain a one-parameter family of distribution functions for fluctuations of a flat curvature PNG. We have the exact values of the means which include Proposition 2.1 as a special case when $w=0$.

Proposition 3.4. *We have for each $w\in\mathbb{R}$,*

$$\int_{-\infty}^{\infty}x\,dH(x;w,-w)=4w^2.\tag{3.31}$$

Proof. By l'Hopital's rule and Lemma 3.1 (iv), (v), we have

$$H(x;w,-w)=\big\{a(x;w)a(x;-w)-y(x)v(x)\big\}\,\mathrm{F}_{\mathrm{GUE}}(x),\tag{3.32}$$

where

$$y(x) = (2u^2 + x - 4w^2)a(x;w)a(x;-w) - (u' + 2wu)b(x;w)a(x;-w) - (u' - 2wu)a(x;w)b(x;-w). \tag{3.33}$$

Then by Lemma 3.1 (iv), (v) again, we obtain

$$y'(x) = a(x;w)a(x;-w), \tag{3.34}$$

which implies that

$$H(x;w,-w) = \{y'(x) - y(x)v(x)\} \, \mathrm{F_{GUE}}(x) = \frac{d}{dx}\Big\{y(x) \, \mathrm{F_{GUE}}(x)\Big\}. \tag{3.35}$$

Hence as in the proof of Proposition 2.1,

$$\lim_{x\to\infty}\left[xH(x;w,-w) - \int_{-\infty}^{x} H(y;w,-w)dy\right] = \lim_{x\to\infty}(xy' - xvy - y) \, \mathrm{F_{GUE}}(x) = 4w^2, \tag{3.36}$$

using the asymptotics of u, v, a, b. □

Remark. The function $G(x;w)$ appeared as $F^{\boxtimes}(x;w)$ in (2.21), (2.22) of [4]. Indeed this function was obtained in a different point selection process. Namely, in the open square $(0,1)\times(0,1)$, let $\delta = \{(t,t) : 0 < t < 1\}$, the diagonal line, and let $\delta^t = \{(t, 1-t) : 0 < t < 1\}$, the anti-diagonal line. We select $4P(t^2)$ points in $(0,1)\times(0,1)\setminus(\delta\cup\delta^t)$, $2P(\alpha t)$ points on $\delta\setminus(\frac{1}{2},\frac{1}{2})$, and $2P(\beta t)$ points on $\delta^t\setminus(\frac{1}{2},\frac{1}{2})$ such that the resulting point configuration is symmetric with respect to both δ and δ^t. Let $L^{\boxtimes}(t;\alpha,\beta)$ be the length of the longest up/right path of a random point configuration in this point selection process. In [4], it is proved that for *any* fixed $\beta \geq 0$,

$$\mathbb{P}\left(\frac{L^{\boxtimes}(t;\alpha,\beta) - 4t}{2t^{1/3}} \leq x\right) \quad\to\quad \begin{cases} \mathrm{F_{GUE}}(x), & 0 \leq \alpha < 1, \\ \mathrm{F_{GOE}}(x)^2, & \alpha = 1, \end{cases} \tag{3.37}$$

$$\mathbb{P}\left(\frac{L^{\boxtimes}(t;\alpha,\beta) - 2(\alpha+\alpha^{-1})t}{\sqrt{2(\alpha-\alpha^{-1})}t^{1/2}} \leq x\right) \quad\to\quad \mathrm{erf}(x), \qquad \alpha > 1, \tag{3.38}$$

$$\mathbb{P}\left(\frac{L^{\boxtimes}(t;\alpha,\beta) - 4t}{2t^{1/3}} \leq x\right) \quad\to\quad G(x;w), \qquad \alpha = 1 - \frac{2w}{t^{1/3}} \tag{3.39}$$

This result is identical to the limits of $L(t)$ when $\alpha_- < 1$ is fixed. Comparing this with theorems 1.1 and 3.3, we see that after scaling, $L^{\boxtimes}(t;\alpha,\beta)$ with any fixed β and $L(t)$ with any fixed $\alpha_- < 1$ have the same statistics in the limit $t\to\infty$. Indeed, one can prove more than that. In finite t, $L(2t)$ with $\alpha_- = 0$ and $L^{\boxtimes}(t;\alpha_+,0)$ are the same (see Remark at the end of Step 1. in Section 5.)

