DIVISION OF THE HUMANITIES AND SOCIAL SCIENCES

CALIFORNIA INSTITUTE OF TECHNOLOGY

PASADENA, CALIFORNIA 91125

LINEARITY OF THE OPTIMAL INCOME TAX: A GENERALIZATION

Gerald H. Kramer

James M. Snyder

SOCIAL SCIENCE WORKING PAPER 534

March 1984

Linearity of the Optimal Income Tax: A Generalization

Gerald H. Kramer and James M. Snyder

Abstract

In an earlier paper [3], we examined the nature of individual and collective preferences over alternative income tax schedules in the context of a simple model in which individuals respond to high tax rates by working in an untaxed "sheltered" sector of the economy. There we established the social optimality of a linear income tax among the set of tax schedules that are continuous, nondecreasing convex functions of income. Here we relax the restrictions on tax schedules, most importantly allowing schedules to have concave (decreasing marginal tax rate) as well as convex (increasing marginal tax rate) regions. In fact, we prove that a linear income tax is socially preferred to any nonlinear lower semi-continuous tax schedule.

Linearity of the Optimal Income Tax: A Generalization

Gerald H. Kramer and James M. Snyder

March 1984

In an earlier paper [3], we examined the nature of individual and collective preferences over alternative income tax schedules in the context of a simple model in which individuals respond to high tax rates by working in an untaxed "sheltered" sector of the economy. One central result established there [Proposition 3.6] was that the optimal income tax (from the point of view of an agent interested in maximizing a social welfare function of the type found in the optimal income taxation literature) is always a linear or flat-rate schedule, in which all incomes are taxed at the same marginal rate. The linearity property does not depend on the distribution of ability or the specific form of the social welfare function (though of course the parameters of the optimal schedule do depend on these data), so in those respects the result is a general one.

However the analysis in [3] did severely constrain the class of admissable schedules (to a subset of the class of continuous, non-decreasing, convex functions of income, with marginal rates not exceeding unity). These assumptions are restrictive in several respects. It would clearly be better to derive properties such as continuity or marginal-rate restrictions as consequences of some more general principle of fairness or optimality, rather than simply impose them as constraints. Further, while these constraints are generally satisfied by the kinds of statutory income tax schedules which are seen in practice, this is not true of the effective incidence of the tax burden, particularly when transfers or other benefits are included; this typically varies with income in a complicated, non-convex fashion (see, for example, [1]). Moreover, from a technical point of view, the convexity assumption, in particular, makes the generality of the linearity result somewhat suspect. The key part of the argument (Propositions 3.4, 3.5) involved showing that any strictly convex schedule can be dominated by certain less-convex (or linear) ones; hence, within the class of convex schedules, any optimal schedule must be linear. But since this optimum occurs on the boundary of the set of admissable schedules, it is not clear that it would still be optimal in some richer admissable set containing the linear schedules as interior rather than boundary members; thus the linearity result might well be an artifact of the convexity assumption.

In the present paper, however, we shall show that the linearity property is in fact quite general, and remains valid with essentially no restrictions on the class of admissable schedules (other than lower semi-continuity). (We also relax another assumption of the earlier analysis, by permitting the government's revenue target to be arbitrary.) The main result to be proven--the Theorem of section 4—shows that any non-linear schedule is dominated by a certain linear schedule.

1. Definitions and Assumptions

We assume a simple one-good economy with a legal "taxable" sector, and an untaxed "sheltered" sector. Worker-consumers allocate their work effort between these two sectors so as to maximize consumption, or after-tax income. A unit of labor pays a (pretax) return of 1 consumption unit in the taxable sector, or a lower but untaxed return in the sheltered sector. The relative return (or "wage") in the tax-sheltered sector, $w(L)$, is a continuous, strictly decreasing function of the total labor supply to that sector, L, with $w(L) \in (0,1)$ for all $L \in [0,N]$.[1]

