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# STRAIGHTFORWARD ELECTIONS, UNANIMITY, AND PHANTOM VOTERS

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ABSTRACT

Nonmanipulable direct revelation social choice functions are characterized for societies where the space of alternatives is a euclidean space and all voters have separable preferences with a global optimum. If a nonmanipulable choice function satisfies a weak unanimity-respecting condition (which is equivalent to having an unrestricted range) then it will depend only on voters' ideal points. Further, such a choice function will decompose into a product of one-dimensional mechanisms in the sense that each coordinate of the chosen point depends only on the respective coordinate of the voter's ideal points. Each coordinate function will also be nonmanipulable and respect unanimity. Such one-dimensional mechanisms are uncompromising in the sense that voters cannot take an extreme position to influence the choice to their advantage. Two characterizations of uncompromising choice functions are presented. One is in terms of a continuity condition, the other in terms of "phantom voters," i.e., those points which are chosen which are not any voter's ideal point. There are many such mechanisms which are not dictatorial. However, if differentiability is required of the choice function, this forces it to be either constant or dictatorial. In the multidimensional case, nonseparability of preferences leads to dictatorship, even if preferences are restricted to be quadratic.


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## I. INTRODUCTION

A social choice mechanism is unlikely to achieve its desired performance if it can be manipulated to the advantage of individual participants. The articulation of this postulate by Hurwicz [3], Gibbard [2], and Satterthwaite [6] stimulated a surge of interest in choice mechanisms, termed straightforward mechanisms, which are immune to such manipulation. The pervasive result has been that straightforward choice mechanisms on sufficiently large domains must be dictatorial. Unfortunately, the reduction of straightforward mechanisms on large domains to dictatorship provides no information about nondictatorial straightforward choice mechanisms on domains which permit their existence.

An important breakthrough in this latter problem has been achieved by Moulin [4].[1] Moulin considers mechanisms which choose a point on the line for each single-peaked preference profile. Assuming that the mechanism is sensitive only to the participants' maximal points, Moulin completely characterizes the wide class of straightforward mechanisms. The object of the present paper is to refine Moulin's characterization, and more importantly, to extend it to multidimensional environments.

Our characterization will be obtained for several multidimensional extensions of the class of single-peaked preferences. Perhaps the most immediate generalization of this class is the class of preferences representable by utility functions $u(x) = \sum_{j=1}^{m} v_j(x_j)$, where each $v_j$ has a unique maximizer from which it decreases monotonically in either direction. These will be called separable preferences. A class of traditional interest to political scientists (see, e.g., Riker and Ordeshook [5]) is the class of <u>quadratic</u> preferences, representable by utilities of the form $u(x) = -(x - p)'A(x - p)$ where A is a symmetric positive definite matrix. The intersection of these two classes, the class of quadratic separable preferences, is representable by utilities of the form $u(x) = -\sum_j a_j(x_j - p_j)^2$, where each $a_j$ is positive. We will consider all three classes, and will use the terminology of political science in calling the participants <u>voters</u> and their unique preference maxima <u>ideal points</u>.

In charaterizating straightforward choice mechanisms we require the mechanisms to respect unanimity in the limited sense that if all voters have the same ideal point then the common ideal point should be chosen. Straightforwardness alone implies this repsect of unanimity on the range of the mechanism (see the remarks at the end of Section II) so this assumption is equivalent to requiring the range of a mechanism to agree with the range of voters' ideal points. On quadratic separable preference domains, straightforward choice mechanisms which respect unanimity in this sense have two characteristic properties. First, such mechanisms are sensitive only to the voters' ideal points. Hence,



even in the multidimensional case, Moulin's assumption can be replaced by our unanimity assumption. Second, such mechanisms decompose into a product of one-dimensional mechanisms in the sense that each coordinate of the point chosen depends only on the respective coordinate of the voters' ideal points. Also, each of the one-dimensional mechanisms is straightforward and respects unanimity as a one-dimensional choice mechanism. This characterization, stated as Theorem 6.1, is extended to the class of separable preference environments in Corollary 7.1. However, the separability restriction is essential. On quadratic preference profiles in which the matrices A, mentioned above, are permitted to have small off-diagonal elements, the only straightforward unanimity-respecting mechanisms are dictatorial (Proposition 7.3).

The multidimensional characterization relies on our characterization for the one-dimensional case which we obtain first. On profiles of preferences representable by utilities of the form $u(x) = -|x - p|$, Proposition 3.1 states that straightforward unanimity respecting mechanisms are <u>uncompromising</u>. That is, if a voter's ideal point lies to the right (respectively left) of the chosen point $p^0$, then any change in the ideal point which leaves it to the right (respectively left) of $p^0$ will not affect the choice. This characterization extends directly to profiles of preferences with a unique ideal point from which they decrease monotonically in each direction (Proposition 7.1). Proposition 4.5 shows that the uncompromising mechanisms constitute precisely the class characterized by Moulin under different assumptions. The uncompromising mechanisms which respect unanimity form the coordinates of the multidimensional mechanisms described above.

The impossibility result expressed in Proposition 7.3 forms an interesting companion piece to the impossibility theorem of Satterthwaite and Sonnenschein [7, Theorem 3]. Using a model which also permits private goods, Satterthwaite and Sonnenschein studied continuously differentiable straightforward mechanisms which satisfy some additional regularity conditions. In the pure public choice case, they showed essentially that if the mechanism is straightforward on a domain large enough to be open in a $C^2$ topology on preference profiles then it must be dictatorial. The regularity conditions they impose are appropriate to the local nature of their approach, which would be applicable, for example, to the study of mechanisms defined only on a neighborhood of a particular utility profile. In contrast, we have taken a more global approach and avoided the imposition of any mathematical structure on the choice mechanism itself. This is in part due to our view, expressed formally in Corollary 4.4, that differentiability alone goes a long way toward eliminating nondictatorial straightforward mechanisms. It may also be worth noting that the domain in our result, the set of quadratic preference profiles, is too small to be open in a $C^2$ topology, although it may be rich enough to provide the perturbations necessary for the Satterthwaite-Sonnenschein result.

## II. SOME NOTATION AND DEFINITIONS

The set of social alternatives is the m-dimensional Euclidean space $R^m$. A preference on $R^m$ is a total, reflexive, transitive binary relation on $R^m$. Given a preference G, its asymmetric part is denoted by $\hat{G}$ and its symmetric part by $\tilde{G}$. A preference G is *star-shaped* if



there exists a point $p \in R^m$, called the *ideal point* of the preference, such that for each $p' \neq p$ and each $0 < \lambda < 1$ we have $p\hat{G}[\lambda p' + (1 - \lambda)p]\hat{G}p'$. (Clearly such a point p is unique.) Given any star-shaped preference G, let I(G) denote its ideal point. A preference G is *separable* if for every j and every $x_j, x_j'$ and for every $k \neq j$ and every $\bar{x}_k, \tilde{x}_k$ we have $(\bar{x}_1,\ldots,\bar{x}_{j-1},x_j,\bar{x}_{j+1},\ldots,\bar{x}_m)G(\bar{x}_1,\ldots,\bar{x}_{j-1},x_j',\bar{x}_{j+1},\ldots,\bar{x}_m) \Leftrightarrow$ $(\tilde{x}_1,\ldots,\tilde{x}_{j-1},x_j,\tilde{x}_{j+1},\ldots,\tilde{x}_m)G(\tilde{x}_1,\ldots,\tilde{x}_{j-1},x_j',\tilde{x}_{j+1},\ldots,\tilde{x}_m)$. Let $S_m$ denote the set of star-shaped separable preferences on $R^m$. A preference G is *quadratic* if it can be represented by a utility of the form $(x_1,\ldots,x_m) \mapsto -\sum_{i,j=1}^{m} a_{ij}(x_i - p_i)(x_j - p_j)$ where the matrix $A = [a_{ij}]$ is symmetric and positive definite. Each quadratic preference is star-shaped with ideal point $p = (p_1,\ldots,p_m)$. A quadratic preference is separable if and only if $a_{ij} = 0$ for $i \neq j$. In this case the utility takes the form $(x_1,\ldots,x_m) \mapsto -\sum_{i=1}^{m} \alpha_i(x_i - p_i)^2$. Let $Q_m$ denote the set of separable quadratic preferences on $R^m$. We will often identify a separable quadratic preference by the pair of parameters $(\alpha,p) \in R^m_{++} \times R^m$ of its utility. When m is clear from the context we will denote $S_m$ by S and $Q_m$ by Q.

There are N voters, indexed by the superscript i. A *profile* of preferences is an element of $(S_m)^N$. A profile $\langle G^i \rangle$ is *unanimous* if $I(G^i) = p$ for all i, i.e, all the voters have the same ideal point. For a given profile $\Pi = \langle G^i \rangle$ denote by $\langle \Pi; G, k \rangle$ or $\langle G^i; G, k \rangle$ the profile $\langle G^1,\ldots,G^{k-1}, G, G^{k+1},\ldots,G^N \rangle$, i.e., the profile obtained from $\Pi$ by substituting G in the $k^{th}$ coordinate. For any $D \subset S_m$ a *social choice function* with domain D is a mapping $C : D^N \to R^m$. The choice function $C$ is <u>dictatorial</u> if for some $k$ and every profile $\langle G^i \rangle$, we have $C(\langle G^i \rangle) = I(G^k)$. The choice function $C$ <u>respects unanimity</u> (or is <u>unanimous</u>) if for every unanimous profile $\Pi = \langle G^i \rangle$ with $I(G^i) = p$ for all $i$, then $C(\Pi) = p$. A choice function is <u>straightforward</u> if for each profile $\Pi = \langle G^i \rangle$ and each $k$, $C(\Pi) G^k \, C(\langle \Pi; G, k \rangle)$ for every $G \in D$. If for some profile $\langle G^i \rangle$ there is a $G$ and $k$ such that $C(\langle G^i; G, k \rangle) \hat{G}^k \, C(\langle G^i \rangle)$ then we say that voter $k$ can <u>manipulate</u> $C$ at $\langle G^i \rangle$ via $G$. Thus a straightforward choice function is nowhere manipulable.

Note that straightforwardness implies that $C$ respects unanimity on its range. For suppose $p \in R^m$ is in the range of $C$. Then $p = (C(\Pi)$ for some profile $\Pi \in S^N$. Consider the profile $\Pi^1 = \langle \Pi; G, 1 \rangle$ where $I(G) = p$. Then $C(\Pi^1) = p$ otherwise voter 1 could manipulate $C$ at $\Pi^1$ via $G^1$. Continuing in this fashion $C(\Pi^N) = p$ where $\Pi^N = \langle G^i \rangle$ has $I(G^i) = p$ for all $i$.

## III. THE ONE-DIMENSIONAL CASE

It will be shown in Theorem 6.1 below that under appropriate hypotheses a choice function on a multidimensional space can be decomposed into a product of choice functions on one-dimensional spaces. Thus we begin by studying choice functions where the space of alternatives is one-dimensional and all the voters have quadratic preferences. In this case a voter's ideal point $p$ completely describes the voter's preference : $xGy \Leftrightarrow |p - x| \leq |p - y|$. Given a choice function $C$ with domain $Q$, define the function $c : R^N \rightarrow R$ via $c(\langle p^i \rangle) = C(\langle G^i \rangle)$ where for each $i$, $G^i \in Q$ and $I(G^i) = p^i$. Such a function $c : R^N \rightarrow R$ will be called a <u>voting mechanism</u>. We will identify the quadratic



preferences with their ideal points and also identify the functions $C$ and $c$. Thus we shall apply terms like straightforward and unanimity respecting to $c$ as well as $C$. A profile $\langle p^i \rangle$ will be denoted $\pi$. A function $c : R^N \rightarrow R$ is called <u>uncompromising</u> if for each $\langle p^i \rangle \in R^N$, and each $j$, if

$$p^o = c(\langle p^i \rangle) \text{ then}$$

$$p^j > p^o \text{ implies } c(\langle p^i; p, j \rangle) = p^o \text{ for all } p \geq p^o, \text{ and}$$

$$p^j < p^o \text{ implies } c(\langle p^i; p, j \rangle) = p^o \text{ for all } p \leq p^o.$$

<u>3.1 Proposition</u>: Suppose $c$ is uncompromising, then $c$ (more properly $C$) is straightforward.

<u>Proof of Proposition 3.1</u>: We have to show that for any profile $\pi = \langle p^i \rangle$ and any voter $k$ that $p^k$ minimizes the function $f(\cdot) = |p^k - c(\langle \pi; \cdot, k \rangle)|$ on $R$. Let $p^o = c(\langle p^i \rangle)$. If $p^o = p^k$ this is immediate. Suppose that $p^k > p^o$. Then since $c$ is uncompromising $p^k$ minimizes $f$ on $[p^o, \infty)$. Let $p' < p^o$ and consider $p^{o\prime} = c(\langle p^i; p', k \rangle)$. If $p^{o\prime} > p^o$, then also $p^{o\prime} > p'$, so that $c(\langle p^i; p, k \rangle) = p^{o\prime}$ for $p \leq p^{o\prime}$. Thus for $p \in [p^o, p^{o\prime}]$ we must have $p^{o\prime} = c(\langle p^i; p, k \rangle) = p^o$, a contradiction. Therefore $p^{o\prime} \leq p^o < p^k$ so $p^k$ minimizes $f$ over all of $R$.

