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EVALUATION OF THE DISTRIBUTION FUNCTION OF THE LIMITED INFORMATION MAXIMUM LIKELIHOOD ESTIMATOR

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by

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1. Introduction

The exact cumulative distribution function (cdf) of the Limited Information Maximum Likelihood (LIML) estimator of the coefficient of an endogenous variable in a single equation in a simultaneous equations system has been studied by Anderson and Sawa [1975], [1977] in the case that the covariance matrix of the reduced form is known. This estimator, which is termed the LIMLK estimator, is mathematically equivalent to the maximum likelihood estimator of the slope coefficient in a linear functional relationship when the covariance matrix is known to within a proportionality factor. Anderson and Sawa [1978] gave extensive numerical tables of the exact cdf of the LIMLK estimator; in [1977] they examined the accuracy of the approximate distributions based on the asymptotic expansion.

In practice with econometric models, however, a priori information of the covariance matrix is usually not available and one must use the LIML estimator which involves estimating the covariances. The exact density of the LIML estimator was investigated by Mariano and Sawa [1972], but their expression is too complicated to permit numerical evaluation.

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In general, it is known that the empirical cdf of an estimator from a Monte Carlo experiment approximates its exact cdf very well. In the case of the LIML estimator, the empirical cdf can be easily generated by using the canonical representation of the estimator that Anderson [1974] derived. (See Section 3 below.) This enables us to study the small sample properties of the LIML estimator satisfactorily.

The purposes of the present paper are to estimate the cdf of the LIML estimator using the empirical distribution and to compare it with the cdf of the LIMLK estimator. The exact distribution of the estimator depends on the values taken by the noncentrality parameter, a standardization of the structural coefficient, the number of excluded exogenous variables, as well as the degrees of the freedom of the reduced form. We can analyze fully the properties of the LIML estimator with systematic computation for a limited number of cases with different values of these four quantities as Anderson and Sawa [1979] did for the TSLS estimator. Thus this study makes it possible to compare the small sample properties of the LIML, LIMLK, and TSLS estimators completely.

Anderson [1974] gave an asymptotic expansion of the cdf of the LIML estimator in term of the noncentrality parameter, which turns out to be the same expression as the LIMLK estimator up to a certain order. (See Anderson [1977].) Hence we can examine the accuracy of the approximate distribution of the LIML estimator based on the asymptotic expansion by simply comparing the cdf's of the LIML and LIMLK estimators and the tables given by Anderson and Sawa [1978].

2. The Model and Estimators

We consider the distribution of the LIML estimator of the structural parameter in an equation

$$y_1 = \beta y_2 + Z_1\gamma_1 + u \ , \tag{2.1}$$

where y_1 and y_2 are T-component column vectors of observations on two endogenous variables, Z_1 is a $T \times K_1$ matrix of observations on K_1 included exogenous variables, β is a scalar parameter, γ_1 is K_1-component column vector of parameters, and u is a T-component vector of unobservable disturbances. The reduced form of the system of structural equation includes

$$(y_1 \ \ y_2) = (Z_1 \ \ Z_2)\begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{pmatrix} + (v_1 \ \ v_2) \tag{2.2}$$

where Z_2 is $T \times K_2$ matrix of observations on K_2 exogenous variables in the system that are excluded from the structural equation (2.1), π_{11} and π_{12} are K_1-component vectors, π_{21} and π_{22} K_2-component vectors of reduced form coefficients, and $(v_1 \ \ v_2)$ is a $T \times 2$ matrix of disturbances. We assume that the rows of $(v_1 \ \ v_2)$ are independently normally distributed, each having mean 0 and nonsingular covariance matrix

$$\Omega = \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix} \ . \tag{2.3}$$

Since $u = v_1 - \beta v_2$, the components of u are independently normally distributed with means 0 and variances $\sigma^2 = \omega_{11} - 2\beta\omega_{12} + \beta^2\omega_{22}$. The matrix $(\pi_{21} \ \ \pi_{22})$ is of rank one, π_{22} has at least one nonzero component, and $\pi_{21} = \beta\pi_{22}$. The $T \times K$ matrix $Z = (Z_1 \ \ Z_2)$ is assumed to be of rank K and $T > K$.

Let p_{21} and p_{22} be the least squares estimators for π_{21} and π_{22} and

$$A_{22.1} = Z_2'Z_2 - Z_2'Z_1(Z_1'Z_1)^{-1}Z_1'Z_2 \ . \tag{2.4}$$

Then the Limited Information Maximum Likelihood (LIML) estimator of β is the negative of the ratio of the second to the first component of b satisfying

$$\left(\frac{1}{T}G - \lambda_1\hat{\Omega}\right)b = 0 \ , \tag{2.5}$$

where λ_1 is the smallest root of

$$\left|\frac{1}{T}G - \lambda\hat{\Omega}\right| = 0 \ , \tag{2.6}$$

$$G = \begin{pmatrix} p_{21}' \\ p_{22}' \end{pmatrix} A_{22.1}(p_{21} \ \ p_{22}) \ , \tag{2.7}$$

$$\hat{\Omega} = \frac{1}{T}\begin{pmatrix} y_1' \\ y_2' \end{pmatrix}[I - Z(Z'Z)^{-1}Z'](y_1 \ \ y_2) \ . \tag{2.8}$$

Replacing $\hat{\Omega}$ in (2.5) and (2.6) with Ω, we obtain the LIML estimator when the covariance matrix is known, namely the LIMLK estimator. The LIML estimator will be denoted $\hat{\beta}_{LI}$, and the LIMLK estimator $\hat{\beta}_K$.

We shall consider the distributions of the normalized estimators, that is,

$$(2.9) \qquad \frac{\sqrt{\pi_{22}' A_{22.1} \pi_{22}}}{\sigma} (\hat{\beta} - \beta) \; ,$$

the limiting distribution of which is $N(0,1)$ as $T \to \infty$ with $\pi_{22}' A_{22.1} \pi_{22}/T$ bounded and bounded away from 0. The distribution of (2.9) for either estimator depends on K_2, the number of excluded exogenous variables, on $T - K$, the number of degrees of freedom in $\hat{\Omega}$, on

$$(2.10) \qquad \delta^2 = \frac{\pi_{22}' A_{22.1} \pi_{22}}{\omega_{22}} \; ,$$

the noncentrality parameter associated with (2.1) or alternatively

$$(2.11) \qquad \mu^2 = (1 + \alpha^2)\delta^2 \; ,$$

and on

$$(2.12) \qquad \alpha = \frac{\omega_{22}}{|\Omega|^{\frac{1}{2}}} (\beta - \frac{\omega_{12}}{\omega_{22}}) \; ,$$

the standardized structural coefficient, which measures the difference between the structural parameter (the coefficient of proportion between the systematic parts of y_1 and y_2) and the regression coefficient of one disturbance on the other. The simulation is done for various sets of values of these quantities.

An asymptotic expansion of the distribution of (2.9) for either LIML or LIMLK is (Anderson [1974])

$$(2.13) \qquad \Pr\left\{\frac{\sqrt{\pi_{22}' A_{22.1} \pi_{22}}}{\sigma} (\hat{\beta} - \beta) \le w\right\}$$

$$= \Phi(w) + \{-\frac{\alpha}{\mu} w^2 + \frac{1}{2\mu^2} [-(K_2 - 1)w$$

$$+ (2\alpha^2 - 1) w^3 - \alpha^2 w^5]\} \phi(w) + O(\mu^{-3})$$

where $\Phi(\cdot)$ and $\phi(\cdot)$ are the cdf and density of $N(0,1)$. The mean and variance computed from the first three terms of (2.13) are α/μ and

$$(2.14) \qquad 1 + \frac{K_2 + 2 + 8\alpha^2}{\mu^2} = 1 + \frac{8 + \frac{K_2 - 6}{1 + \alpha^2}}{\delta^2} \; .$$

3. The Monte Carlo Procedure

Using the transformation Q in Anderson [1974], the procedure can be substantially simplified. The standardized LIML estimator (2.9) has the distribution of

$$(3.1)\qquad \frac{\mu\hat{\beta}^*}{1-\alpha\hat{\beta}^*},$$

where

$$(3.2)\qquad \hat{\beta}^* = (g_{22} - \lambda_1 c_{22})^{-1}(g_{12} - \lambda_1 c_{12}),$$

$$(3.3)\qquad \lambda_1 = \frac{c_{11}g_{22} + c_{22}g_{11} - 2c_{12}g_{12} - \sqrt{D}}{2(c_{11}c_{22} - c_{12}^2)},$$

$$(3.4)\qquad D = (c_{11}g_{22} - c_{22}g_{11})^2 + 4(c_{11}g_{12} - c_{12}g_{11})(c_{22}g_{12} - c_{12}g_{22}).$$

The random variables g_{ij} and c_{ij}, $i,j = 1,2$, can be rewritten

$$(3.5)\qquad g_{11} = \sum_{i=1}^{m} z_{1i}^2, \quad g_{12} = \sum_{i=1}^{m} z_{1i}z_{2i} + \mu z_{11},$$

$$g_{22} = \sum_{i=2}^{m} z_{2i}^2 + (z_{21} + \mu)^2,$$

$$c_{ij} = \sum_{k=1}^{q} w_{ik}w_{jk}, \quad i,j = 1,2,$$

where z_{1i}, z_{2i}, $i = 1,\ldots,m$ and w_{ik}, $i = 1,2$, $k = 1,\ldots,q$ are normally and independently distributed with means 0 and variances 1, $m = K_2$ and $q = T - K$. The standardized LIMLK estimator (2.9) has the distribution of $\mu\hat{\beta}_K^*/(1 - \alpha\hat{\beta}_K^*)$, where

$$(3.6)\qquad \hat{\beta}_K^* = \frac{g_{11} - g_{22} + \sqrt{(g_{11} - g_{22})^2 + 4g_{12}^2}}{2g_{12}}.$$

This expression can be obtained from (3.2) by replacing c_{11} and c_{22} by 1 and c_{12} by 0 .

The simulations are carried out by generating the distribution of the g_{ij}'s and the c_{ij}'s. We observe that g_{22} is distributed as $\chi_{m-1}^2 + \chi_1{}^2(\mu^2)$; $g_{12}/\sqrt{g_{22}}$ is distributed according to $N(0,1)$; $g_{11} - (g_{12})^2/g_{22}$ is distributed as χ_{m-1}^2; c_{11} is distributed as χ_q^2, $c_{12}/\sqrt{c_{11}}$ is distributed according to $N(0,1)$; and $c_{22} - (c_{12}^2)/c_{11}$ is distributed as χ_{q-1}^2. These random variables are independently distributed; here $\chi_1^2(\mu^2)$ is the non-central χ^2 random variable with 1 degree of freedom and μ^2 the noncentrality parameter. Hence the LIML estimator can be expressed as a function of four χ^2's, two normal random variables, and one non-central χ^2 random variables.

In the case of $\hat{\beta}_K^*$ the estimator is a function of two χ^2's, one normal, and one non-central χ^2 random variable.

Since both the LIML and LIMLK estimators depend only on $N(0,1)$, χ_n^2, and $\chi_1^2(\mu^2)$, it is important to generate these random numbers efficiently in our Monte Carlo simulation. The standard normal random number is generated by a standard technique, namely, adding twelve uniform random numbers and subtracting 6. To generate the random variable χ_n^2, the following procedure is efficient when the number of degrees of freedom n is small. First let U_i be the random variable with the uniform distribution on $(0,1)$. Then $V = 2\sum_{i=1}^{n} \log U_i$ is distributed as χ_{2n}^2. For χ_{2n+1}^2 we add to the sum one χ_1^2 variable, which is the square of a standard normal variate. When n is large, however, the above procedure is not efficient. In this case, the approximation of Wilson-Hilferty is accurate enough for practical purpose. The distribution of

$$z = \frac{\sqrt{9n}}{2}\left\{\left(\frac{\chi_n^2}{n}\right)^{\frac{1}{3}} - \left(1 - \frac{2}{9n}\right)\right\} \tag{3.7}$$

has approximately the standard normal distribution. Therefore we can get a χ_n^2 random number from one normal random number. We use this approximation when $q \geq 10$ and $m \geq 10$.

For each set of values of K_2, $T - K$, and δ^2, 20,000 sets of g_{11}, g_{22}, g_{12}, c_{11}, c_{22} and c_{12} were generated. For each such set of g_{ij}'s and c_{ij}'s the corresponding $\hat{\beta}^*$ and $\hat{\beta}_K^*$ were calculated. From them one can estimate the marginal cdf's of $\hat{\beta}^*$ and $\hat{\beta}_K^*$ for $\alpha = 0$.

On the basis of 20,000 replications we can calculate from the distribution of the Kolmogorov-Smirnov statistic that the empirical cdf of an estimator is within .01 of the true cdf everywhere with probability more than .99. To evaluate the accuracy of our Monte Carlo experiments, we compared the empirical and exact cdf's of the LIMLK estimator. The latter has been studied and tabulated extensively by Anderson and Sawa [1978]. The maximum absolute value of the difference between the empirical and the exact cdf's is tabulated in Table 1.1 for the case of $\alpha = 0$.

Table 1.1

Maximum Absolute Difference between Empirical and Exact cdf's of LIMLK estimator* for $\alpha = 0$

	Noncentrality Parameter				
	30	50	100	300	1000
$K_2 = 3$	0.7D-02	0.5D-02	0.4D-02	**	**
$K_2 = 10$	0.8D-02	0.9D-02	0.5D-02	0.7D-02	**
$K_2 = 30$	**	0.7D-02	0.5D-02	0.7D-02	0.7D-02

*The cdf's were calculated at values of the argument -7.0(.5)-2.0(.2)-1.0(.1)1.0(.2)2.0(.5)7.0

**We did not perform the simulation in these cases.

Table 1.1 shows that the maximum value is .009, less than one percent, and there is no systematic bias in our simulation. This agrees with the above theoretical consideration. On the whole, we confirm that our experiments are accurate and sampling errors are small.

4. Estimation of the CDF of the LIML Estimator

The empirical cdf is an estimator of the distribution function. However, it may be possible to find a more efficient estimator of the cdf of the LIML estimator since we already know the exact distribution of the LIMLK estimator. We can make use of the fact that $\hat{\beta}^*$ and $\hat{\beta}_K^*$ are calculated together and hence have a joint distribution with a good deal of correlation. The joint empirical densities of some pairs are sketched in Figures 1.1 to 1.6. (These resulted from an earlier simulation study with other values of $T - K$ and K_2.) The association between the two estimates is high if K_2 is small, δ^2 is large, and $T - K$ is large. The association decreases as K_2 increases, $T - K$ decreases, and δ^2 decreases. The effect of K_2 seems most important.

First, by Monte Carlo simulation we obtain the empirical cdf of the LIML and LIMLK estimators for $\alpha = 0.0$ and $K_2 = 3$, $T - K = 10, 30$, $\delta^2 = 10, 30, 50, 100$; $K_2 = 10$, $T - K = 10, 30, 100$, $\delta^2 = 30, 50, 100, 300$; $K_2 = 30$, $T - K = 30, 100, 300$, $\delta^2 = 50, 100, 300, 1000$.

Second, we get the empirical densities of the LIML and LIMLK estimators by numerical differentiation. We expect that the sampling errors of an empirical density in intervals are almost independent but the values of the empirical cdf in intervals are highly dependent.

Third, in order to reduce the sampling errors of the empirical density, we use a smoothing procedure. We tried fitting $\phi(x) \sum_{i=0}^{n} c_i H_{2i}(x)$ to the densities by least squares, where $\phi(\cdot)$ is the density of the standard normal distribution, the c_i's are coefficients and the $H_{2i}(\cdot)$ is the 2i-th order Hermite polynominal. We used only even polynomials since both the LIML and LIMLK estimators have symmetric densities when $\alpha = 0.0$. However, we found that the fit was not satisfactory in the tails, especially in the case of $K_2 = 30$. Then we modified the procedure by fitting $\phi(x/k) \sum_{i=0}^{n} c_i H_{2i}(x/k)$ where $k^2 = 1 + K_2/\delta^2$. A possible justification for adjusting the scale parameter k is that

k^2 is the asymptotic variance of the LIML estimator when K_2 is large. (Kunitomo [1978]) We chose $n = 3$ because use of more polynomials caused some instability in tail areas. As a result, the goodness of fit is satisfactory in each case.

Fourth, let $\hat{f}_{LI}(x) = \phi(x/k) \sum_{i=0}^{n} \hat{c}_{i1} H_{2i}(x/K)$ and $\hat{f}_K(x) = \phi(x/k) \sum_{i=0}^{n} \hat{c}_{i2} H_{2i}(x/k)$ be the estimated densities of the LIML estimator and the LIMLK estimator, respectively. Then an estimate of the difference of the cdf's of these estimators is given by

$$(4.1) \qquad \hat{D}(x) = \int_0^x [\hat{f}_{LI}(t) - \hat{f}_K(t)]dt$$

$$= d_0[\Phi(\tfrac{x}{k}) - \tfrac{1}{2}] + \sum_{i=1}^{n} d_i H_{2i-1}(\tfrac{x}{k}) \phi(\tfrac{x}{k}) ,$$

where $d_0 = k(\hat{c}_{01} - \hat{c}_{02})$ and $d_i = k(\hat{c}_{i2} - \hat{c}_{i1})$. The values of d_i, $i = 0,1,2,3$, are tabulated in Table 1.2.

Table 1.2

Estimated Coefficients of the Differences Between the cdf's of LIML and LIMLK Estimators for $\alpha = 0.0$

$K_2 = 3$

	δ^2	10	30	50	100
T - K = 10	D-0	-0.715D-02	0.293D-03	-0.271D-02	-0.237D-03
	D-1	-0.398D-01	-0.241D-01	-0.836D-02	-0.214D-02
	D-2	-0.140D-01	-0.442D-02	-0.147D-02	0.266D-02
	D-3	-0.195D-02	-0.217D-03	-0.131D-03	0.414D-03
T - K = 30	D-0	-0.162D-02	-0.144D-02	-0.192D-03	0.104D-02
	D-1	-0.117D-01	-0.554D-02	-0.341D-02	-0.382D-02
	D-2	-0.480D-02	-0.471D-03	0.175D-02	0.166D-03
	D-3	-0.817D-03	0.244D-03	0.540D-03	0.172D-03

$K_2 = 10$

	δ^2	30	50	100	300
T - K = 10	D-0	-0.239D-01	-0.143D-01	0.287D-02	-0.227D-02
	D-1	-0.123D+00	-0.912D-01	-0.913D-01	-0.144D-01
	D-2	-0.144D-01	0.388D-02	-0.308D-01	0.392D-02
	D-3	-0.811D-03	0.267D-02	-0.404D-02	0.439D-03
T - K = 30	D-0	-0.638D-02	-0.319D-02	0.555D-03	-0.262D-03
	D-1	-0.435D-01	-0.555D-01	-0.219D-01	-0.118D-01
	D-2	-0.314D-02	-0.156D-01	-0.485D-02	-0.303D-02
	D-3	-0.577D-04	-0.265D-02	-0.610D-03	-0.394D-03
T - K = 100	D-0	-0.115D-02	-0.212D-02	0.118D-02	0.368D-03
	D-1	-0.947D-02	-0.525D-02	-0.163D-01	0.133D-03
	D-2	0.293D-02	0.164D-02	-0.116D-01	0.245D-02
	D-3	0.797D-03	0.721D-03	-0.183D-02	0.261D-03

Table 1.2 (continued)

Estimated Coefficients of the Differences Between the cdf's of LIML and LIMLK Estimators for $\alpha = 0.0$

$K_2 = 30$

	δ^2	50	100	300	1000
T - K = 30	D-0	-0.180D-01	-0.177D-02	-0.220D-02	0.139D-02
	D-1	-0.184D+00	-0.144D+00	-0.634D-01	-0.149D-01
	D-2	-0.308D-01	-0.253D-01	-0.134D-01	0.552D-03
	D-3	-0.366D-02	-0.279D-02	-0.153D-02	0.152D-03
T - K = 100	D-0	-0.620D-02	-0.132D-02	0.461D-02	-0.166D-02
	D-1	-0.478D-01	-0.283D-01	-0.230D-01	0.117D-02
	D-2	0.897D-02	0.768D-02	-0.593D-02	0.504D-02
	D-3	0.176D-02	0.153D-02	-0.103D-02	0.105D-02
T - K = 300	D-0	-0.125D-02	-0.610D-03	0.986D-03	0.933D-04
	D-1	-0.703D-02	0.321D-02	-0.437D-02	-0.419D-02
	D-2	0.974D-02	0.111D-01	0.187D-02	-0.125D-02
	D-3	0.152D-02	0.162D-02	0.369D-03	-0.112D-03

Fifth, an estimate of the cdf of the LIML estimator is given

(4.2) $$F_{LI}(x) = F_K(x) + \hat{D}(x) \quad ,$$

where $F_K(\cdot)$ is the exact distribution of the LIMLK which is already known. Note that $F_{LI}(x)$ is defined for all values of x.