Each individual supplies a fixed amount of labor to the economy, but this amount varies across individuals (or equivalently, all supply the same amount but its productivity varies across individuals). $F(n)$ is the number of individuals who supply n or fewer units. F is nonatomic, and its support is an interval $[\underline{n},\bar{n}] \subset [0,1]$. The total amount of labor available to the economy is

$$N \equiv \int_{\underline{n}}^{\bar{n}} n \cdot dF(n) > 0.$$

A tax schedule is a lower semi-continuous function $T:[0,1] \rightarrow [-1,+1]$ which specifies the net tax liability or credit $T(x)$ due on the pretax (taxable) income x. Lower semi-continuity ensures the existence of an optimal labor allocation for all individuals; we place no other restrictions on the form of T, so admissable schedules may be discontinuous, increasing or decreasing functions of income, or whatever. A schedule is feasible if it satisfies the government's and all individuals' budgets constraints. Thus, a feasible schedule is one for which $T(x) \leq x$ for all $x \in [0,1]$, and $R_T \leq G$, where R_T is the total tax revenue raised by the schedule T (we show below that this is unique for any T), and G is the (exogenously given) government revenue target.

2. Preliminary Results

Given a tax schedule T and wage $w \in (0,1)$ an individual of type n who allocates $x \in [0,n]$ of his work effort to the taxed sector and $n - x$ to the untaxed sector earns a post-tax income of

(1) $$x - T(x) + w(n - x) = -[T(x) - (1 - w) \cdot x] + w \cdot n.$$

An optimal allocation is one which maximizes this, or equivalently, which minimizes the quantity in square brackets. Let

(2) $$\hat{x}_T(n;w) \equiv \{\hat{x} \in [0,n]: \hat{x} - T(\hat{x}) + w \cdot (n - \hat{x}) \geq x - T(x) + w \cdot (n - x) \text{ for } \forall x \in [0,n]\}$$

be the set of allocations which are optimal for n at the wage w, under the schedule T. In view of the observation above, evidently

Comment 1. $\hat{x} \in \hat{x}_T(n;w)$ if and only if $\hat{x}$ minimizes $T(x) - (1 - w) \cdot x$ over $x \in [0,n]$.

From lower semi-continuity, $\hat{x}_T(n;w)$ is non-empty and compact for all n, w, T. A more explicit characterization, which will be useful below, is as follows: suppose the function $T(x) - (1 - w) \cdot x$ possesses a minimum x' on some interval of the form $[0,x^*)$. Let $\underline{x}$ be

the smallest such minimum (by lower semi-continuity $\underline{x} \equiv \min\{x' = \arg\min_{[0,x^*)} T(x) - (1 - w) \cdot x\}$ is well defined), and let $\bar{x}$ define the largest interval on which this is still a minimum (i.e. $\bar{x} \equiv \sup\{x^{**}: T(x) - (1 - w) \cdot x \geq T(x') - (1 - w) \cdot x' \text{ for all } x \in [0,x^{**})\}$.

We shall say the interval $[\underline{x},\bar{x})$ is w-critical for T. Clearly any T and w define a unique (possibly empty) set of disjoint critical intervals.

Now for any $n \in [\underline{x},\bar{x})$, from Comment 1 evidently $\hat{x} \in \hat{x}_T(n;w)$ if and only if $T(\hat{x}) - (1 - w) \cdot \hat{x} = \min_{x \in [0,n]} T(x) + (1 - w) \cdot x = T(\underline{x}) + (1 - w) \cdot \underline{x}$; thus $\hat{x} = \underline{x}$ is always optimal (though not necessarily uniquely so) for all such n. For $n = \bar{x}$, either $T(\bar{x}) - (1 - w) \cdot \bar{x} = T(\underline{x}) - (1 - w) \cdot \underline{x}$, in which case the above equality again defines the optima, or else there is a discontinuity at $\bar{x}$ with (from lower semi-continuity) $T(\bar{x}) + (1 - w)T(\bar{x}) < T(\underline{x}) + (1 - w) \cdot \underline{x}$, so from Comment 1, $\hat{x} = \bar{x} = n$ uniquely. On the other hand if n does not belong to the closure of any critical interval, then it must be true that $T(x) - (1 - w) \cdot x$ has no minimum on $[0,n)$. By lower semi-continuity it must have a minimum on $[0,n]$, which must therefore be at n, so $\hat{x} = n$ uniquely, again by Comment 1. Thus, summarizing:

Comment 2. The correspondence $\hat{x}_T$ is as follows: If $n \notin [\underline{x},\bar{x}]$ for every w-critical interval $[\underline{x},\bar{x})$, then

(i) $\hat{x}_T(n;w) = \{n\}$.

