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TOURNAMENT METHODS IN CHOICE THEORY

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Abstract

Choice procedures using the notion of "tournament matrix" are investigated in the framework of general choice theory. Tournament procedures of multicriterial choice are introduced and studied. New characteristic conditions for describing some tournament and other essentially nonclassical choice functions are obtained. The comparison of tournament and graph-dominant choice mechanisms is established.

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I. Introduction

In the general choice theory there are two directions with the same long time historical development and both of them are equally logical. One of them is the extremized choice in a criterial space and its generalization namely the choice of nondominant vertices in a graph. The other one is tournament mechanisms of choice. These directions had been independently developing that is practically not interacting with one another and the first one had been considerably better developed. While the analysis of extremized procedures and their generalizations was dealt with in numerous publications and summarized in a number of monographs and surveys, the second direction found inessential reffection in the scientific literature. A slight progress in this direction can be observed only with respect to some comparatively easy facts. In connection with this there appeared a tradition to call the graphdominant mechanisms classically rational ones, while tournament mechanisms are to be referred as an example of nonclassical mechanisms. However from up to the point of internal logic and an importance of application these methods are of equal worth. Therefore the need arose of "laying bridges" between these two directions, i.e. the necessity of finding methods for comparing graphdominant and tournament mechanisms of choice and choice functions generated by them. In particular, it gave rise to the following question: which of the choice functions, generated by tournament rules, can be equivalently generated by the graph-dominant mechanisms of choice? And vice versa, which of the graph-dominant choice functions can be generated by the tournament mechanisms?

This paper is devoted to the study of the tournament mechanisms of choice from the position of the general choice theory and comparison of these mechanisms with the graphdominant mechanisms.

2. Statement of the problem

Choice is studied within the framework of a formal model (see M.A.Aizerman and A.V.Malishevki (1981)). A finite set of variants $A = \{x_i, \ i = 1,2,\dots,N\}$ is given. Any of its nonempty subsets $X \subseteq A$ can be represented for choice. The act of choice consists in isolating the subset $Y \subseteq X$ from X under some rule. The totality of the pairs $\{(X,Y)\}$ $\forall X \subseteq A$ generates a choice function $C(\cdot)$. The choice mechanism is assigned as follows: some structure is fixed on the set A (for example, a graph, criterion, tournament matrix) then the rule, indicating how to find $Y \subseteq X$ using this structure at each representation $X \subseteq A$ is given. Various choice mechanisms, generating the same choice function is referred as equivalent ones. The classes of choice mechanisms are called equivalent if any mechanism from one class has its equivalen in the other class and *vice versa*.

A directed graph Γ or, which is the same, a binary relation is used as a structure in the graph-dominant mechanism. When representing the set of variants $X \subseteq A$, the vertex-generated subgraph Γ_x with variants $x_i \in X$ being its vertices, is isolated from the graph Γ. The choice rule consists in isolating the subset $Y \subseteq X$ of all nondominant variants - vertices of the subgraph Γ_x - i.e. those without any arc coming to them from other vertices of the graph Γ_x:

(1) $\quad Y = \{x_i \in X \mid \text{there exists no } x_j \in X \text{, such that } x_j \Gamma x_i\}$,

where $x_j \Gamma x_i$ denotes that there is an arc from the vertex x_j

to the vertex x_i in the graph Γ. The mapping $X \to Y$ realized by such a mechanism is referred to as the graphdominant choice function $C_{GD}(\cdot)$. By the analysis of all possible graphs Γ, the class of such choice functions is formed and henceforth if will be denoted via Q_{GD}.

Graphdominant choice functions with nonempty choice (i.e. $C(X) \neq \emptyset \quad \forall X \subseteq A$) form the subclass $\tilde{Q}_{GD}$ of the class Q_{GD}; and the choice functions from $\tilde{Q}_{GD}$ are generated by dominant rule (1) on acyclic graphs. Those are namely such choice functions that are usually referred to as classically rational in the literature (see Richter (1971), Plott (1976) and Mirkin (1979)).

In the classical theory, moreover the choice mechanism under rule (1) on the graphs, mechanism of choice of the Pareto-optimal variants in the criterial space $\{\varphi_\nu\}$, $\nu = \overline{1,n}$ is also widely used:

$$(2) \qquad Y = \{x_i \in X \mid \text{there exists no } x_j \in X \text{, such that } [\varphi_\nu(x_j) \geq \varphi_\nu(x_i) \ \forall \nu = \overline{1,n} \text{ and } \exists \nu_0 : \varphi_{\nu_0}(x_j) > \varphi_{\nu_0}(x_i)]\}.$$

Here, the estimate of a variant x_i under the criterion φ_ν is denoted as $\varphi_\nu(x_i)$. With $n \geq 1$ this mechanism is equivalent to the choice under rule (1) on the transitive graph Γ defined by the relation:

$$x_i \Gamma x_j \Longleftrightarrow [\varphi_\nu(x_i) \geq \varphi_\nu(x_j) \ \forall \nu = \overline{1,n}] \wedge [\exists \nu_0 : \varphi_{\nu_0}(x_i) > \varphi_{\nu_0}(x_j)].$$

The class of choice mechanisms under rule (2) in all possible criterial spaces is equivalent to the class of choice mechanisms under rule (1) on the all possible transitive and acyclic graphs; and the class containing choice functions of the Pareto-optimal variants Q_{PAR} is a subclass $\tilde{Q}_{G\mathcal{D}}$.

A tournament matrix is used as a structure in the tournament mechanisms of choice, i.e. an integer-valued square matrix of the pairwise compared variants $T = \| t_{ij} \|$, $i,j = \overline{1,N}$ meeting the following conditions: for any $i,j = \overline{1,N}$ $t_{ij} \geq 0$, $t_{ii} = 0$ and if $i \neq j$, then $t_{ij} + t_{ji} = n$. The rows and columns of this matrix correspond to the variants from the set A. A total score table of one-round or multiround tournament may be used as an example of a tournament matrix. The integer t_{ij} is interpreted in this example as a number of points won by a sportsman x_i from a sportsman x_j. In the tournament mechanisms of choice the number n denotes the number of tournament rounds, and the square submatrix T_X of the tournament matrix T consisting of all t_{ij} such that $x_i, x_j \in X$, corresponds to the representation $X \subseteq A$.

The choice tournament rules are arranged as follows: a numerical index is introduced and according to its value the variants are ranked (in many problems the aim is variants ranking but not the choice). Then the variants with the highest rank are included in the choice (and only these variants).

Further two rules of choice on tournament matrices and, respectively, two classes of choice functions are considered: total score choice rule (Copeland (1951)) and guaranteed result rule, based on maximization of minimum score number.

The total score choice rule is widely known. For example, it is used for identification of the winners in the round sports tournaments. The maxmin rule of a guaranteed result is not yet widely used, although it is applied to real problems. The Appendix carries the data showing that one of the multicriterial procedures of choice (see Nogin (1976)) as well as the procedure utilized in the dynamical voting theory (see Kramer (1977)) are reduced to the maxmin choice on a tournament matrix.

In case of the total score rule with representation $X \subseteq A$ on a submatrix T_X the total (sum) of the row elements $S_X(x_i)$ is calculated for each variant $x_i \in X$ (the total score ("winnings") won by a variant x_i from the rest variants, included in the representation X). The variants are ranked with respect to $S_X(x_i)$ and the variants $x_i \in X$ having the maximum total sum, are included into the choice:

$$(3) \qquad Y = \{ x_i \in X \mid S_X(x_i) = \max_{x_j \in X} S_X(x_j) \} .$$

The choice function, generated by this mechanism will be referred as a total score choice function and denoted via $C_{sum}(\cdot)$. When applying the maxmin tournament rule with the representation $X \subseteq A$ instead of the sum $S_X(x_i)$ we define the number $M_X(x_i)$, equal to a minimal element in i - row of a submatrix T_X, located beyond its main diagonal. In this case the choice rule has the following form:

$$(4) \qquad Y = \{ x_i \in X \mid M_X(x_i) = \max_{x_j \in X} M_X(x_j) \} .$$

The choice function, generated by such mechanism will be denoted as $C_M(\cdot)$.

Each of the two above tournament mechanisms on the all possible tournament matrices corresponds to a certain class of choice functions: Q_{sum} is for rule (3) of the total score and Q_M is for maxmin rule (4).

It should be noted that the choice under rules (4) and (3) is not empty, i.e. $\forall C(\cdot) \in Q_{sum}$ and $\forall C(\cdot) \in Q_M$ holds $C(X) \neq \emptyset \quad \forall X \subseteq A$.

Further the tournament procedures of multicriterial choice (section 3) are presented and subjected to study from the angle of the general choice theory.

Section 4 deals with the formulation of new characteristic conditions required for description of nonclassical choice functions.

The problem of how are classes Q_{GD}, Q_{SUM} and Q_M mutually related in the space $\mathcal{C}$ of all choice functions will be dealt upon in Section 5. The answer to this question allows us to judge to what extent the mechanisms of one of these three types can be equivalently substituted for the mechanisms of other type.

3. Criterial tournaments

The Borda voting rule (see Borda (1781), Young and Levenglick (1978)), widely applied in the voting theory, can be described in terms of tournament matrices of a specified type. In fact let n of ranking $R_1, R_2, \ldots, R_n$ of variants $x_1, x_2, \ldots, x_N$ are given. Simplify the description the rankings is counted on to be strict. Under the Borda rule, the ranking R, aggregating the rankings $R_1, \ldots, R_n$ is formed as follows. Variants x_i in the ranking R are arranged in the increasing order of sums $\sum_{\nu=1}^{n} z_\nu(x_i)$ where $z_\nu(x_i)$ is a rank of a variant x_i in the ranking R_ν.

The square matrix $T_\nu = \| t_{ij}^\nu \|$ is corresponded to each ranking R_ν $(\nu \in \{1, \ldots, n\})$, with the matrix constructed as follows: $z_\nu(x_i) < z_\nu(x_j) \Leftrightarrow t_{ij}^\nu = 1,\ t_{ji}^\nu = 0,\ t_{ii} = 0$ $\forall i,j = \overline{1,N}$. This matrix T_ν is a transitive tournament matrix (see Moon (1968)). The rank of a variant x_i in the ranking R_ν is connected with the sum of elements in i - row $S_A^\nu(x_i) = \sum_{j=1}^{N} t_{ij}^\nu$ of the matrix T_ν in the following way:

$$z_\nu(x_i) = N - S_A^\nu(x_i).$$

Hence:

$$\sum_{\nu=1}^{n} z_\nu(x_i) = N \cdot n - \sum_{\nu=1}^{n} S_A^\nu(x_i). \tag{5}$$

Let us consider the tournament matrix $T = \| t_{ij} \|$, such that $t_{ij} = \sum_{\nu=1}^{n} t_{ij}^\nu$ $\forall i,j = \overline{1,N}$. Matrix T can be interpreted as a matrix of n - round tournament in which each "round" T_ν is a transitive tournament. For this matrix the sum of elements in i - row is

$$S_A(x_i) = \sum_{j=1}^{N} t_{ij} = \sum_{j=1}^{N} \sum_{\nu=1}^{n} t_{ij}^\nu = \sum_{\nu=1}^{N} S_A^\nu(x_i).$$

Expression (5) implies

$$S_A(x_i) = N \cdot n - \sum_{\nu=1}^{n} z_\nu(x_i)$$

and consequently $\sum_{\nu=1}^{n} z_\nu(x_i) < \sum_{\nu=1}^{n} z_\nu(x_j) \Leftrightarrow S(x_i) > S(x_j)$.

Thus, the ranking under the Borda rule coincides with the ranking under the total score rule (3) on a specific tournament matrix T.