We can estimate the variance of $\hat{F}_{LI}(x)$ or equivalently of $\hat{D}(x)$ at any x from the variance matrix of d_0, d_1, d_2, d_3. The covariance matrix is estimated by the program for regression of the difference of the empirical densities on $\phi(x/k)H_{2i}(x/k)$, $i = 0,1,2,3$. Because the variance of the difference of the empirical densities is not constant over the intervals, this estimated covariance matrix is approximate. Table 1.3 gives the estimated standard deviation of $\hat{F}_{LI}(x)$ at $x = 1$. The standard deviation is 0 at $x = 0$ and approaches 0 as $x \to \infty$. The largest value in Table 1.3 is 0.00228; many of the other values are considerably smaller. Roughly speaking, we have assurance at the 99% confidence level that the error in our estimate is less than .005; this is considerably smaller than the value of .01 from the Kolmogorov-Smirnov statistic. However, it should be noted that the standard deviation at some other x's can be larger than the above values in Table 1.3 .

Table 1.3

Estimated Standard Deviations of cdf's at $x = 1.0$ for $\alpha = 0.0$

	δ^2	10	30	50	100
$K_2 = 3$	T - K = 10	0.110D-02	0.892D-03	0.867D-03	0.686D-03
	T - K = 30	0.101D-02	0.109D-02	0.737D-03	0.776D-03

	δ^2	30	50	100	300
$K_2 = 10$	T - K = 10	0.162D-02	0.174D-02	0.111D-02	0.111D-02
	T - K = 30	0.228D-02	0.182D-02	0.131D-02	0.138D-02
	T - K = 100	0.115D-02	0.112D-02	0.927D-03	0.653D-03

	δ^2	50	100	300	1000
$K_2 = 30$	T - K = 30	0.120D-02	0.154D-02	0.189D-02	0.108D-02
	T - K = 100	0.165D-02	0.155D-02	0.157D-02	0.832D-03
	T - K = 300	0.205D-02	0.137D-02	0.142D-02	0.114D-02

Finally, if $0 < \alpha$ and $0 < \alpha x + \mu$, we have

(4.3) $$\Pr\left\{\frac{\mu\hat{\beta}^*}{1 - \alpha\hat{\beta}^*} \le x\right\} = \Pr\left\{\mu\hat{\beta}^* \le \frac{x}{1 + \frac{\alpha}{\mu} x}\right\} + \Pr\{\mu\hat{\beta}^* > \frac{\mu}{\alpha}\} \quad .$$

Then by using this formula we tabulate the estimated cdf of the LIML estimator for $\alpha = 1.0, 2.0$, and 5.0. In most cases, the second term of (4.3) is found to be negligible numerically (though it does imply a small negative median bias). Note that the probabilities for $\alpha = 0$ are available for all x because we have the fitted function.

5. Discussion of the Distributions

5.1 The Distribution of the LIML Estimator

The distributions are tabulated in standardized terms, that is, of (2.9). The tabulation makes comparisons and interpolation easier. The asymptotic standard deviation (ASD) of $\hat{\beta}$ is

$$(5.1) \qquad \frac{\sigma}{\delta\sqrt{\omega_{22}}} = \frac{\sqrt{1+\alpha^2}\sqrt{|\underset{\sim}{\Omega}|}}{\delta\omega_{22}} .$$

The spread of the distribution of the (unstandardized) estimator increases with $|\alpha|$ and decreases with δ. The estimators with which we wish to compare, LIMLK and TSLS, have the same asymptotic standard deviation. In the remainder of the discussion we consider the normalized distributions (as tabulated).

For $\alpha = 0$ the densities are symmetric. As α increases, there is some slight asymmetry [see (4.3)], but the median is close to 0. For given α, K_2, and $T - K$ the lack of symmetry decreases as δ^2 increases. For given α, δ^2, and $T - K$ the asymmetry increases with K_2. In case of asymmetry the median of $\hat{\beta}_{LI} - \beta$ is slightly negative (the median of $\hat{\beta}_{LI}$ is less than β).

The distributions have relatively long tails (in agreement with the moments not existing). As $\delta^2 \to \infty$, the distributions approach $N(0,1)$; however, for small values of δ^2 there is an appreciable probability outside of 3 or 4 ASD's. As δ^2 increases, the spread of the normalized distribution decreases. For given α, K_2, and δ^2, the spread decreases as $T - K$ increases, corresponding to $\hat{\underset{\sim}{\Omega}}$ being a better estimate of $\underset{\sim}{\Omega}$. The spread tends to increase with K_2 and decrease with α. These observations about the spread agree with the asymptotic variance (2.14).

The above properties of the LIML estimator have been pointed out by Anderson and Sawa [1979] comparing the TSLS estimator with the LIMLK estimator.

5.2 Comparison with the LIMLK Estimator

The LIML and LIMLK estimators differ only in that the covariance matrix $\underset{\sim}{\Omega}$ is estimated to obtain the LIML estimator. The distributions of the two estimators have similar features; the LIML distribution tends to be a little more spread out than the distribution of the corresponding LIMLK estimator.

Figures 2 graph the difference between the estimated cdf of the LIML estimator and the exact cdf of the LIMLK estimator for the case $\alpha = 0.0$. We omit figures for the cases $\alpha = 1.0$, 2.0 and 5.0 because they are similar to those for $\alpha = 0.0$. Some general comments can be made by comparing those figures. The difference of two cdf's decreases as the noncentrality parameter δ^2 increase and as the number of degrees of freedom $T - K$ increases. As the noncentrality parameter increases, the sampling error in $\hat{\Omega}$ becomes less important and hence there is less difference between LIML based on $\hat{\Omega}$ and LIMLK based on Ω. That the difference decreases as $T - K$ increases is due to reducing the sampling error in $\hat{\Omega}$. In each case the LIML estimator has more probability in the tails than the LIMLK estimator because we estimate the covariance matrix, hence increasing the variability of the estimator. When the number of excluded exogenous variables K_2 is as small as 3, the difference between the cdf's is less than .01 and can be ignored for practical purposes, such as testing hypothesis. As K_2 increases, the maximum difference increases (for given δ^2 and $T - K$). When K_2 is as large as 30, as it can be in contemporary econometric models, the cdf of the LIML estimator may differ substantially from that of the LIMLK estimator. However, when $T - K$ is as large as 300, this difference becomes negligible. On the whole, we might say that the LIML estimator and the LIMLK estimator share the same small sample properties as long as K_2 is small or $T - K$ and/or δ^2 are large.

5.3 Comparison with the TSLS Estimator

Anderson and Sawa [1977], [1979] have given tables of the distributions of the TSLS estimator and discussed their properties. These properties agree with the asymptotic expansions of the distributions (Anderson and Sawa [1973]). The mean and variance of the asymptotic expansion (to 3 terms) are

$$(5.2) \qquad -\frac{(K_2 - 2)\alpha}{\mu} = -\frac{(K_2 - 2)\alpha}{\delta\sqrt{1 + \alpha^2}} ,$$

$$(5.3) \qquad 1 - \frac{K_2 - 4 + 4(K_2 - 3)\alpha^2}{\mu^2} = 1 - \frac{4(K_2 - 3) - \frac{3K_2 - 8}{1 + \alpha^2}}{\delta^2} .$$

The most striking feature is that for $\alpha \neq 0$ the distribution is skewed towards the left for $\alpha > 0$ and the distortion increases with α and K_2. Figures 3 show the estimated cdf of the LIML estimator, and the exact cdf's of the LIMLK and TSLS estimators for the case when $\alpha = 1.0$, $\delta^2 = 100.0$, and $K_2 = 3, 10, 30$. The LIML and LIMLK estimators are essentially median-unbiased in each case. On the other hand, the TSLS estimator is biased. As K_2 increases, this bias becomes more serious; for $K_2 = 30$ the median is about -1.6 ASD's. If K_2 is large, the TSLS estimator substantially underestimates the parameter. This fact definitely favors the LIML estimator over the TSLS estimator. However, when K_2 is as small as 3, the TSLS estimator is very similar to the LIML estimator. Particularly the distribution of the LIMLK estimator can be approximated very well by the limiting normal distribution. (When $K_2 = 1$, the TSLS, LIMLK, and LIML estimators are identical.)

The distributions of the LIML and LIMLK estimators are a little more spread out than the distributions of the TSLS estimator. This reflects the fact that the LIML and LIMLK estimators do not have moments of positive integer order. The difference between the asymptotic variance of the LIML and LIMLK estimator and that of the TSLS estimator is $2(K_2 - 1)(1 + 2\alpha^2)/\mu^2 = 2(K_2 - 1)[2 - 1/(1 + \alpha^2)]/\delta^2$. The difference increases with α and K_2 and decreases with δ^2.

The distributions of the LIML and LIMLK estimators approach normality faster than the distribution of the TSLS estimator, due primarily to the bias of the latter.

The conclusion we draw from the comparison is that unless K_2 is small LIML is to be preferred to the TSLS estimator because the latter is so biased, particularly if K_2 is large.

5.4 Errors of Approximations

Anderson and Sawa [1977] gave a small table of the maximum difference between the cdf of the LIMLK estimator and the cdf of the normal distribution, the asymptotic expansion (2.13) combined with (4.3), and a similar expansion to order δ^{-4}. The differences between the cdf of the LIML estimator and the approximations can be expected to be similar.

6. Three Methods of Estimating the cdf of the LIML Estimator

As indicated above, the cdf of the LIML estimator at $\alpha = 0$ could be estimated by the empirical cdf of the Monte Carlo results. Another procedure is to add to the known cdf of the LIMLK estimator the difference between the empirical cdf of the LIML estimator and the empirical cdf of the LIMLK estimator. (Each of these could be improved by using the fact that the densities are symmetric about 0.) The latter method exploits the association between the LIML and LIMLK estimate for each simulated sample.

Tables 3.1 to 3.4 give the estimated LIML cdf on the basis of the two methods and the use of the fitted function for $K_2 = 3$, $\delta^2 = 30$, and 50, $K_2 = 10$, $\delta^2 = 30$ and 50, and $K_2 = 30$, $\delta^2 = 50$. It will be seen that there is little difference between using the difference of the cdf's and the smoothed difference (columns 2 and 3); in fact, in most cases they differ by only 1 or 2 in the third decimal place. The empirical cdf of the LIML estimator tends to differ more from the other two.