On the other hand if $n \in [\underline{x}, \bar{x})$ for some such interval, then

(ii) $\hat{x}_T(n;w) = \{\hat{x} \in [\underline{x}, n]: T(\hat{x}) + (1 - w) \cdot \hat{x} = T(\underline{x}) + (1 - w) \cdot \underline{x}\}$,

while for $n = \bar{x}$ either

(iiia) $\hat{x}_T$ given by (i) if T is discontinuous at $\bar{x}$ with $T(\bar{x}) - (1 - w) \cdot \bar{x} < T(\underline{x}) + (1 - w) \cdot \underline{x}$, or

(iiib) $\hat{x}_T$ given by (ii) otherwise.

The aggregate labor supply to the untaxed sector is

$$(3) \quad \int_{\underline{n}}^{\bar{n}} [n - \hat{x}_T(n;w)]dF(n) \equiv N - \hat{X}_T(n;w);$$

this is evidently non-empty and compact (and convex, from Richter's theorem) for all w. If for some wage $\bar{w}$ there exists an $\bar{X} \in \hat{X}_T(n;\bar{w})$ such that the wage schedule satisfies $w(N - \bar{X}) = \bar{w}$, we shall say $\bar{w}$ and $N - \bar{X}$ are an <u>equilibrium</u> wage and labor supply for the schedule T, respectively. In the Appendix it is shown that for any T such an equilibrium necessarily exists, and is unique. We denote by w_T^* and $N - X_T^*$ the equilibrium wage and labor supply under the schedule T.

From individual maximization, an individual's optimal allocations must all yield the same after-tax income. We may thus denote by $y_T(n)$ the after-tax income distribution induced by T at the equilibrium wage, where

$$(4) \quad y_T(n) = \hat{x} - T(\hat{x}) + w_T^*(n - \hat{x}) \text{ for all } \hat{x} \in \hat{x}_T(n;w_T^*).$$

Since $\hat{x}_T(n;w)$ is not single-valued, individual before-tax incomes and taxes paid are indeterminate, in general. However the aggregates are

unique, since in order to clear the labor market, the individual choices must almost everywhere equal some integrable selection $\hat{x}(n)$ of $\hat{x}_T(n;w_T^*)$ such that $\int\hat{x}(n)dF(n) = X_T^*$ (here and henceforth we omit explicit mention of the limits $\underline{n}, \bar{n}$ of integration; they are intended throughout). Thus aggregate before-tax income is

$$(5) \quad Z_T \equiv \int[\hat{x}(n) + w_T^*(n - \hat{x}(n)] = (1 - w_T^*)X_T^* + w_T^*N.$$

If we denote by

$$(6) \quad Y_T \equiv \int y_T(n)$$

total after-tax income, the total tax revenue collected is then

$$(7) \quad R_T \equiv Z_T - Y_T,$$

and is again unique. In view of this, the following notion is well-defined:

<u>Definition</u>: Two schedules T, T' are <u>equivalent</u> if and only if

(i) $w_T^* = w_{T'}^*$,

(ii) $R_T = R_{T'}$, and

(iii) $y_T(n) = y_{T'}(n)$ for all n.

Since equivalent schedules induce the same after-tax income distribution, and raise the same revenue, their welfare implications clearly are the same.

3. Simple Schedules

Let us say a schedule T is *simple* if

(8) $T(x) = T(\underline{x}) + (1 - w_T^*)(x - \underline{x})$ for all $x \in [\underline{x},\bar{x})$,

for every w_T^*-critical interval $[\underline{x},\bar{x})$. A simple schedule is thus linear, with slope $(1 - w_T^*)$, over its w_T^*-critical intervals. An alternative and equivalent characterization is as follows:

Comment 3. T is simple if and only if

$$\frac{T(x'') - T(x')}{(x'' - x')} \leq 1 - w_T^*$$

for all x', x'' such that $0 \leq x' < x'' \leq 1$.