In such tournament matrices (and, namely, in matrices which represent the sum of one-round transitive matrices) in addition to relations $t_{ij} + t_{ji} = n,\ t_{ii} = 0,\ t_{ij} \geq 0$ $\forall i,j = \overline{1,N}$ the inequality of a triangle is fulfilled: $t_{ij} + t_{jk} \geq t_{ik}$ $\forall i,j,k = \overline{1,N}$. Let us prove this statement. From $t_{ij}^\nu \in \{0, 1\}$ it follows that inequality $t_{ij}^\nu + t_{jk}^\nu \geq t_{ik}^\nu$ may be violated only in one case: $t_{ij}^\nu = 0$, $t_{jk}^\nu = 0$, $t_{ik}^\nu = 1$. But since all matrices $T_\nu = \| t_{ij}^\nu \|$ $(\nu = 1, \ldots, n)$ are transitive, then if $t_{ij}^\nu = 0$ and $t_{jk}^\nu = 0$, then $t_{ik}^\nu = 0$ for any $i,j,k = \overline{1,N}$. By summing up all matrices T_ν $(\nu = \overline{1,n})$ we obtain: $\sum_{\nu=1}^{n} t_{ij}^\nu + \sum_{\nu=1}^{n} t_{jk}^\nu \geq \sum_{\nu=1}^{n} t_{ik}^\nu$, i.e. $t_{ij} + t_{jk} \geq t_{ik}$ for all $i,j,k = \overline{1,n}$.

It is easy to show in example that a general type tournament matrix does not satisfy this property.

Tournament matrices which are the sum of transitive one-round tournament matrices, may be used in problems of multicriterial choice. For each criterion φ_ν $(\nu \in \{1,\dots,n\})$ we construct a transitive matrix observing the following rule:

$$t_{ij}^{\nu} = 1 \quad \text{if} \quad \varphi_\nu(x_i) > \varphi_\nu(x_j),$$
$$t_{ij}^{\nu} = 0 \quad \text{if} \quad \varphi_\nu(x_i) < \varphi_\nu(x_j),$$

and as it was previously done we shall consider the matrix $T = \|t_{ij}\|$, where $t_{ij} = \sum_{\nu=1}^{n} t_{ij}^{\nu}$ $\forall i,j = \overline{1,N}$. (To simplify the statement it is assumed that there are no variants with coinciding criterial estimates, i.e. with $x_i \neq x_j$:

$\varphi_\nu(x_i) \neq \varphi_\nu(x_j)$ for all $\nu \in \{1,\dots,n\}$

Such matrices which are the sum of transitive tournament matrices, are referred to as criterial tournament matrices or matrices of n-round criterial tournaments.

In terms of criterial tournament matrices the choice of Pareto-optimal variants can be described as follows. The square submatrix T_X of matrix T corresponds to representation $X \subseteq A$. Variants $x_i \in X$ are Pareto-optimal, with a row in matrix T_X with the exception of the main diagonal containing no number t_{ij} equal to zero.

Criterial tournament matrices form the subclass Σ_{TOUR}^{CR} of the class of all possible tournament matrices: $\Sigma_{TOUR}^{CR} \subset \Sigma_{TOUR}$. The choice on criterial matrices under rule (3) generates the class of choice functions $Q_{SUM}^{CR} \subseteq Q_{SUM}$, and under rule (4) it generates the class $Q_M^{CR} \subseteq Q_M$. The question may arise, wheather the choice-functions, generated by these rules on criterial tournament matrices satisfy any specified properties, isolating them among the functions generated on the general-type tournament matrices. In another words what part of the class Q_{SUM} is formed by the class Q_{SUM}^{CR} and what part of the class Q_M is formed by the class Q_M^{CR}.

<u>Theorem 1</u>. Let an arbitrary tournament matrix of a general type $T \in \Sigma_{TOUR}$ is given. Then there exists a criterial tournament matrix $T^{CR} \in \Sigma_{TOUR}^{CR}$, such that:

a) choice functions, generated on matrices T and T^{CR} under rule (3) coincide, and

b) choice functions, generated on these matrices under rule (4) coincide.

From Theorem 1 it follows that despite of the criterial matrices being a part of the set of all tournament matrices, the classes of choice functions on criterial matrices and general-type tournament matrices coincide both under rule (3) and (4), i.e. $Q_{SUM} = Q_{SUM}^{CR}$, $Q_M = Q_M^{CR}$.

<u>Proof</u>:

1°. First, let us prove that choice functions, generated under rule (3) on tournament matrices $T^1 = \|t_{ij}^1\|$ and $T^2 = d_1 \cdot T^1 + d_2 \cdot E$ (where $d_1 > 0$, $d_2 \geq 0$ are integers; $E = \|e_{ij}\|$, $e_{ij} = 1$ $\forall i,j = \overline{1,N}$ $(i \neq j)$, $e_{ii} = 0$ $\forall i = \overline{1,N}$ coincide, the same refers to choice function generated under rule (4) on these matrices T^1 and T^2.

Fix any representation $X \subseteq A$ and analyse the respective submatrices T_X^1 and T_X^2 of matrices T^1 and T^2. Denote cardinality of representation X via $N_X = |X|$. Then

$$S_X^2(x_i) = \sum_{x_j \in X} t_{ij}^2 = \sum_{x_j \in X} (d_1 \cdot t_{ij}^1 + d_2) = d_1 \cdot S_X^1(x_i) + N_X d_2$$

It is obvious that the values $S_X^1(x_i)$ and $S_X^2(x_i)$ amount to maximum with the same $x_i \in X$. Similarly, values $M_X^1(x_i)$ and $M_X^2(x_i) = d_1 \cdot M_X^1(x_i) + d_2$ also amount to maximum with the same $x_i \in X$. Hence, the choice on submatrices T_X^1 and T_X^2 under rules (3) and (4) coincides for any representations $X \subseteq A$

2°. Let us prove that with $T=\|t_{ij}\|$ being a matrix of n-round tournament of a general type there exists an integer $d>0$ such that matrix $T'=T+dE$ is a matrix of a criterial $(n+2d)$-round tournament.

Let us consider the case with n being odd and even individually.

a) n is even.

Set-up on matrix T the criterial estimates $\varphi_\nu(x_i)$ of variants $x_i \in A$ as follows. Take any fixed pair of numbers $t_{k\ell}$ and $t_{\ell k}$ which is symmetrical with respect to the main diagonal of matrix T and construct n of criteria φ_ν with respect to these numbers. In criteria whose number is equal to $t_{k\ell}$ the variant x_k is better than variant x_ℓ. In criteria whose number is equal to $t_{\ell k}$, the variant x_k is worse than variant x_ℓ. The remaining variants from the set $A \setminus \{x_k \cup x_\ell\}$ in any $\frac{n}{2}$ criteria are arranged in a arbitrary fixed order R and in the remaing $\frac{n}{2}$ criteria it is done in the reverse order R^{-1}. Further, in any $\frac{n}{2}$ criteria (to be more exact those arranged in the order R) variants x_k and x_ℓ are better than the rest variants from $A \setminus \{x_k \cup x_\ell\}$ and in the remaining $\frac{n}{2}$ criteria (those arranged in the order R^{-1}) variants x_k and x_ℓ are worse than the remaining variants.

Fig.1 shows the above construction of n criteria for the case when $t_{k\ell} > t_{\ell k}$; for the case with $t_{k\ell} = t_{\ell k}$ criteria are constructed in a similar way.

For this n criteria, obtained with respect to numbers $t_{k\ell}$ and $t_{\ell k}$, we construct matrix of n-round criterial tournament $T^{(k,\ell)}=\|t_{ij}^{(k,\ell)}\|$ $(i,j=\overline{1,N})$. Index (k,ℓ) of the matrix $T^{(k,\ell)}$ signifies that this matrix is constructed with respect to numbers $t_{k\ell}$ and $t_{\ell k}$. It is seen from the const-

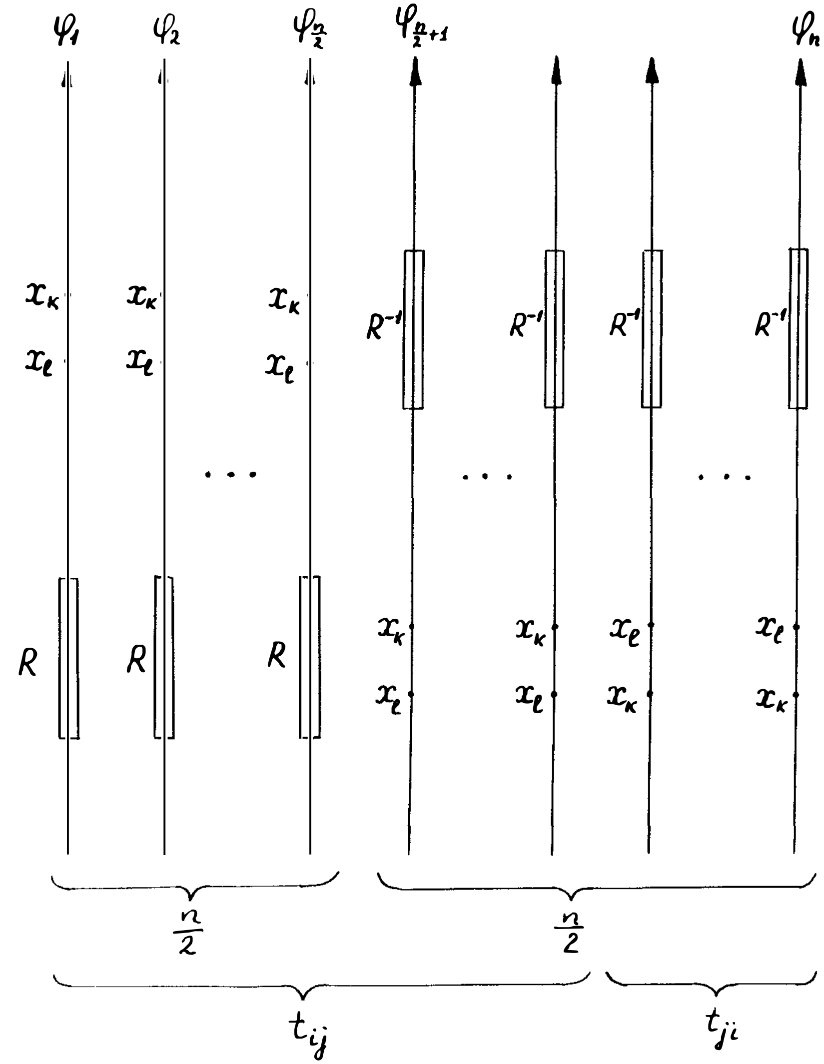

Figure 1

ruction of criteria that in the matrix $T^{(k,\ell)}$ we have $t^{(k,\ell)}_{k\ell} = t_{k\ell}$; $t^{(k,\ell)}_{\ell k} = t_{\ell k}$; $t^{(k,\ell)}_{ii} = 0$; $t^{(k,\ell)}_{ij} = \frac{n}{2}$ for all $i,j = \overline{1,N}$ ($i \neq k$, $i \neq \ell$, $j \neq k$, $j \neq \ell$, $i \neq j$). Hence, the matrix $T^{(k,\ell)}$ has the form shown in Fig. 2. The submatrix $T_X^{(k,\ell)}$ of the matrix $T^{(k,\ell)}$ corresponds to representations $X \subseteq A$.

For each of the remaining pairs of numbers t_{ij} and t_{ji} ($i,j = \overline{1,N}$, $i \neq j$) symmetrical with respect to the main diagonal of matrix T we construct n criterial estimates and then we construct our matrices $T^{(i,j)}$ with respect to them. The total dimension of the constructed criterial space $\{\varphi_\nu\}$ is equal to $n\,\frac{N(N-1)}{2}$.

We denote the sum of all matrices $T^{(k,\ell)}$ $(k,\ell = \overline{1,N}, k \neq \ell)$ as $T' = \| t'_{ij} \|$:

$$T' = \sum_{\substack{k,\ell = \overline{1,N} \\ k \neq \ell}} T^{(k,\ell)} .$$

The construction of matrices $T'^{(k,\ell)}$ allows us to see that by fixing the arbitrary $x_i, x_j \in A$ we obtain:

$$t'_{ij} = t_{ij} + \frac{n}{2}\left[\frac{N(N-1)}{2} - 1\right]$$, i.e.