Figure 1.1

The Joint Density of LIML and LIMLK

T - K = 46, K_2 = 3, α = 0.0, δ^2 = 100.0

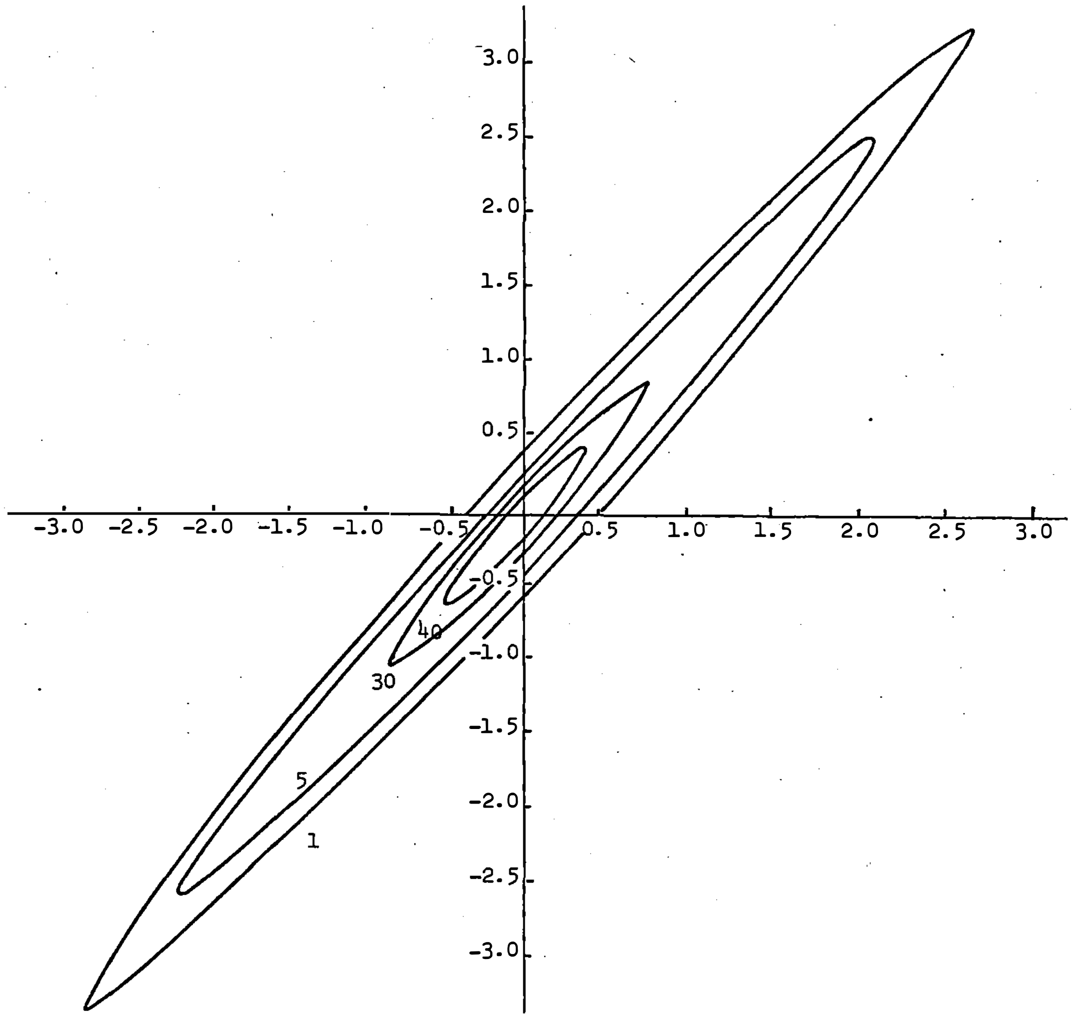


Figure 1.2

The Joint Density of LIML and LIMLK

$T - K = 20,\ K_2 = 3,\ \alpha = 0.0,\ \delta^2 = 100.0$

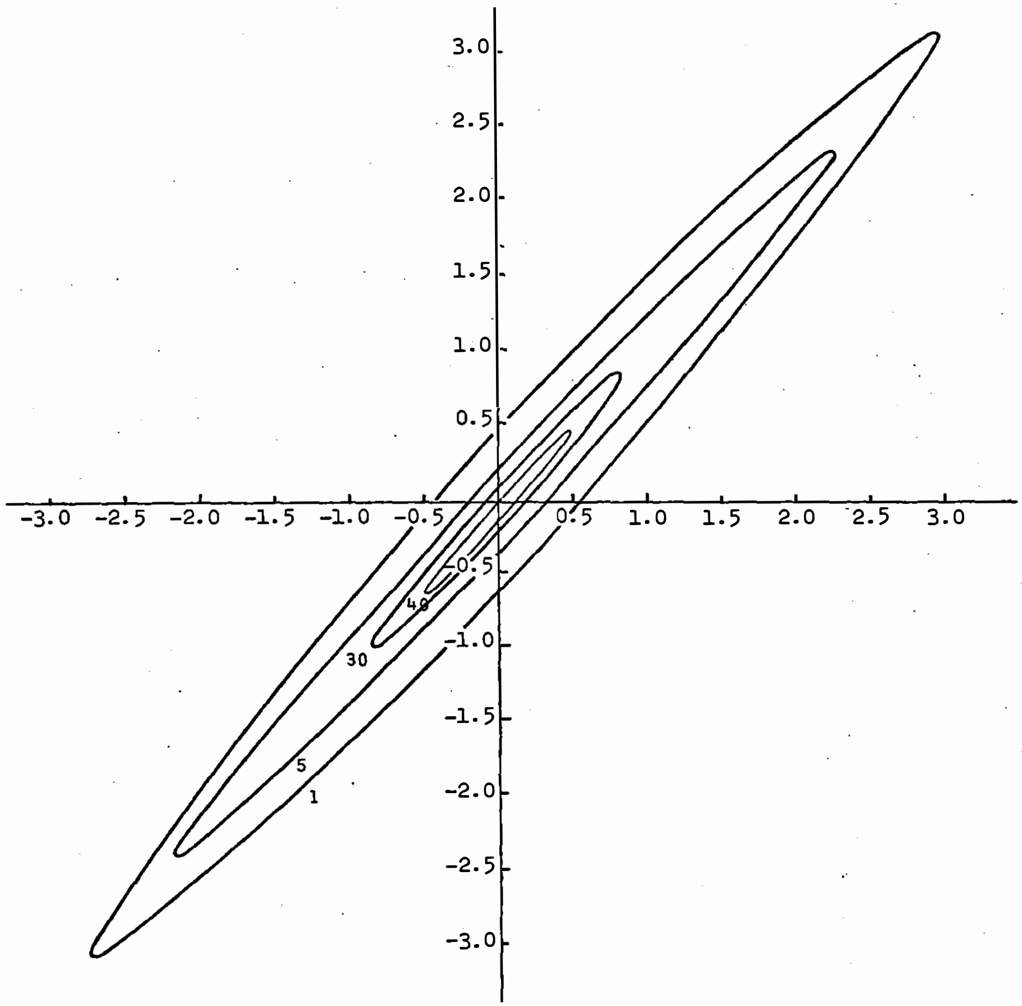


Figure 1.3

The Joint Density of LIML and LIMLK

$T - K = 46,\ K_2 = 3,\ \alpha = 0.0,\ \delta^2 = 10.0$

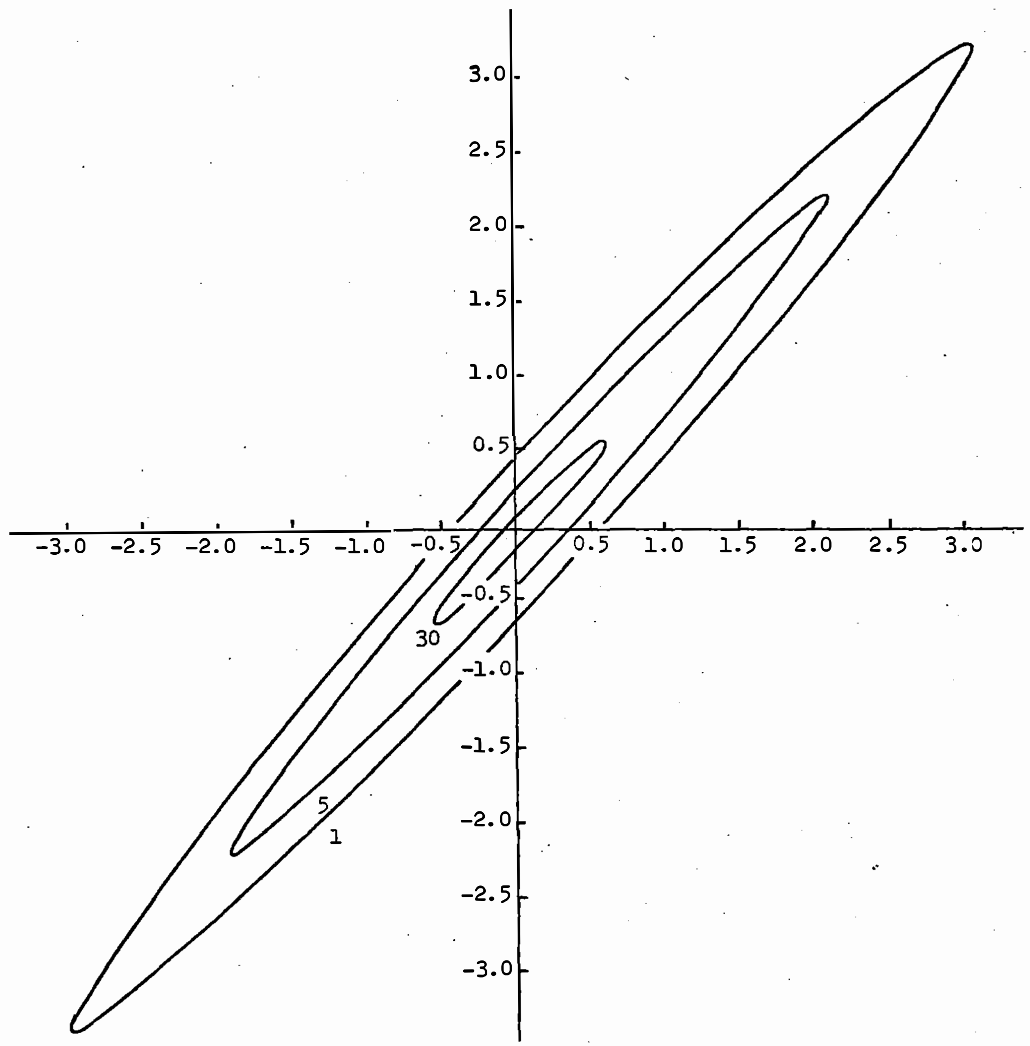


Figure 1.4

The Joint Density of LIML and LIMLK

T - K = 20, K_2 = 3, α = 0.0, δ^2 = 10.0

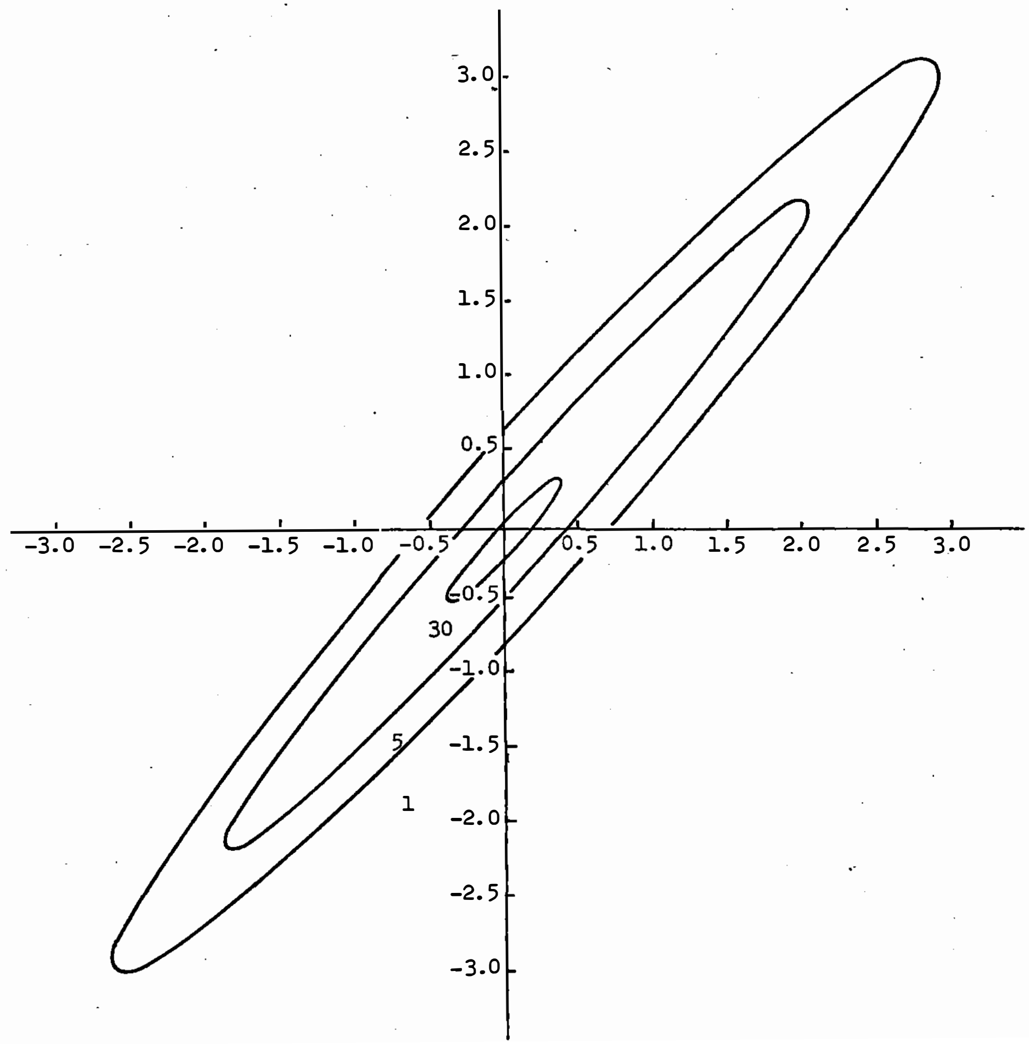


Figure 1.5

The Joint Density of LIML and LIMLK

$T - K = 20,\ K_2 = 26,\ \alpha = 0.0,\ \delta^2 = 100.0$

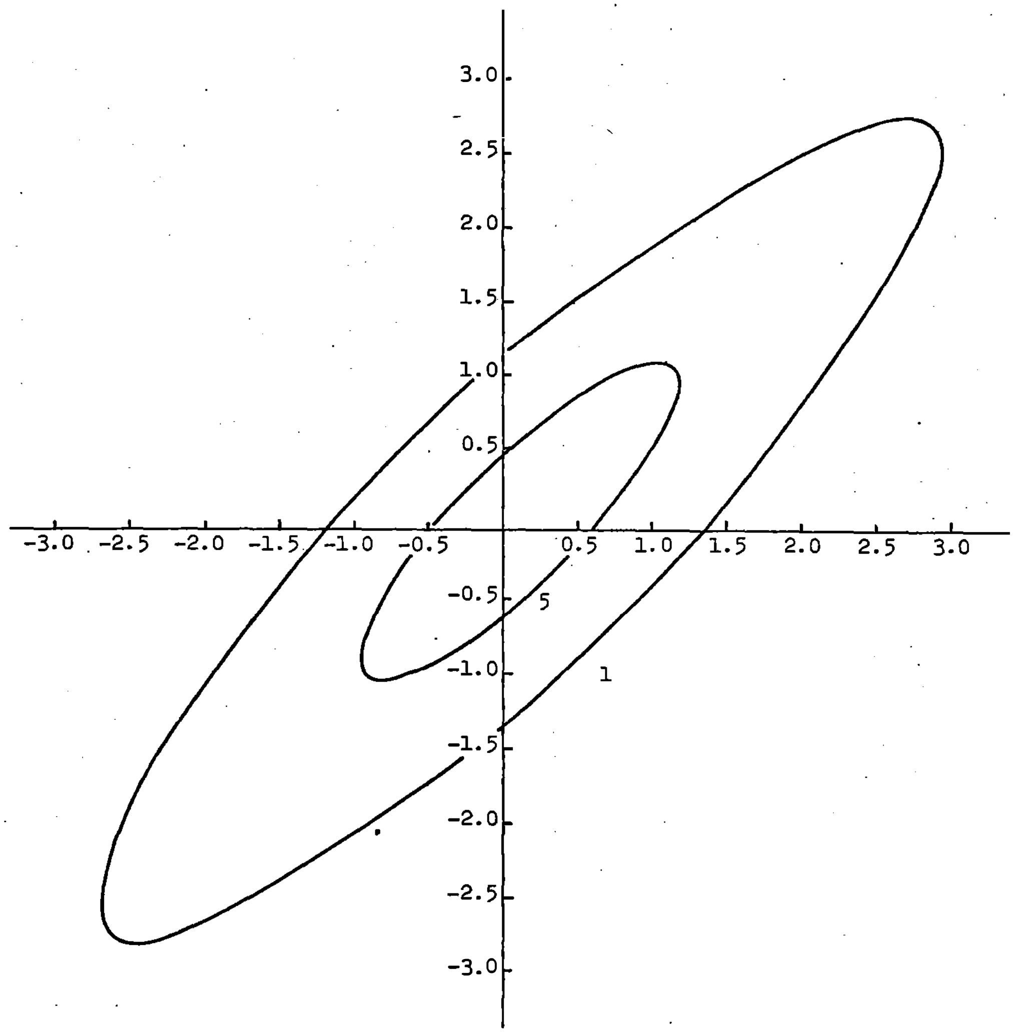


Figure 1.6

The Joint Density of LIML and LIMLK

$T - K = 20,\ K_2 = 26,\ \alpha = 0.0,\ \delta^2 = 10.0$

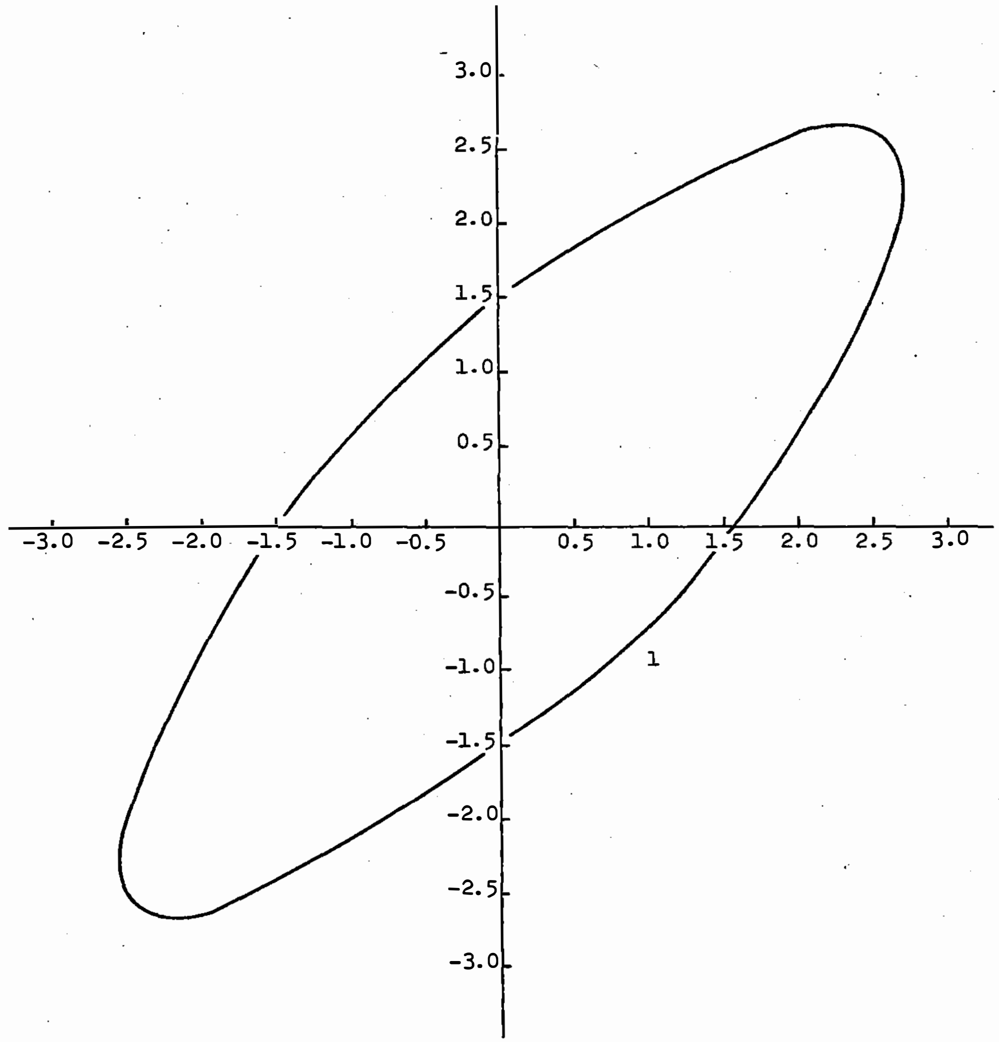


Figure 2.1

Estimated Difference of cdf's of LIML and LIMLK

$T - K = 10,\ K_2 = 3,\ \alpha = 0.0$

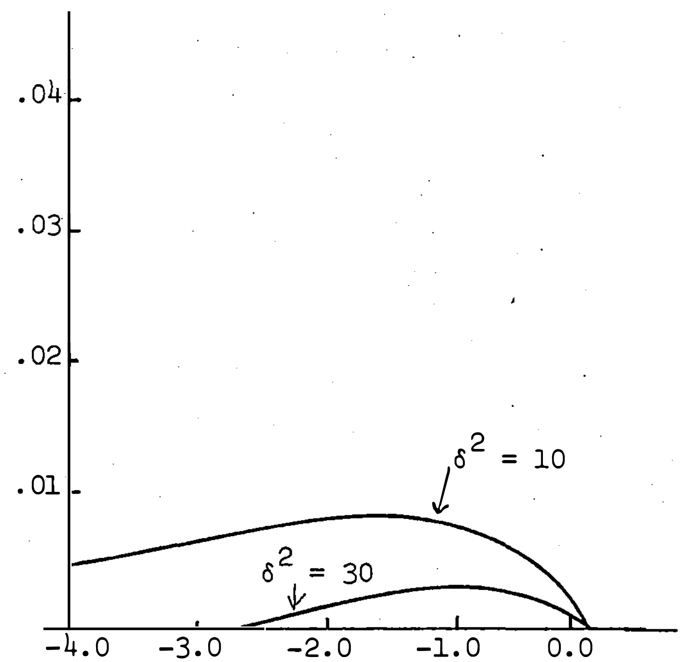


Figure 2.2

Estimated Difference of cdf's of LIML and LIMLK

$T - K = 30,\ K_2 = 3,\ \alpha = 0.0$

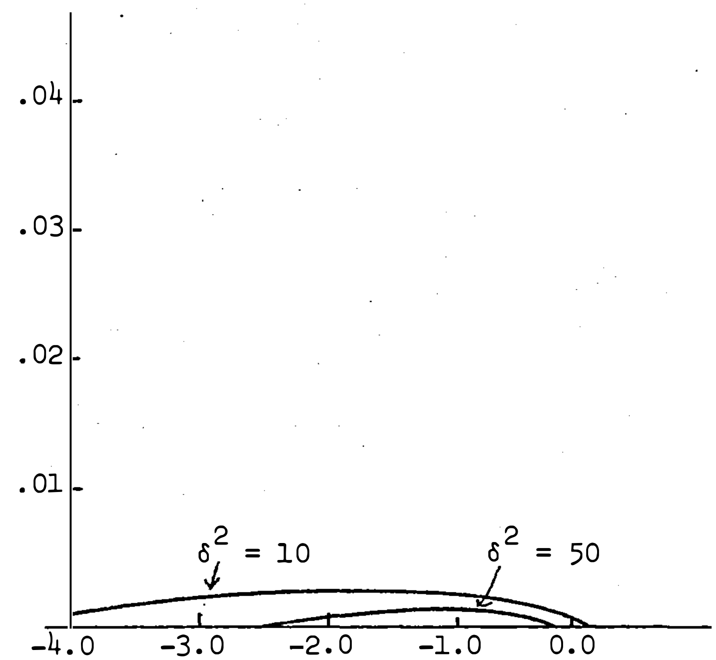


Figure 2.3

Estimated Difference of cdf's of LIML and LIMLK

$T - K = 10,\ K_2 = 10,\ \alpha = 0.0$

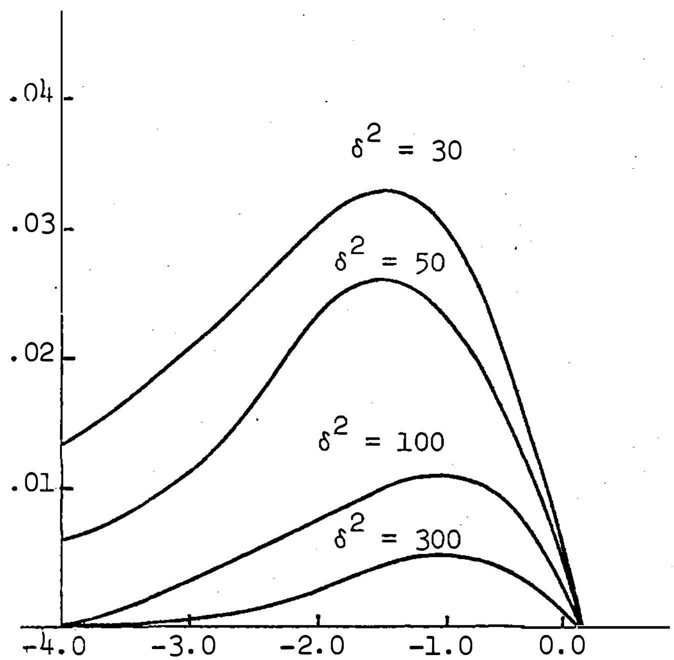


Figure 2.4

Estimated Difference of cdf's of LIML and LIMLK

$T - K = 30,\ K_2 = 10,\ \alpha = 0.0$

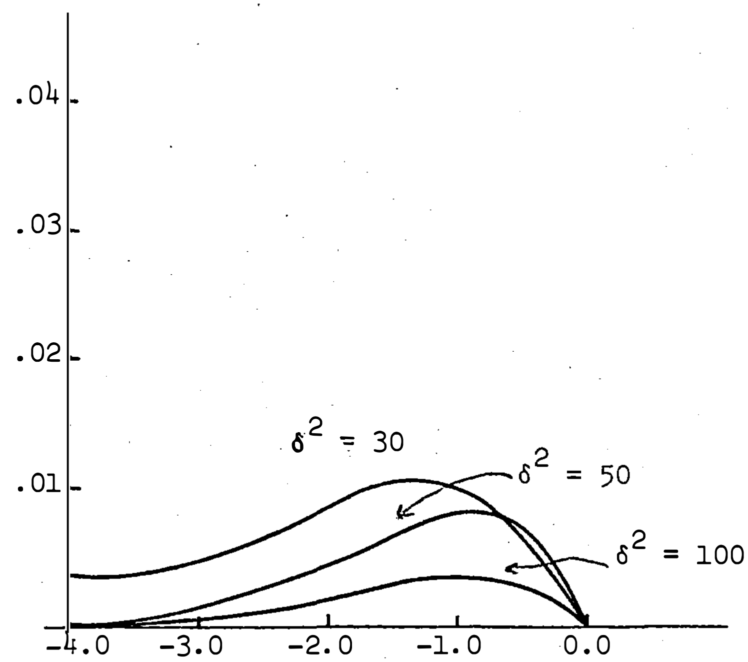


Figure 2.5

Estimated Difference of cdf's of LIML and LIMLK

$T - K = 100,\ K_2 = 10,\ \alpha = 0.0$

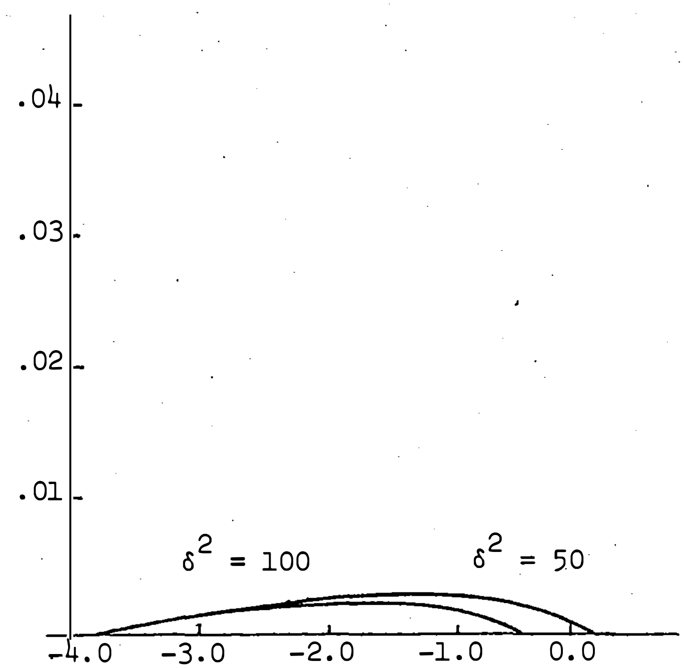


Figure 2.6

Estimated Difference of cdf's of LIML and LIMLK

$T - K = 30,\ K_2 = 30,\ \alpha = 0.0$

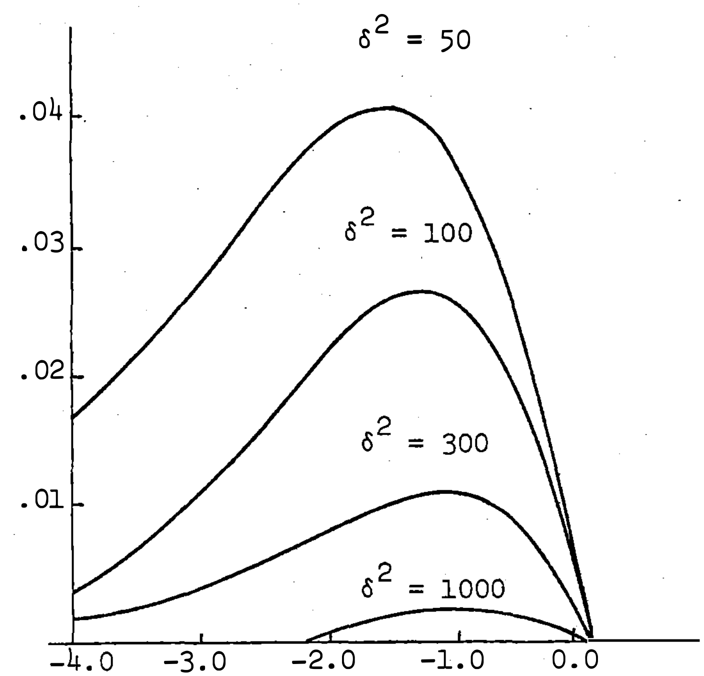


Figure 2.7

Estimated Difference of cdf's of LIML and LIMLK

T - K = 100, K_2 = 30, α = 0.0

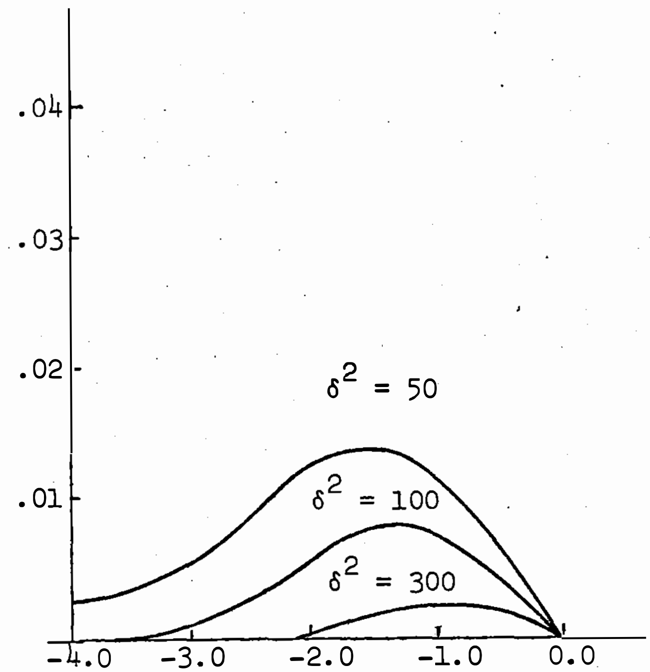


Figure 2.8

Estimated Difference of cdf's of LIML and LIMLK

T - K = 300, K_2 = 30, α = 0.0

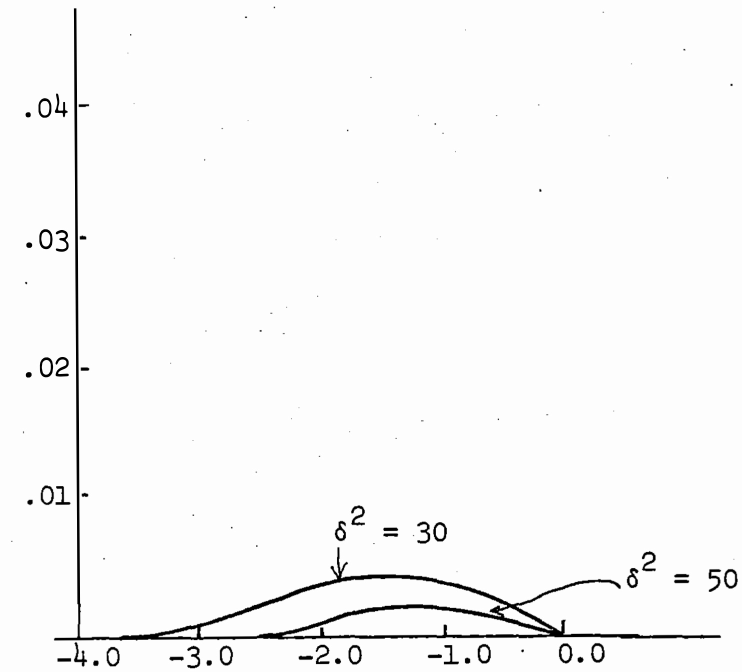
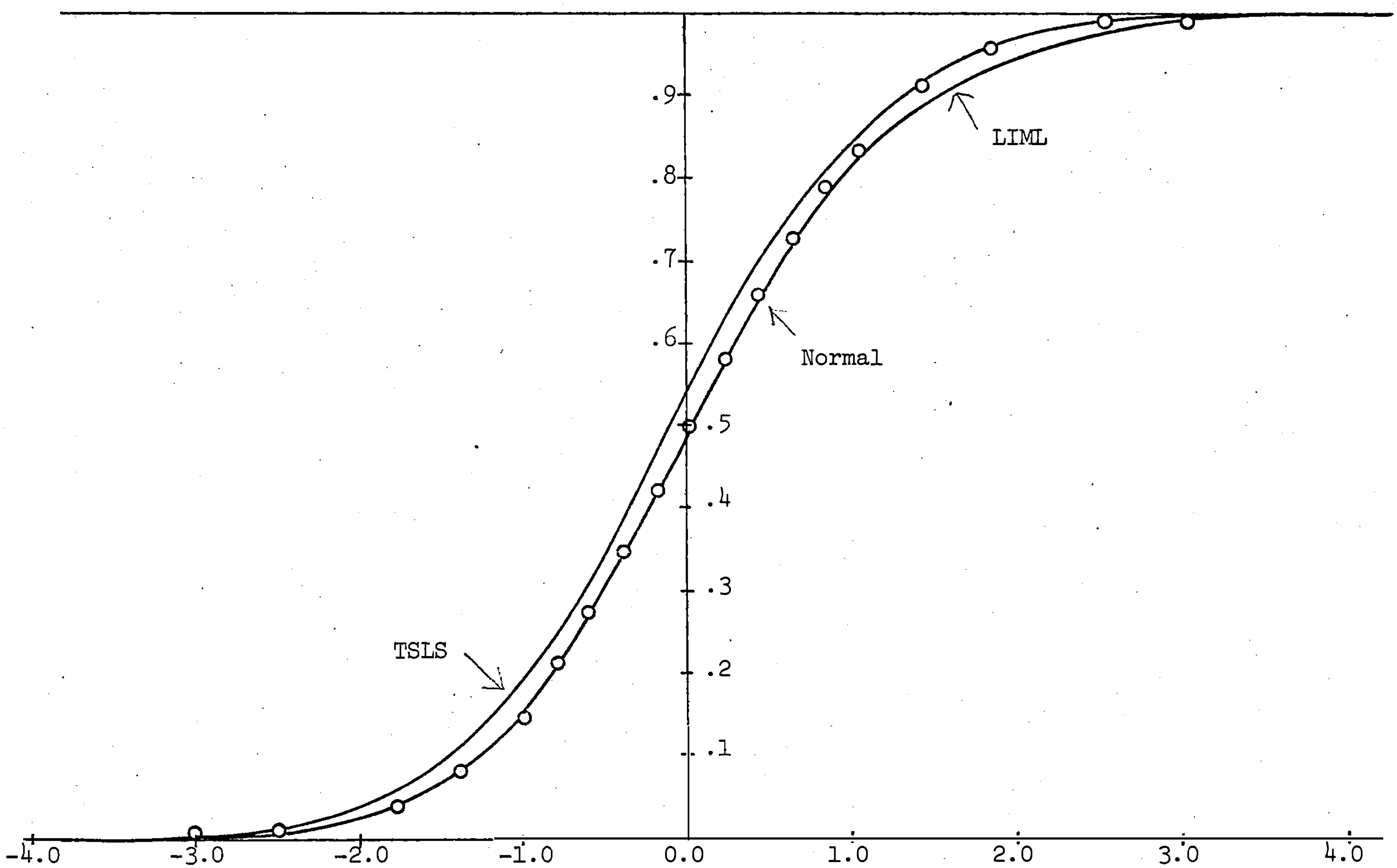


Figure 3.1

cdf's

T - K = 10, K_2 = 3, α = 1.0, δ^2 = 100.0

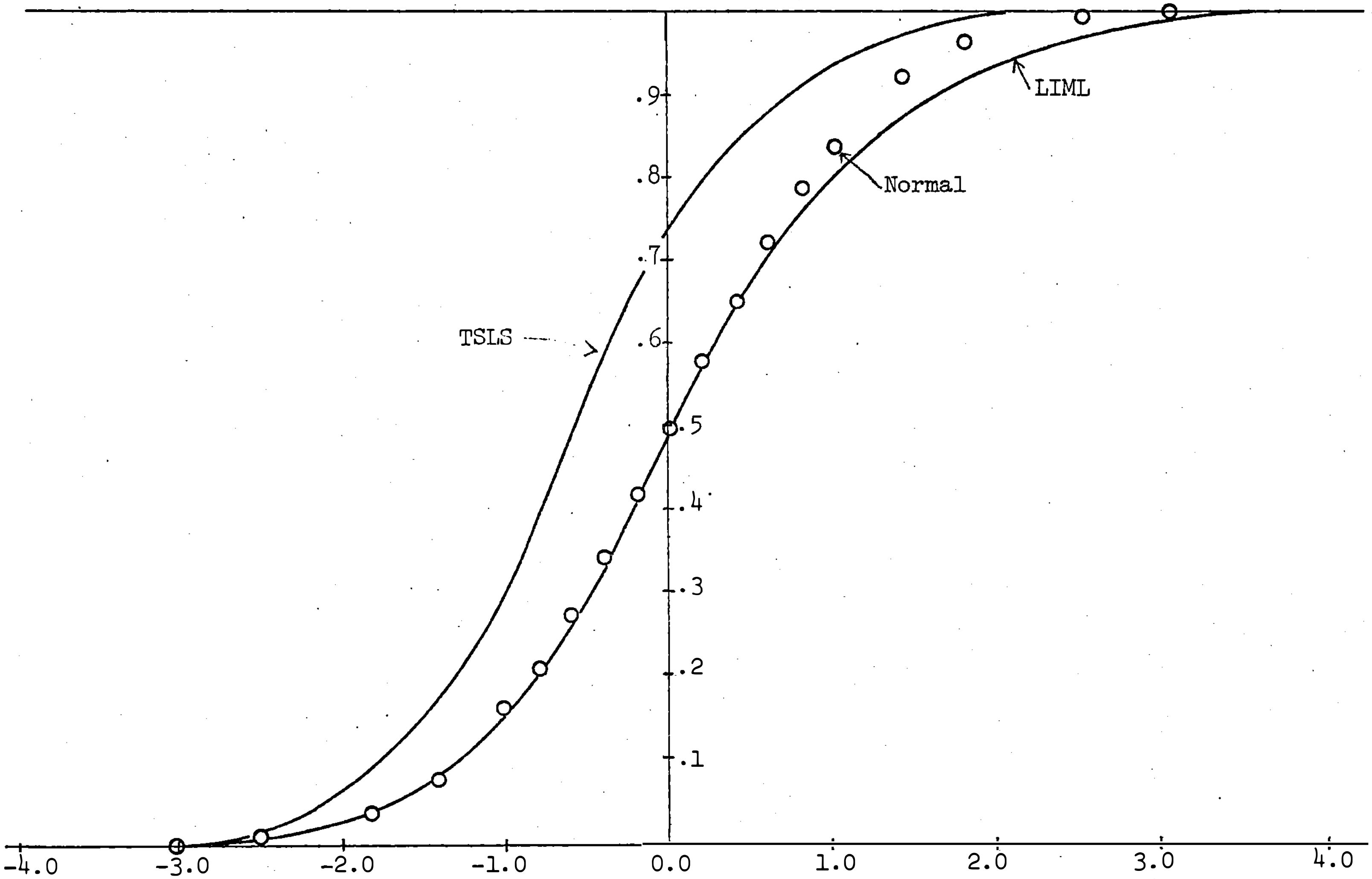


Note: The cdf of LIMLK is indistinguishable from the cdf of LIML.

Figure 3.2

cdf's

$T - K = 10,\ K_2 = 10,\ \alpha = 1.0,\ \delta^2 = 100.0$



Note: The cdf of LIMLK is indistinguishable from the cdf of LIML.

Figure 3.3

cdf's

$T - K = 30,\ K_2 = 30,\ \alpha = 1.0,\ \delta^2 = 100.0$

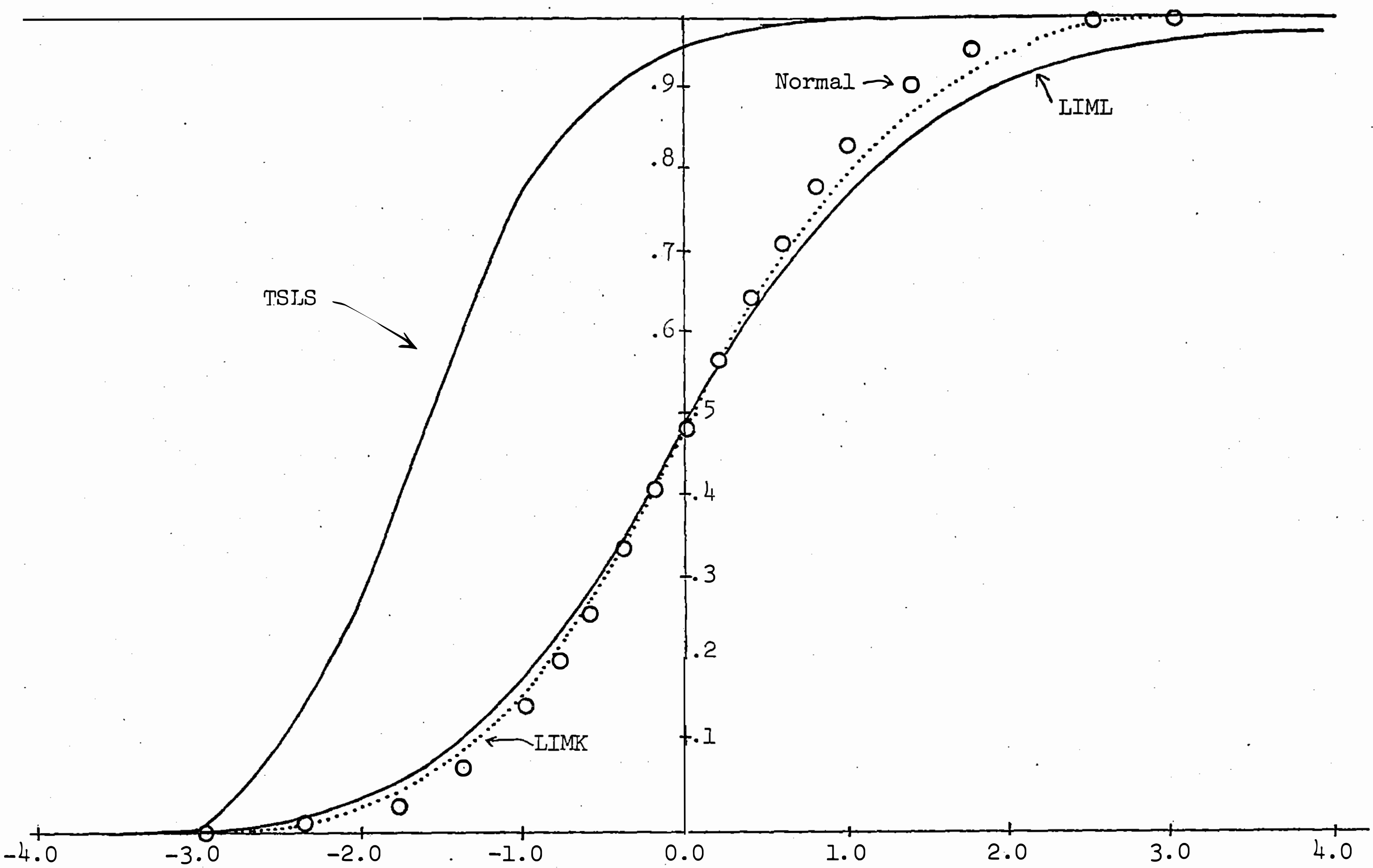


Table 2.1

Estimated Distribution Function of LIML Estimator
$T - K = 10,\ K_2 = 3,\ \alpha = 0.0$

w/δ^2	10	30	50	100	∞