Proof. If the inequality holds everywhere it clearly holds (with equality) on every w_T^*-critical interval, so T is simple. Conversely, if the inequality fails for some x', x'', let $\underline{x} = \min\{x^* \in [0,x'']: T(x^*) - (1 - w_T^*)x^* \leq T(x) - (1 - w_T^*)x$ for all $x \in [0,x'')\}$. Evidently this defines a w_T^*-critical interval $[\underline{x},\bar{x})$ which contains x'', and $T(\underline{x}) - (1 - w_T^*)\underline{x} \leq T(x') - (1 - w_T^*)x' < T(x'') - (1 - w_T^*)x''$, so T is not simple. QED

Comment 3 implies, in particular, that the marginal rate of a simple schedule cannot exceed $1 - w_T^*$. Simple schedules have the following convenient property:

Comment 4. If T is simple then $y_T(n) = n - T(n)$ for all n.

Proof. Suppose $n \in [\underline{x},\bar{x})$ for some w_T^*-critical interval $[\underline{x},\bar{x})$. Since T is simple, $T(n) = T(\underline{x}) + (1 - w_T^*)(n - \underline{x})$, so from (ii) of Comment 2, $\hat{x} = n$ is optimal, whence $y_T(n) = n - T(n)$ for all such n. If n does not belong to (the closure of) any w_T^*-critical interval $\hat{x} = n$ is also optimal (uniquely optimal) for n from the rest of Comment 2, implying the same conclusion. QED

The importance of simple schedules lies in the following fact:

Proposition 1. For any T there exists a simple schedule T' which is equivalent to T. Moreover T' is feasible if T is.

Proof. We construct T' by linearizing T at over its w_T^*-critical intervals. Without loss of generality we can suppose there is just one such interval $[\underline{x},\bar{x})$. T' is then defined by

$$T'(x) = \begin{cases} T(\underline{x}) + (1 - w_T^*)(x - \underline{x}) & \text{for all } x \in [\underline{x},\bar{x}), \\ T(x) & \text{otherwise.} \end{cases}$$

Evidently $T'(x) - (1 - w_T^*)x = T'(\underline{x}) - (1 - w_T^*)\underline{x}$ for all $x \in [\underline{x},\bar{x})$, and $[\underline{x},\bar{x})$ is the unique w_T^*-critical interval of T'. Comment 2 then implies

$$\hat{x}_{T'}(n;w_T^*) = [\underline{x},n] \supset \hat{x}_T(n;w_T^*) \text{ for } n \in [\underline{x},\bar{x}),$$
$$\hat{x}_{T'}(n;w_T^*) = \{n\} = \hat{x}_T(n;w_T^*) \text{ for } n \notin [\underline{x},\bar{x}], \text{ and either}$$
$$\hat{x}_{T'}(n;w_T^*) = [\underline{x},n] \supset \hat{x}_T(n;w_T^*), \text{ or}$$
$$\hat{x}_{T'}(n;w_T^*) = \{n\} = \hat{x}_T(n;w_T^*), \text{ for } n = \bar{x}.$$

Thus $\hat{x}$ optimal for n at w_T^* under T is also optimal under T' at the same wage, so $\hat{X}_{T'}(w_T^*) \supset \hat{X}_T(w_T^*)$, implying $w_{T'}^* = w_T^*$ and $X_{T'}^* = X_T^*$. This implies that $[\underline{x},\bar{x})$ is the unique $w_{T'}^*$-critical interval of T', and that

T' is simple.

We next show that both schedules induce the same after-tax income distribution. (ii) of Comment 2 implies that for $n \in [\underline{x},\bar{x})$, $\hat{x} = \underline{x}$ is optimal under T at w^*_T and under T' at at $w^*_{T'}$, (since $[\underline{x},\bar{x})$ is critical in either case) and $T'(\underline{x}) = T(\underline{x})$ by construction, so $y_T(n) = \underline{x} - T(\underline{x}) + w^*_T(n - \underline{x}) = \underline{x} - T'(\underline{x}) + w^*_{T'}(n - \underline{x}) = y_{T'}(n)$. Similarly, for $n \notin [\underline{x},\bar{x}]$, $\hat{x} = n$ is (uniquely) optimal under T at w^*_T and T' at $w^*_{T'}$, , and $T(n) = T'(n)$, so $y_T(n) = n - T(n) = n - T'(n) = y_{T'}(n)$ for all such n. The same conclusion is readily verified for $n = \bar{x}$, whence $y_T(n) = y_{T'}(n)$, all n.