4 Lattice directed polymer and exclusion process

In this section, we analyze a certain lattice directed polymer problem which is closely related to the point selection model discussed above.

Let $\mathbb{N} = \{1,2,3,\cdots\}$ and $\mathbb{N}^* = \{0,1,2,\cdots\}$. Let $0 < q < 1$, $\alpha_\pm \geq 0$ be fixed numbers such that $\alpha_\pm\sqrt{q} < 1$. We denote by $g(q)$ the geometric distribution with parameter q :

$$\mathbb{P}(g(q) = k) = (1-q)q^k, \qquad k = 0,1,2,\cdots. \tag{4.1}$$

When a random variable w has distribution $g(q)$, we use the notation $w \sim g(q)$. At each site $(i,j) \in \mathbb{N}^* \times \mathbb{N}^*$, we attach a random variable $w(i,j)$ where

$$\begin{aligned} w(i,j) &\sim g(q), \qquad (i,j) \in \mathbb{N} \times \mathbb{N}, && (4.2) \\ w(i,0) &\sim g(\alpha_+ \sqrt{q}), \qquad i \in \mathbb{N}, && (4.3) \\ w(0,j) &\sim g(\alpha_- \sqrt{q}), \qquad j \in \mathbb{N}, && (4.4) \\ w(0,0) &= 0. && (4.5) \end{aligned}$$

We call a collection π of sites in $\mathbb{N}^* \times \mathbb{N}^*$ an up/right path if when $(i,j) \in \pi$, either $(i+1,j) \in \pi$, or $(i,j+1) \in \pi$. Let $Path(N)$ be the set of all up/right paths from $(0,0)$ to (N,N). Define

$$X(N) = \max\{ \sum_{(i,j)\in\pi} w(i,j) : \pi \in Path(N)\}. \tag{4.6}$$

The special case $\alpha_+ = \alpha_- = 0$ was introduced by Johansson in [9]; the above model adds a special row and column to his model. In [9], it is proved that

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{X(N) - \mu(q)N}{\sigma(q)N^{1/3}} \le x \right) = \mathrm{F}_{\mathrm{GUE}}(x), \qquad \alpha_+ = \alpha_- = 0, \tag{4.7}$$

where

$$\mu(q) = \frac{2\sqrt{q}}{1-\sqrt{q}}, \qquad \sigma(q) = \frac{q^{1/6}(1+\sqrt{q})^{1/3}}{1-\sqrt{q}}. \tag{4.8}$$

It is shown in [9] that this directed polymer model can be interpreted as a growth model in 2-dimensional space, or as a discrete exclusion process. The corresponding exclusion process in our case is the following. We use the notation $+$ for the location of a particle. If a site is vacant, we use the notation $-$. Initially there are particles at the sites $\{\cdots, -4, -3, -2\} \cup \{0\} \subset \mathbb{Z}$. Hence the initial configuration on $\mathbb{Z}$ can be written as $(\cdots, +, +, +, -, +, -, -, -, \cdots)$ where the leftmost $-$ is at the site -1 and the rightmost $+$ is at the site 0. At each (discrete) time, the rightmost particle jumps to its right site with probability $1 - \alpha_+\sqrt{q}$, and the leftmost *hole* jumps to its left with probability $1 - \alpha_-\sqrt{q}$, while in the 'bulk', each particle jumps to its right (equivalently, a hole jumps to its left) with probability $1 - q$ if its right site is vacant.

As in the point selection model, when $\alpha_\pm$ are small enough, $X(N)$ would have the limiting distribution $\mathrm{F}_{\mathrm{GUE}}$ as in the case when $\alpha_\pm = 0$. For general $\alpha_\pm$, we obtain results parallel to those in Theorems 1.1 and 3.3. After the following changes, we obtain the same limiting results as in Theorems 1.1 and 3.3.