-4.0	.023	0.001	0.002	0.000	0.000
-3.0	.036	0.006	0.006	0.002	0.001
-2.5	.051	0.015	0.013	0.008	0.006
-2.0	.076	0.038	0.032	0.027	0.023
-1.8	.090	0.052	0.047	0.041	0.036
-1.6	.109	0.072	0.066	0.060	0.055
-1.4	.134	0.099	0.092	0.086	0.081
-1.2	.164	0.132	0.126	0.120	0.115
-1.0	.202	0.174	0.168	0.163	0.159
-0.9	.225	0.199	0.193	0.188	0.184
-0.8	.248	0.225	0.220	0.216	0.212
-0.7	.275	0.253	0.250	0.246	0.242
-0.6	.302	0.284	0.281	0.278	0.274
-0.5	.332	0.317	0.315	0.312	0.309
-0.4	.363	0.352	0.349	0.347	0.345
-0.3	.396	0.387	0.386	0.383	0.382
-0.2	.430	0.424	0.423	0.422	0.421
-0.1	.465	0.462	0.461	0.461	0.460
0.0	.500	0.500	0.500	0.500	0.500
0.1	.535	0.538	0.539	0.539	0.540
0.2	.570	0.576	0.577	0.578	0.579
0.3	.604	0.613	0.614	0.617	0.618
0.4	.637	0.648	0.651	0.653	0.655
0.5	.668	0.683	0.685	0.688	0.691
0.6	.698	0.716	0.719	0.722	0.726
0.7	.725	0.747	0.750	0.754	0.758
0.8	.752	0.775	0.780	0.784	0.788
0.9	.775	0.801	0.807	0.812	0.816
1.0	.798	0.826	0.832	0.837	0.841
1.2	.836	0.868	0.874	0.880	0.885
1.4	.866	0.901	0.908	0.914	0.919
1.6	.891	0.928	0.934	0.940	0.945
1.8	.910	0.948	0.953	0.959	0.964
2.0	.924	0.962	0.968	0.973	0.977
2.5	.949	0.985	0.987	0.992	0.994
3.0	.964	0.994	0.994	0.998	0.999
4.0	.977	0.999	0.998	1.000	1.000

Table 2.2

Estimated Distribution Function of LIML Estimator
$T - K = 30,\ K_2 = 3,\ \alpha = 0.0$

w/δ^2	10	30	50	100	∞
-4.0	.019	0.002	0.000	0.000	0.000
-3.0	.032	0.007	0.004	0.002	0.001
-2.5	.046	0.016	0.011	0.003	0.006
-2.0	.070	0.038	0.031	0.026	0.023
-1.8	.085	0.052	0.045	0.040	0.036
-1.6	.103	0.072	0.064	0.069	0.055
-1.4	.128	0.098	0.090	0.085	0.081
-1.2	.158	0.130	0.124	0.119	0.115
-1.0	.197	0.172	0.167	0.162	0.159
-0.9	.219	0.197	0.192	0.187	0.184
-0.8	.243	0.223	0.219	0.215	0.212
-0.7	.270	0.252	0.249	0.245	0.242
-0.6	.298	0.283	0.279	0.277	0.274
-0.5	.329	0.315	0.313	0.311	0.309
-0.4	.360	0.350	0.348	0.346	0.345
-0.3	.394	0.386	0.385	0.383	0.382
-0.2	.429	0.423	0.422	0.422	0.421
-0.1	.464	0.462	0.461	0.461	0.460
0.0	.500	0.500	0.500	0.500	0.500
0.1	.536	0.538	0.539	0.539	0.540
0.2	.571	0.577	0.578	0.578	0.579
0.3	.606	0.614	0.615	0.617	0.618
0.4	.640	0.650	0.652	0.654	0.655
0.5	.671	0.685	0.687	0.689	0.691
0.6	.702	0.717	0.721	0.723	0.726
0.7	.730	0.748	0.751	0.755	0.758
0.8	.757	0.777	0.781	0.785	0.788
0.9	.781	0.803	0.808	0.813	0.816
1.0	.804	0.828	0.833	0.838	0.841
1.2	.842	0.870	0.876	0.881	0.885
1.4	.872	0.902	0.910	0.915	0.919
1.6	.897	0.928	0.936	0.941	0.945
1.8	.916	0.948	0.955	0.960	0.964
2.0	.930	0.962	0.969	0.974	0.977
2.5	.954	0.984	0.989	0.992	0.994
3.0	.968	0.993	0.996	0.998	0.999
4.0	.981	0.998	1.000	1.000	1.000