The case $p^k < p^o$ is proved by a symmetric argument.

Q.E.D.

The following is a partial converse to Proposition 3.1.

3.2 Proposition: Suppose that C is straightforward and respects unanimity. Then c is uncompromising.

Before presenting the proof we give the following example indicating that straightforwardness alone does not imply that c is uncompromising.

3.3 Example Define $C : Q^N \to R$ by

$$C(\langle G^i \rangle) = \begin{cases} -1 & \text{if } \max_i I(G^i) \leq 0 \\ +1 & \text{if } \max_i I(G^i) > 0. \end{cases}$$

This choice function is straightforward since the only way to change the value chosen is to change $\max_i I(G^i)$, and no one has an incentive to do this. The choice function is not unanimous, nor is it uncompromising, for consider the unanimous profile at 0. Then any voter can change the choice from −1 to +1 simply by reporting a positive ideal point.

Note that this choice function is not straightforward if we extend the domain from Q to S. The class S of star-shaped preferences on the line includes asymmetric preferences. Consider a unanimous profile at 0 where one voter prefers +1 to −1, then that voter can manipulate the outcome by reporting a preference in S with a positive ideal point.



Proof of Proposition 3.2

The proof of Proposition 3.2 is divided into several lemmas.

3.4 Lemma: Suppose that c is straightforward. Then for each k and each profile $\langle p^i \rangle$, letting $c(\langle p^i \rangle) = p^o$, we have

a) if $p^k > p^o$, then $c(\langle p^i;p,k \rangle) = p^o$ for all $p \in [p^o, p^k]$

b) if $p^k < p$ , then $c(\langle p^i;p,k \rangle) = p^o$ for all $p \in [p^k, p^o]$.

In other words, if a voter moves toward the chosen point, without passing through it, then the chosen point remains unchanged.

Proof of Lemma 3.4: Let $\langle p^i \rangle$ be given and $p^o = c(\langle p^i \rangle)$. Suppose $p^k > p^o$ for some k. Let $p \in [p^o, p^k]$. See Figure 1.

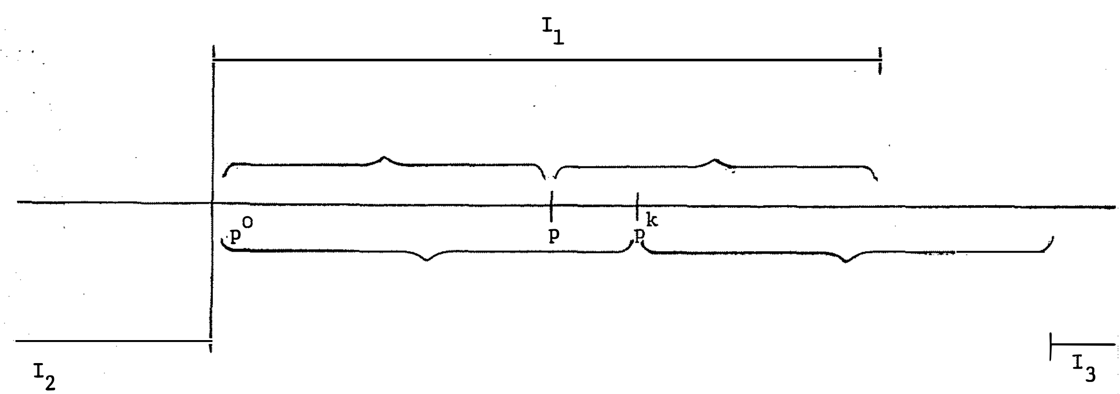


Figure 1

Since c is straightforward $c(\langle p^i;p,k\rangle)$ is at least close to p as $p^o$ is. (Otherwise k could manipulate c at $\langle p^i;p,k\rangle$ via $p^k$.) So $c(\langle p^i;p,k\rangle)$ lies in interval $I_1$ in Figure 1. In particular, $c(\langle p^i;p,k\rangle) \geq p^o$. On the other hand $c(\langle p^i;p,k\rangle)$ must lie at least far from $p^k$ as $p^o$ does. (Else k could manipulate at $\langle p^i\rangle$ via p.) Thus $c(\langle p^i;p,k\rangle)$ must lie in either $I_2$ or $I_3$. But $I_1 \cap (I_2 \cup I_3) = p^o$ so $c(\langle p^i;p,k\rangle) = p^o$.

Q.E.D.

Lemma 3.4 shows that for a straightforward choice function a voter can move towards the chosen point without changing it. The next lemma describes what can happen when a voter moves away from the chosen point.

<u>3.5 Lemma</u>: Suppose that c is straighforward, let $\langle p^i\rangle$ be a profile and put $p^o = c(\langle p^i\rangle)$. Fix some voter k and set

$$p_\ell = \inf\{p : c(\langle p^i;p,k\rangle) = p^o\}$$

$$p_u = \sup\{p : c(\langle p^i;p,k\rangle) = p^o\}.$$

Then

(i) if $\infty < p_\ell < p^o$ then $c(\langle p^i;p,k\rangle) = 2p_\ell - p^o$

for $p \in [2p_\ell - p_o, p_\ell)$,

and

(ii) if $p^o < p_u < \infty$ then $c(\langle p^i;p,k\rangle) = 2p_u - p^o$

for $p \in (p_u, 2p_u - p^o]$.



Refer to Figure 2 to interpret this lemma. From the definition of $p_\ell$ and Lemma 3.4 it follows that for any point $p' \in (p_\ell, p_o]$, we have $c(\langle p^i;p',k\rangle) = p^o$. Lemma 3.5 says that any point $p \in [2p_\ell - p^o, p_\ell)$ yields $c(\langle p^i; p,k\rangle) = 2p_\ell - p_o$. Thus a voter moving from p' through $p_\ell$ causes a jump in the chosen point from $p^o$ to a point symmetrically opposite $p_\ell$:

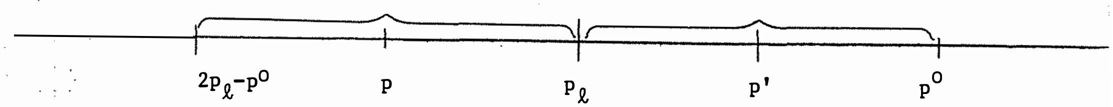


Figure 2

<u>Proof of Lemma 3.5</u>: We will prove (i). The proof of (ii) is symmetric. Define the function $\tilde{c}$ by $\tilde{c}(p) = c(\langle p^i;p,k\rangle)$. For $0 < \varepsilon < (p^o - p_\ell)/2$, consider $\tilde{c}(p_\ell - \varepsilon)$. This point cannot lie to the right of $p_\ell$, for if it did then voter k could move from $p_\ell - \varepsilon$, past $p_\ell$, toward $\tilde{c}(p_\ell - \varepsilon)$ without changing the chosen point (Lemma 3.4). Thus $\tilde{c}(p_\ell - \varepsilon) = \tilde{c}(p_\ell + \delta) = p^o$ for some $\delta > 0$, which contradicts the definition of $p_\ell$.

Another possibility which is ruled out is $\tilde{c}(p_\ell - \varepsilon) \in [2p_\ell - p^o + \varepsilon,\ p_\ell]$, for then k could manipulate at $\langle p^i; p_\ell + \varepsilon/2, k\rangle$ via $p_\ell - \varepsilon$. Thus $\tilde{c}(p_\ell - \varepsilon) \leq 2p_\ell - p^o + \varepsilon$. But $\tilde{c}(p_\ell - \varepsilon)$ must be as close to $p_\ell - \varepsilon$ as $p^o$ is. Thus $\tilde{c}(p_\ell - \varepsilon) \geq 2p_\ell - p_o - 2\varepsilon$. See Figure 3.

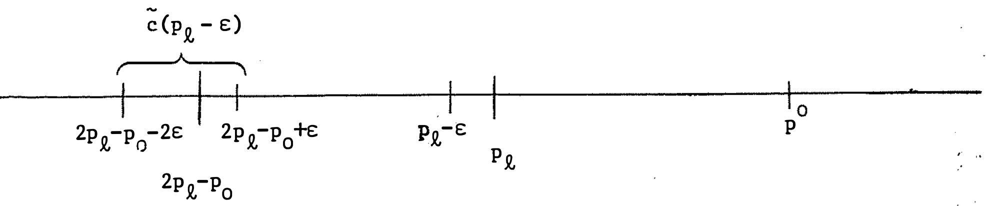


Figure 3

By Lemma 3.4, $\tilde{c}(p) = \tilde{c}(p_\ell - \varepsilon)$ for all $p \in [\tilde{c}(p_\ell - \varepsilon), p_\ell - \varepsilon]$. This must hold for all $\varepsilon > 0$. In particular, for any $0 < \varepsilon^o < \frac{p_o - p_\ell}{2}$ and any $p^*$ with $2p_\ell - p_o + \varepsilon^o < p^* < p_\ell - \varepsilon^o$ we have $\tilde{c}(p^*) \in [2p_\ell - p_o - 2\varepsilon,\ 2p_\ell - p_o + \varepsilon]$ for all $0 < \varepsilon < \varepsilon^o$. Thus $\tilde{c}(p) = 2p_\ell - p_o$ for all $p \in [2p_\ell - p_o,\ p_\ell)$.

Q.E.D.



<u>3.6 Lemma</u>: Suppose that c is straightforward and respects unanimity. Then for each profile $\langle p^i\rangle$, $c(\langle p^i\rangle) \in [\min_i p^i, \max_i p^i]$.

<u>Proof of Lemma 3.6</u>: Suppose by way of contradiction that $p^o = c(\langle p^i\rangle) > \max_i p^i$. Without loss of generality suppose $p^N \leq p^{N-1}, \ldots, \leq p^1$. From Lemma 3.4 moving any voter closer to $p^o$ will not change the outcome, so moving them all to $p^1$ yields $c(\langle p^1\rangle) = p^o$, which violates unanimity. Therefore $c(\langle p^i\rangle) \leq \max_i p^i$. A symmetric argument yields the other half of the conclusion.

Q.E.D.

<u>Proof of Proposition 3.2</u>: Let $\pi = \langle p^i\rangle$ be a profile and set $p^o = c(\pi)$. What needs to be shown is that for any k, if $p^k < p^o$ then $p_\ell = -\infty$ and if $p^k > p^o$ then $p_u = +\infty$, where $p_\ell$, $p_u$ are as defined in Lemma 3.5.

Without loss of generality we can renumber the voters so that $p^1 \leq p^2 \leq, \ldots, \leq p^N$. Suppose for some k that $p^k > p^o$. We will first obtain the desired result for voter N and procede by backward induction.

By Lemma 3.4, $c(\langle\pi; p, N\rangle) = p^o$ for $p \in [p^o, p^N]$. By Lemma 3.5 either $c(\langle\pi; p, N\rangle) = p^o$ for all $p > p^N$ or there exists $p_u$ with $p^o < p^N \leq p_u < \infty$ such that $c(\langle\pi; p, N\rangle) = 2p_u - p^o$ for all $p \in (p_u, 2p_u - p^o]$. But for $p \in (p_u, 2p_u - p^o]$ since $p^N = \max_i p^i$, we have by Lemma 3.6 that $c(\langle\pi; p, N\rangle) \leq p$. Thus it cannot be that $p_u < \infty$.

See Figure 4.

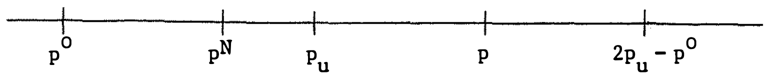


Figure 4

Now suppose $p^k > p^o$, $k < N$ and $c(\langle\pi;p,j\rangle) = p^o$ for all $p > p^o$ and all $j > k$. Let $p_u = \sup\{p : c(\langle\pi;p,k\rangle) = p^o\}$ and suppose by way of contradiction that $p_u < \infty$. By Lemma 3.5 $c(\langle\pi;p,k\rangle) = 2p_u - p^o$ for $p \in (p_u, 2p_u - p^o]$ so by Lemma 3.6, $p_u$ must satisfy $2p_u - p^o \leq p^N$. See Figure 5.