$T' = T + d \cdot E$, where T' is a matrix of $n\,\frac{N(N-1)}{2}$-round criterial tournament.

b) n is odd.

Represent a matrix $E = \| e_{ij} \|$ in the form:

$$E = E^1 + E^2, \quad E^1 = \| e^1_{ij} \|, \quad E^2 = \| e^2_{ij} \|,$$

$$e^1_{ij} = \begin{cases} 1, & \text{if } i < j \\ 0, & \text{if } i \geq j \end{cases} ; \qquad e^2_{ij} = \begin{cases} 1, & \text{if } i > j \\ 0, & \text{if } i \leq j \end{cases} .$$

Let us consider the matrix $\overline{T} = T + E^1 + E^2$. Denote $T^* = T + E^1$. Then $\overline{T} = T^* + E^2$, where $\overline{T} = \| \overline{t}_{ij} \|$, $T^* = \| t^*_{ij} \|$.

$$\begin{array}{c|cccccc|}
 & & x_i & & & x_j & \\
\hline
 & 0 & \frac{n}{2} & \frac{n}{2} & \frac{n}{2} & \frac{n}{2} & \frac{n}{2} \\
x_i & \frac{n}{2} & 0 & \frac{n}{2} & \frac{n}{2} & t_{ij} & \frac{n}{2} \\
 & \frac{n}{2} & \frac{n}{2} & 0 & \frac{n}{2} & \frac{n}{2} & \frac{n}{2} \\
 & \frac{n}{2} & \frac{n}{2} & \frac{n}{2} & 0 & \frac{n}{2} & \frac{n}{2} \\
x_j & \frac{n}{2} & t_{ji} & \frac{n}{2} & \frac{n}{2} & 0 & \frac{n}{2} \\
 & \frac{n}{2} & \frac{n}{2} & \frac{n}{2} & \frac{n}{2} & \frac{n}{2} & 0 \\
\hline
\end{array}$$

Figure 2

Hence, $t^*_{ij} + t^*_{ji} = n+1 \quad \forall i,j = \overline{1,N},\ i \neq j$, i.e. $t^*_{ij} + t^*_{ji}$ is even and matrix E^2 is a matrix of one-round criterial tournament. Thus, the case with n being odd is reduced to the case with n being even.

Q.E.D.

Let points $\varphi_\nu(x_i)$ in some criterial space $\{\varphi_\nu\},\ \nu=\overline{1,n}$ correspond to variants $x_i \in A$. Using criterial estimates of variants $\varphi_\nu(x_i)$ we construct a criterial tournament matrix $T = \|t_{ij}\|$. Let us compare choice functions $C_{sum}(\cdot)$ and $C_M(\cdot)$, generated on the matrix T under total score rule (3) and maxmin rule (4) respectively, with the choice function $C_{PAR}(\cdot)$, generated under choice rule (2) of Pareto-optimal variants in the same criterial space.

<u>Theorem 2.</u> The choice functions $C_{sum}(\cdot)$ and $C_M(\cdot)$,generated under total score rule (3) and maxmin rule (4) respectively, on the criterial tournament matrix, constructed with respect to the values of criterial estimates $\varphi_\nu(x_i)$ of variants $x_i \in A$ in the criterial space $\{\varphi_\nu\},\ \nu=\overline{1,n}$ are embeded into the choice function of Pareto-optimal variants $C_{PAR}(\cdot)$ in this criterial space $\{\varphi_\nu\},\ \nu=\overline{1,n}$, i.e. $C_{sum}(X) \subseteq C_{PAR}(X)$ and $C_M(X) \subseteq C_{PAR}(X) \quad \forall X \subseteq A$.

<u>Proof:</u>

Let us prove the theorem for the function $C_{sum}(\cdot)$ .Assume that the statement of the theorem is wrong. Let $x_i \in C_{sum}(X)$ for some $X \subseteq A$ but $x_i \in C_{PAR}(X)$. Then there exists $x_j \in X$ such that $\varphi_\nu(x_j) \geq \varphi_\nu(x_i) \quad \forall \nu = \overline{1,n}$ and $\exists\ \nu_0$: $\varphi_{\nu_0}(x_j) > \varphi_{\nu_0}(x_i)$. But then $t_{jk} \geq t_{ik} \quad \forall x_k \in X$ and at the same $t_{ji} > t_{ij}$, i.e. $\sum_{x_k \in X} t_{jk} > \sum_{x_k \in X} t_{ik}$ and $x_i \notin C_{sum}(X)$.

The obtained contradiction proves the theorem.

The proof for the function $C_M(\cdot)$ could be done similarly.

Q.E.D.

It should be noted that under maxmin rule (4) the set $X \subseteq A$ with respect to values $M_X(x_i)$ of the variants $x_i \in X$ included in it is separated into "layers". All in all there are $n+1$ of such layers corresponding to values $M_X(x_i)$ from 0 to n (fig. 3) . Some of these layers may be empty. It is directly seen that in these terms the Pareto set $C_{PAR}(X)$ is a totality of all variants $x_i \in X$ which belong to the layers, for which $M_X(x_i)$ are equal respectively: $n, n-1, \ldots, 2, 1$ and only the variants from the layer with $M_X(x_i)=0$ do not belong to the Pareto set.

In this way, with respect to $M_X(x_i)$ the Pareto set $C_{PAR}(X)$ is separated into n layers, some of which may be empty. The variants $x_i \in X$ located in the first nonempty layer having a maximal value $M_X(x_i)$ are included in the choice under rule (4).

Under total-score rule (3) separation of the Pareto set does not occur, (generally speaking) i.e. there exists such $X \subseteq A$ and $x_i, x_j \in X$, that $S_X(x_i) = S_X(x_j)$ but $x_i \in C_{PAR}(X),\ x_j \in X \setminus C_{PAR}(X)$.

Note, that total score choice rule (3) and maxmin rule (4) are particular cases of a one-parametrical family of the choice rules $\{\Pi_q,\ q=[1,\infty]\}$. According to the rule Π_q each variant $x_i \in X$ requires calculation of the exponent value

$$W^q_X(x_i) = Const(q) - \sqrt[q]{\sum_{x_j \in X} (n-t_{ij})^q}$$

and the choice realized making use of formula

$$Y = \{x_i \in X \mid W^q_X(x_i) = \max_{x_j \in X} W^q_X(x_j)\}$$

With $q = 1$ and $Const(1) = nN$ we deal with total-score choice

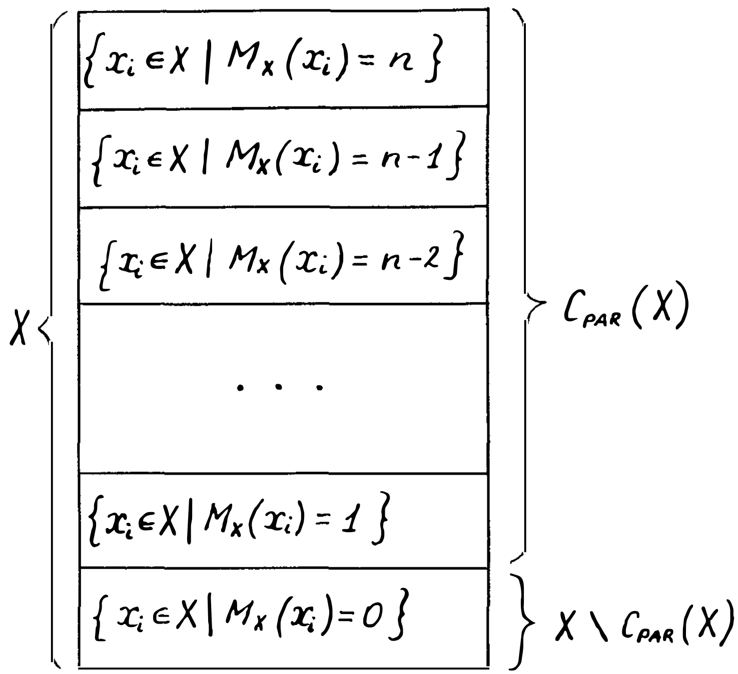

Figure 3

rule (3) and with $q = \infty$ and $Const(\infty) = n$ we pass over to maxmin rule (4). It can be proved (similar to the procedure shown for the choice function $C_{sum}(\cdot)$ in Theorem 2) that the choice function $C^q(\cdot)$ generated by the rule π_q on a criterial tournament matrix, satisfyes the property

$C^q(X) \subseteq C_{PAR}(X) \quad \forall X \subseteq A$ with any $q \in [1, \infty]$.

Remark. We assumed earlier that all criteria φ_ν, $\nu = \overline{1, n}$ are strict, i.e., providing for strict ranking of variants. In terms of tournament matrices it means that a matrix of one-round criterial tournament T_ν , constructed with respect to the criterion φ_ν $(\nu \in \{1, 2, \dots, n\})$ consists only of 0 and 1 . We analyse the case where the criteria φ_ν are weak orders, i.e. coinciding of variants estimates relative to the criterion is feasible . In this case in constructing the matrix $T_\nu = \| t^\nu_{ij} \|$ with respect to (the criterion) φ_ν we obtain :

$$t^\nu_{ij} = \begin{cases} 1, \text{ if } \varphi_\nu(x_i) > \varphi_\nu(x_j) , \\ 0, \text{ if } \varphi_\nu(x_i) < \varphi_\nu(x_j) . \end{cases}$$

However, in addition to it we assume that if $\varphi_\nu(x_i) = \varphi_\nu(x_j)$ then $t^\nu_{ij} = \frac{1}{2}$, i.e. if $\varphi_\nu(x_i) = \varphi_\nu(x_j)$, then it means that variants x_i and x_j ended the game "in a draw" and according to the accepted rules of (sport) tournaments they got a half a score each.

The matrix of n-round criterial tournament $T = \| t_{ij} \|$ again has the form :

$$t_{ij} = \sum_{\nu=1}^{n} t^\nu_{ij} \quad \forall\, i, j = \overline{1, N} .$$

However, if under strict criteria φ_ν it consisted of numbers 0,1,2,..., n , now under nonstrict criteria φ_ν it consists of numbers $0, \frac{1}{2}, 1, 1\frac{1}{2}, 2, 2\frac{1}{2}, \dots, n$. Multiplying the matrix T by 2 we obtaining a matrix of $2n$-round criterial **tournament** $\tilde{T} = \| \tilde{t}_{ij} \|$, where $\tilde{t}_{ij} = 2 t_{ij} \quad \forall i,j = \overline{1,N}$. The matrix $\tilde{T}$ consists of numbers $0,1,2,3,\dots, 2n-1, 2n$. This matrix is a matrix of a criterial $2n$-round tournament with all criteria φ_ν, $\nu = \overline{1,2n}$ being strict. As it was shown in item 1° of the proof in Theorem 1, the choice functions, generated by rule (3) on the tournament matrices $T' = \| t'_{ij} \|$ and $T^2 = d_1 \cdot T^1$ (where $d_1 > 0$) coincide as well as the choice functions generated by rule (4) on these matrices.

Thus, all results obtained for strict criteria φ_ν are extended with no changes made to the case of nonstrict criteria.

4. Characteristic conditions for choice function description

The language of characteristic conditions are widely applied to the description of choice functions classes in the theory of choice (see Arrow (1959), Sen (1971), Aizerman and Malishevski (1981), Chernoff (1954)). There exists an abundant set of characteristic conditions providing for comprehensive description of the graph-dominant choice functions class.

In these characteristic conditions the "behaviour" of choice functions under certain deformations of the representations X is used for description of these functions.

These are the formulations of these characteristic conditions. The terminology of Aizerman'and Malishevski's paper (1981) were used in them. That paper contains the references to the original papers.