(i). Every limit is taken as $N \to \infty$.

(ii). The scaled random variable in Theorem 1.1 (i) and Theorem 3.3 is now

$$\frac{X(N) - \mu(q)N}{\sigma(q)N^{1/3}} \tag{4.9}$$

where $\mu(q)$ and $\sigma(q)$ are defined in (4.8).

(iii). The scaling of $\alpha_\pm$ in Theorem 3.3 is now

$$\alpha_\pm = 1 - \frac{2w_\pm}{\sigma(q)N^{1/3}}. \tag{4.10}$$

(iv). The scaling in Theorem 1.1 (ii) is now

$$\frac{X(N) - \eta(q,\alpha)N}{\rho(q,\alpha)\sqrt{N}}, \qquad 1 < \alpha < \frac{1}{\sqrt{q}}, \tag{4.11}$$

where

$$\eta(q,\alpha) = \frac{\sqrt{q}(\alpha + \alpha^{-1} - 2\sqrt{q})}{(1-\sqrt{q}\alpha)(1-\sqrt{q}\alpha^{-1})}, \qquad \rho(q,\alpha) = \frac{\sqrt{q}(\alpha - \alpha^{-1})^{1/2}(\sqrt{q}^{-1} - \sqrt{q})^{1/2}}{(1-\sqrt{q}\alpha)(1-\sqrt{q}\alpha^{-1})}. \tag{4.12}$$

We note that $\eta(q,\alpha) > \mu(q)$ for $\alpha > 1$.

Remark 1. In the above model (also in [4]), only the paths ending at the diagonal point (N,N) are considered. It is of interest to obtain similar results for path ending at general point (M,N). The difficulty in this general case comes from the fact that we need asymptotics of orthogonal polynomials with respect to the weight function $(1+\sqrt{q}z)^M(1+\sqrt{q}z^{-1})^N$ (see Step 2. of Section 5). This weight function is not real for $|z|=1$, which makes the Riemann-Hilbert method (which we employed to obtain asymptotics) more difficult to analyze. We are planning to come back to this problem in the future.

We note that in [9], Johansson was able to obtain results for this general case when $\alpha_+ = \alpha_- = 0$. He have used different determinant expression (of Fredholm type rather than Toeplitz type) involving different orthogonal polynomials (which is discrete), and did not involve the non-real weight function.

Remark 2. If we take the limit as $q \to 1$, we obtain exponential random variables instead of geometric distribution (see [9]). In order to investigate the exponential random variables case, we set $q = 1 - \frac{1}{L}$ and $l = xL$, and let $L \to \infty$ in the determinantal formula (5.9)-(5.10). This double scaling limit is not carried out yet, which we plan to do in the future.

Remark 3. If we take the limit as $q \to 0$, we have rare events, hence we obtain the Poisson process discussed above. Indeed, by setting $\sqrt{q} = \frac{t}{N}$, and letting $N \to \infty$, we recover the Theorems 1.1 and 3.3 from (4.9)-(4.11).

5 Proofs

In this section, we sketch how one can obtain Theorems 1.1 and 3.3 using the results of [3, 4]. We split the proof into three steps.

Step 1. We will prove the following formula :

$$\mathbb{P}(L(t) \le l) = e^{-(\alpha_+ + \alpha_-)t - t^2}(D'_l - \alpha_+\alpha_- D'_{l-1}) \tag{5.1}$$

where

$$D'_l = \mathbb{E}_{U \in U(l)} \det(1 + \alpha_+ U)(1 + \alpha_- U^\dagger) e^{2t \operatorname{Re}\operatorname{Tr}(U)}. \tag{5.2}$$