Table 2.3

Estimated Distribution Function of LIML Estimator
$T - K = 10,\ K_2 = 10,\ \alpha = 0.0$

w/δ^2	30	50	100	300	∞
-4.0	0.020	0.008	0.000	0.001	0.000
-3.0	0.036	0.018	0.007	0.003	0.001
-2.5	0.054	0.034	0.017	0.009	0.006
-2.0	0.086	0.064	0.040	0.028	0.023
-1.8	0.105	0.082	0.054	0.043	0.036
-1.6	0.127	0.104	0.076	0.062	0.055
-1.4	0.154	0.131	0.103	0.090	0.081
-1.2	0.186	0.164	0.139	0.124	0.115
-1.0	0.226	0.205	0.183	0.169	0.159
-0.9	0.247	0.228	0.208	0.194	0.184
-0.8	0.271	0.252	0.235	0.231	0.212
-0.7	0.296	0.278	0.263	0.250	0.242
-0.6	0.322	0.307	0.293	0.281	0.274
-0.5	0.349	0.336	0.325	0.315	0.309
-0.4	0.378	0.367	0.359	0.350	0.345
-0.3	0.408	0.400	0.393	0.386	0.382
-0.2	0.438	0.432	0.428	0.424	0.421
-0.1	0.469	0.466	0.466	0.462	0.460
0.0	0.500	0.500	0.500	0.500	0.500
0.1	0.531	0.534	0.536	0.538	0.540
0.2	0.562	0.568	0.572	0.576	0.579
0.3	0.592	0.600	0.607	0.614	0.618
0.4	0.622	0.633	0.641	0.650	0.655
0.5	0.651	0.664	0.675	0.685	0.691
0.6	0.678	0.693	0.707	0.719	0.726
0.7	0.704	0.722	0.737	0.750	0.758
0.8	0.729	0.748	0.765	0.779	0.788
0.9	0.753	0.772	0.792	0.807	0.816
1.0	0.774	0.795	0.817	0.831	0.841
1.2	0.814	0.836	0.861	0.876	0.885
1.4	0.846	0.869	0.897	0.910	0.919
1.6	0.873	0.896	0.924	0.938	0.945
1.8	0.895	0.918	0.946	0.957	0.964
2.0	0.914	0.936	0.960	0.972	0.977
2.5	0.946	0.966	0.983	0.992	0.994
3.0	0.964	0.982	0.993	0.997	0.999
4.0	0.980	0.992	1.000	0.999	1.000

Table 2.4

Estimated Distribution Function of LIML Estimator
$T - K = 30,\ K_2 = 10,\ \alpha = 0.0$

w/δ^2	30	50	100	300	∞
-4.0	0.010	0.001	0.000	0.000	0.000
-3.0	0.020	0.009	0.004	0.003	0.001
-2.5	0.036	0.020	0.011	0.009	0.006
-2.0	0.065	0.044	0.033	0.026	0.023
-1.8	0.083	0.061	0.048	0.041	0.036
-1.6	0.105	0.083	0.069	0.060	0.055
-1.4	0.132	0.112	0.095	0.087	0.081
-1.2	0.166	0.147	0.131	0.121	0.115
-1.0	0.208	0.191	0.175	0.165	0.159
-0.9	0.230	0.216	0.200	0.190	0.184
-0.8	0.255	0.242	0.226	0.218	0.212
-0.7	0.281	0.269	0.255	0.247	0.242
-0.6	0.309	0.300	0.286	0.279	0.274
-0.5	0.338	0.330	0.319	0.313	0.309
-0.4	0.369	0.363	0.353	0.348	0.345
-0.3	0.401	0.397	0.389	0.385	0.382
-0.2	0.434	0.430	0.425	0.423	0.421
-0.1	0.466	0.465	0.463	0.461	0.460
0.0	0.500	0.500	0.500	0.500	0.500
0.1	0.534	0.535	0.537	0.539	0.540
0.2	0.566	0.570	0.575	0.577	0.579
0.3	0.599	0.603	0.611	0.615	0.618
0.4	0.631	0.637	0.647	0.652	0.655
0.5	0.662	0.670	0.681	0.687	0.691
0.6	0.691	0.700	0.714	0.721	0.726
0.7	0.719	0.731	0.745	0.753	0.758
0.8	0.745	0.758	0.774	0.782	0.788
0.9	0.770	0.784	0.800	0.810	0.816
1.0	0.792	0.809	0.825	0.835	0.841
1.2	0.834	0.853	0.869	0.879	0.885
1.4	0.868	0.888	0.905	0.913	0.919
1.6	0.895	0.917	0.931	0.940	0.945
1.8	0.917	0.939	0.952	0.959	0.964
2.0	0.935	0.956	0.967	0.974	0.977
2.5	0.964	0.980	0.989	0.992	0.994
3.0	0.980	0.991	0.996	0.997	0.999
4.0	0.990	0.999	1.000	1.000	1.000

Table 2.5

Estimated Distribution Function of LIML Estimator
T - K = 100, K_2 = 10, α = 0.0

w/δ^2	30	50	100	300	∞
-4.0	0.006	0.002	0.000	0.000	0.000
-3.0	0.016	0.008	0.004	0.002	0.001
-2.5	0.030	0.019	0.011	0.008	0.006
-2.0	0.058	0.042	0.032	0.025	0.023
-1.8	0.075	0.059	0.045	0.039	0.036
-1.6	0.097	0.079	0.066	0.058	0.055
-1.4	0.124	0.107	0.092	0.085	0.081
-1.2	0.158	0.141	0.127	0.120	0.115
-1.0	0.200	0.183	0.172	0.164	0.159
-0.9	0.222	0.208	0.197	0.189	0.184
-0.8	0.248	0.234	0.224	0.217	0.212
-0.7	0.275	0.262	0.253	0.246	0.242
-0.6	0.304	0.293	0.284	0.278	0.274
-0.5	0.333	0.323	0.317	0.312	0.309
-0.4	0.365	0.357	0.352	0.347	0.345
-0.3	0.398	0.392	0.388	0.384	0.382
-0.2	0.432	0.427	0.425	0.422	0.421
-0.1	0.465	0.463	0.462	0.461	0.460
0.0	0.500	0.500	0.500	0.500	0.500
0.1	0.535	0.537	0.538	0.539	0.540
0.2	0.568	0.573	0.575	0.578	0.579
0.3	0.602	0.608	0.612	0.616	0.618
0.4	0.635	0.643	0.648	0.653	0.655
0.5	0.667	0.677	0.683	0.688	0.691
0.6	0.696	0.707	0.716	0.722	0.726
0.7	0.725	0.738	0.747	0.754	0.758
0.8	0.752	0.766	0.776	0.783	0.788
0.9	0.778	0.792	0.803	0.811	0.816
1.0	0.800	0.817	0.828	0.838	0.841
1.2	0.842	0.859	0.873	0.880	0.885
1.4	0.876	0.893	0.908	0.915	0.919
1.6	0.903	0.921	0.934	0.942	0.945
1.8	0.925	0.941	0.955	0.961	0.964
2.0	0.942	0.958	0.968	0.975	0.977
2.5	0.970	0.981	0.989	0.993	0.994
3.0	0.984	0.992	0.996	0.998	0.999
4.0	0.994	0.998	1.000	1.000	1.000

Table 2.6

Estimated Distribution Function of LIML Estimator
T - K = 30, K_2 = 30, α = 0.0

w/δ^2	50	100	300	1000	∞
-4.0	0.023	0.005	0.002	0.000	0.000
-3.0	0.047	0.017	0.006	0.002	0.001
-2.5	0.069	0.034	0.015	0.007	0.006
-2.0	0.108	0.066	0.038	0.026	0.023
-1.8	0.128	0.086	0.054	0.040	0.036
-1.6	0.152	0.109	0.075	0.060	0.055
-1.4	0.182	0.139	0.102	0.087	0.081
-1.2	0.215	0.175	0.139	0.122	0.115
-1.0	0.253	0.217	0.182	0.165	0.159
-0.9	0.274	0.241	0.206	0.191	0.184
-0.8	0.296	0.265	0.234	0.218	0.212
-0.7	0.319	0.291	0.262	0.248	0.242
-0.6	0.342	0.318	0.293	0.280	0.274
-0.5	0.367	0.346	0.325	0.313	0.309
-0.4	0.393	0.376	0.358	0.349	0.345
-0.3	0.419	0.406	0.392	0.385	0.382
-0.2	0.446	0.437	0.427	0.423	0.421
-0.1	0.472	0.469	0.464	0.461	0.460
0.0	0.500	0.500	0.500	0.500	0.500
0.1	0.528	0.531	0.636	0.539	0.540
0.2	0.554	0.563	0.573	0.577	0.579
0.3	0.581	0.594	0.608	0.615	0.618
0.4	0.607	0.624	0.642	0.651	0.655
0.5	0.633	0.654	0.675	0.687	0.691
0.6	0.658	0.682	0.707	0.720	0.726
0.7	0.681	0.709	0.738	0.752	0.758
0.8	0.704	0.735	0.766	0.782	0.788
0.9	0.726	0.759	0.794	0.809	0.816
1.0	0.747	0.783	0.818	0.835	0.841
1.2	0.785	0.825	0.861	0.878	0.885
1.4	0.818	0.861	0.898	0.913	0.919
1.6	0.848	0.891	0.925	0.940	0.945
1.8	0.872	0.914	0.946	0.960	0.964
2.0	0.892	0.934	0.962	0.974	0.977
2.5	0.931	0.966	0.985	0.993	0.994
3.0	0.953	0.983	0.994	0.998	0.999
4.0	0.977	0.995	0.998	1.000	1.000

Table 2.7

Estimated Distribution Function of LIML Estimator
T - K = 100, K_2 = 30, α = 0.0

w/δ^2	50	100	300	1000	∞
-4.0	0.009	0.001	0.000	0.000	0.000
-3.0	0.025	0.007	0.002	0.002	0.001
-2.5	0.045	0.020	0.009	0.007	0.006
-2.0	0.082	0.050	0.029	0.026	0.023
-1.8	0.102	0.069	0.045	0.040	0.036
-1.6	0.127	0.092	0.065	0.060	0.055
-1.4	0.157	0.121	0.093	0.086	0.081
-1.2	0.191	0.157	0.129	0.121	0.115
-1.0	0.231	0.199	0.173	0.163	0.159
-0.9	0.254	0.224	0.198	0.189	0.184
-0.8	0.277	0.249	0.226	0.216	0.212
-0.7	0.301	0.276	0.255	0.246	0.242
-0.6	0.327	0.305	0.287	0.277	0.274
-0.5	0.354	0.335	0.319	0.311	0.309
-0.4	0.382	0.367	0.353	0.347	0.345
-0.3	0.411	0.399	0.389	0.384	0.382
-0.2	0.440	0.432	0.425	0.422	0.421
-0.1	0.470	0.466	0.462	0.461	0.460
0.0	0.500	0.500	0.500	0.500	0.500
0.1	0.530	0.534	0.537	0.539	0.540
0.2	0.560	0.568	0.575	0.578	0.579
0.3	0.589	0.601	0.611	0.616	0.618
0.4	0.618	0.633	0.647	0.653	0.655
0.5	0.646	0.665	0.681	0.689	0.691
0.6	0.673	0.695	0.713	0.723	0.726
0.7	0.699	0.724	0.745	0.754	0.758
0.8	0.723	0.751	0.774	0.784	0.788
0.9	0.746	0.776	0.802	0.811	0.816
1.0	0.769	0.801	0.827	0.837	0.841
1.2	0.809	0.843	0.871	0.879	0.885
1.4	0.843	0.879	0.907	0.914	0.919
1.6	0.873	0.908	0.935	0.940	0.945
1.8	0.898	0.931	0.955	0.960	0.964
2.0	0.918	0.950	0.971	0.974	0.977
2.5	0.955	0.980	0.991	0.993	0.994
3.0	0.975	0.993	0.998	0.998	0.999
4.0	0.991	0.999	1.000	1.000	1.000

Table 2.8

Estimated Distribution Function of LIML Estimator
T - K = 300, K_2 = 30, α = 0.0

w/δ^2	50	100	300	1000	∞
-4.0	0.006	0.001	0.000	0.000	0.000
-3.0	0.020	0.006	0.002	0.002	0.001
-2.5	0.037	0.017	0.009	0.007	0.006
-2.0	0.072	0.044	0.029	0.025	0.023
-1.8	0.091	0.063	0.045	0.038	0.036
-1.6	0.116	0.085	0.065	0.059	0.055
-1.4	0.146	0.115	0.092	0.085	0.081
-1.2	0.181	0.150	0.128	0.120	0.115
-1.0	0.221	0.193	0.171	0.163	0.159
-0.9	0.245	0.218	0.196	0.189	0.184
-0.8	0.268	0.244	0.224	0.215	0.212
-0.7	0.294	0.272	0.253	0.245	0.242
-0.6	0.321	0.300	0.285	0.277	0.274
-0.5	0.348	0.331	0.318	0.311	0.309
-0.4	0.378	0.364	0.352	0.347	0.345
-0.3	0.407	0.397	0.387	0.384	0.382
-0.2	0.438	0.430	0.424	0.422	0.421
-0.1	0.468	0.465	0.462	0.461	0.460
0.0	0.500	0.500	0.500	0.500	0.500
0.1	0.532	0.535	0.538	0.539	0.540
0.2	0.562	0.570	0.576	0.578	0.579
0.3	0.593	0.603	0.613	0.616	0.618
0.4	0.622	0.636	0.648	0.653	0.655
0.5	0.652	0.669	0.682	0.689	0.691
0.6	0.679	0.700	0.715	0.723	0.726
0.7	0.706	0.728	0.747	0.755	0.758
0.8	0.732	0.756	0.776	0.785	0.788
0.9	0.755	0.782	0.804	0.811	0.816
1.0	0.779	0.807	0.829	0.837	0.841
1.2	0.819	0.850	0.872	0.880	0.885
1.4	0.854	0.885	0.908	0.915	0.919
1.6	0.884	0.915	0.935	0.941	0.945
1.8	0.909	0.937	0.955	0.962	0.964
2.0	0.928	0.956	0.971	0.975	0.977
2.5	0.963	0.983	0.991	0.993	0.994
3.0	0.980	0.994	0.998	0.998	0.999
4.0	0.994	0.999	1.000	1.000	1.000

Table 2.9

Estimated Distribution Function of LIML Estimator
$T - K = 10,\ K_2 = 3,\ \alpha = 1.0$

w/δ^2	10	30	50	100	∞
-4.0	.005*	.000	.002	0.000	0.000
-3.0	.006	.000	.002	0.000	0.001
-2.5	.008	.000	.003	0.002	0.006
-2.0	.014	.006	.010	0.011	0.023
-1.8	.021	.015	.019	0.021	0.036
-1.6	.032	.030	.034	0.038	0.055
-1.4	.052	.053	.059	0.063	0.081
-1.2	.085	.088	.094	0.098	0.115
-1.0	.136	.135	.139	0.144	0.159
-0.9	.166	.164	.167	0.171	0.184
-0.8	.199	.195	.198	0.201	0.212
-0.7	.236	.229	.231	0.233	0.242
-0.6	.272	.264	.266	0.268	0.274
-0.5	.311	.302	.303	0.304	0.309
-0.4	.353	.341	.341	0.342	0.345
-0.3	.394	.381	.381	0.380	0.382
-0.2	.432	.421	.421	0.420	0.421
-0.1	.468	.460	.460	0.460	0.460
0.0	.505	.500	.500	0.500	0.500
0.1	.542	.539	.539	0.539	0.540
0.2	.575	.575	.576	0.578	0.579
0.3	.604	.611	.612	0.615	0.618
0.4	.634	.645	.647	0.650	0.655
0.5	.662	.676	.680	0.683	0.691
0.6	.689	.705	.710	0.715	0.726
0.7	.712	.733	.739	0.745	0.758
0.8	.733	.759	.766	0.773	0.788
0.9	.753	.782	.790	0.799	0.816
1.0	.772	.803	.813	0.822	0.841
1.2	.802	.841	.852	0.863	0.885
1.4	.830	.872	.883	0.896	0.919
1.6	.852	.898	.909	0.922	0.945
1.8	.871	.918	.929	0.942	0.964
2.0	.888	.935	.945	0.957	0.977
2.5	.918	.962	.972	0.982	0.994
3.0	.940	.979	.985	0.992	0.999
4.0	.963	.993	.995	0.999	1.000

*Note: Since the exact cdf of the LIMLK estimator is not available in this case, we transform the empirical cdf of the LIML estimator for the case of $\alpha = 0.0$ by (4.3).

Table 2.10

Estimated Distribution Function of LIML Estimator
$T - K = 30,\ K_2 = 3,\ \alpha = 1.0$

w/δ^2	10	30	50	100	∞
-4.0	.003*	.001	.000	0.000	0.000
-3.0	.003	.001	.000	0.000	0.001
-2.5	.005	.002	.001	0.002	0.006
-2.0	.009	.007	.008	0.011	0.023
-1.8	.015	.015	.018	0.021	0.036
-1.6	.025	.030	.033	0.037	0.055
-1.4	.045	.053	.057	0.062	0.081
-1.2	.079	.087	.092	0.097	0.115
-1.0	.127	.133	.138	0.143	0.159
-0.9	.156	.162	.166	0.170	0.184
-0.8	.188	.193	.197	0.200	0.212
-0.7	.224	.227	.230	0.232	0.242
-0.6	.264	.263	.264	0.267	0.274
-0.5	.306	.300	.301	0.303	0.309
-0.4	.347	.339	.340	0.341	0.345
-0.3	.383	.380	.380	0.380	0.382
-0.2	.423	.420	.420	0.420	0.421
-0.1	.461	.460	.460	0.460	0.460
0.0	.499	.500	.500	0.500	0.500
0.1	.535	.539	.539	0.539	0.540
0.2	.569	.576	.577	0.578	0.579
0.3	.602	.612	.613	0.615	0.618
0.4	.631	.646	.648	0.651	0.655
0.5	.659	.678	.681	0.684	0.691
0.6	.686	.706	.712	0.716	0.726
0.7	.713	.734	.741	0.746	0.758
0.8	.735	.760	.767	0.774	0.788
0.9	.756	.784	.791	0.800	0.816
1.0	.775	.805	.814	0.823	0.841
1.2	.808	.843	.853	0.864	0.885
1.4	.835	.874	.885	0.897	0.919
1.6	.856	.899	.911	0.923	0.945
1.8	.875	.919	.931	0.943	0.964
2.0	.891	.935	.947	0.958	0.977
2.5	.922	.962	.973	0.982	0.994
3.0	.941	.978	.986	0.992	0.999
4.0	.966	.992	.997	0.999	1.000

*ibid.