Heritage condition (H):

$$X' \subseteq X \Rightarrow C(X') \supseteq C(X) \cap X'$$

Concordance condition (C)

$$X = X' \cup X'' \Rightarrow C(X) \supseteq C(X') \cap C(X'')$$

Independence of rejecting the outcast variants (O)

$$C(X) \subseteq X' \subseteq X \Rightarrow C(X') = C(X)$$

Constancy of residual choice (K)

$$X' \subseteq X, \; X' \cap C(X) \neq \emptyset \Rightarrow C(X') = C(X) \cap X'.$$

It is known (for example, Aizerman and Malishevski (1981)) that simultaneous fulfilment of conditions H and C exactly isolates the class of choice functions Q_{GD}; simultaneous fulfilment of conditions H,C,O and the condition of non-emptiness of choice isolates the class Q_{PAR}, fulfilment of the condition K exactly isolates the class of extremal choice functions with respect to the scalar criterion.

In addition to these conditions the class Q_{GD} can be isolated with the necessity and sufficiency of the other system of conditions referred to as the Condorcet conditions (Aizerman (1984)):

Direct Condorcet condition(DCC):

(6) $$x_i \in C(\{x_i, x_j\}) \quad \forall x_j \in X \Rightarrow x_i \in C(X).$$

Reverse Condorcet condition (RCC):

(7) $$x_i \in C(X) \Rightarrow x_i \in C(\{x_i, x_j\}) \quad \forall x_j \in X.$$

These characteristic conditions are widely used for describing various classes of choice functions, including those which are not graph dominant (see Plott (1973), Aizerman and Malishevski (1981)).

However, almost all these characteristic conditions turned to be inapplicable in describing the classes of choice functions on the tournament matrices Q_{Sum} and Q_M : for example, some functions $C(\cdot)\in Q_{SUM}$ satisfy the condition H, and some of them do not. The same refers to the rest of the above conditions with the exception of the direct Condorcet condition (6). All choice functions from the class Q_M satisfy this condition. Let us prove it.

Let us fix a orbitrary $X\subseteq A$. Let it be $x_i\in C_M(\{x_i, x_j\})$ $\forall x_j\in X$. Then in the matrix T: $t_{ij}\geq t_{ji}$ $\forall x_j\in X$. Hence it follows that $\min_{x_j\in X} t_{ij}\geq \min_{x_\ell\in X} t_{k\ell}$ $\forall x_k\in X$, i.e. $M_X(x_i)\geq M_X(x_j)$ $\forall x_j\in X$ and $x_i\in C_M(X)$.

Other characteristic conditions are required to describe the classes of the choice functions Q_{SUM} and Q_M.

New characteristic conditions are to be formed as follows. The formulation of these conditions include two choice functions: the $C(\cdot)$ - function under study and some other $C^*(\cdot)$. Assume that we look for a characteristic condition for the choice function $C(\cdot)$ which at all representations $X\subseteq A$ is embedded in the function $C^*(\cdot)$, i.e. $C(X)\subseteq C^*(X)$ $\forall X\subseteq A$; and choice function $C^*(\cdot)$ which will be referred to as an embracing one belonging to the known class of the choice functions.

<u>Definition 1.</u> We shall say that the choice function $C(\cdot)$ with respect to the embracing function $C^*(\cdot)$ satisfies the condition:

Independence of rejecting the variants which do not belong to the embracing function $C^*(\cdot)$ (condition IR), if

$$C(X)=C(X\setminus X'), \quad \text{where } X'\cap C^*(X)=\emptyset;$$

Representability of superposition (condition RS), if

$$C(X)=C(C^*(X)) \quad \forall X\subseteq A,$$

Inverse representability of superposition (IRS), if

$$C(X)=C^*(C(X)) \quad \forall X\subseteq A;$$

Commutativity of superposition (condition CS) if

$$C(C^*(X))=C^*(C(X)) \quad \forall X\subseteq A;$$

Strong commutativity of superposition (SCS), if

$$C(X)=C(C^*(X))=C^*(C(X)) \quad \forall X\subseteq A.$$

The characteristic conditions introduced by definition 1 can be simply interpreted. For example, when a choice function of the Pareto-optimal variants $C_{PAR}(\cdot)$ is considered as an embracing function $C^*(\cdot)$ the condition IR establishes that variant, which doesnot belong to the Pareto set, have no influence on the choice. For functions $C(\cdot)$ intended for isolating a part of the Pareto-set, this condition is natural.

Below follows the theorems, establishing the relations between the introduced characteristic conditions depending on a class to which the embracing choice function $C^*(\cdot)$ belongs.

<u>Theorem 3</u>. Let $C^*(\cdot)$ satisfies the condition O. For the choice function $C(\cdot)$ to satisfy the condition RS with respect to $C^*(\cdot)$, it is necessary and sufficient that the function $C(\cdot)$ satisfies the condition IR with respect to $C^*(\cdot)$.

<u>Proof:</u>

1. Sufficiency. Let $C(X)=C(X\setminus X')$, where $X'\cap C^*(X)=\emptyset$. The following will be assumed as X': $X'=X\setminus C^*(X)$. (Here the condition $X'\cap C^*(X)=\emptyset$ is fulfilled). Then $X\setminus X'=$ $=X\setminus(X\setminus C^*(X))=C^*(X)$ (due to $C^*(X)\subseteq X$). Hence $C(X)=$ $=C(C^*(X))$.

2. Necessity. Let $C(X)=C(C^*(X))\ \forall X\subseteq A$.

Let us fix an arbitrary $X\subseteq A$. From the fact that $C^*(\cdot)\in O$ it follows that in rejecting any $X'\subseteq X\setminus C^*(X)$ the following is fulfilled: $C^*(X)=C^*(X\setminus X')$, where $X'\cap C^*(X)=\varnothing$. For the set $X\setminus X'$, where $X'\cap C^*(X)=\varnothing$, correlation $X'\cap C^*(X)=\varnothing$ is fulfilled under the condition of the theorem. Then $C(X\setminus X')=$ $=C(C^*(X\setminus X'))=C(C^*(X))=C(X)$. Hence $C(X\setminus X')=$ $=C(X)$, where $X'\cap C^*(X)=\varnothing$.

Q.E.D.

Theorem 3 gives an answer to the question: when a two-stage choice function $C(C^*(\cdot))$ can be equivalently represented by one choice function $C(\cdot)$ (under the condition, when $C(X)\subseteq C^*(X)\ \forall X\subseteq A$).

In case when $C^*(\cdot)\in O$, it is necessary and sufficient that choice function $C(\cdot)$ satisfies the condition IR with respect to $C^*(\cdot)$.

<u>Theorem 4.</u> For the choice function $C(\cdot)$ to satisfy the condition IRS with respect to $C^*(\cdot)$ it is sufficient for it to satisfy the condition H. The Proof of the theorem is evident. Fix an arbitrary $X\subseteq A$. Since $C(X)\subseteq C^*(X)$ and $C^*(\cdot)\in H$, it follows straightway that $C^*(C(X))=C(X)$

Q.E.D.

From theorems 3 and 4 it follows:

<u>Corollary 1.</u> Let the embracing function $C^*(\cdot)$ satisfies the conditions H and O. Then for the function $C(\cdot)$ to satisfy the condition SCS it is necessary and sufficient that $C(\cdot)$ satisfies the condition IR with respect to $C^*(\cdot)$. In this case conditions IR, RS, CS and SCS are equivalent.

It should be noted that the assertion of Theorem 4 and

corollary 1 can be strengthened by substituting in their formulation instead the condition H for the condition of choice maintenance CM which has the following form: the choice function $C(\cdot)$ satisfies the condition CM if from $X'\subseteq C(X)$ it follows that $X'=C(X')$. It is easy to see that the condition CM is the weakening of the condition H.

As it is known (see Aizerman and Malishevski (1981)) the choice function of the Pareto-optimal variants $C_{PAR}(\cdot)$ satisfies the conditions H, C and O. It follows from corollary 1 that for the choice function $C(\cdot)$ isolating a part of the Pareto set in a criterial space $\{\varphi_\nu\}$ the conditions IR, RS, CS and SCS are equivalent if the choice function of the Pareto-optimal variants in this criterial space is referred to as the embracing one. In particular these functions isolating a part of the Pareto set are the choice functions $C^q(\cdot)$ generated by the rules Π_q from the family $\{\Pi_q,\ q\in[1,\infty]\}$ on the tournament criterial matrices.

<u>Theorem 5.</u> The choice function $C_M(\cdot)$ generated by maximin rule (4) on the tournament criterial matrix, constructed with respect to criterial estimates $\varphi_\nu(x_i)$ of variants $x_i\in A$ in the criterial space $\{\varphi_\nu\}$, $\nu=\overline{1,n}$, satisfies the conditions IR, RS, CS and SCS if the choice function of the Pareto-optimal variants in the same criterial space $\{\varphi_\nu\}$, $\nu=\overline{1,n}$ serves as an embracing function.

<u>Proof</u>

From Corollary 1 it follows that it is sufficient to prove that $C_M(\cdot)$ satisfies one of the conditions IR, RS, CS and SCS with respect to $C_{PAR}(\cdot)$. Let us prove that $C_M(\cdot)$ satisfies the condition IR.

We analyse the arbitrary representation $X\subseteq A$. $C_M(X)$ and $C_{PAR}(\cdot)$ correspond to it, and $C_M(X)\subseteq C_{PAR}(X)$. Now prove that rejecting of variants $x_j \notin C_{PAR}(X)$ does not change the values of $M_X(x_i)$ and $M_X(x_k)$ of variants $x_i \in C_M(X)$, $x_k \in C_{PAR}(X) \setminus C_M(X)$. In rejecting any variants the values $M_X(x_i)$ and $M_X(x_k)$ cannot be reduced (with respect to a defining of the value $M_X(\cdot)$). The values $M_X(x_i)$ and $M_X(x_k)$ do not increase in rejecting the variants $x_j \in C_{PAR}(X)$, i.e. $M_X(x_i) = M_{X\setminus\{x_j\}}(x_i)$, $M_X(x_k) = M_{X\setminus\{x_j\}}(x_k)$. Since for any $x_j \notin C_{PAR}(X)$ there exists $x_\ell \in C_{PAR}(X)$, such that $\varphi_\nu(x_\ell) > \varphi_\nu(x_j)$ $\forall \nu = \overline{1,n}$, i.e. for any $x_i \in C_M(X)$, $x_k \in C_{PAR}(X) \setminus C_M(X)$, $x_j \in C_{PAR}(X)$ we fulfil $t_{ij} \geq t_{i\ell}$ and $t_{kj} \geq t_{k\ell}$, where $x_\ell \in C_{PAR}(X)$. Hence, in rejecting $x_j \notin C_{PAR}(X)$ the values $M_X(x_i)$ and $M_X(x_k)$ do not increase. Thus, $M_X(x_i) = M_{X\setminus X'}(x_i)$, $M_X(x_k) = M_{X\setminus X'}(x_k)$, where $x_i \in C_M(X)$, $x_k \in C_{PAR}(X) \setminus C_M(X)$, $X' \cap C_{PAR}(X) = \emptyset$, i.e. the choice function $C_M(\cdot)$ satisfies the condition IR with respect to $C_{PAR}(\cdot)$.

Q.E.D.

It can be proved that other choice functions from the family $\{C^q(\cdot), q \in [1, q_0], q_0 < \infty\}$ do not satisfy these characteristic conditions.

Fig. 4 shows how the domains corresponding to conditions IR, RS, CS, IRS and SCS are placed in the space $\mathcal{C}$ of all possible choice functions for the cases, when the embracing choice function $C^*(\cdot)$ belongs to various choice functions classes.

For the case, when the embracing function $C^*(\cdot)$ is arbitrary, the location of domains in the space of the choice functions $\mathcal{C}$ is shown in fig. 4a.