Suppose $\alpha_+\alpha_- < 1$. In the lattice directed polymer model discussed in Section 4, add a random variable $w(0,0) \sim g(\alpha_+\alpha_-)$ at the site $(0,0)$. Let $X^+(N)$ be defined by the formula (4.6) with this new random variable added. Now this is a special case of the model $\square$ of (7.5)-(7.7) discussed in Section 7 of [3] : take $W = W' = \{0, 1, \cdots, N\} \subset \mathbb{N}^*$, and take $q_0 = \alpha_-$, $q'_0 = \alpha_+$ and $q_j = q'_j = \sqrt{q}$ for $1 \leq j \leq N$ (in [3], we have taken $W, W' \subset \mathbb{N}$, but simple translation makes no change.) From Theorem 7.1 (7.30) of [3], we obtain

$$\begin{aligned}\mathbb{P}(X^+(N) \leq l) =&(1-\alpha_+\alpha_-)(1-\alpha_+\sqrt{q})^N(1-\alpha_-\sqrt{q})^N(1-q)^{N^2} \\ &\cdot \mathbb{E}_{U\in U(l)} \det\big\{(1+\alpha_+U)(1+\alpha_-U^\dagger)(1+\sqrt{q}U)^N(1+\sqrt{q}U^\dagger)^N\big\}.\end{aligned} \tag{5.3}$$

Now we set $\sqrt{q} = \frac{t}{N}$, and take the limit $N \to \infty$. Then inside the square $\{1, \cdots, N\}^2$, we obtain a Poisson process of parameter t^2 : $\mathbb{P}(w = 0) = 1 - \frac{t^2}{N^2}$, $\mathbb{P}(w = 1) = (1 - \frac{t^2}{N^2})\frac{t^2}{N^2}$ and $\mathbb{P}(w \geq 2) = \frac{t^4}{N^4}$. The probability that there are k points in the square $\{1, \cdots, N\}^2$ is equal to

$$\binom{N}{k}\left(1-\frac{t^2}{N^2}\right)^{N-k}\left\{(1-\frac{t^2}{N^2})\frac{t^2}{N^2}\right\}^k \to \frac{e^{-t^2}(t^2)^k}{k!}, \tag{5.4}$$

disregarding the events of having more than one points at one site which has probability 0 in the limit. Similarly we obtain Poisson process with parameter α_+t on the bottom edge (not including the origin), and Poisson process with parameter α_-t on the left edge (not including the origin). At the origin, we have a geometric distribution with parameter $\alpha_+\alpha_-$. Let $L^+(t)$ be the length of the longest up/right path in this process. This is related to $L(t)$ by

$$L^+(t) = L(t) + \chi, \qquad \chi \sim g(\alpha_+\alpha_-), \tag{5.5}$$

since any up/right path will include the $g(\alpha_+\alpha_-)$ points in the lower-left corner. From (5.3), we obtain

$$\mathbb{P}(L^+(t) \leq l) = (1-\alpha_+\alpha_-)e^{-(\alpha_++\alpha_-)t-t^2}D'_l \tag{5.6}$$

where D'_l is defined in (5.1).

In order to obtain the formula for $L(t)$, set

$$Q(x) = \sum_{l\geq 0} \mathbb{P}(L(t) \leq l)x^l, \qquad Q^+(x) = \sum_{l\geq 0} \mathbb{P}(L^+(t) \leq l)x^l. \tag{5.7}$$

Then using (5.5), we obtain

$$Q^+(x) = \sum_{0\leq k\leq l} \mathbb{P}(L(t) \leq l-k)\,\mathbb{P}(\chi = k) = (1-\alpha_+\alpha_-)(1-\alpha_+\alpha_-x)^{-1}Q(x). \tag{5.8}$$

Thus by comparing the coefficients, we have (5.1) for $\alpha_+\alpha < 1$.

Observe that the first quantity in the right-hand side in (5.1) is entire in α_+, α_-, and the second quantity is polynomial in α_+, α_-, hence is entire. Since both sides of (5.1) agree analytically for $\alpha_+\alpha_- < 1$ and the right-hand side is entire, they agree in general $\alpha_\pm$ where they both converge and are defined. Thus (5.1) holds for $0 \leq \alpha_\pm$. Similar consideration yields the following formulae for the probability of the lattice directed polymer problem :

$$\mathbb{P}(X(N) \leq l) = (1-\alpha_+\sqrt{q})^N(1-\alpha_-\sqrt{q})^N(1-q)^{N^2}(T'_l - \alpha_+\alpha_-T'_{l-1}) \tag{5.9}$$

where

$$T'_l = E_{U\in U(l)} \det\{(1+\alpha_+U)(1+\alpha_-U^\dagger)(1+\sqrt{q}U)^N(1+\sqrt{q}U^\dagger)^N\}, \tag{5.10}$$

and $\alpha_\pm$ are subject to the constraint $0 \le \alpha_\pm < 1/\sqrt{q}$.