Hence $Y_T = Y_{T'}$, and since $X^*_T = X^*_{T'}$, it follows that aggregate before-tax incomes, and revenues collected, are also the same, i.e. $Z_T = Z_{T'}$, $R_T = R_{T'}$. To show that T' is feasible if T is, it remains only to show that $T(x) \leq x$, all x implies $T'(x) \leq x$, all x. This follows immediately from the construction of T', since $T'(x) = T(x)$ for $x \notin [\underline{x},\bar{x})$, and $T'(x) = T(\underline{x}) + (1 - w^*_T)(x - \underline{x}) \leq T(x)$ for $x \in [\underline{x},\bar{x})$. QED

It is nearly, but not quite, true that every schedule has a <u>unique</u> simple equivalent. In particular, if $\bar{n} < 1$ the portion of any schedule which applies to $x \in (\bar{n},1]$ is irrelevant, since taxable incomes in this range cannot occur. Thus if T is equivalent to T' as constructed above, it is also equivalent to every simple schedule which coincides with T' on $(0,\bar{n}]$. If we define a "canonical" simple schedule as a simple schedule T' such that $T'(x) = T(\bar{n}) + (1 - w^*_{T'})(x - \bar{n})$ for $x > \bar{n}$, however, it follows from the construction above that every schedule is equivalent to a unique canonical simple schedule.

In view of this, we may henceforth confine attention to simple schedules.

4. Fairness and Optimality

Following the optimal taxation literature, we can suppose individual welfare to be measured by a continuous, strictly increasing, strictly concave function $u: [-1,+1] \to \mathbb{R}_+$, the same for all individuals, so that $u(y)$ is the welfare of an individual with after-tax income y. Social welfare is the sum of individual welfare levels, so the social utility of the schedule T is

$$\int u(y_T(n))dF(n).$$

An <u>optimal</u> schedule is one which maximizes this over the set of feasible schedules. Optimality in this sense clearly depends on the particular welfare function chosen, and is not invariant under monotone (or even concave) transformations of u. However we may define a partial ordering of schedules which is independent of the choice of u: in particular, if T and T' are feasible schedules such that

$$(9) \quad \int u(y_T(n))dF(n) > \int u(y_{T'}(n))dF(n),$$

for <u>every</u> (continuous, strictly increasing, strictly concave) function u, then we can say the schedule T is <u>unambiguously fairer</u> than T'.

It is well known that this relationship can be directly related to income redistributions. In particular, the following is an immediate consequence of Atkinson [2], pp. 245-48:

Proposition 2 (Atkinson). Let T and T' be feasible schedules such that $R_T = R_{T'}$ and $Y_T = Y_{T'}$. If there exists an $n^* \in (\underline{n},\bar{n})$ such that

$y_T(n) \geq y_{T'}(n)$ for all $n < n^*$, and

$y_T(n) \leq y_{T'}(n)$ for all $n > n^*$,

with strict inequality on some set of positive (F) measure, then T is unambiguously fairer than T'.

(In fact in the Theorem below, the "unambiguously fairer" relationship could alternatively be defined directly in terms of these redistributive inequalities, since these are what are used in the proof.) The main result to be established is the following:

Theorem. If T is a feasible simple schedule which is not linear over $(\underline{n},\bar{n})$ then there exists a (feasible) linear schedule which is unambiguously fairer than T.

Proof. Let $T' = \alpha + \beta x$ be the linear schedule with $\beta = 1 - w_T^*$ and $\alpha = w_T^* N - Y_T$. It is readily verified that this schedule induces the same equilibrium ([0,1) is w_T^*-critical for T', so $\hat{X}_{T'}(w_T^*) = [0,N]$, whence $X_T^* \in \hat{X}_{T'}(w_T^*)$, so $w_{T'}^* = w_T^*$ and $X_{T'}^* = X_T^*$). Hence T' is simple, so $y_{T'}(n) = n - T'(n) = w_T^*(n - N) + Y_T$ for all n (from Comment 4), whence $Y_{T'} = \int[w_T^*(n - N) + Y_T] = Y_T$, and $R_T = Z_T - Y_T = (1 - w_T^*)X_T^* + w_T^* N = Z_T - Y_{T'} = R_{T'}$.