If $C^*(\cdot)$ satisfies the condition O, then the domains

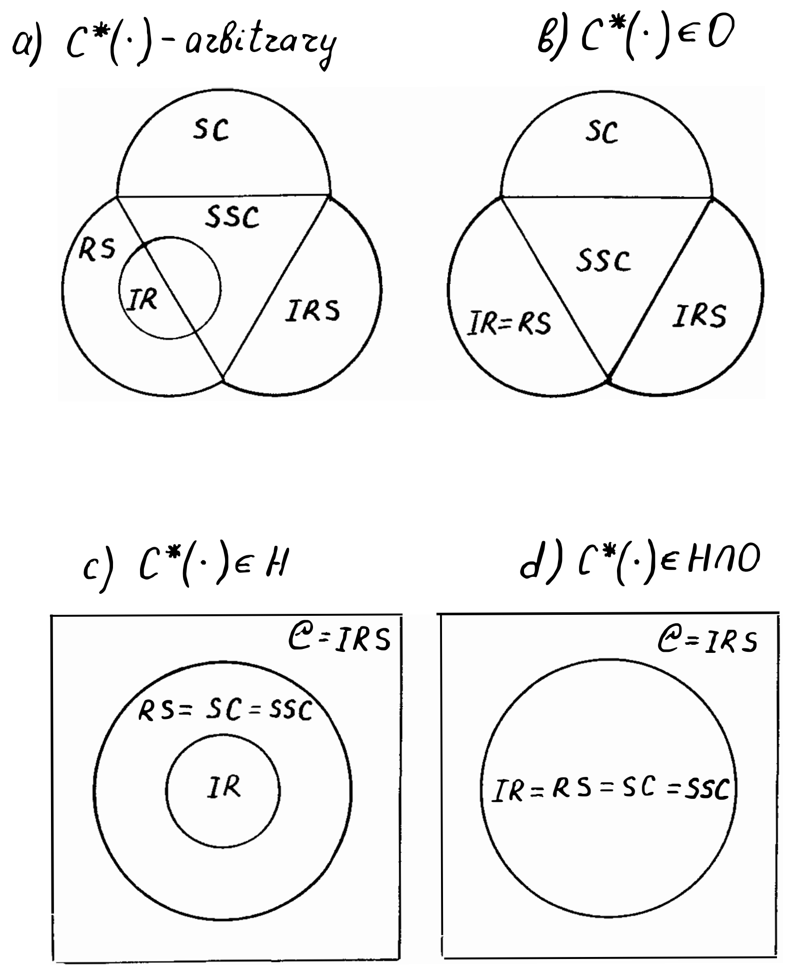

Figure 4

IR and RS coincide as it follows from theorem 3 (fig. 4b).

If $C^*(\cdot)$ satisfies the condition H then, with respect to Theorem 4 the condition IRS is fulfilled for all functions $C(\cdot)$, i.e. the domain IRS occupies the entire space of choice functions $\mathcal{C}$. In this case, the domains CS, SCS and RS coincide (fig. 4c).

And, finally, in the case when $C^*(\cdot)$ satisfies both conditions O and H then, as it follows from corollary 1, the domain IRS still occupies the entire space of choice functions $\mathcal{C}$ and the domains IR, RS, CS and SCS in this space coincide (fig. 4d).

The characteristic conditions introduced above may be useful for describing other essentially non-classical mechanisms of choice. The choice procedure based on the notion of an "ideal" point (see Salukvadze (1972), Yu and Leitman (1974) may serve as an example. According to this procedure the best variants are the nearest (in terms of the Euclidean metrix) to the "ideal" point. "Ideal" point is the point a in the criterial space $\{\varphi_\nu\}$, $\nu = \overline{1,n}$ with the maximal criterial estimates of variants included in the given representation $X \subseteq A$ with respect to each criterion: $\varphi_\nu(a) = \max_{x_i \in X} \varphi_\nu(x_i)$, $\nu = 1, 2, \ldots, n$. The function $C_a(\cdot)$ isolates a part of the Pareto set and it does not satisfy the conditions H, C and O. It is easy to check that choice function $C_a(\cdot)$ satisfies the new characteristic conditions IR, RS, CS and SCS if the choice function of Pareto-optimal variants $C_{PAR}(\cdot)$ is used as an embracing function in the same criterial space.

5. Mutual relations of choice functions classes

Fig.5 shows the class Q_{GD}, Q_{sum} and Q_M, all possible intersections of these classes and their complements

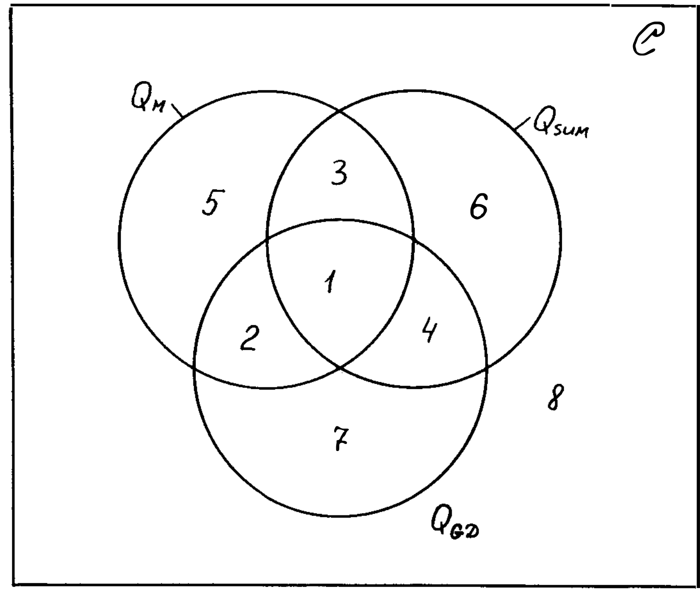

Figure 5

denoted through numbers 1, 2, ..., 8.

Our task is to find out which of these intersections are not empty and to characterize the mechanisms, generating the choice functions, included into each of the domains isolated by them. First of all we are interested in the intersection of the classes: $Q_{G\mathcal{D}} \cap Q_{SUM}$ and $Q_{G\mathcal{D}} \cap Q_M$.

The fact that the function $C(\cdot)$ belongs to the class $Q_{G\mathcal{D}} \cap Q_M$ means that the function may be generated either by dominant rule (1) on a certain graph or by maxmin rule (4) on a certain tournament matrix, and the fact that the function $C(\cdot)$ belongs to the class $Q_{G\mathcal{D}} \cap Q_M$ means that the function can be generated either by dominant rule (1) on a certain graph, or by total score rule (3) on a certain tournament matrix.

<u>Definition 2.</u> The graph Γ^{MAJ} such that $x_i \Gamma^{MAJ} x_j \iff t_{ij} > \frac{n}{2}$ will be referred to as a majority graph of the tournament matrix $T = \| t_{ij} \|$. Γ^{MAJ}

<u>Lemma 1.</u> If the majority graph of the tournament matrix T is acyclic then the choice function, generated by rule (1) on the graph Γ^{MAJ}, coincides with the choice function generated by rule (4) on the matrix T.

<u>Proof.</u>

The subgraph Γ_X^{MAJ} of the graph Γ^{MAJ} and submatrix T_X of the matrix T corresponds to an arbitrary representation $X \subseteq A$. Two cases are possible:

1) There exists a variant - vertex x_{i^*} such that $\forall x_j \in X$ $(j \neq i^*)$: $x_{i^*} \Gamma_X^{MAJ} x_j$. Then according to (1): $C_{G\mathcal{D}}(X) = \{x_{i^*}\}$.

For the submatrix T_X it is true that

$\forall x_i \in X \ (i \neq i^*)$: $t_{i^*i} > \frac{n}{2} \Rightarrow M_X(x_{i^*}) = \min_{x_i \in X} t_{i^*i} > \frac{n}{2}$, and for all $i \neq i^*$ it is true that $M_X(x_i) < \frac{n}{2}$, i.e. $t_{ii^*} = n - t_{i^*i} < \frac{n}{2}$. According to (4) in this case $C_M(X) = \{x_{i^*}\}$ and it means $C_M(X) = C_{G\mathcal{D}}(X)$.

2) The subgraph Γ_X^{MAJ} does not contain a vertex, dominant over all the remaining ones: $|C_{G\mathcal{D}}(X)| \geqslant 2$. Then according to (1) from $x_i, x_j \in C_{G\mathcal{D}}(X)$ it follows that $x_i \bar{\Gamma}_X^{MAJ} x_j$, $x_j \bar{\Gamma}_X^{MAJ} x_i$ and it means that in the matrix T: $t_{ij} = t_{ji} = \frac{n}{2}$, and for all $x_i \in C_{G\mathcal{D}}(X)$, $x_j \notin C_{G\mathcal{D}}(X)$ the following is true: $x_i \Gamma_X^{MAJ} x_j$ and $t_{ij} > \frac{n}{2}$. Therefore, in this case $M_X(x_i) = \frac{n}{2}$ for $x_i \in C_{G\mathcal{D}}(X)$ and if $x_i \notin C_{G\mathcal{D}}(X)$ then $M_X(x_i) < \frac{n}{2}$. Due to arbitrariness of X the following holds:

$C_M(\cdot) = C_{G\mathcal{D}}(\cdot)$.

Q.E.D.

Next theorem gives an answer to the question which of the graphdomanant functions can be generated by maxmin rule (4) on some tournament matrix.

<u>Theorem 6.</u> The class $Q_{G\mathcal{D}} \cap Q_M$ (domain 1 $\cup$ 2 in fig.5) contains all choice functions, generated by dominant rule (1) on acyclic graphs and only on them.

<u>Proof.</u>

Let the arbitrary acyclic graph Γ is given. Define the matrix T: if $x_i \Gamma x_j$, then $t_{ij} = 2$, $t_{ji} = 0$; if $x_i \bar{\Gamma} x_j$ and $x_j \bar{\Gamma} x_i$, then $t_{ij} = t_{ji} = 1$; $t_{ii} = 0$ $\forall x_i \in A$. It is easy to observe that T is a matrix of a two-round tournament, and the graph Γ coincides with its majority graph: $\Gamma = \Gamma^{MAJ}$.

From lemma 1 it follows that the choice function $C_{G\mathcal{D}}(\cdot)$

generated by rule (1) on the graph Γ coincides with the choice function under rule (4) on the matrix T. Hence, any graph-dominant function on the acyclic graph belongs to the domain $Q_{GD} \cap Q_M$.

It was stated above that the choice under rule (4) is always nonempty. It is known (see Mirkin (1979)) that for nonemptiness of the choice under dominant rule (1) acyclicity of graph Γ on which the choice is performed is necessary and sufficient. Hence all choice functions generated by rule (1) on the graphs which contain cycles, are located beyound class Q_M. Thus, graph-dominant choice functions on the graphs containing cycles cannot belong to the domain $Q_{GD} \cap Q_M$. This completes the proof.

Thus any choice function under the dominant rule on the acyclic graph can be represented as a choice function under the maxmin rule on a certain tournament matrix.

The counter question rises: what are the properties of tournament matrices on which the maxmin rule generates choice functions from the class $Q_{GD} \cap Q_M$, i.e. choice functions which can be simultaneously generated also by dominant rule (1) on some graphs.

Theorem 7. For the choice function generated by maxmin rule (4) on the tournament matrix T to coincide with the choice function, generated by dominant rule (1) on a certain acyclic graph Γ it is necessary and sufficient that the majority graph Γ^{MAJ} of the matrix T chould be acyclic. This choice function is generated by the dominant rule on the majority graph Γ^{MAJ}.

Proof.

Sufficiency. Let Γ^{MAJ} of the matrix T be acyclic.

Then according to lemma 1, $C(\cdot)$ coincides with the choice function under rule 1 on the graph Γ^{MAJ}.

Necessity. Let us prove that the function $C^{MAJ}(\cdot)$ generated by rule (1) on the graph Γ^{MAJ} is an accompanying function $C^{ACCOMP}(\cdot)$ (Aizerman (1984)) for $C(\cdot)$; it means that for these functions the following holds: $C^{ACCOMP}(\{x_i, x_j\}) = C^{MAJ}(\{x_i, x_j\}) = C(\{x_i, x_j\})$.