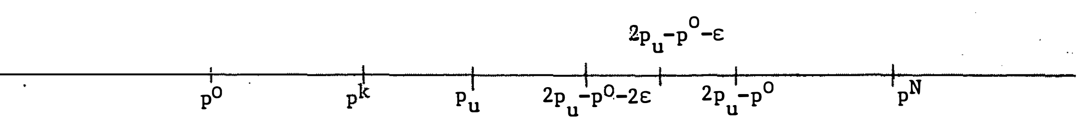


Figure 5



Let $0 < \varepsilon < (p_u - p_o)/2$ and set $p^\varepsilon = 2p_u - p^o - \varepsilon$. We next construct a family of profiles obtained by moving individual k to some point in $(p^o, 2p_u - p_o]$ and the others, one at a time to $p^\varepsilon$. Let $\pi_N(p) = \langle\langle\pi;p,k\rangle; p^\varepsilon, N\rangle$, i.e., the profile obtained by moving k to p and N to $p^\varepsilon$. Next define for $k < j < N$, $\pi_j(p) = \langle\pi_{j+1}(p); p^\varepsilon, j\rangle$, i.e., the profile obtained from $\pi_{j+1}$ by moving one more voter to $p^\varepsilon$.

For $p \in (p_u, 2p_u - p^o]$, $c(\langle\pi;p,k\rangle) = 2p_u - p^o$ so $c(\pi_N(p)) = c(\langle\langle\pi;p,k\rangle; p^\varepsilon, N)$ must be as close to $p^\varepsilon$ as $2p_u - p^o$, otherwise N could manipulate at $\pi_N(p)$ via $p^N$. Thus in particular $c(\pi_N(p)) \geq p^\varepsilon - \varepsilon = 2p_u - p - 2\varepsilon$ for each $p \in (p_u, 2p_u - p^o]$. Also by Lemma 3.4 it must be that $c(\pi_N(p)) = p^o$ for $p \in [p^o, p_u)$. Set $j^* = \min\{j : p^j \geq 2p_u - p^o\}$. Then for each $j \geq j^*$ it also follows that $c(\pi_j(p)) \geq p^\varepsilon - \varepsilon$ for $p \in (p_u, 2p_u - p^o]$ and $c(\pi_j(p)) = p^o$ for $p \in [p^o, p_u)$. So $\sup\{p : c(\pi_{j^*}(p)) = p^o\} < \infty$. By Lemma 3.6, $c(\pi_{j^*}(p)) \leq p^\varepsilon$ for $p \leq p^\varepsilon$. Thus according to Lemma 3.5, $\sup\{p : c(\pi_{j^*}(p)) = p^o\} \leq \frac{p^\varepsilon + p^o}{2} < p_u$. On the other hand $c(\pi_{j^*}(p)) = p^o$ for all $p \in [p^o, p_u)$, a contradiction. Thus the conclusion of the proposition follows.

Q.E.D.

## IV. CHARACTERIZATION OF UNCOMPROMISING VOTING MECHANISMS

If a voting mechanism is uncompromising it will not be influenced by extreme positions taken by the voters. There is no attempt to compromise by "splitting the difference" in positions. We now give a characterization of uncompromising voting mechanisms.

Given a voting mechanism c, a point $p \in R$ is called a phantom voter with ideal point p if there is a profile $\langle p^i \rangle$ with $c(\langle p^i \rangle) = p$ and $p \neq p^i$ for all i. A voting mechanism which has no phantom voters and thus always chooses a point which is the ideal point of some voter is called an elective voting mechanism. Denote by P the set of phantom voters. If P is finite we will also let P denote the cardinality of P. We will also identify N with $\{1,\ldots,N\}$, and P with $\{N+1,\ldots,N+P\}$. Define the elect correspondence $e : R^N \to N \cup P$ via

$$e(\langle p^i \rangle) = \{k \in N : c(\langle p^i \rangle) = p^k\}$$
$$\cup \{N+j \in P : c(\langle p^i \rangle) = p^{N+j}\}$$

that is, e associates to each profile the set of voters, phantom or otherwise, whose ideal point is chosen at that profile. From the definition of phantom voter, e is always nonempty-valued. We say that e has a closed graph if for each $v \in N \cup P$, $\{\langle p^i \rangle \in R^N : v \in e(\langle p^i \rangle)\}$ is closed. (This is equivalent to endowing $N \cup P$ with the discrete topology and asking that $\{(\langle p^i \rangle, v) : v \in e(\langle p^i \rangle)\}$ be closed.)

We have the following characterization.

4.1 Proposition: A voting mechanism c is uncompromising if and only if

(i) there are at most $2^N$ phantom voters

and

(ii) e has a closed graph.



Proof of Proposition 4.1: ($\Longrightarrow$) Suppose that c is uncompromising and let $p^o$ be a phantom voter. Let $\langle p^i \rangle$ be a profile with $c(\langle p^i \rangle) = p^o$ and $p^i \neq p^o$ for any $i \in N$. Set $A = \{i \leq N : p^i < p^o\}$. Then $p^i > p^o$ for each $i \in N \setminus A$. Since c is uncompromising, $c(\langle p'^i \rangle) = p$ for any profile $\langle p'^i \rangle$ with $p'^i < p^o$ for $i \in A$ and $p'^i > p^o$ for $i \notin A$. Thus to each phantom voter can be associated a subset A of N (not necessarily unique) in such a way that distinct phantom voters cannot be associated with the same subset. (If the set A were associated with two distinct phantom voters $p^o$ and $p^{oo}$ then for a profile $\langle p^i \rangle$ with $\max_{i \in A} p^i < p^o \wedge p^{oo}$ and $\min_{i \in N \setminus A} p^i > p^o \vee p^{oo}$ we have $p^o = c(\langle p^i \rangle) = p^{oo}$, a contradiction). This proves the necessity of (i).

To prove the necessity of (ii) first number the phantom voters from N + 1 to N + P (by (i) there are only finitely many). For $N < i \leq N + P$, let $p^i$ denote the ideal point of that phantom voter. Let $\{\pi_n = \langle p_n^i \rangle\}$ be a sequence of profiles converging to $\pi_o = \langle p_o^i \rangle$ with $j \in e(\pi_n)$ for all n and some fixed j. We need to show $j \in e(\pi_o)$, i.e., $c(\pi_o) = p_o^j$, where $p_o^j = p^j$ if $j \in P$.

Pick $j' \in e(\pi_o)$ and choose M sufficiently large so that for each $i \leq N$ and all $n \geq M$,

a) if $\left\{\begin{matrix} p_0^i > p_0^j \\ p_0^i < p_0^j \end{matrix}\right\}$ then $\left\{\begin{matrix} p_0^i > p_n^j \ \& \ p_n^i > p_n^j \\ p_0^i < p_n^j \ \& \ p_n^i < p_n^j \end{matrix}\right\}$

and

b) if $\left\{\begin{matrix} p_0^i > p_0^{j'} \\ p_0^i < p_0^{j'} \end{matrix}\right\}$ then $\left\{\begin{matrix} p_n^i > p_0^{j'} \\ p_n^i < p_0^{j'} \end{matrix}\right\}$

For each $n \geq M$ define the new profile $\pi_n' = \langle p_n'^i \rangle$ via

$$p_n'^i = \begin{cases} p_0^i & \text{if } p_0^i \neq p_0^j \\ p_n^i & \text{if } p_0^i = p_0^j. \end{cases}$$

For each phantom voter $i > N$, let $p_n^i = p^i$.

Since $c$ is uncompromising and $c(\pi_n) = p_n^j$, it follows from (a) that $c(\pi_n) = c(\pi_n')$, as the only difference between $\pi_n$ and $\pi_n'$ is the possible replacement of $p_n^i$ by $p_0^i$, but $p_0^i \lessgtr p_n^j$ as $p_n^i \lessgtr p_n^j$. Thus $c(\pi_n') = p_n^j$.

Now suppose $p_0^j \neq p_0^{j'} = c(\pi_0)$. Then since $c$ is uncompromising it follows from (b) and the definition of $\pi_n'$ that $c(\pi_n') = c(\pi_0)$ and hence $p_n^j = p_0^{j'}$, a contradiction. Thus $p_0^j = p_0^{j'} = c(\pi_0)$ so $j \in e(\pi_0)$. This proves the necessity of (ii).

($\Leftarrow$) To prove that (i) and (ii) imply that $c$ is uncompromising let $\pi = \langle p^i \rangle$ be a profile and let $p^o = c(\langle p^i \rangle)$. Let $k \in N$ and suppose that $p^k > p$. Let $j \in N \cup P$ with $p^j = p^o$ so that $j \in e(\langle p^i \rangle)$.



Let

$$A = \{p > p^o : j \in e(\langle \pi; p, k \rangle)\}$$

and

$$B = \{p > p^o : k \in e(\langle \pi; p, k \rangle) \text{ or } j \notin e(\langle \pi; p, k \rangle)\}.$$

Then $A \cup B = (p^o, \infty)$, $A \cap B = \phi$, and $A$ is closed in $(p^o, \infty)$ by (ii). The set $B$ is also closed in $(p^o, \infty)$. For let $\{p_n\} \subset B$ with $p_n \to \hat{p} \in (p^o, \infty)$ and without loss of generality let $i \in e(p_n)$ for all $n$. (We could always find some $i$ and a convergent subsequence with this property.) There are two possibilities. The first is that $i = k$. Then since $e$ has closed graph we must have $k = i \in e(\langle \pi; \hat{p}, k \rangle)$ so $\hat{p} \in B$. The other possibility is that $i \neq k$. Then by the definition of $B$, $p^i \neq p^j$, but since $e$ has closed graph $i \in e(\langle \pi; \hat{p}, k \rangle)$ so $c(\langle \pi; \hat{p}, k \rangle) = p^i \neq p^j$ so $j \notin e(\langle \pi; \hat{p}, k \rangle)$. Thus $\hat{p} \in B$ and so $B$ is closed in $(p^o, \infty)$. Since $(p^o, \infty)$ is connected, one of $A$ or $B$ is empty, but $p^k \in A$ so $B$ is empty. Thus $j \in e(\langle \pi; p, k \rangle)$ and so $c(\langle \pi; p, k \rangle) = p^j = p^o$ for all $p > p^o$. The case $p^k < p^o$ is symmetric.

Q.E.D.

The following are immediate corollaries of Proposition 4.1

<u>4.2 Corollary</u> Suppose $c : R^N \to R$ is uncompromising. Then $c$ is continuous.

Proof of Corollary 4.2: By Proposition 4.1 it follows that the elect correspondence $e : R^N \to V = \{1,\ldots,N + P\}$ has a closed graph. Let $\pi_n = \langle p_n^i \rangle \to \pi = \langle p^i \rangle$. Let $V' = \{i \in V : i \in e(\pi_n)$ infinitely often$\}$. Then for each $i \in V'$, $\{\pi_n : i \in e(\pi_n)\}$ is a subsequence and hence converges to $\pi$. When $i \in e(\pi_n)$ then $c(\pi_n) = p_n^i$. Since e has closed graph, $i \in e(\pi)$ for all $i \in V'$. Thus $c(\pi_n) \to p^i = c(\pi)$ for each $i \in V'$, and so c is continuous.

Q.E.D.

4.3 Corollary: Suppose $c : R^N \to R$ is uncompromising. For each profile $\pi = \langle p^i \rangle$ and each k, there exist $p_a \leq p_b$ such that

$$c(\langle \pi;p,k \rangle) = \begin{cases} p_a & p \leq p_a \\ p & p_a \leq p \leq p_b \\ p_b & p \geq p_b, \end{cases}$$

and $p_a$ is equal to either $-\infty$ or $p^j$ for some $j \in N \cup P$, $j \neq k$, and $p_b$ is equal to either $+\infty$ or $p^j$ for some $j \in N \cup P$, $j \neq k$.



Proof of Corollary 4.3: Since c is uncompromising, $\{p \in R : c(\langle \pi;p,k \rangle) < p\}$ is an interval of the form $(p_b,\infty)$, where $p_b = \infty$ is possible. Likewise $\{p \in R : c(\langle \pi;p,k \rangle) > p\}$ is an interval of the form $(-\infty,p_a)$ where $p_a = -\infty$ is possible. Since $j \in e(\langle \pi;p,k \rangle)$ for some $j \in N \cup P$ and if $p \neq c(\langle \pi;p,k \rangle)$ we cannot have $j = k$, the conclusion follows.

Q.E.D.

If we imagine the voters arrayed on a line with some voter holding the chosen point, Corollary 4.3 tells us that for an uncompromising mechanism the only way the chosen point can move is for the voter to carry it along and hand it to another voter or place it in the invisible hands of a phantom voter. Note that this gives rise to choice functions which are decidedly not smooth. In fact the only smooth uncompromising choice functions are dictatorial.

4.4 Corollary: If $c : R^N \to R$ is everywhere differentiable and uncompromising then it is dictatorial, i.e., there is some $k \in N$ such that $c(\langle p^i \rangle) \equiv p^k$ or there is some phantom voter at $p^o$ such that $c(\langle p^i \rangle) \equiv p^o$.