Since T is simple, $T(x'') \leq T(x') + (1 - w_T^*)(x'' - x')$ from Comment 3, whence

$T'(x'') - T(x'') \geq T'(x') - T(x') + (1 - w_T^*)(x'' - x')$

for all $x'' > x'$; thus $T'(x) - T(x)$ is nondecreasing on [0,1]. Moreover, since T is nonlinear over $(\underline{n},\bar{n})$ by hypothesis, there must exist $n' \in (\underline{n},\bar{n})$ such that

$$T'(x) - T(x) > T'(\underline{n}) - T(\underline{n}) \text{ for } x \in (n',\bar{n}).$$

Evidently $T'(\underline{n}) - T(\underline{n}) \geq 0$ would imply $T'(x) - T(x) \geq 0$ for all $x > \underline{n}$ with strict inequality for $x > n'$, which from Comment 4 would imply $Y_{T'} < Y_T$, a contradiction; hence $T'(\underline{n}) < T(\underline{n})$. Similarly $T'(\bar{n}) - T(\bar{n}) \leq 0$ would imply $Y_{T'} < Y_T$, again a contradiction, so $T'(\bar{n}) > T(\bar{n})$. Hence there must exist $n^* \in (\underline{n},\bar{n})$ such that $T'(n) \leq T(n)$, whence $Y_{T'}(n) \geq Y_T(n)$ (from Comment 4) for $n < n^*$, and $y_{T'}(n) \leq y_T(n)$ for $n \geq n^*$ (using lower semi-continuity). Moreover the inequality must be strict for n sufficiently close to $\underline{n}$, or to $\bar{n}$. Hence T' is unambiguously fairer than T. QED

From this (and Proposition 1) it follows immediately that every optimal schedule is equivalent to a linear schedule. Moreover it is straightforward to show that for any u there exists an optimal schedule. In particular, if G is not too high there clearly exist feasible schedules; hence, from the Theorem, the set L of linear simple schedules which are feasible is non-empty, and is easily shown to be a compact subset of the parameter space $\mathbb{R}^2$. Since $y_T(n)$ is bounded, $\int u(y_T(n))dF(n)$ has a supremum on L; moreover $T^i \in L$, $T^i \to T$

implies $T \in L$, and $y_{T^i}(n) \to y_T(n)$ for all n from Comment 4, so the supremum is a maximum in L and hence, from the Theorem, in the set of all feasible schedules.

The other results in [3] on optimal schedules also remain valid without qualification. In particular, if the wage schedule satisfies the regularity condition of Proposition 3.7 the optimal (simple, linear) schedule will be unique for any u, and for any distribution of u's, there will exist a linear schedule which is a majority equilibrium in the set of feasible schedules.[2]

APPENDIX

Here we prove the existence and uniqueness of a market equilibrium. In what follows, $\underline{x}_T(n;w) \equiv \min \hat{x}_T(n;w)$, $\bar{x}_T(n;w) \equiv \max \hat{X}_T(n;w)$, $\underline{X}_T(n;w) \equiv \min \hat{X}_T(n;w)$, and $\bar{X}_T(n;w) \equiv \max \hat{X}_T(n;w)$.

<u>Lemma 1</u>. For each $n \in [0,1]$, $\underline{x}_T(n;\cdot)$ and $\bar{x}_T(n;\cdot)$ satisfy

(i) if $w'' \leq w'$ then $\underline{x}_T(n,w'') \leq \bar{x}_T(n,w'')$

$\leq \underline{x}_T(n,w') \leq \bar{x}_T(n,w')$, and

(ii) $\underline{x}_T(n;\cdot)$ is right-hand continuous, and

$\bar{x}_T(n;\cdot)$ is left-hand continuous.