Table 2.11

Estimated Distribution Function of LIML Estimator
$T - K = 10$, $K_2 = 10$, $\alpha = 1.0$

w/δ^2	30	50	100	300	∞
-4.0	.012	.007	0.000	0.000	0.000
-3.0	.012	.007	0.000	0.000	0.001
-2.5	.017	.010	0.006	0.004	0.006
-2.0	.034	.027	0.019	0.018	0.023
-1.8	.047	.041	0.031	0.030	0.036
-1.6	.067	.062	0.049	0.049	0.055
-1.4	.094	.088	0.076	0.076	0.081
-1.2	.130	.124	0.112	0.111	0.115
-1.0	.177	.169	0.159	0.157	0.159
-0.9	.203	.195	0.186	0.182	0.184
-0.8	.232	.224	0.216	0.211	0.212
-0.7	.262	.254	0.247	0.242	0.242
-0.6	.294	.287	0.280	0.275	0.274
-0.5	.328	.320	0.315	0.310	0.309
-0.4	.363	.356	0.352	0.346	0.345
-0.3	.397	.392	0.388	0.384	0.382
-0.2	.431	.428	0.425	0.423	0.421
-0.1	.466	.464	0.463	0.461	0.460
0.0	.500	.500	0.500	0.500	0.500
0.1	.533	.535	0.537	0.538	0.540
0.2	.565	.569	0.572	0.576	0.579
0.3	.596	.602	0.607	0.614	0.618
0.4	.625	.634	0.641	0.649	0.655
0.5	.653	.664	0.673	0.683	0.691
0.6	.678	.692	0.703	0.715	0.726
0.7	.703	.718	0.732	0.745	0.758
0.8	.726	.742	0.759	0.774	0.788
0.9	.747	.764	0.784	0.800	0.816
1.0	.766	.786	0.807	0.825	0.841
1.2	.802	.822	0.848	0.866	0.885
1.4	.831	.853	0.881	0.900	0.919
1.6	.855	.878	0.908	0.927	0.945
1.8	.875	.898	0.930	0.948	0.964
2.0	.893	.916	0.945	0.963	0.977
2.5	.923	.945	0.972	0.986	0.994
3.0	.940	.964	0.984	0.995	0.999
4.0	.964	.983	0.996	0.999	1.000

Table 2.12

Estimated Distribution Function of LIML Estimator
$T - K = 30$, $K_2 = 10$, $\alpha = 1.0$

w/δ^2	30	50	100	300	∞
-4.0	.003	.000	0.000	0.000	0.000
-3.0	.003	.000	0.000	0.000	0.001
-2.5	.005	.002	0.003	0.004	0.006
-2.0	.017	.013	0.014	0.017	0.023
-1.8	.027	.024	0.025	0.028	0.036
-1.6	.046	.042	0.043	0.047	0.055
-1.4	.072	.068	0.069	0.073	0.081
-1.2	.108	.106	0.104	0.108	0.115
-1.0	.157	.154	0.151	0.153	0.159
-0.9	.185	.182	0.178	0.179	0.184
-0.8	.215	.213	0.208	0.208	0.212
-0.7	.247	.245	0.239	0.239	0.242
-0.6	.281	.279	0.273	0.273	0.274
-0.5	.317	.314	0.309	0.308	0.309
-0.4	.354	.351	0.346	0.344	0.345
-0.3	.390	.389	0.384	0.383	0.382
-0.2	.427	.426	0.422	0.422	0.421
-0.1	.463	.463	0.462	0.460	0.460
0.0	.500	.500	0.500	0.500	0.500
0.1	.536	.536	0.538	0.539	0.540
0.2	.569	.571	0.575	0.577	0.579
0.3	.602	.606	0.611	0.615	0.618
0.4	.633	.638	0.646	0.651	0.655
0.5	.663	.669	0.679	0.686	0.691
0.6	.690	.699	0.710	0.717	0.726
0.7	.716	.726	0.740	0.748	0.758
0.8	.740	.751	0.767	0.777	0.788
0.9	.763	.775	0.792	0.803	0.816
1.0	.783	.798	0.815	0.828	0.841
1.2	.820	.837	0.856	0.870	0.885
1.4	.851	.870	0.889	0.903	0.919
1.6	.876	.896	0.916	0.930	0.945
1.8	.897	.932	0.937	0.950	0.964
2.0	.915	.948	0.952	0.964	0.977
2.5	.945	.965	0.978	0.987	0.994
3.0	.964	.981	0.990	0.995	0.999
4.0	.982	.994	0.998	1.000	1.000

Table 2.13

Estimated Distribution Function of LIML Estimator
T - K = 100, K_2 = 10, α = 1.0

w/δ^2	30	50	100	300	∞
-4.0	.001	.001	0.000	0.000	0.000
-3.0	.001	.001	0.000	0.000	0.001
-2.5	.001	.002	0.003	0.003	0.006
-2.0	.011	.012	0.013	0.016	0.023
-1.8	.021	.022	0.024	0.027	0.036
-1.6	.039	.039	0.040	0.045	0.055
-1.4	.064	.065	0.066	0.071	0.081
-1.2	.100	.100	0.101	0.107	0.115
-1.0	.149	.147	0.148	0.152	0.159
-0.9	.177	.174	0.175	0.178	0.184
-0.8	.207	.205	0.205	0.207	0.212
-0.7	.240	.237	0.237	0.238	0.242
-0.6	.275	.272	0.271	0.272	0.274
-0.5	.311	.307	0.307	0.307	0.309
-0.4	.349	.345	0.345	0.343	0.345
-0.3	.387	.384	0.383	0.382	0.382
-0.2	.425	.423	0.422	0.421	0.421
-0.1	.462	.461	0.461	0.460	0.460
0.0	.500	.500	0.500	0.500	0.500
0.1	.537	.538	0.539	0.539	0.540
0.2	.572	.574	0.575	0.578	0.579
0.3	.605	.610	0.612	0.616	0.618
0.4	.637	.644	0.647	0.652	0.655
0.5	.668	.676	0.681	0.686	0.691
0.6	.696	.705	0.712	0.718	0.726
0.7	.722	.733	0.742	0.749	0.758
0.8	.747	.759	0.769	0.778	0.788
0.9	.770	.783	0.795	0.804	0.816
1.0	.791	.806	0.818	0.829	0.841
1.2	.828	.844	0.860	0.871	0.885
1.4	.859	.876	0.893	0.905	0.919
1.6	.884	.902	0.919	0.932	0.945
1.8	.905	.923	0.940	0.952	0.964
2.0	.923	.939	0.955	0.966	0.977
2.5	.952	.967	0.979	0.988	0.994
3.0	.971	.982	0.990	0.996	0.999
4.0	.988	.995	0.998	1.000	1.000

Table 2.14

Estimated Distribution Function of LIML Estimator
T - K = 30, K_2 = 30, α = 1.0

w/δ^2	50	100	300	1000	∞
-4.0	.009	0.001	0.001	0.000	0.000
-3.0	.015	0.005	0.003	0.000	0.001
-2.5	.027	0.014	0.008	0.004	0.006
-2.0	.051	0.035	0.025	0.020	0.023
-1.8	.069	0.051	0.038	0.033	0.036
-1.6	.091	0.073	0.058	0.052	0.055
-1.4	.121	0.102	0.085	0.079	0.081
-1.2	.158	0.138	0.121	0.114	0.115
-1.0	.204	0.184	0.167	0.158	0.159
-0.9	.229	0.210	0.192	0.184	0.184
-0.8	.256	0.239	0.221	0.212	0.212
-0.7	.284	0.268	0.251	0.243	0.242
-0.6	.314	0.299	0.283	0.276	0.274
-0.5	.344	0.331	0.317	0.310	0.309
-0.4	.375	0.364	0.353	0.346	0.345
-0.3	.406	0.397	0.388	0.383	0.382
-0.2	.438	0.432	0.425	0.422	0.421
-0.1	.469	0.467	0.463	0.460	0.460
0.0	.500	0.500	0.500	0.500	0.500
0.1	.531	0.534	0.541	0.539	0.540
0.2	.559	0.566	0.574	0.577	0.579
0.3	.588	0.598	0.609	0.615	0.618
0.4	.616	0.629	0.644	0.651	0.655
0.5	.642	0.658	0.676	0.686	0.691
0.6	.667	0.686	0.707	0.719	0.726
0.7	.691	0.713	0.737	0.750	0.758
0.8	.713	0.738	0.764	0.779	0.788
0.9	.734	0.761	0.791	0.806	0.816
1.0	.753	0.783	0.814	0.831	0.841
1.2	.788	0.821	0.856	0.874	0.885
1.4	.818	0.855	0.890	0.908	0.919
1.6	.843	0.882	0.917	0.935	0.945
1.8	.866	0.905	0.939	0.955	0.964
2.0	.884	0.923	0.954	0.970	0.977
2.5	.917	0.954	0.979	0.991	0.994
3.0	.939	0.971	0.989	0.997	0.999
4.0	.961	0.988	0.997	1.000	1.000

Table 2.15

Estimated Distribution Function of LIML Estimator
T - K = 100, K_2 = 30, α = 1.0

w/δ^2	50	100	300	1000	∞
-4.0	.003	0.000	0.000	0.001	0.000
-3.0	.003	0.000	0.000	0.001	0.001
-2.5	.008	0.004	0.002	0.004	0.006
-2.0	.027	0.021	0.017	0.020	0.023
-1.8	.044	0.035	0.029	0.032	0.036
-1.6	.066	0.056	0.049	0.052	0.055
-1.4	.096	0.084	0.075	0.078	0.081
-1.2	.134	0.120	0.111	0.113	0.115
-1.0	.180	0.167	0.158	0.156	0.159
-0.9	.207	0.193	0.184	0.182	0.184
-0.8	.235	0.222	0.213	0.210	0.212
-0.7	.266	0.253	0.244	0.241	0.242
-0.6	.298	0.285	0.277	0.273	0.274
-0.5	.330	0.319	0.312	0.308	0.309
-0.4	.364	0.354	0.348	0.344	0.345
-0.3	.398	0.390	0.385	0.382	0.382
-0.2	.432	0.427	0.423	0.421	0.421
-0.1	.467	0.464	0.462	0.460	0.460
0.0	.500	0.500	0.500	0.500	0.500
0.1	.533	0.536	0.538	0.539	0.540
0.2	.565	0.571	0.576	0.578	0.579
0.3	.596	0.605	0.612	0.616	0.618
0.4	.626	0.638	0.648	0.653	0.655
0.5	.655	0.669	0.682	0.688	0.691
0.6	.681	0.699	0.713	0.722	0.726
0.7	.707	0.727	0.744	0.752	0.758
0.8	.731	0.753	0.772	0.781	0.788
0.9	.753	0.778	0.799	0.808	0.816
1.0	.774	0.800	0.823	0.833	0.841
1.2	.811	0.839	0.866	0.876	0.885
1.4	.842	0.873	0.900	0.909	0.919
1.6	.868	0.900	0.927	0.935	0.945
1.8	.891	0.922	0.948	0.955	0.964
2.0	.909	0.940	0.964	0.970	0.977
2.5	.943	0.969	0.986	0.990	0.994
3.0	.963	0.984	0.995	0.997	0.999
4.0	.984	0.997	1.000	1.000	1.000

Table 2.16

Estimated Distribution Function of LIML Estimator
T - K = 300, K_2 = 30, α = 1.0

w/δ^2	50	100	300	1000	∞
-4.0	.000	0.000	0.000	0.000	0.000
-3.0	.000	0.000	0.000	0.001	0.001
-2.5	.003	0.002	0.002	0.004	0.006
-2.0	.019	0.016	0.017	0.019	0.023
-1.8	.034	0.030	0.029	0.031	0.036
-1.6	.055	0.050	0.049	0.051	0.055
-1.4	.085	0.077	0.075	0.077	0.081
-1.2	.123	0.114	0.110	0.112	0.115
-1.0	.170	0.160	0.156	0.156	0.159
-0.9	.198	0.187	0.182	0.182	0.184
-0.8	.227	0.217	0.211	0.209	0.212
-0.7	.258	0.248	0.242	0.240	0.242
-0.6	.231	0.280	0.275	0.273	0.274
-0.5	.324	0.315	0.310	0.308	0.309
-0.4	.359	0.351	0.347	0.344	0.345
-0.3	.394	0.388	0.383	0.382	0.382
-0.2	.430	0.425	0.422	0.421	0.421
-0.1	.465	0.463	0.461	0.460	0.460
0.0	.500	0.500	0.500	0.500	0.500
0.1	.535	0.537	0.539	0.539	0.540
0.2	.567	0.573	0.577	0.578	0.579
0.3	.600	0.607	0.614	0.616	0.618
0.4	.630	0.641	0.649	0.653	0.655
0.5	.660	0.673	0.683	0.688	0.691
0.6	.687	0.703	0.715	0.722	0.726
0.7	.714	0.731	0.746	0.753	0.758
0.8	.739	0.758	0.774	0.782	0.788
0.9	.761	0.783	0.801	0.808	0.816
1.0	.783	0.806	0.825	0.833	0.841
1.2	.821	0.846	0.867	0.876	0.885
1.4	.852	0.879	0.901	0.910	0.919
1.6	.879	0.907	0.927	0.936	0.945
1.8	.902	0.929	0.948	0.956	0.964
2.0	.920	0.946	0.963	0.971	0.977
2.5	.953	0.974	0.986	0.991	0.994
3.0	.972	0.988	0.995	0.997	0.999
4.0	.990	0.999	1.000	1.000	1.000

Table 2.17

Estimated Distribution Function of LIML Estimator
T - K = 10, K_2 = 3, α = 2.0

w/δ^2	10	30	50	100	∞
-4.0	.001*	.000	.000	0.000	0.000
-3.0	.001	.000	.000	0.000	0.001
-2.5	.001	.000	.000	0.001	0.006
-2.0	.002	.002	.006	0.008	0.023
-1.8	.002	.007	.013	0.017	0.036
-1.6	.006	.018	.026	0.033	0.055
-1.4	.017	.039	.049	0.057	0.081
-1.2	.045	.074	.083	0.091	0.115
-1.0	.094	.122	.131	0.138	0.159
-0.9	.129	.151	.158	0.166	0.184
-0.8	.165	.184	.190	0.196	0.212
-0.7	.203	.220	.224	0.229	0.242
-0.6	.247	.257	.261	0.265	0.274
-0.5	.290	.297	.300	0.302	0.309
-0.4	.334	.337	.338	0.340	0.345
-0.3	.377	.378	.379	0.379	0.382
-0.2	.419	.420	.420	0.419	0.421
-0.1	.462	.460	.460	0.460	0.460
0.0	.502	.500	.500	0.500	0.500
0.1	.538	.539	.539	0.539	0.540
0.2	.574	.575	.576	0.578	0.579
0.3	.607	.611	.612	0.615	0.618
0.4	.636	.644	.647	0.649	0.655
0.5	.664	.675	.679	0.683	0.691
0.6	.690	.704	.709	0.714	0.726
0.7	.713	.731	.737	0.743	0.758
0.8	.734	.757	.763	0.771	0.788
0.9	.754	.779	.787	0.796	0.816
1.0	.773	.800	.809	0.819	0.841
1.2	.803	.837	.848	0.859	0.885
1.4	.830	.867	.878	0.891	0.919
1.6	.852	.892	.904	0.917	0.945
1.8	.871	.912	.924	0.938	0.964
2.0	.887	.928	.940	0.953	0.977
2.5	.917	.958	.968	0.979	0.994
3.0	.939	.974	.982	0.991	0.999
4.0	.961	.991	.993	0.998	1.000

*ibid.

Table 2.18

Estimated Distribution Function of LIML Estimator
T - K = 30, K_2 = 3, α = 2.0

w/δ^2	10	30	50	100	∞
-4.0	.001*	.001	.000	0.000	0.000
-3.0	.001	.001	.000	0.000	0.001
-2.5	.001	.001	.000	0.001	0.006
-2.0	.001	.003	.004	0.008	0.023
-1.8	.002	.008	.011	0.017	0.036
-1.6	.005	.019	.025	0.032	0.055
-1.4	.016	.039	.047	0.056	0.081
-1.2	.044	.073	.081	0.090	0.115
-1.0	.097	.120	.129	0.137	0.159
-0.9	.129	.149	.157	0.165	0.184
-0.8	.164	.182	.189	0.195	0.212
-0.7	.202	.218	.223	0.228	0.242
-0.6	.246	.256	.259	0.264	0.274
-0.5	.291	.296	.298	0.301	0.309
-0.4	.335	.335	.337	0.339	0.345
-0.3	.377	.377	.378	0.379	0.382
-0.2	.418	.419	.419	0.419	0.421
-0.1	.460	.460	.460	0.460	0.460
0.0	.499	.500	.500	0.500	0.500
0.1	.536	.539	.539	0.539	0.540
0.2	.572	.576	.577	0.578	0.579
0.3	.605	.612	.613	0.615	0.618
0.4	.635	.645	.648	0.650	0.655
0.5	.664	.677	.680	0.684	0.691
0.6	.689	.705	.711	0.715	0.726
0.7	.714	.732	.739	0.744	0.758
0.8	.735	.758	.764	0.772	0.788
0.9	.755	.781	.788	0.797	0.816
1.0	.774	.802	.810	0.820	0.841
1.2	.807	.839	.849	0.860	0.885
1.4	.833	.869	.880	0.892	0.919
1.6	.856	.893	.906	0.918	0.945
1.8	.875	.913	.926	0.939	0.964
2.0	.890	.929	.942	0.954	0.977
2.5	.920	.958	.969	0.979	0.994
3.0	.938	.974	.983	0.991	0.999
4.0	.961	.990	.995	0.998	1.000

*ibid.

Table 2.19

Estimated Distribution Function of LIML Estimator
$T - K = 10,\ K_2 = 10,\ \alpha = 2.0$

w/δ^2	30	50	100	300	∞
-4.0	.012	.007	0.000	0.001	0.000
-3.0	.012	.007	0.000	0.001	0.001
-2.5	.014	.008	0.004	0.003	0.006
-2.0	.024	.020	0.015	0.015	0.023
-1.8	.034	.031	0.025	0.027	0.036
-1.6	.049	.049	0.042	0.045	0.055
-1.4	.074	.074	0.067	0.071	0.081
-1.2	.108	.109	0.103	0.106	0.115
-1.0	.155	.156	0.150	0.153	0.159
-0.9	.184	.183	0.178	0.178	0.184
-0.8	.214	.212	0.209	0.208	0.212
-0.7	.247	.245	0.241	0.240	0.242
-0.6	.282	.278	0.275	0.273	0.274
-0.5	.317	.313	0.311	0.308	0.309
-0.4	.354	.351	0.349	0.345	0.345
-0.3	.391	.388	0.386	0.383	0.382
-0.2	.427	.425	0.424	0.422	0.421
-0.1	.465	.463	0.462	0.461	0.460
0.0	.500	.500	0.500	0.500	0.500
0.1	.534	.536	0.537	0.539	0.540
0.2	.568	.571	0.573	0.576	0.579
0.3	.599	.605	0.608	0.614	0.618
0.4	.629	.636	0.642	0.649	0.655
0.5	.657	.666	0.673	0.682	0.691
0.6	.684	.694	0.704	0.715	0.726
0.7	.708	.720	0.732	0.745	0.758
0.8	.731	.744	0.759	0.773	0.788
0.9	.751	.767	0.783	0.799	0.816
1.0	.771	.787	0.806	0.823	0.841
1.2	.805	.823	0.846	0.864	0.885
1.4	.834	.853	0.879	0.898	0.919
1.6	.858	.878	0.906	0.925	0.945
1.8	.877	.897	0.927	0.946	0.964
2.0	.894	.914	0.943	0.960	0.977
2.5	.923	.943	0.970	0.984	0.994
3.0	.942	.961	0.983	0.994	0.999
4.0	.962	.980	0.994	0.999	1.000

Table 2.20

Estimated Distribution Function of LIML Estimator
$T - K = 30,\ K_2 = 10,\ \alpha = 2.0$

w/δ^2	30	50	100	300	∞
-4.0	.003	.000	0.000	0.000	0.000
-3.0	.003	.000	0.000	0.000	0.001
-2.5	.003	.000	0.001	0.003	0.006
-2.0	.009	.007	0.010	0.014	0.023
-1.8	.015	.015	0.019	0.025	0.036
-1.6	.028	.029	0.036	0.043	0.055
-1.4	.052	.054	0.060	0.068	0.081
-1.2	.086	.090	0.095	0.103	0.115
-1.0	.135	.140	0.142	0.149	0.159
-0.9	.165	.169	0.170	0.175	0.184
-0.8	.197	.200	0.201	0.205	0.212
-0.7	.231	.235	0.233	0.237	0.242
-0.6	.268	.270	0.268	0.271	0.274
-0.5	.306	.307	0.305	0.306	0.309
-0.4	.345	.346	0.343	0.343	0.345
-0.3	.384	.385	0.382	0.382	0.382
-0.2	.423	.423	0.421	0.421	0.421
-0.1	.462	.462	0.461	0.460	0.460
0.0	.500	.500	0.500	0.500	0.500
0.1	.537	.537	0.538	0.540	0.540
0.2	.572	.573	0.576	0.577	0.579
0.3	.605	.608	0.612	0.615	0.618
0.4	.637	.640	0.647	0.651	0.655
0.5	.667	.671	0.679	0.685	0.691
0.6	.695	.701	0.717	0.717	0.726
0.7	.721	.728	0.740	0.748	0.758
0.8	.746	.753	0.767	0.776	0.788
0.9	.767	.777	0.791	0.802	0.816
1.0	.787	.799	0.814	0.826	0.841
1.2	.823	.857	0.854	0.968	0.885
1.4	.854	.870	0.887	0.901	0.919
1.6	.879	.896	0.914	0.928	0.945
1.8	.899	.917	0.934	0.948	0.964
2.0	.916	.935	0.950	0.962	0.977
2.5	.945	.963	0.976	0.985	0.994
3.0	.963	.979	0.988	0.994	0.999
4.0	.981	.992	0.997	1.000	1.000