Proof of Corollary 4.4: Fix $\pi = \langle p^i \rangle$ and for each $k \in N$ define $\tilde{c}_k(\cdot)$ $c(\langle \pi;\cdot,k \rangle)$. It follows from Corollary 4.3 that if $c_k$ is everywhere differentiable it is either constant at $p^o = c(\pi)$ or is the identity on R. Observe that at most one $\tilde{c}_k$ is not constant. For suppose $\tilde{c}_k$ and $\tilde{c}_j$ are both the identity, and suppose $p^k \leq p^j$. Then $\tilde{c}(p^k - 1) = p^k - 1$

and $c(p^j + 1) = p^j + 1$. Since c is uncompromising, we have $p^k - 1 = c(\langle\langle \pi; p^k - 1, k\rangle; p^j + 1, j\rangle) = c(\langle\langle \pi; p^j + 1, j\rangle; p^k - 1, k\rangle) = p^j + 1$, a contradiction.

Next we show that if $\tilde{c}_k$ is the identity, then k is indeed a dictator. To do this it suffices to show that for any $p \in \mathbb{R}$ and $j \in N$ that $c(\langle\langle \pi; p, j\rangle; \cdot, k\rangle)$ is the identity, for by iteration we can achieve any profile for the other voters. Given p, choose $p' > |p| \vee |p^j|$. Now $\tilde{c}_k(p') = p'$ and since c is uncompromising $c(\langle\langle \pi; p', k\rangle; p, j\rangle = p'$. Either $c(\langle\langle \pi; \cdot, k\rangle; p, j\rangle)$ is the identity or it is constant at p'. The latter is ruled out for then $c(\langle\langle \pi; -p', k\rangle; p, j\rangle = p'$, but c is uncompromising so $p' = c(\langle \pi; -p', k\rangle) = \tilde{c}_k(-p') = -p'$, a contradiction.

Finally we show that if $\tilde{c}_k$ is constant for every $k \in N$, then there is a phantom dictator at $p^o = c(\langle p^i \rangle)$. We need only show that for any p and any j that $c(\langle\langle \pi; p, j\rangle; \cdot, k\rangle)$ is still constant at $p^o$. But, if it weren't constant then it must be the identity and by the above argument k would be a dictator, a contradiction.

Q.E.D.

We now present an alternative characterization of uncompromising voting mechanisms, which is closely related to Moulin's Proposition 3 [4].

<u>4.5 Proposition</u>: A voting mechanism c is uncompromising if and only if for each $S \subset N$ there are constants $C_S$ satisfying $-\infty \leq C_S \leq \infty$



for all S, and $C_\phi < \infty$, $C_N > -\infty$, and $S \subset T \Rightarrow C_S \leq C_T$, such that c can be written in the form

$$c(\langle p^i \rangle) = \max_{S \subset N} \left\{ \min_{i \in S}\{p^i\} \wedge C_S \right\}.$$

<u>Proof of Proposition 4.5</u>: We begin with sufficiency. Although we could appeal to Proposition 4.1 above by showing that a function of the above form has a closed-graph elect correspondence, we present a more direct proof.

First note that the restrictions on the constants imply that c takes on only finite real values. Let

$$c(\langle p^i \rangle) = p^o = \max_{S \subset N} \left\{ \min_{i \in S}\{p^i\} \wedge C_S \right\}.$$

Suppose that $p^k < p^o$ for some $k \in N$ and let $p \leq p^o$. For the profile $\langle \pi; p, k\rangle$, $\min_{i \in S}\{p^i\}$ may change for some S, e.g., $S = \{k\}$. Since $p \leq p^o$, though, no such minimum will increase above $p^o$. Thus $c(\langle \pi; p, k\rangle) = p^o$ still. For the case where $p^k > p^o$ for some $k \in N$, let $p > p^o$ and set $\langle p'^i \rangle = \langle \pi; p, k\rangle$. If $\min_{i \in S}\{p^i\} \neq \min_{i \in S}\{p'^i\}$ then it must be that both $\min_{i \in S}\{p^i\} > p^o$ and $\min_{i \in S}\{p'^i\} > p^o$ (as $p > p^o$) and hence it must be that $C_S \leq p^o$ and so still $c(\langle \pi; p, k\rangle) = p^o$. Clearly $c(\langle \pi; p^o, k\rangle) = p^o$. Thus c is uncompromising.

The proof of necessity is based on the observation in the proof of Proposition 4.1 that to every phantom voter we can associate a subset of N. Let c be uncompromising with $P \leq 2^N$ phantom voters with ideal points $p^{N+1}<\ldots<p^{N+P}$. (Again we will identify P and $\{N+1,\ldots,N+P\}$). For a given phantom voter j let $\langle p^i \rangle$ be a profile such that $c(\langle p^i \rangle) = p^j$ and $p^i \neq p^j$, $i \in N$. Let $S = \{i \in N : p^i > p^j\}$ and set $C_S = p^j$. Note that $C_S$ is well defined for such S. Suppose, however, for some S, there is no phantom voter j and profile $\langle p^i \rangle$ such that $c(\langle p^i \rangle) = p^j$, $p^i \neq p^j$ for $i \in N$, with $S = \{i \in N : p^i > p^j\}$. In this case we have not yet defined $C_S$. To do so, choose a profile $\langle p^i \rangle$ with $p^i < p^{N+1}$ for $i \in N\backslash S$ and $p^i > p^{N+P}$ for $i \in S$. If there are no phantom voters use 0 for both $p^{N+1}$ and $p^{N+P}$. Then by hypothesis no phantom voter's ideal point is chosen so $e(\langle p^i \rangle) \subset N\backslash S$ or $e(\langle p^i \rangle) \subset S$.[2] If $e(\langle p^i \rangle) \subset S$ set $C_S = \infty$ and if $e(\langle p^i \rangle) \subset N\backslash S$ set $C_S = -\infty$. Thus to each $S \subset N$ we have assigned a $C_S$. Note that since e is nonempty-valued we have $C_\phi \neq \infty$ and $C_N \neq -\infty$. Also if $S \subset T$ it is easy but tedious to show that $C_S \leq C_T$. Define $h(\langle p^i \rangle)$ by

$$h(\langle p^i \rangle) = \max_{S\subset N}\{\min_{i\in S}\{p^i\} \wedge C_S\}.$$

We now show that $h(\cdot) \equiv c(\cdot)$.

First suppose that $c(\langle p^i \rangle) = p^j$ for some phantom voter j, and some profile $\langle p^i \rangle$ with $p^i \neq p^j$ for all $i \in N$. Let $\bar{S} = \{i \in N : p^i > p^j\}$. Then $C_{\bar{S}} = p^j$. If $T \not\subset \bar{S}$ then there is some $k \in T$ with $p^k < p^j$ so $\min_{i\in T}\{p^i\} \wedge C_T < p^j$. If $T \subset \bar{S}$ then $C_T \leq C_{\bar{S}} = p^j$. Thus $c(\langle p^i \rangle) = \max_{S\subset N}\{\min_{i\in S}\{p^i\} \wedge C_S\} = h(\langle p^i \rangle)$. Next consider the case in which $c(\langle p^i \rangle) = p^k$ for some $k \in N$ and some profile $\langle p^i \rangle$ with $p^i \neq p^k$ for all $i \neq k$ and $p^i \neq p^j$ for all $i \in N$ and all phantom voters j. Let $S \subset \{i \in N : p^i > p^k\}$. We claim that $C_S < p^k$. If there are no



phantom voters this is immediate. If there are phantom voters define $\pi' = \langle p'^i \rangle$ by

$$p'^i = \begin{cases} (p^k \wedge p^{N+1}) - 1 & \text{if } p^i < p^k \\ p^k & \text{if } i = k \\ (p^k \vee p^{N+P}) + 1 & \text{if } p^i > p^k. \end{cases}$$

Since c is uncompromising $c(\langle p'^i \rangle) = c(\langle p^i \rangle) = p^k$. We now move voter k's ideal point to $\hat{p} = (p^k \wedge p^{N+1}) - 1$. By Corollary 4.3, $c(\langle \pi';\hat{p},k \rangle)$ must either equal $\hat{p}$ or $p^j$ for some phantom voter j where $p^j < p^k$. In the first case $C_{\bar{S}} = -\infty$, in the latter $C_{\bar{S}} = p^j < p^k$, where $\bar{S} = \{i \in N : p^i \gtrsim p^k\}$. Since $S \subset \bar{S}$, $C_S < p^k$. Let $S' = \bar{S} \cup \{k\}$. Then we claim $C_{S'} > p^k$. Again consider the profile $\langle p'^i \rangle$, but move voter k's ideal point to $\tilde{p} = (p^k \vee p^{N+P}) + 1$. Then $c(\langle \pi';\tilde{p},k \rangle)$ is either equal to $\tilde{p}$ or $p^j$ for some phantom voter j with $p^j > p^k$. In the first case $C_{S'} = +\infty$, in the latter $C_{S'} = p^j > p^k$. Thus $\min_{i\in S'}\{p^i\} \wedge C_{S'} = p^k$. If $S \not\subset S'$ then there is $i \in S$ with $p^i < p^k$. Thus $p^k = \max_{S\subset N}\{\min_{i\in S}\{p^i\} \wedge C_S\} = h(\langle p^i \rangle)$.

We have thus shown that $c(\cdot)$ and $h(\cdot)$ agree on an open dense set of profiles. As both are continuous, we must have $c(\cdot) \equiv h(\cdot)$.

Q.E.D.

While Propositions 4.1 and 4.5 characterize all the uncompromising voting mechanisms, they are not usually written in the in the form presented in Proposition 4.5. Some commonplace examples of uncompromising voting mechanisms are dictatorships, constants (which have a phantom dictator), median voter, or any other order statistic. Another example is the mechanisms adopted in 1954 by the Iranian Consortium to determine Iran's total annual oil output. Annually, each member company's role was weighted by its fixed share of the total output, and the output chosen, p, was the highest level such that the sum of the shares of members voting for levels as high as p was at least 0.7 [1, pp. 103-108].

Note that in Proposition 4.5 the phantom voters correspond to the finite $C_S$'s. If these are all finite and distinct then the voting mechanism will have exactly $2^N$ phantom voters which shows that the bound in Proposition 4.1 cannot be improved on. The choice function is unanimous if and only if $C_\phi = -\infty$ and $C_N = +\infty$, so unanimous choice functions have at most $2^N - 2$ phantom voters. This bound is also attainable for unanimous mechanisms. Finally note that if all $C_S$'s $= \pm\infty$ then there are no phantom voters and the mechanism is elective.

## V. INCARNATION OF PHANTOM VOTERS

We will justify the terminology "phantom voter" by showing that any uncompromising voting mechanism can be extended to an elective uncompromising voting mechanism whose set of voters includes the phantom voters of the original voting mechanism.

<u>5.1 Proposition</u>: Let c be an uncompromising voting mechanism, $c : R^N \to R$ with P phantom voters $\{a^j : j = N+1,\dots,N+P\}$. Then there is an elective uncompromising voting mechanism $c^* : R^{N+P} \to R$ such that $c(\cdot) = c^*(\cdot, a^{N+1}, \dots, a^{N+P})$.



<u>Proof of Proposition 5.1</u>: By Proposition 4.5, c can be written in the form $c(p^1,\dots,p^N) = \max_{S \subset N} \{\min_{i \in S}\{p^i\} \wedge C_S\}$, where $C_S = \pm\infty$, or $a^j$ for some $j \in P$.

Set $V = N \cup P$ and for each $S^* \subset V$, let $Re(S^*) = S^* \cap N$ denote the set of real voters in $S^*$ and $Ph(S^*) = S^* \cap P$ denote the set of phantom voters in $S^*$.

$$\text{Define} \quad C^*_{S^*} = \begin{cases} +\infty \text{ if } \left[C_{Re(S^*)} = +\infty\right] \text{ or} \\ \quad\quad \left[C_{Re(S^*)} = a^j \ \& \ j \in Ph(S^*)\right] \\ -\infty \quad \text{otherwise.} \end{cases}$$

Note that $S^* \subset T^* \Rightarrow C^*_{S^*} \leq C^*_{T^*}$, $C^*_\phi = -\infty$, and $C^*_N = +\infty$.

Define $c^* : R^{N+P} \to R$ by

$$c^*(p^1,\dots,p^{N+P}) = \max_{S^* \subset V} \{\min_{i \in S^*} \{p^i\} \wedge C^*_{S^*}\}.$$

By Proposition 4.5 $c^*$ is uncompromising and since $C^*_{S^*} = \pm\infty$ for all $S^* \subset V$, $c^*$ is elective.

Now $\quad c^*(p^1,\dots,p^N,a^{N+1},\dots,a^{N+P})$

$$= \max_{S^*: C^*_{S^*} = \infty} \left\{\min_{i \in S^*} \{p^i\}\right\}$$

$$= \max \left\{ \left\{ \min_{i \in S^*}\{p^i\} \right\} : [C_{Re(S^*)} = \infty] \text{ or } [j \in Ph(S^*) \ \& \ C_{Re(S^*)} = a^j] \right\}$$

$$= \max_{S \subset N} \left\{ \min_{i \in S} \{p^i\} \wedge C_S \right\}$$

$$= c(p^1, \ldots, p^N).$$

Q.E.D.