<u>Proof</u>. For any w, $\underline{x}_T(n,w) \leq \bar{x}_T(n,w)$ by definition. For notational convenience, write $x_m = \underline{x}_T(n,w')$. We show that $T(x) - (1 - w'')x > T(x_m) - (1 - w'')x_m \ \forall\ x \in (x_m,n)$ and thus $x_T(n,w'') \leq x_m$, from which (i) follows. If $x \in (x_m,n)$, and $w'' > w$, then $x > x_m$ implies $w''x - w''x_m > w'x - w'x_m$, so

$$T(x) - T(x_m) + x_m - x + w''x - wx_m >$$
$$T(x) - T(x_m) + x_m - x + w'x - w'x_m, \text{ so}$$
$$[T(x) - (1 - w'')x] - [T(x_m) - (1 - w'')x_m] >$$
$$[T(x) - (1 - w'')x] - [T(x_m) - (1 - w')x_m].$$

But x_m minimizes $T(x') - (1 - w')x'$ over $[0,n]$, so the right side is nonnegative, and $T(x) - (1 - w'')x > T(x_m) - (1 - w'')x_m$ as desired.

We now show that $\bar{x}_T(n;\cdot)$ is left-hand continuous; the proof for $\underline{x}_T(n;\cdot)$ is analogous.

Let $w \in (0,1)$, write $\bar{x}_T(n;w) = x^m$, and fix $\varepsilon > 0$. We must find δ such that $w' \varepsilon (w - \delta, w)$ implies $\bar{x}_T(n;w') - x^m < \varepsilon$ (by (i), $\bar{x}_T(n;w') \geq x^m$, $\forall\, w' < w$). Now if $x^m + \varepsilon > n$ then clearly $\bar{x}_T(n,w') < x^m + \varepsilon$ for all $w' < w$ and we may choose any δ, so suppose $x^m + \varepsilon \leq n$.

Let $s(\varepsilon) = \min\limits_{[x^m + \varepsilon, n]} \dfrac{T(x) - T(x^m)}{x - x^m}$. Clearly $s(\varepsilon)$ exists, since $[x^m + \varepsilon, n]$ is compact and $\dfrac{T(x) - T(x^m)}{x - x^m}$ is continuous. Also,

$\forall\, x \in (x^m, n]$ $T(x) - (1 - w)x > T(x^m) - 1 - w)x^m$, or

$\dfrac{T(x) - T(x^m)}{x - x^m} > 1 - w$, so $s(\varepsilon) > 1 - w$.

Let $\delta = s(\varepsilon) - 1 + w > 0$, and let $w' \in (w - \delta, w) = (1 - s(\varepsilon), w)$. Then $\forall\, x \varepsilon\, [x^m + \varepsilon, n]$, $\dfrac{T(x) - T(x^m)}{x - x^m} \geq s(\varepsilon) > 1 - w$, or $T(x) - (1 - w')x > T(x^m) - (1 - w')x^m$. So, $\bar{x}_T(n;w') < x^m + \varepsilon$ and $\bar{x}_T(n;\cdot)$ is left-hand continuous at w. $w \in (0,1)$ was arbitrary, so $\bar{x}_T(n;\cdot)$ is left-hand continuous on $(0,1)$. QED

<u>Lemma 2</u>. $\underline{X}_T$ and $\bar{X}_T$ satisfy

(i) if $w'' > w'$ then $\underline{X}_T(w') \leq \bar{X}_T(w') \leq \underline{X}_T(w'') \leq \bar{X}_T(w'')$, and

(ii) $\underline{X}_T$ is right-hand continuous, and $\bar{X}_T$ is left-hand continuous.

<u>Proof</u>. For all $w \in (0,1)$, $\underline{X}_T(w) = \int_0^1 \underline{x}_T(n;w)dF(n)$ and $\bar{X}_T(w) = \int_0^1 \bar{x}_T(n;w)dF(n)$, so (i) follows directly from Lemma 1.