In fact, with $t_{ij} > \frac{n}{2}$ and hence with $t_{ij} > t_{ji}$ in accord with rule (4): $C^{MAJ}(\{x_i, x_j\}) = \{x_i\}$; according to definition 2 in this case $x_i \Gamma^{MAJ} x_j$, and according to (1): $C(\{x_i, x_j\}) = \{x_i\} = C^{MAJ}(\{x_i, x_j\})$. And in a similar way with $t_{ij} = \frac{n}{2}$ we obtain $C(\{x_i, x_j\}) = \{x_i, x_j\} = C^{MAJ}(\{x_i, x_j\})$.

From the Condorcet principle it follows (see Aizerman (1984)) that the choice function $C(\cdot)$ cannot be equal to any graph-dominant function, which differs from its accompanying one. Since $C(\cdot)$ is a function with a nonempty choice, then in the case when Γ^{MAJ} contains a cycle and the accompanying function is not a function with a nonempty choice, the $C_M(\cdot)$ cannot be graph dominant.

Q.E.D.

The statement of theorem 7 means that the class $Q_{GD} \cap Q_M$ contains all choice functions, generated by maxmin rule (4) on all possible tournament matrices with acyclic majority graphs, and only them.

From theorem 6 and 7 it follows that the class of graphdominant choice mechanisms on all acyclic graphs is equivalent to the class of maxmin mechanisms on all tournament matrices whose majority graphs are acyclic.

Let us pay attention to the difference of the cases with the n - number of tournament rounds being even and with n -being odd.

If n is add, the majority graph Γ^{MAJ} satisfies an additional condition of completness: $[x_i \Gamma^{MAJ} x_j] \cup [x_j \Gamma^{MAJ} x_i] \neq \emptyset$

It is known that the acyclic complete graph is a graph of a strict linear order, i.e. the case of odd n corresponds to a particular type of an acyclic graph. General case of theorem 7 is realized for tournament matrices with even numbers of rounds.

We pass over to consideration of the class $Q_{GD} \cap Q_{SUM}$ (domain 1 $\cup$ 4 in fig. 5). Due to nonemptiness of the choice under rule (3) the choice functions generated by the dominant rule on graphs containing the cycles, cannot be generated by tournament mechanisms and, hence, these functions do not belong to the class $Q_{GD} \cap Q_{SUM}$. However, unlike the class $Q_{GD} \cap Q_M$ the class $Q_{GD} \cap Q_{SUM}$ does not contain all graphdominant functions on the acyclic graphs.

Lemma 2. The choice function, generated by rule (1) on the acyclic graph Γ, which contain a subgraph on three vertices with a single arc (fig. 6), does not belong to the class $Q_{GD} \cap Q_{SUM}$ (i.e. it cannot be generated by total score rule (3) on no tournament matrices).

If we consider that the presence of arc $x_i \Gamma x_j$ means that x_i is more preferable than x_j and the absence of arcs between x_i and x_j is interpreted as relation of incomparability and it is denoted via $x_i \bar{I} x_j$, then the condition of lemma 2 means that relation $\bar{I}$ is transitive:

(8) $\quad x_i \bar{I} x_j\,,\ x_j \bar{I} x_k \Rightarrow x_i \bar{I} x_k\,,$

i.e. it is the relation of equivalency.

Proof.

Let us consider the representations $X = \{x_1, x_2, x_3\}$, $X_1 =$

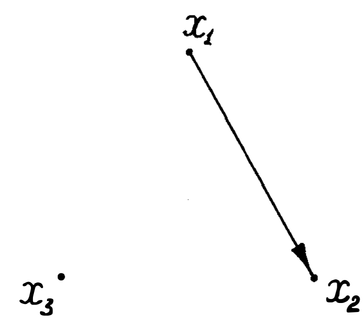

Figure 6

$=\{x_1,x_2\}$, $X_2=\{x_1,x_3\}$, $X_3=\{x_2,x_3\}$. For the choice on these representations under rule (1) on the graph shown in fig. 6 to coincide with the choice under rule (3) on certain matrix $T=\|t_{ij}\|$ the following relations should be fulfilled:

$$C_{GD}(\{x_1,x_2\})=\{x_1\} \Rightarrow t_{12}>\frac{n}{2},\ t_{21}<\frac{n}{2};$$

$$C_{GD}(\{x_1,x_3\})=\{x_1,x_3\} \Rightarrow t_{13}=t_{31}=\frac{n}{2};$$

$$C_{GD}(\{x_2,x_3\})=\{x_2,x_3\} \Rightarrow t_{23}=t_{32}=\frac{n}{2}$$

and simultaneously to be fulfilled is the relation:

$$C_{GD}(\{x_1,x_2,x_3\})=\{x_1,x_3\} \Rightarrow S_x(x_1)=S_x(x_3),$$

which is impossible because

$$S_x(x_1)=t_{12}+t_{13}>n,$$

$$S_x(x_3)=t_{31}+t_{32}=n.$$

Thus, there exists no such tournament matrix T.

This completes the proof of the lemma.

Q.E.D.

Definition 3. The acyclic graph Γ is said to be block-chain (BC-graph), if the set of its vertices may be devided into the subsets-blocks $B_1, B_2, \ldots, B_E$, such that

1. $\exists x_i\in B_\mu, \exists x_j\in B_\nu$ such that $x_i\Gamma x_j \Rightarrow B_\mu \Gamma B_\nu$,
   $\exists x_i\in B_\mu, \exists x_j\in B_\nu$ such that $x_i\overline{\Gamma} x_j \Rightarrow B_\mu \overline{\Gamma} B_\nu$.

Here and further $B_\mu\Gamma B_\nu$ means that $\forall x_k\in B_\mu$, $\forall x_\ell\in B_\nu$: $x_k\Gamma x_\ell$; and $B_\mu\overline{\Gamma}B_\nu$ means that $\forall x_k\in B_\mu$, $\forall x_\ell\in B_\nu$: $x_k\overline{\Gamma}x_\ell$.

2. For any $\nu\in\{1,2,\ldots,E-1\}$: $B_\nu\Gamma B_{\nu+1}$.

3. $\mu\le\nu \Rightarrow B_\nu\overline{\Gamma}B_\mu$.
4. $B_\nu I B_\mu$, $B_\mu I B_\rho \Rightarrow B_\nu I B_\rho$, where $I=\overline{\Gamma\cap\Gamma^{-1}}$.

Lemma 3. The acyclic graph Γ which does not contain a subgraph on three vertices with a single arc (or, and which is the same, satisfying the condition (8)) is BC-graph.

Proof:

The set of vertices, chosen under rule (1) on the graph Γ from the representation $X=A$ will be considered as a block B_1; the chosen set from the representation $X=A\setminus B_1$ will be considered as a block B_2; the chosen set from the representation $X=A\setminus\bigcup_{\nu\in\{1,\ldots,\mu\}}B_\nu$ will be considered as a block B_μ.

It follows from the acyclicity of graph Γ that the totality of all blocks provides the partition of the set A.

Let us demonstrate, that vertices belonging to one block are dominated by the same set of vertices and dominate the same set of vertices. Let it be not this way and for $x_i, x_j\in B_\mu$ there exists $x_k\in B_\nu$, $\nu>\mu$ such that $x_k\Gamma x_i$, $x_k\overline{\Gamma}x_j$. Then on the vertices x_i, x_j, x_k there appears a subgraph, shown in fig. 6, i.e. in this case the graph Γ does not satisfy the condition of lemma 2. Hence, the set of vertices, dominating over x_i and x_j, coincide. For the sets dominated by vertices x_i and x_j, the proof is similar. Thus, the graph Γ may be represented by the aggregated graph G.

Since each block B_μ consists of only nondominated on the μ-step vertices then with $\nu\ge\mu$ for $x_i\in B_\nu$, $x_j\in B_\mu$ it holds true: $x_i\overline{\Gamma}x_j$, i.e. there are no arcs passing from the blocks with greater numbers to the blocks with lower numbers.

On the $(\mu-1)$-step the representation X consists of the set of vertices belonging to blocks $B_{\mu-1}, B_\mu$ and all B_ν $(\nu>\mu)$. On this step the block B_μ is not chosen. Hence, there exists at least one arc that goes to the vertex B_μ. Since there are no arcs going from vertices B_ν, $\nu>\mu$ to B_μ then $B_{\mu-1}\, G\, B_\mu$. Thus, all blocks B_μ form an acyclic chain on the graph G.

The transitivity of noncomparability J follows directly from (8).

In this way the graph Γ satisfies the definition of the BC-graph.

Q.E.D.

Of interest is the form of graphs satisfying condition (8) or the same referres (as it follows from lemma 3) to BC-graphs.

A particular case of this kind of graphs are the graphs of the weak order in which condition (8) is additionally extended by the condition of transitivity of the graph Γ:

$$x_i \Gamma x_j,\ x_j \Gamma x_k \Rightarrow x_i \Gamma x_k .$$

The blocks themselves form an acyclic transitive chain, i.e.
$\nu>\eta \Rightarrow B_\eta G B_\nu,\ B_\nu \bar{G} B_\eta$.

A general-type BC-graph differs from a graph of weak order only in the acyclic chain formed by the blocks, being not obligatory transitive. It means that relation $\nu>\eta \Rightarrow B_\eta G B_\nu$ is obligatory to be realized only for the neighbouring block (i.e. with $\nu=\eta+1$) and transitively closure arcs between the blocks $B_\eta G B_\nu$ with $\nu>\eta+1$ may be not available. Thus blocks B_η and B_ν with $\nu>\eta+1$ may be connected by the relation of noncomparability J, and the relation

of noncomparability J is transitive:
$B_\eta J B_\nu,\ B_\nu J B_\mu \Rightarrow B_\eta J B_\mu$.

Fig. 7 illustrates the example of a BC-graph (the arrows correspond to the relation G and dotted lines correspond to the noncomparability relation J).

<u>Lemma 4.</u> Graphdominant mechanism of choice on any BC-graph Γ is equivalent to the choice mechanism under total score rule (3) on some tournament matrix T.

<u>Proof.</u> Let an arbitrary BC-graph Γ is given.

We are to prove that a choice function generated by rule (1) on this graph coincide with a choice function generated by rule (3) on the tournament matrix $T=\| t_{ij} \|$, which is to be defined as follows:

$$t_{\mu\nu} = \begin{cases} \frac{n}{2} + P\, 2^{-\nu} / \prod_{\eta=1}^{\nu} N_\eta & \text{with } B_\mu G B_\nu , \\ n/2 & \text{with } B_\mu J B_\nu , \\ \frac{n}{2} - P\, 2^{-\nu} / \prod_{\eta=1}^{\nu} N_\eta & \text{with } B_\nu G B_\mu . \end{cases}$$

Here t_{ij} is the same for all $x_i \in B_\mu$, $x_j \in B_\nu$ and therefore for t_{ij} we introduce the notation $t_{\mu\nu}$; $N_\nu = |B_\nu|$ is the number of vertices in block B_ν; values n and P are selected so that all $t_{\mu\nu}$ are integer and positive.

Let us assume that for the representation $X \subseteq A$ nonempty intersections will be the intersections $X \cap B_\lambda$ for all values $\lambda_1 < \lambda_2 < \ldots < \lambda_L$ and for them $K_\lambda = |X \cap B_\lambda| > 0$ with the first z nonempty blocks connected by the equivalency relation J and it is true for the block B_{λ_z} that $B_{\lambda_z} G B_{\lambda_{z+1}}$. Under (1) on the subgraph Γ_X the choice $Y_{G\mathcal{D}} = X \cap \left(\bigcup_{\ell=1}^{z} B_{\lambda_\ell} \right)$. We define the choice on the submatrix T_X under rule (3). From the statement that according to (9) with $\mu<\nu$ the values

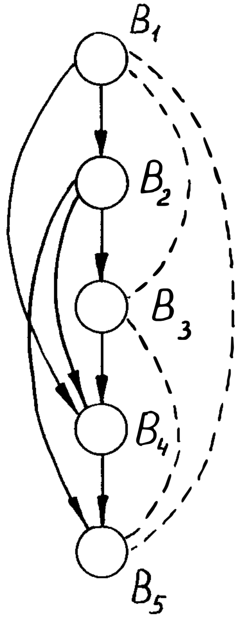

Figure 7

$t_{\mu\nu}$ depend only on the second index and $t_{\nu\mu} = t_{\mu\nu}$ is true for the layers linked by the relation J, it follows that $S_x(x_{\lambda_1}) = S_x(x_{\lambda_2}) = \ldots = S_x(x_{\lambda_z})$ where $S_x(x_\lambda)$ means $S_x(x_i)$ for all $x_i \in B_\lambda$.