## VI. THE MULTIDIMENSIONAL CASE

We now consider the case in which the space of alternatives is more than one-dimensional. The chief result is that a choice function $C : (Q_m)^N \to R^m$ which is straightforward and respects unanimity can be decomposed into a product of one-dimensional choice functions, each of which is straightforward and unanimous and depends only on the location of the voters' ideal points in that coordinate.

6.1 Theorem: A choice function $C : (Q_m)^N \to R^m$ is straightforward and unanimity respecting if and only if there are voting mechanisms $c_1, \ldots, c_m : R^N \to R$ which are straightforward and unanimity-respecting such that

$$C(<G^i>) = (c_1(<I_1(G^i)>), \ldots, c_m(<I_m(G^i)>))$$



where $I_j(G^i)$ is the $j^{th}$ coordinate of voter i's ideal point.

The proof of necessity in Theorem 6.1 requires several preliminary results. Sufficiency is immediate from separability. Recall that we have identified a separable quadratic preference with its pair of parameters $(\alpha, p)$.

6.2 Proposition: Suppose that C respects unanimity and is straightforward. Then

$$\min_i \{p_j^i\} \le C_j(<\alpha^i, p^i>) \le \max_i \{p_j^i\}$$

for each $j = 1, \ldots, m$, and each $<\alpha^i, p^i> \in (Q_m)^N$.

Proof of Proposition 6.2: Let $v(x;\alpha,p) = -\sum_j \alpha_j (x_j - p_j)^2$, a utility for $(\alpha,p)$ and set $E(x;\alpha,p) = \{z \in R^m : v(z;\alpha,p) \ge v(x;\alpha,p)\}$, the ellipsoid of points at least as good as x under $(\alpha,p)$. Let $<\alpha^i, p^i>$ be given and set $p^o = C(<\alpha^i, p^i>)$ and suppose by way of contradiction that $p_1^o < \min_i \{p_1^i\} \equiv \mu$. For each i set $p'^i = \lambda^i p^o + (1-\lambda^i) p^i$ where $\lambda^i p_1^o + (1-\lambda^i) p_1^i = \mu$. Then for each i, $E(p^o;\alpha^i,p'^i) \subset E(p^o;\alpha^i,p^i)$. See Figure 5.

If $p \in E(p^o;\alpha^i,p'^i)$ and $p \ne p^o$, then $v(p;\alpha^i,p^i) > v(p^o;\alpha^i,p^i)$.

Since C is straightforward this implies that $C(\alpha^1, p'^1; \alpha^2, p^2; \ldots; \alpha^N, p^N) = p^o$, $C(\alpha^1, p'^1; \alpha^2, p'^2; \alpha^3, p^3; \ldots; \alpha^N, p^N) = p^o$, etc., so that $C(<\alpha^i, p'^i>) = p^o$.

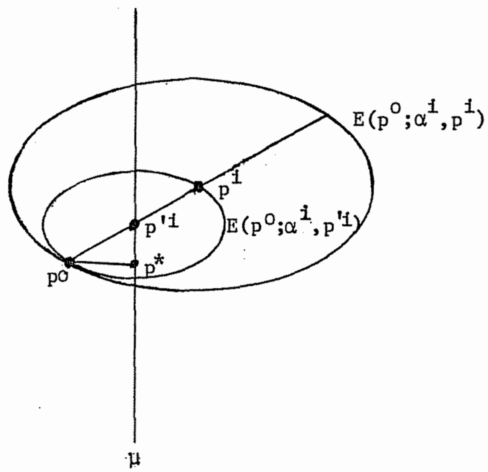


Figure 5

Let $p^* = (\mu, p_2^o, \ldots, p_m^o)$. We now consider what happens when we move each of the voters to $p^*$ and change the parameters of their preferences. For each $i$, define $x^i(\varepsilon^1, \ldots, \varepsilon^i) = C(\alpha'^1, p^*; \ldots; \alpha'^i, p^*; \alpha^{i+1}, p'^i; \ldots; \alpha^N, p'^N)$ where $\alpha'^i = (\varepsilon^i, \alpha_2^i, \ldots, \alpha_N^i)$. Define $x^o = p^o$. We claim the following.

For every $\delta^i > 0$ there is a $\delta^{i-1} > 0$ and $\varepsilon^i > 0$ such that if

(a) $|x_1^{i-1}(\varepsilon^1, \ldots, \varepsilon^{i-1}) - \mu| \geq |p_1^o - \mu| - \delta^{i-1}$

and

(b) $|x_j^{i-1}(\varepsilon^1, \ldots, \varepsilon^{i-1}) - p_j^*| < \delta^{i-1}\; j = 2, \ldots, m$



then

$$|x_1^i\ (\varepsilon^1, \ldots, \varepsilon^i) - \mu| \geq |p_1^o - \mu| - \delta^i$$

and

$$|x_j^i(\varepsilon^1, \ldots, \varepsilon^i) - p_j^*| < \delta^i \qquad j = 2, \ldots, m.$$

Thus for $0 < \delta^N < |p_1^o - \mu|/2$ there are $\varepsilon^1, \ldots, \varepsilon^N > 0$ such that $|x_1^N(\varepsilon^1, \ldots, \varepsilon^N) - \mu| > 0$, but since C respects unanimity we must have $x^N = p^*$ and so $x_1^N = \mu$. This contradiction establishes the Proposition. We now proceed to prove the above claim.

Since C is straightforward we must have that

(i) $v(x^{i-1}; \alpha^i, p'^i) \geq v(x^i; \alpha^i, p'^i)$

and

(ii) $v(x^i; \alpha'^i, p^*) \geq v(x^{i-1}; \alpha'^i, p^*)$.

From (ii), and (b) we have

$$\sum_j \alpha_j'^i (x_j^i - p_j^*)^2 \leq \sum_j \alpha_j'^i (x_j^{i-1} - p_j^*)^2$$

$$= \varepsilon^i (x_1^{i-1} - \mu)^2 + \sum_{j>1} \alpha_j^i (x_j^{i-1} - p_j^*)^2$$

$$\leq \varepsilon^i (x_1^{i-1} - \mu)^2 + \delta^{i-1} \sum_{j>1} \alpha_j^i, \quad \text{for } 0 < \delta^{i-1} < 1$$

Put $\gamma(\varepsilon,\delta) = \{[\varepsilon(x_1^{i-1} - \mu)^2 + \delta \sum_{j>1} \alpha_j^i]/ \min_{j>1} \alpha_j^i\}^{1/2}$.

Then the above inequality implies $|x_j^i - p_j^*| \leq \gamma(\varepsilon^i,\delta^{i-1})$ for each $j > 1$. Also $\gamma(\varepsilon,\delta)$ can be made arbitrarily small by choosing $\varepsilon$, $\delta$ small enough. From (i) above $\sum_j \alpha_j^i(x_j^i - p_j'^i)^2 \geq \sum_j \alpha_j^i(x_j^{i-1} - p_j'^i)^2$ and since $p_1'^i = \mu$ we have

$$\alpha_1^i[(x_1^i - \mu)^2 - (x_1^{i-1} - \mu)^2] \geq \sum_{j>1} \alpha_j^i[(x_j^{i-1} - p_j'^i)^2 - (x_j^i - p_j'^i)^2]$$

$$= \sum_{j>1} \alpha_j^i [(x_j^{i-1} - p_j^* + p_j^* - p_j'^i)^2 - (x_j^i - p_j^* + p_j^* - p_j'^i)^2]$$

$$= \sum_{j>1} \alpha_j^i [(x_j^{i-1} - p_j^*)^2 + 2(x_j^{i-1} - p_j^*)(p_j^* - p_j'^i) - (x_j^i - p_j^*)^2 - 2(x_j^i - p_j^*)(p_j^* - p_j'^i)].$$

$$\geq \{\sum_{j>1} \alpha_j^i(\delta^{i-1})^2\} - 2\{\sum_{j>1} \alpha_j^i(p_j^* - p_j'^i)[\delta^{i-1} + \gamma(\varepsilon^i,\delta^{i-1})]\} - \{\sum_{j>1} \alpha_j^i \gamma(\varepsilon^i,\delta^{i-1})^2\}.$$

Dividing through by $\alpha_1^i$, we can choose $\varepsilon^i$, $\delta^{i-1}$, $\eta$ small enough so that

$$|x_1^i - \mu| \geq |x_1^{i-1} - \mu| - \eta \geq |p_1^o - \mu| - \delta^{i-1} - \eta \geq |p_1^o - \mu| - \delta^i.$$

Q.E.D.



6.3 Proposition: Suppose that C is straightforward and respects unanimity. Then for each j there is a function $c_j^* : \mathbb{R}^N \to \mathbb{R}$ such that for any $\langle\alpha^i,p^i\rangle$ with $p_{j'}^i = p_{j'}^{i'}$ for all i, i' and all $j' \neq j$, $C_j(\langle\alpha^i,p^i\rangle) = c_j^*(\langle p_j^i\rangle)$. Moreover, for each j, $c_j^*$ is straightforward and respects unanimity.

Proof of Proposition 6.3: Let $p^o \in R^m$ and let C' denote the restriction of C to the set $Q' = \{\langle\alpha^i,p^i\rangle \in Q^N : p_j^i = p_j^o$ for all i and all $j > 1\}$. Proposition 6.2 implies that $C_j'(\cdot) \equiv p_j^o$ for each $j > 1$. For each i let $Q'^i = \{(\alpha^i,p^i) : p_j^i = p_j^o$ for each $j > 1\}$, and let $R' = \{p \in R^m : p_j = p_j^o$ for each $j > 1\}$. For any i, let $(\alpha^i,p^i) \in Q'^i$ and let $u^i$ denote the utility function determined by these characteristics. For any x, x' $\in$ R', $u^i(x) \geq u^i(x')$ if and only if $|x_1 - p_1^i| \leq |x_1' - p_1^i|$. The function $C_1' : \Pi_i Q'^i \to R$ is a voting mechanism. Since C is straightforward and respects unanimity, $C_1'$ inherits these properties. Proposition 6.4 below implies that $C_1'(\langle\alpha^i,p^i\rangle) = C_1'(\langle\alpha'^i,p'^i\rangle)$ whenever $p_1^i = p_1'^i$ for each i, so define $c_1^* : R^N \to R$ by $c_1^*(\langle p_1^i\rangle) = C_1'(\langle\alpha^i,p^i\rangle)$ for any $\langle\alpha^i,p^i\rangle \in R_{++}^{mN} \times R'^N$. Then $c_1^*$ is straightforward and respects unanimity.

Let $p' \in R^m$ and define $c_1^{**} : R^N \to R$ exactly as $c_1^*$ is defined but with p' in place of $p^o$. We need to show that $c_1^{**} = c_1^*$. Let $\langle p_1^i\rangle \in R^N$ and let $p_1^* = c_1^*(\langle p_1^i\rangle)$. Suppose that voters are indexed so that $p_1^1 \leq p_1^2,\ldots, \leq p_1^{j^*} \leq p_1^* \leq p_1^{j^*+1} \leq ,\ldots, \leq p_1^N$ for some $j^* \geq 0$. See Figure 6. Let $a = p_1^1$, $b = p_1^N$ and for each i, let