Table 2.21

Estimated Distribution Function of LIML Estimator
T - K = 100, K_2 = 10, α = 2.0

w/δ^2	30	50	100	300	∞
-4.0	.001	.001	0.000	0.000	0.000
-3.0	.001	.001	0.000	0.000	0.001
-2.5	.000	.001	0.002	0.002	0.006
-2.0	.004	.006	0.009	0.013	0.023
-1.8	.010	.014	0.018	0.024	0.036
-1.6	.021	.027	0.033	0.041	0.055
-1.4	.044	.051	0.057	0.066	0.081
-1.2	.078	.085	0.092	0.102	0.115
-1.0	.127	.133	0.139	0.148	0.159
-0.9	.157	.162	0.167	0.174	0.184
-0.8	.189	.192	0.198	0.204	0.212
-0.7	.224	.227	0.231	0.236	0.242
-0.6	.262	.263	0.266	0.270	0.274
-0.5	.300	.300	0.303	0.305	0.309
-0.4	.340	.340	0.342	0.342	0.345
-0.3	.381	.380	0.381	0.381	0.382
-0.2	.421	.420	0.421	0.420	0.421
-0.1	.461	.460	0.460	0.460	0.460
0.0	.500	.500	0.500	0.500	0.500
0.1	.538	.539	0.539	0.540	0.540
0.2	.575	.576	0.577	0.578	0.579
0.3	.608	.612	0.613	0.616	0.618
0.4	.641	.646	0.648	0.652	0.655
0.5	.672	.678	0.681	0.686	0.691
0.6	.701	.707	0.713	0.718	0.726
0.7	.727	.735	0.742	0.749	0.758
0.8	.752	.761	0.769	0.777	0.788
0.9	.774	.785	0.794	0.803	0.816
1.0	.795	.807	0.817	0.827	0.841
1.2	.831	.844	0.857	0.869	0.885
1.4	.862	.876	0.891	0.903	0.919
1.6	.887	.902	0.917	0.930	0.945
1.8	.907	.922	0.937	0.950	0.964
2.0	.924	.939	0.953	0.964	0.977
2.5	.953	.966	0.977	0.986	0.994
3.0	.970	.981	0.989	0.995	0.999
4.0	.987	.994	0.997	1.000	1.000

Table 2.22

Estimated Distribution Function of LIML Estimator
T - K = 30, K_2 = 30, α = 2.0

w/δ^2	50	100	300	1000	∞
-4.0	.009	0.001	0.001	0.000	0.000
-3.0	.012	0.005	0.002	0.001	0.001
-2.5	.021	0.010	0.007	0.004	0.006
-2.0	.038	0.027	0.022	0.018	0.023
-1.8	.051	0.040	0.034	0.030	0.036
-1.6	.069	0.059	0.052	0.049	0.055
-1.4	.096	0.086	0.079	0.076	0.081
-1.2	.132	0.123	0.115	0.111	0.115
-1.0	.178	0.170	0.160	0.155	0.159
-0.9	.205	0.197	0.186	0.182	0.184
-0.8	.233	0.225	0.215	0.210	0.212
-0.7	.264	0.256	0.246	0.241	0.242
-0.6	.296	0.289	0.279	0.274	0.274
-0.5	.329	0.322	0.314	0.309	0.309
-0.4	.363	0.357	0.350	0.345	0.345
-0.3	.398	0.393	0.386	0.382	0.382
-0.2	.432	0.429	0.424	0.422	0.421
-0.1	.466	0.465	0.462	0.460	0.460
0.0	.500	0.500	0.500	0.500	0.500
0.1	.533	0.538	0.537	0.540	0.540
0.2	.564	0.569	0.575	0.578	0.579
0.3	.595	0.602	0.610	0.616	0.618
0.4	.625	0.633	0.645	0.651	0.655
0.5	.652	0.663	0.677	0.686	0.691
0.6	.678	0.692	0.709	0.719	0.726
0.7	.702	0.719	0.738	0.750	0.758
0.8	.725	0.744	0.766	0.779	0.788
0.9	.746	0.767	0.792	0.806	0.816
1.0	.766	0.789	0.816	0.831	0.841
1.2	.801	0.827	0.856	0.873	0.885
1.4	.830	0.859	0.890	0.907	0.919
1.6	.855	0.886	0.917	0.934	0.945
1.8	.975	0.907	0.937	0.954	0.964
2.0	.892	0.924	0.954	0.969	0.977
2.5	.923	0.954	0.978	0.990	0.994
3.0	.941	0.971	0.989	0.997	0.999
4.0	.961	0.986	0.996	1.000	1.000

Table 2.23

Estimated Distribution Function of LIML Estimator
T - K = 100, K_2 = 30, α = 2.0

w/δ^2	50	100	300	1000	∞
-4.0	.003	0.000	0.000	0.000	0.000
-3.0	.003	0.000	0.000	0.001	0.001
-2.5	.003	0.002	0.001	0.004	0.006
-2.0	.015	0.013	0.014	0.018	0.023
-1.8	.027	0.025	0.025	0.030	0.036
-1.6	.044	0.042	0.043	0.049	0.055
-1.4	.070	0.069	0.069	0.075	0.081
-1.2	.107	0.105	0.105	0.110	0.115
-1.0	.155	0.152	0.151	0.153	0.159
-0.9	.183	0.179	0.178	0.180	0.184
-0.8	.213	0.209	0.207	0.208	0.212
-0.7	.245	0.241	0.239	0.239	0.242
-0.6	.280	0.275	0.275	0.271	0.274
-0.5	.315	0.310	0.309	0.307	0.309
-0.4	.352	0.347	0.345	0.343	0.345
-0.3	.389	0.385	0.383	0.381	0.382
-0.2	.426	0.424	0.422	0.421	0.421
-0.1	.464	0.462	0.461	0.460	0.460
0.0	.500	0.500	0.500	0.500	0.500
0.1	.535	0.537	0.538	0.540	0.540
0.2	.570	0.574	0.577	0.579	0.579
0.3	.603	0.609	0.613	0.617	0.618
0.4	.635	0.642	0.649	0.653	0.655
0.5	.665	0.674	0.683	0.688	0.691
0.6	.692	0.705	0.715	0.722	0.726
0.7	.718	0.733	0.745	0.752	0.758
0.8	.743	0.759	0.774	0.781	0.788
0.9	.765	0.784	0.800	0.808	0.816
1.0	.786	0.806	0.824	0.833	0.841
1.2	.823	0.845	0.866	0.875	0.885
1.4	.854	0.877	0.900	0.908	0.919
1.6	.880	0.904	0.927	0.934	0.945
1.8	.900	0.925	0.947	0.954	0.964
2.0	.917	0.942	0.963	0.969	0.977
2.5	.948	0.970	0.986	0.989	0.994
3.0	.966	0.985	0.996	0.997	0.999
4.0	.984	0.997	0.999	1.000	1.000

Table 2.24

Estimated Distribution Function of LIML Estimator
T - K = 300, K_2 = 30, α = 2.0

w/δ^2	50	100	300	1000	∞
-4.0	.000	0.000	0.000	0.000	0.000
-3.0	.000	0.000	0.000	0.001	0.001
-2.5	.000	0.000	0.001	0.004	0.006
-2.0	.007	0.009	0.014	0.017	0.023
-1.8	.018	0.020	0.025	0.029	0.036
-1.6	.034	0.037	0.043	0.048	0.055
-1.4	.060	0.062	0.069	0.074	0.081
-1.2	.096	0.099	0.104	0.109	0.115
-1.0	.144	0.145	0.149	0.153	0.159
-0.9	.173	0.173	0.176	0.180	0.184
-0.8	.204	0.203	0.205	0.207	0.212
-0.7	.237	0.236	0.237	0.238	0.242
-0.6	.273	0.270	0.271	0.271	0.274
-0.5	.309	0.306	0.307	0.307	0.309
-0.4	.347	0.344	0.344	0.343	0.345
-0.3	.385	0.383	0.381	0.381	0.382
-0.2	.424	0.422	0.421	0.421	0.421
-0.1	.462	0.461	0.460	0.460	0.460
0.0	.500	0.500	0.500	0.500	0.500
0.1	.537	0.538	0.539	0.540	0.540
0.2	.572	0.576	0.578	0.579	0.579
0.3	.607	0.611	0.615	0.617	0.618
0.4	.639	0.645	0.650	0.653	0.655
0.5	.670	0.678	0.684	0.688	0.691
0.6	.699	0.709	0.717	0.722	0.726
0.7	.725	0.737	0.747	0.753	0.758
0.8	.751	0.764	0.776	0.782	0.788
0.9	.774	0.789	0.802	0.808	0 816
1.0	.795	0.812	0.826	0.833	0.841
1.2	.833	0.852	0.867	0.875	0.885
1.4	.865	0.883	0.901	0.909	0.919
1.6	.890	0.910	0.927	0.935	0.945
1.8	.911	0.932	0.947	0.955	0.964
2.0	.928	0.948	0.963	0.970	0.977
2.5	.959	0.975	0.985	0.990	0.994
3.0	.976	0.989	0.995	0.997	0.999
4.0	.991	0.999	1.000	1.000	1.000

Table 2.25

Estimated Distribution Function of LIML Estimator
T - K = 10, K_2 = 3, α = 5.0

w/δ^2	10	30	50	100	∞
-4.0	.001*	.000	.002	0.000	0.000
-3.0	.001	.000	.002	0.000	0.001
-2.5	.001	.000	.002	0.000	0.006
-2.0	.001	.001	.005	0.006	0.023
-1.8	.001	.004	.010	0.015	0.036
-1.6	.001	.014	.022	0.030	0.055
-1.4	.007	.033	.044	0.053	0.081
-1.2	.031	.067	.078	0.088	0.115
-1.0	.077	.115	.126	0.135	0.159
-0.9	.111	.145	.154	0.165	0.184
-0.8	.149	.179	.186	0.194	0.212
-0.7	.190	.215	.221	0.227	0.242
-0.6	.233	.253	.259	0.263	0.274
-0.5	.280	.294	.298	0.301	0.309
-0.4	.326	.336	.337	0.340	0.345
-0.3	.376	.377	.378	0.379	0.382
-0.2	.420	.419	.420	0.419	0.421
-0.1	.464	.460	.460	0.460	0.460
0.0	.507	.500	.500	0.500	0.500
0.1	.544	.539	.539	0.539	0.540
0.2	.577	.575	.576	0.578	0.579
0.3	.609	.611	.612	0.615	0.618
0.4	.639	.644	.647	0.649	0.655
0.5	.667	.675	.679	0.682	0.691
0.6	.694	.704	.708	0.713	0.726
0.7	.717	.731	.736	0.743	0.758
0.8	.739	.756	.762	0.770	0.788
0.9	.758	.777	.786	0.795	0.816
1.0	.776	.798	.808	0.818	0.841
1.2	.806	.835	.846	0.857	0.885
1.4	.832	.865	.876	0.890	0.919
1.6	.855	.890	.901	0.916	0.945
1.8	.874	.910	.922	0.936	0.964
2.0	.890	.926	.938	0.952	0.977
2.5	.919	.955	.965	0.978	0.994
3.0	.939	.972	.981	0.990	0.999
4.0	.963	.989	.993	0.998	1.000

*ibid.

Table 2.26

Estimated Distribution Function of LIML Estimator
T - K = 30, K_2 = 3, α = 5.0

w/δ^2	10	30	50	100	∞
-4.0	.001*	.001	.000	0.000	0.000
-3.0	.001	.001	.000	0.000	0.001
-2.5	.001	.001	.000	0.000	0.006
-2.0	.001	.002	.003	0.006	0.023
-1.8	.001	.005	.008	0.015	0.036
-1.6	.001	.014	.021	0.029	0.055
-1.4	.008	.033	.042	0.052	0.081
-1.2	.032	.066	.076	0.087	0.115
-1.0	.079	.113	.124	0.134	0.159
-0.9	.112	.143	.153	0.164	0.184
-0.8	.151	.177	.185	0.193	0.212
-0.7	.191	.213	.220	0.226	0.242
-0.6	.236	.252	.257	0.262	0.274
-0.5	.285	.293	.296	0.300	0.309
-0.4	.332	.334	.336	0.339	0.345
-0.3	.378	.376	.377	0.379	0.382
-0.2	.423	.418	.419	0.419	0.421
-0.1	.465	.460	.460	0.460	0.460
0.0	.507	.500	.500	0.500	0.500
0.1	.544	.539	.539	0.539	0.540
0.2	.580	.576	.577	0.578	0.579
0.3	.610	.612	.613	0.615	0.618
0.4	.639	.645	.648	0.650	0.655
0.5	.668	.677	.680	0.683	0.691
0.6	.696	.705	.710	0.714	0.726
0.7	.718	.732	.738	0.744	0.758
0.8	.749	.757	.763	0.771	0.788
0.9	.759	.779	.787	0.796	0.816
1.0	.777	.800	.809	0.819	0.841
1.2	.808	.837	.847	0.858	0.885
1.4	.835	.867	.878	0.891	0.919
1.6	.858	.892	.903	0.917	0.945
1.8	.877	.912	.924	0.937	0.964
2.0	.894	.927	.940	0.953	0.977
2.5	.922	.955	.967	0.978	0.994
3.0	.942	.972	.982	0.990	0.999
4.0	.964	.989	.995	0.998	1.000

*ibid.

Table 2.27

Estimated Distribution Function of LIML Estimator
T - K = 10, K_2 = 10, α = 5.0

w/ δ^2	30	50	100	300	∞
-4.0	.012	.007	0.000	0.001	0.000
-3.0	.012	.007	0.000	0.001	0.001
-2.5	.013	.007	0.002	0.003	0.006
-2.0	.021	.017	0.014	0.014	0.023
-1.8	.029	.027	0.023	0.026	0.036
-1.6	.042	.043	0.038	0.043	0.055
-1.4	.064	.068	0.063	0.069	0.081
-1.2	.098	.103	0.098	0.104	0.115
-1.0	.145	.149	0.146	0.151	0.159
-0.9	.173	.176	0.174	0.177	0.184
-0.8	.205	.206	0.205	0.206	0.212
-0.7	.238	.239	0.238	0.238	0.242
-0.6	.274	.273	0.272	0.271	0.274
-0.5	.311	.310	0.309	0.308	0.309
-0.4	.349	.348	0.347	0.344	0.345
-0.3	.387	.386	0.385	0.382	0.382
-0.2	.425	.424	0.423	0.422	0.421
-0.1	.464	.463	0.462	0.461	0.460
0.0	.500	.500	0.500	0.500	0.500
0.1	.535	.536	0.537	0.539	0.540
0.2	.569	.572	0.574	0.576	0.579
0.3	.601	.606	0.608	0.614	0.618
0.4	.632	.637	0.643	0.649	0.655
0.5	.660	.667	0.684	0.682	0.691
0.6	.687	.696	0.704	0.715	0.726
0.7	.712	.722	0.732	0.744	0.758
0.8	.735	.745	0.759	0.773	0.788
0.9	.755	.768	0.784	0.799	0.816
1.0	.774	.788	0.806	0.823	0.841
1.2	.808	.824	0.846	0.863	0.885
1.4	.837	.854	0.878	0.898	0.919
1.6	.860	.878	0.905	0.924	0.945
1.8	.879	.897	0.926	0.945	0.964
2.0	.895	.913	0.942	0.960	0.977
2.5	.924	.943	0.968	0.984	0.994
3.0	.942	.960	0.983	0.994	0.999
4.0	.962	.979	0.993	0.998	1.000

Table 2.28

Estimated Distribution Function of LIML Estimator
T - K = 30, K_2 = 10, α = 5.0

w/ δ^2	30	50	100	300	∞
-4.0	.003	.000	0.000	0.000	0.000
-3.0	.003	.000	0.000	0.000	0.001
-2.5	.003	.000	0.000	0.003	0.006
-2.0	.006	.005	0.008	0.013	0.023
-1.8	.001	.012	0.017	0.024	0.036
-1.6	.022	.024	0.031	0.041	0.055
-1.4	.042	.048	0.056	0.066	0.081
-1.2	.076	.083	0.090	0.101	0.115
-1.0	.124	.133	0.138	0.147	0.159
-0.9	.154	.162	0.166	0.174	0.184
-0.8	.187	.194	0.197	0.203	0.212
-0.7	.222	.229	0.230	0.235	0.242
-0.6	.260	.265	0.265	0.269	0.274
-0.5	.300	.303	0.303	0.306	0.309
-0.4	.340	.343	0.341	0.342	0.345
-0.3	.380	.383	0.381	0.381	0.382
-0.2	.421	.422	0.420	0.421	0.421
-0.1	.461	.462	0.461	0.460	0.460
0.0	.500	.500	0.500	0.500	0.500
0.1	.538	.537	0.538	0.540	0.540
0.2	.573	.574	0.577	0.577	0.579
0.3	.607	.609	0.612	0.615	0.618
0.4	.640	.641	0.648	0.651	0.655
0.5	.670	.672	0.680	0.685	0.691
0.6	.698	.703	0.711	0.717	0.726
0.7	.724	.730	0.740	0.747	0.758
0.8	.749	.755	0.767	0.770	0.788
0.9	.770	.778	0.792	0.802	0.816
1.0	.790	.800	0.814	0.826	0.841
1.2	.827	.838	0.854	0.867	0.885
1.4	.856	.871	0.886	0.901	0.919
1.6	.881	.896	0.913	0.927	0.945
1.8	.901	.917	0.934	0.947	0.964
2.0	.917	.933	0.949	0.962	0.977
2.5	.946	.963	0.975	0.985	0.994
3.0	.964	.978	0.988	0.994	0.999
4.0	.882	.992	0.997	0.999	1.000

Table 2.29

Estimated Distribution Function of LIML Estimator
T - K = 100, K_2 = 10, α = 5.0

w/δ^2	30	50	100	300	∞
-4.0	.001	.001	0.000	0.000	0.000
-3.0	.001	.001	0.000	0.000	0.001
-2.5	.000	.001	0.001	0.002	0.006
-2.0	.002	.004	0.008	0.012	0.023
-1.8	.005	.011	0.016	0.023	0.036
-1.6	.015	.022	0.029	0.039	0.055
-1.4	.035	.045	0.053	0.064	0.081
-1.2	.068	.078	0.087	0.100	0.115
-1.0	.116	.126	0.134	0.146	0.159
-0.9	.146	.155	0.163	0.173	0.184
-0.8	.179	.186	0.194	0.202	0.212
-0.7	.215	.221	0.228	0.234	0.242
-0.6	.254	.258	0.263	0.268	0.274
-0.5	.294	.297	0.301	0.305	0.309
-0.4	.335	.337	0.340	0.341	0.345
-0.3	.377	.378	0.380	0.380	0.382
-0.2	.419	.419	0.420	0.420	0.421
-0.1	.460	.460	0.460	0.460	0.460
0.0	.500	.500	0.500	0.500	0.500
0.1	.539	.539	0.539	0.540	0.540
0.2	.576	.577	0.577	0.578	0.579
0.3	.610	.613	0.613	0.616	0.618
0.4	.644	.647	0.649	0.652	0.655
0.5	.675	.679	0.682	0.686	0.691
0.6	.704	.709	0.713	0.718	0.726
0.7	.730	.737	0.742	0.748	0.758
0.8	.755	.762	0.769	0.777	0.788
0.9	.777	.786	0.794	0.803	0.816
1.0	.798	.808	0.817	0.827	0.841
1.2	.834	.845	0.857	0.868	0.885
1.4	.864	.877	0.890	0.902	0.919
1.6	.889	.902	0.916	0.929	0.945
1.8	.909	.922	0.937	0.949	0.964
2.0	.925	.938	0.952	0.964	0.977
2.5	.954	.966	0.977	0.986	0.994
3.0	.971	.980	0.990	0.995	0.999
4.0	.988	.993	0.998	0.999	1.000