Let us estimate $S_x(x_{\lambda_z})$:

$$S_x(x_{\lambda_z}) = \sum_{\ell=1}^{z-1} K_{\lambda_\ell}\cdot\frac{n}{2} + (K_{\lambda_z}-1)\cdot\frac{n}{2} + K_{\lambda_{z+1}}\left(\frac{n}{2} + P\cdot 2^{-\lambda_{z+1}} \Big/ \prod_{\ell=1}^{z+1} N_{\lambda_\ell}\right) + \sum_{\ell=z+2}^{L} K_{\lambda_\ell}\cdot t_{\lambda_z\lambda_\ell} \geq \sum_{\ell=1}^{z-1} K_{\lambda_\ell}\cdot\frac{n}{2} + (K_{\lambda_z}-1)\cdot\frac{n}{2} + K_{\lambda_{z+1}}\left(\frac{n}{2} + P\cdot 2^{-\lambda_{z+1}} \Big/ \prod_{\ell=1}^{z+1} N_{\lambda_\ell}\right) + \sum_{\ell=z+2}^{L} K_{\lambda_\ell}\cdot\frac{n}{2} = \frac{n}{2}\sum_{\ell=1}^{L} K_{\lambda_\ell} - \frac{n}{2} + K_{\lambda_{z+1}}\cdot P\cdot 2^{-\lambda_{z+1}} \Big/ \prod_{\ell=1}^{z+1} N_{\lambda_\ell}.$$

Let us estimate $S_x(x_{\lambda_q})$ for $q > z$:

$$S_x(x_{\lambda_q}) = \sum_{\ell=1}^{q-1} K_{\lambda_\ell} t_{\lambda_\ell\lambda_q} + (K_{\lambda_q}-1)\cdot\frac{n}{2} + \sum_{\ell=q+1}^{L} K_{\lambda_\ell} t_{\lambda_q\lambda_\ell} < \sum_{\ell=1}^{q-1}\frac{n}{2}\cdot K_{\lambda_\ell} + (K_{\lambda_q}-1)\cdot\frac{n}{2} + \sum_{\ell=q+1}^{L} K_{\lambda_\ell}\cdot t_{\lambda_q\lambda_\ell} \leq \sum_{\ell=1}^{q-1}\frac{n}{2}\cdot K_{\lambda_\ell} + (K_{\lambda_q}-1)\cdot\frac{n}{2} + \sum_{\ell=q+1}^{L} K_{\lambda_\ell}\cdot\left(\frac{n}{2} + P\cdot 2^{-\lambda_\ell} \Big/ \prod_{u=1}^{\ell} N_{\lambda_u}\right) = \frac{n}{2}\sum_{\ell=1}^{L} K_{\lambda_\ell} - \frac{n}{2} + P\cdot\sum_{\ell=q+1}^{L}\left(K_{\lambda_\ell}\cdot 2^{-\lambda_\ell} \Big/ \prod_{u=1}^{\ell} N_{\lambda_u}\right).$$

In order to prove that $S_x(x_{\lambda_z}) > S_x(x_{\lambda_q})$ it is sufficient to prove that

$$K_{\lambda_{z+1}}\cdot 2^{-\lambda_{z+1}} \Big/ \prod_{\ell=1}^{z+1} N_{\lambda_\ell} > \sum_{\ell=q+1}^{L}\left(K_{\lambda_\ell}\cdot 2^{-\lambda_\ell} \Big/ \prod_{u=1}^{\ell} N_{\lambda_u}\right),$$

i.e.

$$K_{\lambda_{z+1}} > \sum_{\ell=q+1}^{L}\left(K_{\lambda_\ell} \Big/ \prod_{u=z+1}^{\ell} N_{\lambda_u}\right)\cdot 2^{-(\lambda_\ell - \lambda_{z+1})}.$$

Taking into account that $K_{\lambda_\ell} \leq N_{\lambda_\ell}$

$$\sum_{\ell=q+1}^{L}\left(K_{\lambda_\ell} \Big/ \prod_{u=z+1}^{\ell} N_{\lambda_u}\right)\cdot 2^{-(\lambda_\ell-\lambda_{z+1})} \leq \sum_{\ell=q+1}^{L} 2^{-(\lambda_\ell-\lambda_{z+1})} < \sum_{\lambda=1}^{L} 2^{-\lambda} < 1.$$

From the above and evident nonequality $K_{\lambda_{z+1}} \geq 1$ it follows that $S_x(x_{\lambda_z}) > S_x(x_{\lambda_q})$.

From relations $S_x(x_{\lambda_1}) = \ldots = S_x(x_{\lambda_z}) > S_x(x_{\lambda_q})$ with $q > z$ it follows that the choice under rule (3) on the submatrix T_x is equal to $X \cap \left(\bigcup_{\ell=1}^{z} B_{\lambda_\ell} \right)$, i.e. it coincides with the graphdominant choice. From the arbitrariness of the representation $X \subseteq A$ it follows that $C_{G\mathcal{D}}(\cdot) = C_{sum}(\cdot)$, i.e. the mechanisms are equivalent. This completes the proof of lemma 4.

Q.E.D.

The assertions of lemmas 2,3 and 4 may be reduced to one theorem.

<u>Theorem 8</u>. Let there be a choice function $C(\cdot)$, generated by dominant rule (1) on the graph Γ. For this choice function to be generated by total score rule (3) on a certain tournament matrix it is necessary and sufficient that the graph Γ is to be a BC-graph.

<u>Proof:</u>

<u>Necessity</u>: Let there be the tournament matrix T on which the choice function $C(\cdot)$ is generated by rule (3). Then due to nonemptiness of the choice under rule (3) the graph Γ is acyclic. According to lemma 2 in the graph Γ there are no vertex- generated subgraphs with a single arc (fig. 6). According to lemma 3 the graph Γ is a BC-graph.

<u>Sufficiency</u> follows directly from lemma 4.

Q.E.D.

For the inverse problem, consisting in defining the particular features of tournament matrices, on which the choice under total score rule can be represented by a graph dominant choice function on a certain graph, there had been no so complete and, at the same time, simple a solution as in the case of maxmin rule (4). Acyclicity requirement for a majority graph of a tournament matrix is necessary only in this case. Only necessary is also a requirement of using a BC-graph as a majority graph. Full solution can be obtained by one of general criteria of graphdominance, for example the Richter criterion (see Richter (1971)). However in this case, the solution of the problem, representing a choice mechanism under the total score rule on a given matrix in the form of a graphdominant choice on a certain graph, requires the analysis of a choice function on the entire set of representations $X \subseteq A$ or, and which is the same, scanning all submatrices of the tournament matrix. It is not the conditions which require the scanning of any submatrices but those requiring the scanning of only 3x3-matrices may prove to be useful.

The sufficient condition is

$$t_{ik} \geq t_{ij} \quad \text{for all} \quad k > i ,$$

and the necessary condition is

$$t_{ik} + t_{ij} \geq t_{jk} + t_{ji} \quad \text{for all} \quad k > i$$

Theorems 6-8 allow us to characterize the domains shown in fig. 5.

Domain (1) (according to theorem 8) contains all choice functions, generated by dominant rule (1) on the BC-graphs (and only then).

Domain (1) $\cup$ (2) (according to theorem 6) contains all functions, generated by dominant rule (1) on the acyclic graphs (and only them); on the other hand (according to theorem 7) this domain is filled with the choice functions, generated by maxmin rule (4) on the tournament matrices, whose majority graphs are acyclic (and only them).

Since in the union of domains (1) $\cup$ (2) $\cup$ (4) there are only the choice functions, generated by rule (1) on the acyclic graphs (Mirkin (1979)) and according to theorem 6 all these functions are located in domain $1 \cup 2$, domain 4 is empty.

Domain $3 \cup 5$ contains the functions, generated by the maxmin rule on the tournaments matrices whose majority graphs contain at least one cycle.

Domain 7 consists of all functions, generated by rule (1) on the graphs, containing at least one cycle.

With respect to domains 3, 5, 6 and 8 we shall only prove that each of these domains is not empty.

For domains 3 and 8 this task turns to be not complicated: for domain $3 = Q_M \setminus Q_{SUM} \setminus Q_{GD}$ there estist matrices, for example, the matrix shown in fig. 8 on which a choice under rule (3) and choice under rule (4) generate the same function $C(\cdot)$, belonging to the intersection $Q_M \cap Q_{SUM}$, and from the obvious presence of a cycle in the majority graph of these matrix it follows that $C(\cdot)$ belongs to domain 3 but not to domain 1. It is not comparatively dificult to show that to domain $8 = \mathcal{C} \setminus Q_{GD} \setminus Q_{SUM} \setminus Q_M$ ($\mathcal{C}$ is the set of all choice functions) belongs the function $C(\cdot)$ such that $C(\{x_1, x_2\}) = \{x_1, x_2\}$, $C(\{x_1, x_3\}) = \{x_1, x_3\}$, $C(\{x_2, x_3\}) = \{x_2, x_3\}$, $C(\{x_1, x_2, x_3\}) = \emptyset$. It is obvious that with an empty choice available we have $C(\cdot) \notin Q_M$, $C(\cdot) \notin Q_{SUM}$ and through violation of the Condorcet principle it follows that $C(\cdot) \in Q_{GD}$.

For the rest of the domains the proof of nonemptiness turns out to be more complicated. Thus, in order to form an example of the function $C(\cdot)$ from domain $5 = Q_M \setminus Q_{GD} \setminus Q_{SUM}$ one should not only find a matrix T (under majority graph contains cycles) on which this function is generated under rule (4) but it requires to prove non-existence of the matrix T' with the choice under rule (3) coinciding with $C(\,)$. And

	x_1	x_2	x_3
x_1	0	1	0
x_2	0	0	1
x_3	1	0	0

Figure 8

similarly, for domain (6) $= Q_{SUM} \setminus Q_{GD} \setminus Q_{M}$ one should construct the matrix T with its majority graph containg the cycles and there should be no matrix T' with its choice under rule (4) coinciding with the choice on T under rule (3).

Domain (5) . Fig. 9 contains a tournament matrix T . The choice function, constructed on this matrix under the maxmin rule acquires the following values:

$$C_M(\{x_2, x_4\}) = \{x_2\},\ C_M(\{x_1, x_2, x_4\}) = \{x_1, x_2, x_4\},$$

$$C_M(\{x_2, x_3, x_4\}) = \{x_2, x_3, x_4\},\ C_M(\{x_1, x_2, x_3, x_4\}) = \{x_1, x_2, x_3, x_4\}.$$

The majority graph Γ^{MAJ} has cycles and hence (from theorem 7) there is no graphdominant function which equivalent to $C_M(\cdot)$. Now we are to prove the nonexistence of a tournament matrix T' with its choice realized under the total score rule coinciding with function $C_M(\cdot)$.