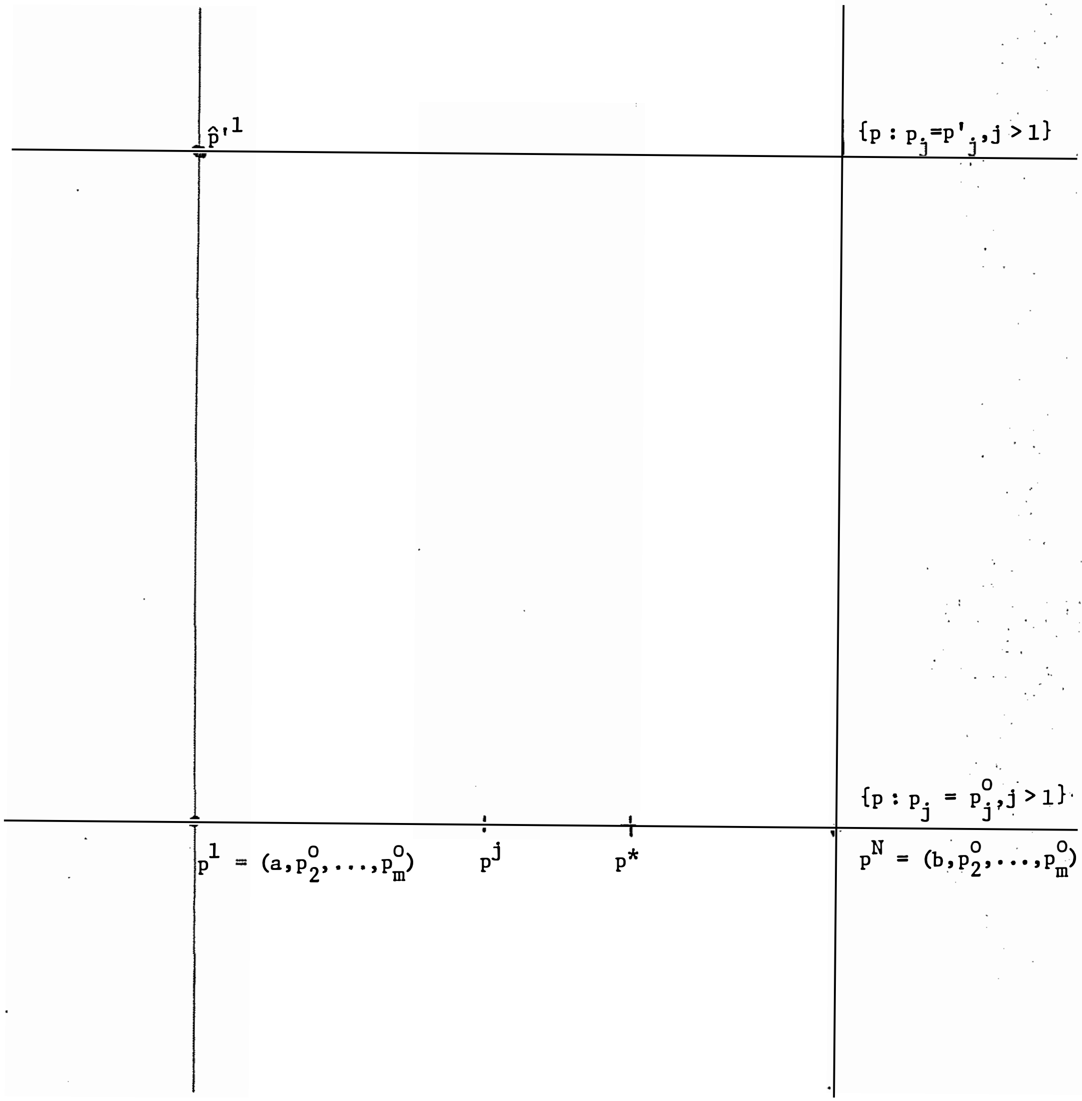


Figure 6



$$\hat{p}^i_1 = \begin{cases} a & \text{if } p^i_1 < p^*_1 \\ p^*_1 & \text{if } p^i_1 = p^*_1 \\ b & \text{if } p^i_1 > p^*_1, \end{cases}$$

and let $\hat{p}^i_j = p^o_j$ for each $j > 1$. By Proposition 3.2, $c_1(<\hat{p}^i_1>) = p^*_1$. For each $i$, let $\varepsilon^i > 0$ and let $\alpha^i = (1, \varepsilon^i, \ldots, \varepsilon^i)$. Then by Proposition 6.4 $C(<\alpha^i, \hat{p}^i>) = (p^*_1, p^o_2, \ldots, p^o_m)$. Let $\hat{p}'^1 = (a, p'_2, \ldots, p'_m)$ and let $x^1 = c(\alpha^1, \hat{p}'^1; \alpha^2, \hat{p}^2; \ldots; \alpha^N, \hat{p}^N)$. We suppress the dependence of $x^1$ on $\varepsilon^1$. By Proposition 6.2 $x^1_1 \in [a,b]$ and for each $j > 1$, $\min\{p^o_j, p'_j\} \le x^i_j \le \max\{p^o_j, p'_j\}$.

Since c is straightforward,

$$v(p^*; \alpha^1, \hat{p}^1) \ge v(x^1; \alpha^1, \hat{p}^1)$$

so

$$(p^*_1 - a)^2 \le (x^1_1 - a)^2 + \varepsilon^1 \sum_{j>1} (x^1_j - p^o_j)^2.$$

Thus $(x^1_1 - a)^2 \ge (p^*_1 - a)^2 - \varepsilon^1 \sum_{j>1} (x^1_j - p^o_j)^2$. Hence for any $\delta^1 > 0$ there is some $\varepsilon^1 > 0$ with $|x^1_1 - a| \ge |p^*_1 - a| - \delta^1$. Since $x^1_1 \in [a,b]$, it must be that $x^1_1 \ge p^*_1 - \delta^1$. Also, since C is straightforward,

$$v(x^1; \alpha^1, \hat{p}'^1) \ge v(p^*; \alpha^1, \hat{p}'^1)$$

or

$$(x^1_1 - a)^2 \le (p^*_1 - a)^2 + \varepsilon^1 \sum_{j>1} (p^o_j - p'_j)^2.$$

Hence for any $\delta^1 > 0$ there is some $\varepsilon^1 > 0$ such that $|p_1^* - x_1^1| < \delta^1$.

For each $i > 1$ let $\hat{p}_1'^{i} = \hat{p}_1^i$ and $\hat{p}_j'^{i} = p_j'$ for $j > 1$ and set $x^i = C(\alpha^1,\hat{p}'^1;\ldots;\alpha^i,\hat{p}'^i;\alpha^{i+1},\hat{p}^i;\ldots;\alpha^N,\hat{p}^N)$. By the above sort of argument we have that for each i and each $\delta^i > 0$ there exist $\varepsilon^{i'} > 0$, $1 \leq i' \leq i$ with $|x_1^i - p_1^*| < \delta^i$. (Again, we have suppressed the dependence of $x^i$ on $\varepsilon^1,\ldots,\varepsilon^i$.) In particular, for any $\delta^N > 0$ there exist $\varepsilon^i > 0$, $1 \leq i \leq N$ with $|x_1^N - p_1^*| < \delta^N$. Since $x_1^N = c_1^{**}(\langle\hat{p}_1^i\rangle)$, which is independent of $\langle\alpha^i\rangle$ by Proposition 6.4 and thus independent of $\langle\varepsilon^i\rangle$, we have $x_1^N = p_1^*$. By Proposition 3.2 $c_1^{**}(\langle\hat{p}_1^i\rangle) = c_1^{**}(\langle p_1^i\rangle)$, so $c_1^{**}(\langle p_1^i\rangle) = c_1^*(\langle p_1^i\rangle)$. Since $\langle p_1^i\rangle$ was chosen arbitrarily, this proves that $c_1^{**} = c_1^*$, so $c_1^*$ exists as asserted. The existence of $c_j^*$ for $j > 1$ is obtained exactly analogously.

Q.E.D.

The proof of Proposition 6.3 relies on Proposition 6.4 below, which is of some interest in its own right and hence is presented in somewhat greater generality than necessary for just the proof of Proposition 6.3. For each $i \leq N$ let $A^i$ be an abstract set of characteristics for voter i, and let $f^i : A^i \to R$ be a function which associates with each characteristic $a^i$ a preference relation G on R of the form pGp' if and only if $|p^o - p| \leq |p^o - p'|$, where $p^o = f^i(a^i)$. For example, let $A^i = R_{++} \times R$, where each element $(a_1,a_2)$ parametrizes a utility function $u^i : R \to R$ defined by $u^i(p) = -a_1(a_2 - p)^2$. In this case $f^i$ is the projection $(a_1,a_2) \mapsto a_2$.



For each i and each $p \in R$, let $A_p^i = (f^i)^{-1}(p)$. We will assume that $A_p^i \neq \phi$ for each $p \in R$ and each i. Let $A = \Pi_i A^i$. Then a voting mechanism $c : A \to R$ is straightforward if for each $\langle a^i\rangle \in A$ and each k, $a^k$ minimizes $|c(\langle a^i; \cdot,k\rangle) - f^k\langle a^k\rangle|$ on $A^k$; and c respects unanimity if $c(\langle a^i\rangle) = p$ whenever $f^i(a^i) = p$ for all i.

<u>6.4 Proposition</u>: Suppose that $c : A \to R$ is straightforward and respects unanimity. Then for each $\langle a^i\rangle, \langle a'^i\rangle \in A$ with $f^i(a^i) = f(a'^i)$ for all i, $c(\langle a^i\rangle) = c(\langle a'^i\rangle)$.

<u>Proof of Proposition 6.4</u>: Let $g : R^N \to A$ such that for each $\langle p^i\rangle \in R^N$, $f[g(\langle p^i\rangle)] = \langle p^i\rangle$. Then $c(g(\cdot))$ is straightforward and respects unanimity. Corollary 4.2 implies that $c(g(\cdot))$ is continuous. Let $\langle p^{oi}\rangle \in R^N$ and let $\langle a^i\rangle \in A$ with $f^i(a^i) = p^{oi}$ for each i. Define g' by

$$g'(\langle p^i\rangle) = \begin{cases} \langle a^i\rangle & \text{if } \langle p^i\rangle = \langle p^{oi}\rangle \\ g(\langle p^i\rangle) & \text{otherwise.} \end{cases}$$

Since $c(g'(\cdot))$ is also continuous, it follows that $c(\langle a^i\rangle) = c[g(\langle p^{oi}\rangle)]$, which completes the proof.

Q.E.D.

<u>6.5 Example</u>: The following example indicates that if $c$ does not respect unanimity then $c(\langle a^i;\cdot,k\rangle)$ need not be constant on $A^k_p$. For each $i$, let $A^i = R_{++} \times R$ and let $f^i : (a^i_1, a^i_2) \mapsto a^i_2$. Define $c : A \to R$ by

$$c(\langle a^i\rangle) = \begin{cases} 1 & \text{if there is some } i \text{ with } a^i_2 > 0 \text{ or} \\ & a^i_2 = 0 \text{ and } a^i_1 > 1; \text{ and} \\ -1 & \text{otherwise.} \end{cases}$$

Then $c$ is straightforward but does not respect unanimity, and $c(\langle a^i;\cdot,a^k_2;k\rangle)$ is not constant if $a^k_2 = 0$ and $a^j_2 < 0$ for all $j \neq k$.

<u>Proof of Theorem 6.1</u>: Let $c^*_1,\ldots,c^*_m$ be the one-dimensional voting mechanisms established in Proposition 6.3 and let $\langle\alpha^i,p^i\rangle \in Q^N$. Set $p^o = G(\langle\alpha^i,p^i\rangle)$. It suffices to show that $p^o_1 = c^*_1(\langle p^i_1\rangle)$.

Suppose we have indexed the voters so that for some $0 \leq i_1 \leq i_2 \leq N$ we have



(i) for $i \leq i_1$ $\quad p^i_1 < p^o_1$

and for $i,i' \leq i_1, i \geq i'$ implies $p^i_1 \geq p^{i'}_1$

(ii) for $i_1 < i \leq i_2$ $\quad p^i_1 > p^o_1$

and for $i_1 < i,i' \leq i_2, i \geq i'$ implies $p^i_1 \leq p^{i'}_1$

(iii) for $i > i_2$ $\quad p^i_1 = p^o_1$.

See Figure 7.

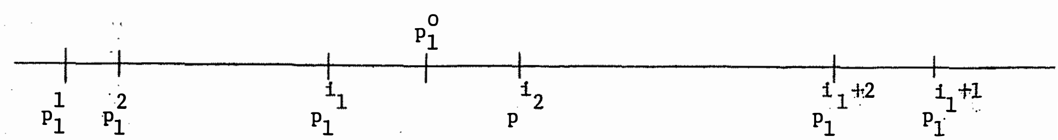


Figure 7

Since C is straightforward, $p^o = C(\alpha^1,p^1;\ldots;\alpha^{i_2},p^{i_2};\alpha^{i_2+1},p^o;\ldots;\alpha^N,p^o)$, (otherwise for some $i > i_2$, i could manipulate at $(\alpha^1,p^1;\ldots;\alpha^{i-1},p^{i-1};\alpha^i,p^o;\ldots;\alpha^N,p^o)$ via $(\alpha^i,p^i)$.)

For each i define $p'^i$ by $p'^i_j = p^o_j$ for $j > 1$ and

$$p'^i_1 = \begin{cases} p_1^{i_1} & \text{if } i \le i_1 \\ p_1^{i_2} & \text{if } i_1 < i \le i_2 \\ p_1^o & \text{if } i > i_2 \end{cases}$$

For each $i \le i_2$, let $\varepsilon^i > 0$ and set $\alpha'^i = (\varepsilon^i,1,\ldots,1)$. Also for each $i \le i_2$ let

$$\hat{p}^i = C(\alpha'^1,p'^1;\ldots;\alpha'^i,p'^i;\alpha^{i+1},p^{i+1};\ldots;\alpha^{i_2},p^{i_2};\alpha^{i_2+1},p^o;\ldots;\alpha^N,p^o),$$

again suppressing the dependence of $\hat{p}^i$ on $\varepsilon^1,\ldots,\varepsilon^i$.

Then $\hat{p}_1^{i_2} = c_1^*(\langle p'^i_1\rangle) = c_1^*(\langle p_1^i\rangle)$, where the first equality follows from Proposition 6.3 and the second from Proposition 3.2.