Remark. It is the formula (5.1) that makes connection with the process ⊠ mentioned in Remark of Section 3. In [3] (4.16), it is proved that $\mathbb{P}(L^{\boxtimes}(t;\alpha,0) \le 2l)$ is given by (5.1) with $\alpha_+ = \alpha$ and $\alpha_- = 0$. It is not clear why these two processes should be the same.

Step 2. Let

$$D_l = E_{U\in U(l)} e^{2t \operatorname{Re}\operatorname{Tr}(U)}, \tag{5.11}$$

which is in another form, the $l\times l$ Toeplitz determinant $\det(c_{j-k})_{0\le j,j<l}$ where c_j is the Fourier coefficient of $e^{2t\cos\theta}$. In [2], the asymptotics of D_l in the double scaling limit as $t, l \to \infty$ was studied; hence the second step in the proof of Theorem 1.1 is to eliminate the term $\det(1+\alpha_+U)(1+\alpha_-U^\dagger)$ from the integrand in (5.1). This step is established in Theorem 3.2 of [3]. Let $\pi_n(z) = z^n + \cdots$ be the n^{th} monic orthogonal polynomial with respect to the weight $e^{t(z+z^{-1})}dz/(2\pi i z)$ on the unit circle $|z| = 1$ where $z \in \mathbb{C}$:

$$\int_{|z|=1} \pi_n(z)\overline{\pi_m(z)} e^{t(z+z^{-1})} \frac{dz}{2\pi i z} = \delta_{nm} N_n, \tag{5.12}$$

for some constants N_n. Note that since the weight function is real, all coefficients of π_n are real. We define

$$\pi_n^*(z) = z^n \pi(z^{-1}). \tag{5.13}$$

In Theorem 3.2 of [3], using the Weyl integration formula for $U(l)$ and familiar Vandermonde type argument together with the relations between orthogonal polynomials on the unit circle and those on the interval, it is proved that for $\alpha_+\alpha_- \ne 1$,

$$D'_l = \frac{\pi_l^*(-\alpha_+)\pi_l^*(-\alpha_-) - \alpha_+\alpha_-\pi_l(-\alpha_+)\pi_l(-\alpha_-)}{1-\alpha_+\alpha_-} D_l \tag{5.14}$$

For $\alpha_+\alpha_- = 1$, l'Hopital's rule applies implying with $\alpha_+ = \alpha$, $\alpha_- = 1/\alpha$,

$$D'_l = \big\{(1-l)\pi_l(-\alpha)\pi_l(-\alpha^{-1}) - a\pi'_l(-\alpha)\pi_l(-\alpha^{-1}) - \alpha^{-1}\pi_l(-\alpha)\pi'_l(-\alpha^{-1})\big\} D_l. \tag{5.15}$$

We obtain similar formulae for the lattice directed polymer problem: replace the weight $e^{t(z+z^{-1})}$ by $(1+\sqrt{q}z)^N(1+\sqrt{q}z^{-1})^N$.

Step 3. The remaining task is to obtain the asymptotics of D_l and $\pi_l(-\alpha)$ as $l, N \to \infty$ in a proper rate. This is obtained in [2, 4] by applying the Deift-Zhou steepest descent method (see [6, 5]) to the Riemann-Hilbert problem for the orthogonal polynomials $\pi_l(z)$. These asymptotic results are summarized in Section 5 of [4]. Theorem 1.1 follows by plugging in these asymptotics into (5.14), (5.15); we omit the calculations. There are similar asymptotic results for the lattice directed polymer problem which yield Theorem 3.3; see Proposition 3.2 and subsequent remarks in [1].

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