Let us try to construct such a matrix $T' = \|t'_{ij}\|$:

$$C_{sum}(\{x_1, x_2, x_4\}) = \{x_1, x_2, x_4\} \Rightarrow$$

$$t'_{12} + t'_{14} = t'_{21} + t'_{24} = t'_{41} + t'_{42} \Rightarrow t'_{12} = t'_{41} = t'_{24} ;$$

$$C_{sum}(\{x_2, x_3, x_4\}) = \{x_2, x_3, x_4\} \Rightarrow t'_{32} = t'_{43} = t'_{24} ;$$

$$C_{sum}(\{x_1, x_2, x_3, x_4\}) = \{x_1, x_2, x_3, x_4\} \Rightarrow$$

$$t'_{21} + t'_{23} + t'_{24} = t'_{41} + t'_{42} + t'_{43} \Rightarrow$$

$$n - t'_{12} + n - t'_{32} + t'_{24} = t'_{41} + n - t'_{24} + t'_{43} \Rightarrow$$

$$n - t'_{24} = t'_{24} \Rightarrow t'_{24} = \frac{n}{2} .$$

But $C'_{sum}(\{x_2, x_4\}) = \{x_2\} \Rightarrow t'_{24} > \frac{n}{2}$.

This contradiction proves that the above function $C_M(\cdot)$ cannot be generated by the choice under rule (3) on any tournament matrix. Thus, domain (5) in fig.5 is nonempty.

	x_1	x_2	x_3	x_4
x_1	0	9	6	1
x_2	1	0	1	9
x_3	4	9	0	1
x_4	9	1	9	0

Figure 9

$$\begin{array}{c|c|c|c|}
 & x_1 & x_2 & x_3 \\ \hline
x_1 & 0 & 1 & 0 \\ \hline
x_2 & 1 & 0 & 1 \\ \hline
x_3 & 2 & 1 & 0 \\ \hline
\end{array}$$

Figure 10

Domain 6 . The choice under maxmin rule on a tournament matrix satisfies the Condorcet condition (6). Fig. 10 illustrates a tournament matrix, whose choice function generated by the total score rule , does not satisfy condition (6). In fact $x_2 \in C_{sum}(\{x_1, x_2\})$, $x_2 \in C_{sum}(\{x_2, x_3\})$, but $x_2 \notin C_{sum}(\{x_1, x_2, x_3\})$.

Violation of condition (6) shows that this choice function cannot be represented by the choice under maxmin rule on any tournament matrix. Hence, domain 6 in fig. 5 is nonempty.

Appendix

Let us prove that the procedure of variants choice proposed in the paper Nogin (1976) and criterial interpretation of the procedure proposed in the paper Kramer (1977) on the dynamic voting theory can be equivalently represented by the tournament choice under maxmin rule (4).

In his paper Nogin (1976) proposed the method of choice of z -optimal variants in a multicriterial problem of choice. The variant $x_i \in X$ is referred to as z-optimal ($z \in \{1,2, \ldots, n\}$) if it belongs to the Pareto set relative to any z-dimensional subspace of the criterial space $\{\varphi_\nu\}$, $\nu = \overline{1,n}$. Obviously, 1 -optimal variant (if it exists) provides maximum to all criteria. The set of n-optimal variants coincides with the Pareto set and with $1 \leq z < n$, each z-optimal variant is $(z+1)$ -optimal.

Thus, the set of z-optimal variants, with decreasing (number z) from n to 1, gradually narrows from the Pareto set to the set of unanimously-extremal variants. The variants with the least possible z are included into the choice, i.e. in this case the choice rule has the following form :

(10) $$Y = \{x_i \in X \mid x_i \in \cap C_{PAR}(X) \text{ in all possible totalities } \{\varphi_\nu\}, \nu = \overline{1,z} \text{ and } \cap C_{PAR}(X) \text{ in}$$

all possible totalities $\{\varphi_\nu\}$, $\nu=\overline{1,\tau-1}$ is empty$\}$.

Let us show that the choice rule of τ-optimal variants (10) and maxmin rule (4) generates in the criterial space $\{\varphi_\nu\}$, $\nu=\overline{1,n}$ coinciding choice functions.

An arbitrary $X \subseteq A$ is fixed. Let it be $x_i \in C_{\tau\text{-OPT}}(X)$ and $\tau=\tau_0$. Then the variant x_i belongs to the Pareto set in all τ_0-dimensional criterial subspaces of the criterial space $\{\varphi_\nu\}$, $\nu=\overline{1,n}$ and simultaneously there exists $(\tau-1)$-dimensional subspace in which the variant $x_i \in X$ does not belong to the Pareto set. Hence, in this (τ_0-1)-dimensional subspace there exists a variant $x_j \in X$, dominating over x_i in this space under τ_0-1 criteria. (Note, that all criteria φ_ν, $\nu=\overline{1,n}$ are assumed to be strict). The definition of τ-optimal variants choice rule (10) also implies that in X there is no variant x_k dominating over x_i under τ_0 criteria, since these τ_0 of the criteria would have formed a τ_0-dimensional subspace in which the variant x_i would belong to the Pareto set.

In terms of the criterial tournament matrix $T=\|t_{ij}\|$ the above stated means that $M_X(x_i)=n-(\tau_0-1)$. Variants x_ℓ which do not belong to $C_{\tau\text{-OPT}}(X)$ are dominanted not less than under τ_0 criteria, i.e. $M_X(x_\ell) \le n-\tau_0$. Hence, $M_X(x_i)= \max\limits_{x_j\in X} M_X(x_j)$, i.e. $x_i \in C_M(X)$.

Let us prove now that if $x_i \in C_M(X)$ then $x_i \in C_{\tau\text{-OPT}}(X)$. Let it be $x_i \in C_M(X)$ and $M_X(x_i)=\ell$ $(\ell>0)$. Then there exists $x_j \in X$, which dominate over x_i under $n-\ell$ criteria, i.e. there exists $(n-\ell)$-dimensional subspace of the criterial space $\{\varphi_\nu\}$, $\nu=\overline{1,n}$ in which the variant $x_j \in X$ does not belong to the Pareto set. And since there exists no variant $x_k \in X$ dominating over a variant x_i under $n-\ell+1$ criteria, then in any $(n-\ell+1)$-dimensional subspace of a n-dimensional criterial space $\{\varphi_\nu\}$, $\nu=\overline{1,n}$ the variant x_i belongs to the Pareto set. Hence, the variant $x_i \in X$ will be τ-optimal if we assume $\tau=\ell+1$, i.e. $x_i \in C_{\tau\text{-OPT}}(X)$.

Now we determine the relations of maxmin rule (4) with one of the procedures, used in the dynamical voting theory. In the papers devoted to dynamical aspects of the voting theory the following formal model is considered (see Plott (1967), McKelvey (1974)), Aizerman (1981)). There are n voters. We introduce continuum space $\mathcal{X}$ of"view points" with the Euclidean metric and assume that each voter ν is characterized by a point $\Lambda_\nu \in \mathcal{X}$ of this space - its "ideal". Ideals do not change with time, i.e. points Λ_ν, $\nu=1,\dots,n$ are fixed. Parallel with voters the programs Ψ and Ω are assumed. Any points Λ_Ψ, Λ_Ω of the space $\mathcal{X}$ may be assumed as programs. Every voter indicates which of the programs he prefers. Voters behaviour may be assumed as follows: at representing programs Ψ and Ω a ν-voter estimates the distance to Λ_Ψ and Λ_Ω from his ideal $\Lambda_\nu \in \mathcal{X}$ and votes for the program to which this distance is shorter. In case of distances equality a ν-voter abstains from voting.

In this case the Pareto set $Pa\tau$ is to be a part of the space $\mathcal{X}$, restricted by a convex cover pulled on the ideals of voters $\Lambda_1,\dots,\Lambda_n$ (fig. 11). The notion of the Pareto set defines as following. If there is program Ψ which do not belong to the domain $Pa\tau$ (fig. 11) then there exists such a program Ω, belonging to the Pareto set $Pa\tau$, which is preferred by all voters $\Lambda_1,\dots,\Lambda_n$. If we assume a distance from some program Ψ to the ideal of a ν-voter: $\rho(\Psi,\Lambda_\nu)$ as a criterial estimate of the program Ψ under a ν-criterion: $\varphi_\nu(\Psi)=-\rho(\Psi,\Lambda_\nu)$ then each program Ψ can be represented by a point in a n-dimensional criterial space $\{\varphi_\nu\}$, $\nu=\overline{1,n}$. It this case the notion of the Pareto

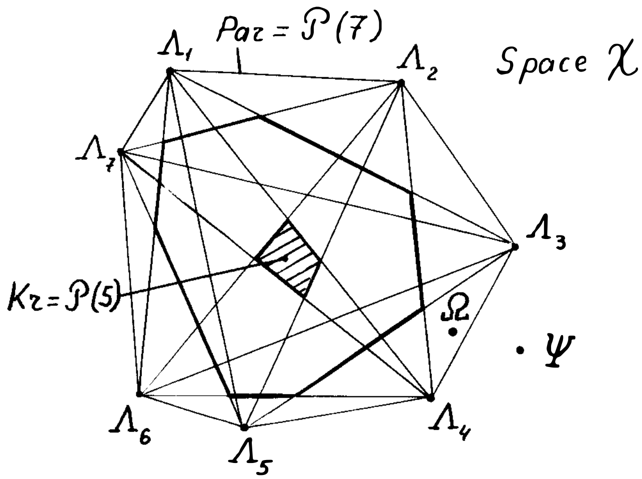

Figure 11

set Par coincides with the notion of the Pareto set $C_{PAR}(X)$ in the multicriterial choice problems.

In the paper Kramer (1977), regarded dynamical aspects of the voting theory, the central subspace possessing particular properties is isolated under a certain rule. This domain will be referred as the Kramer domain Kr (fig. 11). Kramer analysed a strengthened majority voting system in which a program wins if it is supported by the maximal possible number of voters (but not just more than a half of voters as it is the case with the usual majority). It was proved that in this case a winning trajectory monotonicly approaches the domain Kr . However, as the winning trajectory gets inside the domain Kr it may be thrown out of it and even outside the domain Par . Hereafter the winning trajectory approachs monotonicly the domain Kr again and so on.

Let us see what the domain Kr represent. Denote via $\mathcal{P}(r)$ $(r = n, n-1, n-2, \dots)$ the domain of the space χ consisting of programs which cannot be beaten by other programs from χ by any group of r voters:

$$\mathcal{P}(r) = \{ \Psi \in \chi \mid \text{there exists no } \Omega \in \chi \text{ such that } \Omega \succ_y \Psi \text{ for any group of } r \text{ voters} \}.$$

(Here $\Omega \succ_y \Psi$ denotes that the program Ω is more preferable for a voter Λ_y than program Ψ . Then $\mathcal{P}(n) = Par$ and $\mathcal{P}(r-1) \subseteq \mathcal{P}(r)$.

For any r the domain $\mathcal{P}(r)$ is constructed as shown in fig. 11: we consider all combinations from r voters and for each of them a convex cover is pulled on the corresponding ideals.

The domain $\mathcal{P}(\tau)$ is the intersection of all these covers. With τ gradually decreasing from n to 1 beginning with some $\tau = \tau_0$ the domain $\mathcal{P}(\tau_0)$ becomes empty: $\mathcal{P}(\tau_0) = \emptyset$, but $\mathcal{P}(\tau_0 + 1) \neq \emptyset$.

The nonempty domain of the space $\mathcal{X}$ corresponding to the least $\tau = \tau_{min}$ is to be referred as the Kramer domain $K\tau$:

$$K\tau = \mathcal{P}(\tau_{min}) = \mathcal{P}(\tau_0 + 1), \text{ where } \mathcal{P}(\tau_0) = \emptyset.$$

The domain $K\tau$ consists of the programs for which the maximal number of voters who vote contra these programs is minimal.

The procedure of constructing the domain $K\tau$ and the fact that the notion (in a definite sense) of the Pareto set $Pa\tau$ in the dynamical voting theory coincides with the notion of the Pareto set $C_{PAR}(X)$ in the multicriterial choice problems imply that the procedure of isolating τ-optimal variants (10) (and hence, maxmin rule (4)) is a transfer of the procedure of isolating the domain $K\tau$ to a multicriterial choice problem. And from the fact that (and it was proved above) the rule of choice of τ-optimal variants (10) and maxmin rule (4) generate coinciding choice functions in the space $\{\varphi_\nu\}$, $\nu = \overline{1,n}$, it follows that these choice mechanisms are equivalent.

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