We will now show that $\hat{p}_1^{i_2} = p_1^o$, which will complete the proof. For each j, let $\Delta_j = (\max_i p_j^i - \min_i p_j^i)^2$. Let $1 \le i \le i_1$, $\delta > 0$ and suppose that $|\hat{p}_j^{i-1} - p_j^o| \le \delta$ for each j, where $\hat{p}^0 \equiv p^o$. Since C is straightforward, $v(\hat{p}^i;\alpha'^i,p'^i) \ge v(\hat{p}^{i-1};\alpha'^i,p'^i)$ or else i could manipulate via $(\alpha^i,p^i)$.



Thus

$$\sum_j \alpha'^i_j(\hat{p}^i_j - p'^i_j)^2 \le \sum_j \alpha'^i_j(\hat{p}^{i-1}_j - p'^i_j)^2,$$

so

$$\varepsilon^i(\hat{p}_1^i - p_1^{i_1})^2 + \sum_{j>1}(\hat{p}_j^i - p_j^o)^2$$

$$\le \varepsilon^i(\hat{p}_1^{i-1} - p_1^{i_1})^2 + \sum_{j>1}(\hat{p}_j^{i-1} - p_j^o)^2$$

$$\le \varepsilon^i[(p_1^o - p_1^{i_1})^2 + 2\delta|p_1^o - p_1^{i_1}| + \delta^2] + (m-1)\delta^2,$$

so

$$\sum_{j>1}(\hat{p}_j^i - p_j^o)^2$$

$$\le \delta^2(m-1) + \varepsilon^i[(p_1^o - p_1^{i_1})^2 - (\hat{p}_1^i - p_1^{i_1})^2 + 2\delta|p_1^o - p_1^{i_1}| + \delta^2]$$

$$\le \delta^2(m-1) + \varepsilon^i[\Delta_1 + 2\delta(\Delta_1)^{1/2} + \delta^2],$$

and

$$(\hat{p}_1^i - p_1^{i_1})^2 \le (p_1^o - p_1^{i_1})^2 + 2\delta(\Delta_1)^{1/2} + (\delta^2/\varepsilon^i)(m-1) + \delta^2.$$

Hence for any $\delta^i > 0$ there exist $\delta > 0$ and $\varepsilon^i > 0$ with $|\hat{p}_1^i - p_1^{i_1}| \le |p_1^o - p_1^{i_1}| + \delta^i$ and $|\hat{p}_j^i - p_j^o| \le \delta^i$ for each $j > 1$. Also, since C is straightforward, $v(\hat{p}^{i-1};\alpha^i,p^i) \ge v(\hat{p}^i;\alpha^i,p^i)$ or i could manipulate via $(\alpha'^i,p'^i)$. Thus $\sum_j \alpha_j^i(\hat{p}_j^{i-1} - p_j^i)^2 \le \sum_j \alpha_j^i(\hat{p}_j^i - p_j^i)^2$ so

$$\alpha_1^i[(\hat{p}_1^i - p_1^i)^2 - (\hat{p}_1^{i-1} - p_1^i)^2]$$

$$\ge \sum_{j>1} \alpha_j^i[(\hat{p}_j^{i-1} - p_j^i)^2 - (\hat{p}_j^i - p_j^i)^2]$$

$$\ge \sum_{j>1} \alpha_j^i[(|p_j^o - p_j^i| - \delta)^2 - (|p_j^o - p_j^i| + \hat{\delta}^i)^2]$$

$$= -\sum_{j>1} \alpha_j^i[2\delta|p_j^o - p_j^i| + 2\hat{\delta}^i|p_j^o - p_j^i| + \delta^2 - (\hat{\delta}^i)^2].$$

Hence

$$(\hat{p}_1^i - p_1^i)^2 \ge (\hat{p}_1^{i-1} - p_1^i)^2 - \{\sum_{j>1} \alpha_j^i[\quad]\}/\alpha_1^i$$

$$\ge (p_1^o - p_1^i)^2 - 2\delta\,|p_1^o - p_1^i| + (\delta)^2 - \{\sum_{j>1} \alpha_j^i[\quad]\}/\alpha_1^i.$$

Hence for any $\delta^i > 0$ there exists $\hat{\delta}^i > 0$, $\delta > 0$ and $\varepsilon^i > 0$ with

a) $|\hat{p}_1^i - p_1^i| \ge |p_1^o - p_1^i| - \delta^i$,



and choosing $\hat{\delta}^i < \delta^i$, as shown above

b) $|\hat{p}_1^i - p_1^{i_1}| \le |p_1^o - p_1^{i_1}| + \delta^i$,

and

c) $|\hat{p}_j^i - p_j^o| \le \delta^i$ for each $j > 1$.

If $p_1^i < p_1^{i_1}$, $\delta^i$ can be chosen small enough so that (a) and (b) imply that $|\hat{p}_1^i - p_1^o| \le \delta^i$. If $p_1^i = p_1^{i_1}$, then $p_1^{i_1} = \min\{p_1^i, p_1^{i+1}, \ldots, p_1^N\}$, so Proposition 6.2 implies that $\hat{p}_1^i \ge p_1^i$. This inequality together with (a) implies that $|\hat{p}_1^i - p_1^o| \le \delta^i$. Thus we have proved by induction that for any $\delta > 0$, there exist $\varepsilon^1 > 0, \ldots, \varepsilon^{i_1} > 0$ such that $|\hat{p}_1^{i_1} - p_1^o| \le \delta$. A symmetric argument shows that for any $\delta > 0$ there exist $\varepsilon^1 > 0, \ldots, \varepsilon^{i_2} > 0$ such that $|\hat{p}_1^{i_2} - p_1^o| \le \delta$. But $\hat{p}_1^{i_2} = c_1^*(\langle p'^i \rangle)$, which is independent of the choice of $\langle \alpha'^i \rangle$, and hence of $\langle \varepsilon^i \rangle$, so $\hat{p}_1^{i_2} = p_1^o$.

Q.E.D.

We now present an example to show that by restricting the domain of the choice function further, it may no longer be possible to decompose the choice function into one-dimensional choice functions. The domain restriction is that all indifference surfaces be spheres. In this case there are no natural coordinates (in terms of preferences), and there are choice functions for which no choice of coordinates allows a decomposition.

<u>6.6 Example</u>: Let $D \subset Q_2$ be the set of quadratic preferences on the plane representable by a utility of the form $(x_1,x_2) \mapsto -(x_1 - p_1)^2 - (x_2 - p_2)^2$. We can identify every such preference with its ideal point $(p_1,p_2)$.

Given a point $p \in R^2$, let $K(p)$ denote the cone $\{x \in R^2 : x_1 \leq p_1 \; \& \; x_2 \leq p_2 \; \& \; x_2 - x_1 \leq p_2 - p_1\}$. See Figure 8.

Let $N = 2$ and define $C : D^2 \to R^2$ by $C(G^1,G^2)$ maximizes $-(x_1 - p_1^2)^2 - (x_2 - p_2^2)^2$ over $K(p^1)$ where $p^i = I(G^i)$. That is, $C(G^1,G^2)$ is voter 2's favorite point in $K(p^1)$. Clearly voter 2 cannot manipulate this choice function and it is easily seen that neither can voter 1. Also in a situation as depicted in Figure 8 each coordinate of the chosen point C depends on all of $p_1^1, p_2^1, p_1^2, p_2^2$. Nor can C be factored by transforming coordinates so that rays $r_1$ and $r_2$ lie on the new coordinate axes, for such a decomposition fails if $p^2 \in K(p^1)$.

Also note that in general, the straightforward choice functions do not generate Pareto efficient outcomes. For example consider the case of two voters on the plane and suppose voter 1 chooses coordinate 1 and voter 2 chooses coordinate 2. Then the chosen point will be a corner of the rectangle determined by their ideal points, whereas the set of Pareto efficient points will be a diagonal arc. Note that in the case of only one dimension Lemma 3.6 implies that straightforward unanimous functions choose Pareto efficient points.



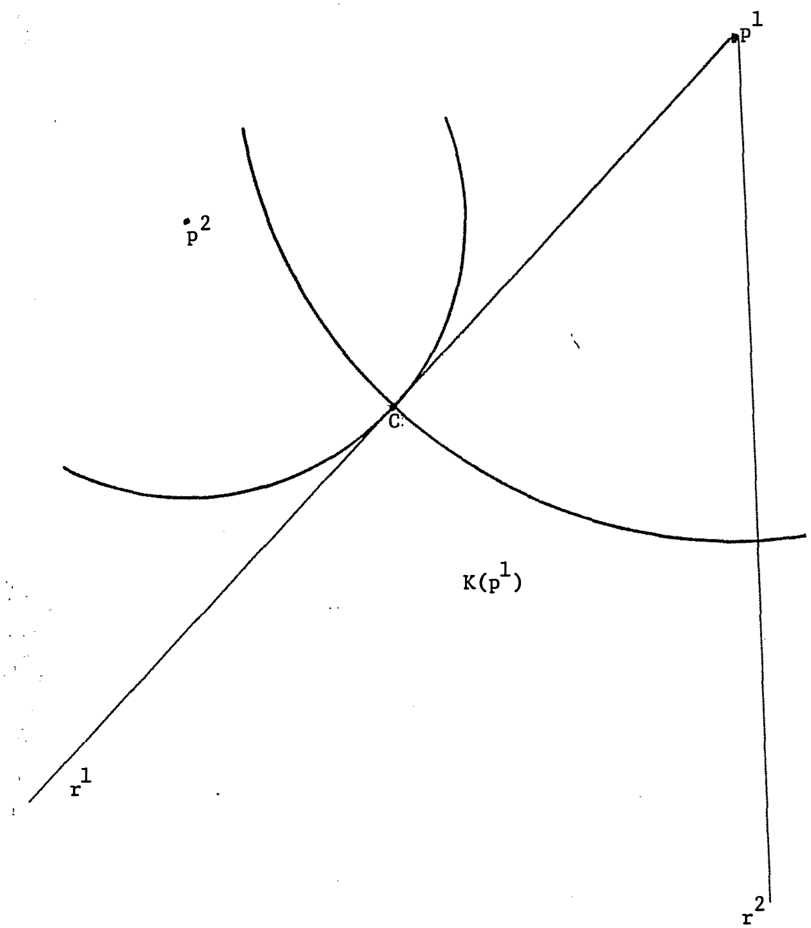


Figure 8

## VII. EXTENSIONS

We now procede to extend the preceding results from choice functions with domain $Q^N$ to those with domain $S^N$. We also show that enlarging the domain to include nonseparable preferences forces straightforward choice functions to be dictatorial. The following is a Corollary of Theorem 6.1.

7.1 Corollary: A choice function $C : (S_m)^N \to R^m$ is straightforward and respects unanimity if and only if there are functions $c_1^*, \ldots, c_m^* : R^N \to R$, which are uncompromising and respect unanimity such that

$$C_j(\langle G^i \rangle) = c_j^*(\langle I_j(G^i) \rangle),\ j = 1, \ldots, m \text{ for all } \langle G^i \rangle \in S_m^N.$$

Proof of Corollary 7.1: We will use the notation $(G,p)$ to denote a preference in $S_m$ with ideal point $p$.