To see that $\bar{X}_T$ is left-hand continuous, let $w \in (0,1)$ and let $\{w_j\}$ be

a sequence in $(0,w]$ with $w_j \to w$. Then $\lim\limits_{w_j \to w} \bar{x}_T(w_j) = \lim\limits_{w_j \to w} \int_0^1 \bar{x}_T(n;w_j)dF(n) = \int_0^1 \lim\limits_{w_j \to w} \bar{x}_T(n;w_j)dF(n)$ (by the Lesbeque Dominated Convergence Theorem) $= \int_0^1 \bar{x}_T(n,w)dF(n)$ (by Lemma 1) $= \bar{X}_T(n;w)$. A similar argument shows that $\underline{X}_T$ is right-hand continuous. QED

Recall that an equilibrium is a pair $(\bar{w}, N - \bar{X})$ satisfying $\bar{X} \in \hat{X}_T(n;w)$ and $w(N - \bar{X}) = \bar{w}$.

<u>Proposition 0</u>. For any tax schedule T, a market equilibrium $(\bar{w}, N - \bar{X})$ exists and is unique.

<u>Proof</u>. As noted in the paper (following (3)) $\bar{X}_T$: $(0,1) \to\!\to [0,N]$ has nonempty compact, convex values. Hence $\hat{X}_T(w) = [\underline{X}_T(w), \bar{X}_T(w)]$ $\forall\, w$, so by Lemma 2 $\hat{X}_T$ clearly has closed graph (it is upper hemicontinous and has compact range), so $N - \hat{X}_T$ does also. By assumption, the wage function w: $[0,N] \to (0,1)$ is continuous, so by the Von Neumann Intersection Lemma, an equilibrium $(\bar{w}, N - \bar{X})$ exists. To prove uniqueness, let $(w', N - X')$ be another equilibrium, and suppose $w' \neq \bar{w}$. If $w' > \bar{w}$ then $N - X' < N - \bar{X}$ since w is a strictly decreasing function, so $\bar{X} < X'$. But by Lemma 2 $\bar{X}_T(w') \leq \underline{X}_T(\bar{w})$, so $X' \in \hat{X}_T(w')$ implies that $X' \leq \underline{X}_T(\bar{w}) \leq \bar{X}$, a contradiction. Similarly, we cannot have $w' < \bar{w}$, so $w' = \bar{w}$. And, since w is strictly decreasing, $X' = \bar{X}$. QED

NOTES

1. In the standard optimal taxation framework--e.g., Mirrlees [4]--workers respond to taxes by substituting untaxed leisure for taxable work effort. If we think of untaxed work effort as "leisure" the structure above is essentially the same, except for the utility function. If we denote leisure or untaxed work effort by ℓ, the implicit utility function in our model is (any strictly increasing, strictly concave transformation of):

$$u(y,\ell) = y + w \cdot \ell,$$

where y is after-tax consumption or income and w is the tax-sheltered wage vote. In Mirrlees' notation, consumption is x and time worked is $y = 1 - \ell/n$, so this would become

$$u(x,y) = x + wn(1 - y).$$

Evidently x and y are perfect substitutes in consumption; this utility function violates Mirrlees' assumptions ([4], p. 176) by tending to $wn(1 - y) \geq 0$, not to $-\infty$, as $x \to 0+$ and as $y \to 1-$, so behaves quite differently from his.

2. The results of section 4 of [3], concerning the preferences over schedules by "selfish" individuals interested in maximizing their own after-tax consumption, do require some qualification. Within the present class of admissable schedules, an individual of type n will typically prefer a schedule with a discontinuity (downward) at n. If admissable schedules are required to be non-decreasing functions of income, however, the individually

optimal simple schedules will still be of the form given by Proposition 4.2 of [3], if $G \leq 0$. (For $G > 0$, the preferred schedules for large n cannot be constant for all $x < n$, because of the feasiblity constraint $T(x) \leq x$, all x, so will have a segment of positive slope for low income levels.)

REFERENCES

[1] Anderson, Martin. "The Roller-Coaster Income Tax." The Public Interest 50 (1978):17-28.

[2] Atkinson, A. B. "On the Measurement of Inequality." Journal of Economic Theory 2 (1970):244-63.

[3] Kramer, Gerald H., and James M. Snyder. "Fairness, Self-Interest, and the Politics of the Progressive Income Tax." California Institute of Technology, Social Science Working Paper 498, 1983.

[4] Mirrlees, J. A. "An Exploration in the Theory of Optimum Income Taxation." Review of Economic Studies 38 (1971):175-208.