Table 2.30

Estimated Distribution Function of LIML Estimator
T - K = 30, K_2 = 30, α = 5.0

w/δ^2	50	100	300	1000	∞
-4.0	.009	0.001	0.001	0.000	0.000
-3.0	.011	0.003	0.002	0.000	0.001
-2.5	.019	0.009	0.006	0.003	0.006
-2.0	.034	0.024	0.020	0.018	0.023
-1.8	.044	0.036	0.032	0.029	0.036
-1.6	.060	0.054	0.050	0.048	0.055
-1.4	.084	0.079	0.076	0.075	0.081
-1.2	.118	0.115	0.111	0.109	0.115
-1.0	.164	0.162	0.157	0.154	0.159
-0.9	.191	0.189	0.183	0.180	0.184
-0.8	.220	0.218	0.212	0.209	0.212
-0.7	.252	0.249	0.240	0.240	0.242
-0.6	.285	0.283	0.277	0.274	0.274
-0.5	.320	0.318	0.312	0.308	0.309
-0.4	.356	0.353	0.348	0.345	0.345
-0.3	.392	0.390	0.385	0.382	0.382
-0.2	.428	0.427	0.423	0.421	0.421
-0.1	.464	0.464	0.462	0.460	0.460
0.0	.500	0.500	0.500	0.500	0.500
0.1	.535	0.536	0.538	0.540	0.540
0.2	.567	0.570	0.575	0.578	0.579
0.3	.599	0.604	0.611	0.616	0.618
0.4	.630	0.636	0.646	0.652	0.655
0.5	.658	0.666	0.678	0.687	0.691
0.6	.685	0.695	0.710	0.719	0.726
0.7	.710	0.723	0.739	0.750	0.758
0.8	.734	0.748	0.768	0.779	0.788
0.9	.755	0.771	0.793	0.806	0.816
1.0	.774	0.792	0.817	0.831	0.841
1.2	.809	0.831	0.857	0.873	0.885
1.4	.839	0.862	0.890	0.907	0.919
1.6	.862	0.888	0.917	0.934	0.945
1.8	.882	0.909	0.937	0.954	0.964
2.0	.898	0.926	0.954	0.968	0.977
2.5	.926	0.954	0.978	0.990	0.994
3.0	.944	0.970	0.989	0.997	0.999
4.0	.961	0.985	0.996	1.000	1.000

Table 2.31

Estimated Distribution Function of LIML Estimator
T - K = 100, K_2 = 30, α = 5.0

w/ δ^2	50	100	300	1000	∞
-4.0	.003	0.000	0.000	0.000	0.000
-3.0	.003	0.000	0.000	0.000	0.001
-2.5	.003	0.001	0.001	0.003	0.006
-2.0	.011	0.011	0.012	0.018	0.023
-1.8	.020	0.020	0.023	0.029	0.036
-1.6	.035	0.037	0.041	0.048	0.055
-1.4	.058	0.062	0.066	0.074	0.081
-1.2	.093	0.097	0.101	0.108	0.115
-1.0	.141	0.144	0.148	0.152	0.159
-0.9	.169	0.171	0.175	0.178	0.184
-0.8	.200	0.202	0.204	0.207	0.212
-0.7	.233	0.234	0.236	0.238	0.242
-0.6	.269	0.269	0.271	0.271	0.274
-0.5	.306	0.306	0.307	0.306	0.309
-0.4	.345	0.343	0.343	0.343	0.345
-0.3	.383	0.382	0.382	0.381	0.382
-0.2	.422	0.422	0.421	0.420	0.421
-0.1	.462	0.461	0.461	0.460	0.460
0.0	.500	0.500	0.500	0.500	0.500
0.1	.537	0.538	0.539	0.540	0.540
0.2	.573	0.575	0.577	0.579	0.579
0.3	.607	0.611	0.614	0.617	0.618
0.4	.640	0.645	0.650	0.654	0.655
0.5	.671	0.677	0.684	0.689	0.691
0.6	.699	0.708	0.716	0.722	0.726
0.7	.726	0.737	0.746	0.752	0.758
0.8	.751	0.763	0.775	0.781	0.788
0.9	.774	0.788	0.801	0.808	0.816
1.0	.795	0.809	0.825	0.833	0.841
1.2	.831	0.848	0.867	0.875	0.885
1.4	.862	0.880	0.900	0.908	0.919
1.6	.887	0.906	0.926	0.934	0.945
1.8	.907	0.927	0.947	0.954	0.964
2.0	.923	0.944	0.963	0.969	0.977
2.5	.952	0.970	0.986	0.989	0.994
3.0	.969	0.984	0.996	0.997	0.999
4.0	.984	0.996	0.999	1.000	1.000

Table 2.32

Estimated Distribution Function of LIML Estimator
T - K = 300, K_2 = 30, α = 5.0

w/ δ^2	50	100	300	1000	∞
-4.0	.000	0.000	0.000	0.000	0.000
-3.0	.000	0.000	0.000	0.000	0.001
-2.5	.000	0.000	0.001	0.003	0.006
-2.0	.004	0.007	0.012	0.017	0.023
-1.8	.011	0.016	0.023	0.028	0.036
-1.6	.025	0.032	0.041	0.047	0.055
-1.4	.047	0.055	0.066	0.073	0.081
-1.2	.082	0.091	0.100	0.107	0.115
-1.0	.130	0.137	0.146	0.152	0.159
-0.9	.159	0.165	0.173	0.178	0.184
-0.8	.191	0.196	0.202	0.206	0.212
-0.7	.225	0.229	0.234	0.237	0.242
-0.6	.262	0.265	0.269	0.271	0.274
-0.5	.300	0.302	0.305	0.306	0.309
-0.4	.340	0.340	0.342	0.343	0.345
-0.3	.379	0.380	0.380	0.381	0.382
-0.2	.420	0.420	0.420	0.420	0.421
-0.1	.460	0.460	0.460	0.460	0.460
0.0	.500	0.500	0.500	0.500	0.500
0.1	.539	0.539	0.540	0.540	0.540
0.2	.575	0.577	0.578	0.579	0.579
0.3	.611	0.613	0.616	0.617	0.618
0.4	.644	0.648	0.651	0.654	0.655
0.5	.676	0.681	0.685	0.689	0.691
0.6	.706	0.712	0.718	0.722	0.726
0.7	.733	0.741	0.748	0.653	0.758
0.8	.759	0.768	0.777	0.782	0.788
0.9	.783	0.793	0.803	0.808	0.816
1.0	.803	0.815	0.827	0.833	0.841
1.2	.841	0.855	0.868	0.875	0.885
1.4	.873	0.886	0.901	0.909	0.919
1.6	.897	0.912	0.927	0.935	0.945
1.8	.918	0.934	0.947	0.955	0.964
2.0	.934	0.950	0.963	0.970	0.977
2.5	.963	0.976	0.986	0.990	0.994
3.0	.979	0.989	0.995	0.997	0.999
4.0	.992	0.999	1.000	1.000	1.000

Table 3.1

Comparisons of Estimators of F_{LI}

	$\delta^2 = 30$						$\delta^2 = 50$					
	T - K = 10, K_2 = 3			T - K = 30, K_2 = 3			T - K = 10, K_2 = 3			T - K = 30, K_2 = 3		
	1	2	3	1	2	3	1	2	3	1	2	3
-4.0	.002	.003	.001	.002	.002	.002	.001	.000	.002	.000	.000	.000
-3.0	.008	.008	.006	.007	.007	.007	.004	.005	.006	.003	.004	.004
-2.5	.017	.018	.015	.015	.016	.016	.011	.012	.013	.010	.012	.011
-2.0	.040	.039	.038	.035	.037	.038	.032	.032	.032	.028	.030	.031
-1.8	.055	.054	.052	.050	.052	.052	.047	.046	.047	.043	.045	.045
-1.6	.076	.075	.072	.070	.071	.072	.067	.065	.066	.064	.064	.064
-1.4	.101	.100	.099	.097	.097	.098	.093	.091	.092	.089	.089	.090
-1.2	.134	.133	.132	.133	.130	.130	.129	.126	.126	.124	.124	.124
-1.0	.177	.174	.174	.177	.173	.172	.171	.169	.168	.165	.168	.167
- .9	.202	.198	.199	.202	.198	.197	.197	.194	.193	.190	.193	.192
- .8	.229	.225	.225	.230	.225	.223	.225	.222	.220	.219	.219	.219
- .7	.258	.255	.253	.258	.253	.252	.255	.251	.250	.249	.249	.249
- .6	.290	.285	.284	.286	.282	.283	.286	.282	.281	.284	.281	.279
- .5	.321	.318	.317	.321	.316	.315	.318	.314	.315	.316	.315	.313
- .4	.358	.354	.352	.355	.352	.350	.353	.349	.349	.350	.348	.348
- .3	.393	.389	.387	.390	.386	.386	.390	.386	.386	.386	.384	.385
- .2	.430	.426	.424	.425	.424	.423	.428	.423	.423	.423	.423	.422
- .1	.467	.466	.462	.463	.463	.462	.465	.462	.461	.460	.461	.461
.0	.506	.502	.500	.503	.501	.500	.503	.498	.500	.499	.502	.500
.1	.543	.539	.538	.542	.540	.538	.541	.537	.539	.535	.539	.539
.2	.575	.576	.576	.579	.576	.577	.578	.577	.577	.573	.579	.578
.3	.612	.614	.613	.615	.613	.614	.617	.614	.614	.609	.615	.615
.4	.645	.649	.648	.652	.651	.650	.653	.651	.651	.646	.652	.652
.5	.678	.684	.683	.686	.685	.685	.688	.686	.685	.680	.687	.687
.6	.710	.717	.716	.717	.719	.717	.722	.721	.719	.716	.721	.721
.7	.740	.746	.747	.746	.748	.748	.752	.752	.750	.747	.751	.751
.8	.768	.775	.775	.775	.777	.777	.781	.780	.780	.777	.782	.781
.9	.794	.799	.801	.802	.806	.803	.808	.807	.807	.802	.809	.808
1.0	.820	.825	.826	.827	.829	.828	.832	.832	.832	.827	.834	.833
1.2	.863	.866	.868	.868	.870	.870	.875	.875	.874	.870	.877	.876
1.4	.899	.899	.901	.901	.902	.902	.909	.909	.908	.905	.911	.910
1.6	.924	.924	.928	.928	.929	.928	.936	.936	.934	.931	.935	.936
1.8	.945	.945	.948	.948	.948	.948	.956	.954	.953	.953	.955	.955
2.0	.961	.961	.962	.963	.963	.962	.969	.968	.968	.968	.970	.969
2.5	.983	.983	.985	.985	.985	.984	.989	.988	.987	.989	.988	.989
3.0	.992	.992	.994	.992	.993	.993	.996	.995	.994	.997	.996	.996
4.0	.997	.998	.999	.998	.998	.998	1.000	1.000	.998	1.000	1.000	1.000

1 = The empirical F_{LI}

2 = F_K + [empirical F_{LI} - empirical F_K]

3 = Estimator (4.2)

Table 3.2

Comparisons of Estimators of F_{LI}

$\delta^2 = 30$

	T - K = 10, K_2 = 10			T - K = 30, K_2 = 10			T - K = 100, K_2 = 10		
	1	2	3	1	2	3	1	2	3
-4.0	.021	.023	.020	.009	.010	.010	.006	.008	.006
-3.0	.039	.038	.036	.022	.022	.020	.017	.016	.016
-2.5	.057	.057	.054	.036	.036	.036	.032	.031	.030
-2.0	.090	.088	.086	.065	.064	.065	.059	.058	.058
-1.8	.108	.107	.105	.082	.082	.083	.076	.076	.075
-1.6	.131	.132	.127	.105	.105	.105	.097	.098	.097
-1.4	.160	.159	.154	.134	.133	.132	.126	.125	.124
-1.2	.194	.192	.186	.168	.168	.166	.161	.160	.158
-1.0	.233	.231	.226	.208	.206	.208	.202	.200	.200
- .9	.255	.250	.247	.232	.228	.230	.227	.224	.222
- .8	.278	.275	.271	.258	.257	.255	.252	.249	.248
- .7	.302	.298	.296	.284	.282	.281	.280	.275	.275
- .6	.327	.323	.322	.312	.310	.309	.309	.303	.304
- .5	.355	.350	.349	.342	.340	.338	.340	.334	.333
- .4	.382	.376	.378	.372	.370	.369	.371	.366	.365
- .3	.409	.404	.408	.403	.400	.401	.402	.398	.398
- .2	.439	.434	.438	.436	.431	.434	.434	.432	.432
- .1	.468	.464	.469	.468	.465	.466	.466	.464	.465
.0	.499	.495	.500	.501	.497	.500	.500	.497	.500
.1	.530	.528	.531	.535	.532	.534	.535	.533	.535
.2	.560	.559	.562	.569	.566	.566	.568	.566	.568
.3	.590	.592	.592	.601	.600	.599	.600	.602	.602
.4	.619	.612	.622	.633	.632	.631	.632	.634	.635
.5	.647	.651	.651	.662	.662	.662	.663	.667	.667
.6	.673	.678	.678	.691	.690	.691	.692	.695	.696
.7	.701	.706	.704	.720	.718	.719	.723	.728	.725
.8	.725	.731	.729	.745	.744	.745	.749	.754	.752
.9	.749	.757	.753	.769	.770	.770	.773	.779	.778
1.0	.771	.779	.774	.790	.792	.792	.795	.801	.800
1.2	.811	.819	.814	.831	.831	.834	.837	.843	.842
1.4	.845	.850	.846	.866	.866	.868	.871	.875	.876
1.6	.873	.875	.873	.894	.894	.895	.901	.902	.903
1.8	.896	.899	.895	.918	.916	.917	.925	.927	.925
2.0	.916	.917	.914	.934	.934	.935	.943	.943	.942
2.5	.946	.946	.946	.964	.963	.964	.971	.970	.970
3.0	.965	.964	.964	.979	.979	.980	.984	.984	.984
4.0	.981	.979	.980	.991	.989	.990	.995	.993	.994

Table 3.3

Comparisons of Estimators of F_{LI}

$\delta^2 = 50$

	T - K = 10, K_2 = 10			T - K = 30, K_2 = 10			T - K = 100, K_2 = 10		
	1	2	3	1	2	3	1	2	3
-4.0	.008	.008	.008	.002	.002	.001	.001	.002	.002
-3.0	.019	.020	.018	.008	.010	.009	.007	.008	.008
-2.5	.033	.034	.034	.021	.021	.020	.017	.019	.019
-2.0	.061	.063	.064	.047	.044	.044	.041	.043	.042
-1.8	.080	.080	.082	.064	.063	.061	.057	.058	.059
-1.6	.103	.104	.104	.087	.086	.083	.078	.079	.079
-1.4	.134	.134	.131	.115	.112	.112	.107	.108	.107
-1.2	.166	.167	.164	.151	.146	.147	.143	.143	.141
-1.0	.207	.208	.205	.195	.189	.191	.184	.185	.183
- .9	.231	.232	.228	.222	.218	.216	.209	.209	.208
- .8	.256	.254	.252	.247	.245	.242	.235	.233	.234
- .7	.281	.278	.278	.274	.271	.269	.265	.262	.262
- .6	.310	.306	.307	.304	.300	.300	.297	.293	.293
- .5	.340	.336	.336	.336	.330	.330	.329	.326	.323
- .4	.368	.367	.367	.369	.363	.363	.359	.358	.357
- .3	.399	.399	.400	.404	.398	.397	.391	.392	.392
- .2	.430	.433	.432	.439	.432	.430	.426	.429	.427
- .1	.464	.469	.466	.470	.464	.465	.461	.465	.463
.0	.495	.500	.500	.504	.498	.500	.496	.502	.500
.1	.526	.533	.534	.540	.535	.535	.532	.538	.537
.2	.560	.567	.568	.574	.569	.570	.566	.574	.573
.3	.594	.600	.600	.606	.604	.603	.601	.608	.608
.4	.626	.635	.633	.639	.637	.637	.635	.644	.643
.5	.658	.667	.664	.672	.669	.670	.669	.679	.677
.6	.688	.694	.693	.703	.702	.700	.701	.708	.707
.7	.718	.726	.722	.731	.731	.731	.732	.740	.738
.8	.743	.748	.748	.759	.761	.758	.760	.765	.766
.9	.769	.773	.772	.785	.785	.784	.788	.792	.792
1.0	.792	.797	.795	.809	.808	.809	.812	.817	.817
1.2	.833	.836	.836	.855	.852	.853	.854	.858	.859
1.4	.866	.871	.869	.890	.886	.888	.889	.893	.893
1.6	.896	.899	.896	.918	.915	.917	.919	.921	.921
1.8	.919	.920	.918	.939	.936	.939	.941	.942	.941
2.0	.939	.938	.936	.956	.954	.956	.959	.958	.958
2.5	.969	.967	.966	.981	.980	.980	.984	.982	.981
3.0	.982	.981	.982	.991	.990	.991	.993	.992	.992
4.0	.994	.994	.992	.998	.998	.999	.999	.999	.998

Table 3.4

Comparisons of Estimators of F_{LI}

$\delta^2 = 50$

	T - K = 30, K_2 = 30			T - K = 100, K_2 = 30			T - K = 300, K_2 = 30		
	1	2	3	1	2	3	1	2	3
-4.0	.023	.023	.023	.010	.011	.009	.006	.007	.006
-3.0	.046	.047	.047	.026	.027	.025	.020	.021	.020
-2.5	.070	.071	.069	.045	.046	.045	.038	.038	.037
-2.0	.107	.110	.108	.080	.081	.082	.072	.070	.072
-1.8	.129	.131	.128	.101	.101	.102	.094	.090	.091
-1.6	.153	.154	.152	.127	.128	.127	.120	.114	.116
-1.4	.183	.185	.182	.156	.157	.157	.152	.146	.146
-1.2	.216	.125	.215	.191	.191	.191	.189	.181	.181
-1.0	.254	.251	.253	.233	.231	.231	.228	.221	.221
- .9	.274	.273	.274	.254	.252	.254	.251	.245	.245
- .8	.296	.295	.296	.279	.278	.277	.274	.267	.268
- .7	.319	.318	.319	.305	.304	.301	.301	.295	.294
- .6	.344	.341	.342	.331	.330	.327	.326	.321	.321
- .5	.369	.365	.367	.357	.355	.354	.353	.346	.348
- .4	.394	.391	.393	.384	.383	.382	.381	.376	.378
- .3	.420	.419	.419	.412	.410	.411	.413	.408	.407
- .2	.447	.445	.446	.442	.442	.440	.445	.441	.438
- .1	.472	.469	.472	.472	.473	.470	.474	.468	.468
.0	.500	.499	.500	.499	.500	.500	.502	.496	.500
.1	.526	.527	.528	.530	.532	.530	.532	.529	.532
.2	.553	.555	.554	.559	.562	.560	.560	.560	.562
.3	.578	.580	.581	.588	.593	.589	.589	.590	.593
.4	.601	.605	.607	.617	.620	.618	.619	.622	.622
.5	.627	.631	.633	.644	.647	.646	.648	.655	.652
.6	.649	.654	.658	.671	.674	.673	.674	.681	.679
.7	.675	.680	.681	.698	.699	.699	.700	.705	.706
.8	.698	.703	.704	.722	.725	.725	.726	.732	.732
.9	.720	.725	.726	.745	.748	.746	.749	.754	.755
1.0	.741	.747	.747	.767	.770	.769	.771	.778	.779
1.2	.779	.783	.785	.806	.809	.809	.814	.819	.819
1.4	.815	.817	.818	.842	.840	.843	.850	.852	.854
1.6	.844	.846	.848	.872	.871	.873	.881	.883	.884
1.8	.870	.875	.872	.898	.898	.898	.906	.907	.909
2.0	.891	.893	.892	.919	.919	.918	.928	.928	.928
2.5	.931	.932	.931	.954	.955	.955	.960	.962	.963
3.0	.954	.954	.953	.973	.972	.975	.977	.978	.980
4.0	.977	.976	.977	.989	.989	.991	.992	.992	994

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