Sufficiency is direct. To prove necessity, note that Theorem 6.1 implies the existence of functions $c_1^*, \ldots, c_m^*$ which are uncompromising and unanimity-respecting such that $C_j(\langle G^i, p^i \rangle) = c_j^*(\langle p_j^i \rangle)$ for all $\langle G^i \rangle \in Q^N$. Define $c^*(\langle p^i \rangle) = (c_1^*(\langle p_1^i \rangle), \ldots, c_m^*(\langle p_m^i \rangle))$. To start an induction argument, let $1 \leq i^o \leq N$, and suppose that $C_j(\langle G^i, p^i \rangle) = c_j^*(\langle p_j^i \rangle)$ whenever $G^i \in Q$ for each $i \geq i^o$. Let $(G^i, p^i) \in S$ for each $i \leq i^o$, and let $(G^i, p^i) \in Q$ for each $i > i^o$. Let $p^o = C(\langle G^i, p^i \rangle)$,



let $G'^{i^o} \in Q$, and let $p' = C(G^1, p^1; \ldots; G^{i^o-1}, p^{i^o-1}; G'^{i^o}, p^{i^o}; G^{i^o+1}, p^{i^o+1}; \ldots; G^N, p^N) = c^*(\langle p^i \rangle)$ by the induction hypothesis. Suppose by way of contradiction that $p' \neq p^o$. Since C is straightforward, $p^o\ G^{i^o}\ p'$ and $p' G'^{i^o} p^o$. In particular, $p' \neq p^{i^o}$ otherwise $i^o$ could manipulate by reporting $G'^{i^o}$ instead of $G^{i^o}$. Also $p^{i^o} \neq p^o$, for if $p^{i^o} = p^o$, then since $p' \neq p^o$, $i^o$ could manipulate by reporting $G^{i^o}$ instead of $G'^{i^o}$. Let A denote the cube determined by $p^o$ and $p^{i^o}$. That is, $A = \{p \in R^M;$ for each $j,\ p_j \not\in [p_j^o, p^{i^o}]\}$.[3] Since $G^{i^o}$ is separable and star-shaped, if $p \in A$ and $p^o\ G^{i^o}\ p$ then $p = p^o$ as $p^o$ is $G^{i^o}$ worst in A so $p' \not\in A$, as $p' \neq p^o$. Renumbering coordinates if necessary suppose that $p_1^o \neq p_1^{i^o}$ and $p' \not\in [p_1^o, p_1^{i^o}]$. Let $G''^{i^o} \in Q$ with the ideal point $p'' = (p_1^o, p_2^{i^o}, \ldots, p_m^{i^o})$ and the coefficient $\alpha^{i^o} = (1, \varepsilon, \ldots, \varepsilon)$ for some $\varepsilon > 0$. Since $c_j^*$ is uncompromising for each j, the induction hypothesis implies that $p' = C(G^1, p^1; \ldots; G^{i^o-1}, p^{i^o-1}; G''^{i^o}, p''; G^{i^o+1}, p^{i^o+1}; \ldots; G^N, p^N)$. But for $\varepsilon$ sufficiently small, $p^o \hat{G}''^{i^o} p'$ contradicting the straightforwardness of C. This proves that $p' = p$, and thus that $C(\langle G^i, p^i \rangle) = c^*(\langle p^i \rangle)$ if $G^i \in Q$ for each $i \geq i^o + 1$. By induction, $C(\langle G^i, p^i \rangle) = c^*(\langle p^i \rangle)$ for all $\langle G^i, p^i \rangle \in S$, which completes the proof.

Q.E.D.

We now consider the case of nonseparable preferences.

7.2 Lemma  Let $c^* : (R^m)^N \to R^m$ be of the form $c^*(p^1,\ldots,p^N) = (c_1^*(p_1^1,\ldots,p_1^N),\ldots,c_m^*(p_m^1,\ldots,p_m^N))$ and suppose that $c^*$ is nondictatorial. Then, renumbering voters and coordinates, if necessary, there exist nonempty open intervals $I^1$ and $I^2$, ideal coordinates, $p_1^2,\ldots,p_1^N$ and $p_2^1,p_2^3,\ldots,p_2^N$ such that for each $p_1^* \in I^1$ and each $p_2^* \in I^2$, $c_1^*(p_1^*,p_1^2,\ldots,p_1^N) = p_1^*$ and $c_2^*(p_2^1,p_2^*,p_2^3,\ldots,p_2^N) = p_2^*$.

Proof: Since $c^*$ is nondictatorial, either

i) $c^*_{j^o}$ is nondictatorial for some $j^o$; or

ii) there are coordinates $j$ and $j'$, and voters $i$ and $i'$, with $i \neq i'$ such that $i$ is a dictator for $c_j^*$ and $i'$ is a dictator for $c_{j'}^*$.

In case (ii) the desired result is immediate. In case (i), renumbering coordinates if necessary, suppose $j^o = 1$. Then by Corollary 4.3 there are voters $i$, $i'$, with $i \neq i'$, nonempty open intervals $I^i \subset R$ and $I^{i'} \subset R$, and points $(p^1,\ldots,p^{i-1},p^{i+1},\ldots,p^N)$ and $(p'^1,\ldots,p'^{i'-1},p'^{i'+1},\ldots,p'^N)$ such that for each $p^* \in I^i$, $c_1^*(p^1,\ldots,p^{i-1},p^*,p^{i+1},\ldots,p^N) = p^*$ and for each $p^* \in I^{i'}$, $c_1^*(p'^1,\ldots,p'^{i'-1},p^*,p'^{i'+1},\ldots,p'^N) = p^*$.

Let $1 \leq i_2 \leq N$, let $I^{i_2}$ be a nonempty open interval, and let $p_2^1,\ldots,p_2^{i''-1}$, $p_2^{i''+1},\ldots,p_2^N \in R$ with $c_2^*(p_2^1,\ldots,p_2^{i''-1},p_2^*,p^{i''+1},\ldots,p^N) = p_2^*$, for each $p_2^* \in I^{i_2}$. Renumbering voters, if necessary, we can take $i_2 = 2$. If $i \neq i_2$, take $i = 1$; and if $i = i_2$, take $i' = 1$.

Q.E.D.



Now let $\bar{Q}_m$ denote the set of all quadratic preferences, not necessarily separable. For each $\varepsilon > 0$ let

$$Q_\varepsilon = \{(A,p) \in \bar{Q}_m : |a_{ij}| < \varepsilon \text{ if } i \neq j\},$$

where $(A,p)$ is identified with the preference represented by the utility $(x_1,\ldots,x_m) \mapsto -(x-p)'A(x-p)$.

7.3 Proposition: Let $\varepsilon > 0$ and suppose that $C : Q_\varepsilon^N \to R^m$ is straightforward and respects unanimity. Then $C$ is dictatorial.

The proof is somewhat intricate, so we first describe the intuition behind it. Suppose there are two voters and two coordinates. We need to show that every nondictatorial choice function $C$ can be manipulated. To make matters as simple as possible, suppose that the function $c^*$ associated with the restriction of $C$ to $Q^N$ is given by

$$c^*(p^1,p^2) = (p_1^1,p_2^2),$$

that is, voter 1 dictates the first coordinate and voter 2 dictates the second. Lemma 7.2 implies that this is a rough approximation to the general nondictatorial case. Let

$$A^1 = \begin{bmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{bmatrix} \qquad A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

for small $\varepsilon > 0$. Given voter 2's characteristics $(A^2,p^2)$ for some $p^2 \in R^2$, we have $c^*(R^2,p^2) = C(Q;A^2,p^2) \subset C(Q_\varepsilon;A^2,p^2)$. Since C is straightforward, there must be no point $p \in C(Q_\varepsilon;A^2,p^2)\setminus C(Q;A^2,p^2)$; otherwise voter 1 could manipulate C if p were his ideal point and his preferences were in Q. Hence

$$C(Q_\varepsilon;\ A^2,\ p^2) = c^*(R^2,p^2) = R \times \{p_2^2\}.$$

If voter 1's preferences are $(A^1, p^1)$, his most preferred point in this set is

$$\hat{p}^1(p^2) = (p_1^1 - \varepsilon(p_2^2 - p_2^1),\ p_2^2).$$

Since C is straightforward we must have

(*) $$C(A^1,\ p^1;\ A^2,\ p^2) = \hat{p}^1(p^2).$$

We have now shown that $C(A^1,\ p^1;\ A^2,p^2) \neq c^*(p^1,p^2)$, but this alone does not make C manipulable. The important feature of (*) is that voter 2 can influence the first coordinate while continuing to determine the second coordinate. Voter 2's most preferred point in $C(A^1,\ p^1;\ A^2,\ R^2)$, according to (*), is obtained by announcing $(A^2;\ \hat{p}^2)$, where $\hat{p}_1^2 = p_1^2$ and $\hat{p}_2^2 = (1/1+\varepsilon^2)[p_1^1 - p_1^2 - \varepsilon^2\ p_2^1 + p_2^2]$. In general, $\hat{p}^2 \neq p^2$ and $\hat{p}^1(\hat{p}^2)$ is preferred to $\hat{p}^1(p^2)$ so voter 2 can manipulate C.



<u>Proof of Proposition 7.3</u>: By Proposition 6.3 there is a voting mechanism $c^* : R^{mN} \to R$ satisfying the conclusion of Proposition 6.3 such that if $(A^i,p^i) \in Q$ for each i, $C(\langle A^i,p^i\rangle) = c^*(\langle p^i\rangle)$. Let $\langle A^i,p^i\rangle \in Q_\varepsilon^N$. Since C is straightforward

(*) $$C(A^1,p^1;\ldots;A^{i-1},p^{i-1};Q_\varepsilon;A^{i+1},p^{i+1};\ldots;A^N,p^N) =$$
$$C(A^1,p^1;\ldots;A^{i+1},p^{i-1};\ Q;A^{i+1},p^{i+1};\ldots;A^N,p^N),$$

for each i. (In particular, C is dictatorial if and only if c* is dictatorial. If c* is dictatorial, then the r.h.s. of (*) is a singleton for each i not the dictator, hence C is dictatorial.) Suppose by way of contradiction that c* is nondictatorial. Let $(p_1^2,\ldots,p_1^N)$, $(p_2^1,p_2^3,\ldots,p_2^N)$, $I^1$, $I^2$ be given by Lemma 7.2. Let $p_1^* \in I^1$ with $p_1^* \neq p_1^2$, let $p_2^* \in I_2$, and let $0 < \delta < \varepsilon$ such that $p_1^* - \delta(p_2^* - p_2^1) \neq p_1^2$, and for each $p_2'^*$ sufficiently near $p_2^*$, $p_1^* - \delta(p_2'^* - p_2^1) \in I^1$. Define $p^{o1}$ by $p_1^{o1} = p_1^*$, $p_2^{o1} = p_2^1$, and $p_j^{o1} = 0$ for each $j > 2$. Let $A^1$ satisfy $a_{jj}^1 = 1$ for each j and if $j \neq k$,

$$a_{jk}^1 = \begin{cases} \delta & \text{if } j = 1 \text{ and } k = 2, \text{ or } j = 2 \text{ and } k = 1; \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Define $p^{o2}$ by $p_1^{o2} = p_1^2$, $p_2^{o2} = p_2^*$, and $p_j^{o2} = 0$ for all $j > 2$; let $\delta > 0$, and let $A^2$ satisfy $a_{jj}^2 = 1$ for all j and $a_{jk} = 0$ if $j \neq k$. For each $i > 2$, let $p_1^{oi} = p_1^i, p_2^{oi} = p_2^i$, and $p_j^i = 0$ for all $j > 2$; and let $a_{jj}^i = 1$

for all j and $a^1_{jk} = 0$ if $j \neq k$. Then (*) implies that the point $\hat{p}(p_2'^*) = (p_1^* - \delta(p_2'^* - p_2^1), p_2'^*, 0, \ldots, 0)$ maximizes $-(p - p^{o1})' A^1 (p - p^{o1})$ on $C(Q; A^2, (p_1^2, p_2'^*, 0, \ldots, 0); A^3, p^{o3}; \ldots; A^N, p^{oN})$ for each $p_2'^*$ sufficiently near $p_2^*$. However, since $\hat{p}_1(p_2^*) \neq p_1^2$, for $p_2'^*$ near $p_2^*$ with $\text{sgn}(p_2'^* - p_2^*) = -\text{sgn}[p_1^2 - \hat{p}_1(p_2^*)]$ we have

$$-[\hat{p}(p_2'^*) - p^{o2}]' A^2 [\hat{p}(p_2'^*) - p^{o2}] > -[\hat{p}(p_2^*) - p^{o2}]' A^2 [\hat{p}(p_2^*) - p^{o2}].$$

Hence voter 2 can manipulate via $(A^2, (p_1^2, p_2'^*, 0, \ldots, 0))$ instead of $(A^2, p^{o2})$, so C is not straightforward. This contradiction proves that C is dictatorial.

Q.E.D.

## VIII. OPEN PROBLEMS

Our characterization of straightforward unanimity respecting choice mechanisms leaves several open problems. The most obvious question is: what happens if the unanimity assumption is dropped? Dropping the unanimity assumption is equivalent to restricting the range of the mechanism, and in economic environments such restrictions arise as feasibility constraints. It seems unlikely that a transparent characterization can be developed to cover all range restrictions, but there are interesting special cases. For example, it should be possible to characterize the mechanisms whose range is an affine



set. Another question of this nature is : what range constraints preserve the continuity of all straightforward mechanisms? Example 3.3 shows that if the domain of preference profiles is further restricted to preferences representable by utilities $u(x) = -\|x - p\|$, additional straightforward choice mechanisms arise. Their characterization is clearly desirable. Corollary 7.1 and Proposition 7.3 seem to indicate that separability is the property most responsible for permitting the wide class of nondictatorial mechanisms we have characterized. However, this implication needs to be more precise. Any quadratic preference, and in particular any preference considered in Proposition 7.3, is separable under a suitable change of coordinates, namely the coordinate change that diagonalizes the quadratic form. In a political science context, these coordinates represent a voter's perception of the issues in an election. This raises the question of whether nondictatorial mechanisms disappear in Proposition 7.3 because the voters' perceptions of the issues are permitted to differ across voters, or because these perceptions are in addition permitted to vary across profiles.

## FOOTNOTES

1. We are indebted to Eric Maskin for this reference.

2. The definition of the elect correspondence e precedes Proposition 4.1 above.

3. The notation $[p_j^o, p_j^{io}]$ is an abbreviation of $[\min\{p_j^o, p_j^{io}\}, \max\{p_j, p_j^{io}\}]$.



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