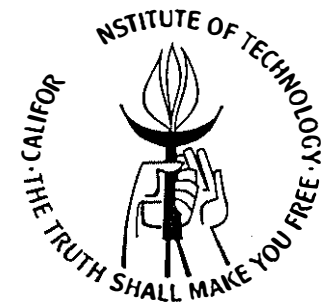


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OPTIMAL RESEARCH FOR COURNOT OLIGOPOLISTS

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SOCIAL SCIENCE WORKING PAPER 563

February 1985

ABSTRACT

We extend the classical Cournot model to take account of uncertainty in either the cost function or the demand function. By undertaking research, firms can acquire private (asymmetric) information to help resolve their uncertainty and make a more informed production decision. The model is a two stage game: in the first stage research levels are chosen, and in the second stage, conditional on private research outcomes, production decisions are made.

We find that for a linear, continuous information structure there is a unique Nash equilibrium to the game. In the equilibrium there may be an inefficient amount of aggregate research and there may be incomplete pooling as well.

The model specializes to the classical case when the cost of research is zero (and each firm gains essentially the same information by doing an infinite amount of research) or when the cost of research is so high no firm undertakes research.

OPTIMAL RESEARCH FOR COURNOT OLIGOPOLISTS

1. Introduction

We study an oligopolistic industry in which there is uncertainty either in the cost function or the demand function. By undertaking research prior to their production decision, firms can acquire private information which may help them resolve the uncertainty and make a more informed decision on the level of production. We characterize equilibrium behavior that will emerge in such a model, where firms first choose a research level and then choose a level of production based on their private data. We then compare the behavior of the oligopoly with an efficiency standard, where the amount of research and the total output maximizes net expected social benefits—expected consumer plus producer surplus.

Our model is an extension of the classical model of an oligopoly, and specializes to the classical model in two ways. First, as the cost of research increases, the amount of research declines. When the cost of research is sufficiently high, firms undertake no research and the model specializes to the classical model of an oligopoly without research. Second, if the cost of research decreases to zero, each firm does an infinite amount of research. Each firm obtains essentially the same information, and the model again specializes to the classical one.

Some of our results do not depend upon the cost of research and are the same as in the classical oligopoly model. We find, for

example, that as the number of firms increases, the total expected equilibrium output increases monotonically and converges to the expected efficient output.

However, in the extension, when the cost of research is positive (to prevent infinite research) and low enough to permit some research, we find new and contrasting results. First, the total amount of research undertaken by the oligopoly can be either less or more than the efficient amount of research—more if the cost of research is low and less if the cost of research is high. Second, we find incomplete pooling of information. And third, as a consequence of these sources of inefficiency, we find that the expected net benefits in the oligopoly equilibrium of n firms converge as n increases, to a level below the expected benefits of the efficiency standard.

Our model is similar to those of Clarke [1984], Gal-Or [1984] and Novshek and Sonnenschein [1982] in the second stage game of output decisions given private signals. More explicitly, in all these papers, firms face a linear uncertain demand, and each observes a private signal for the state of the demand. These papers then address questions of incentives for information sharing. There are two principal differences between this paper and the ones cited. First, we focus on research incentives when information is costly and there is no sharing. (In a model with a normal information structure Gal-Or [1983] shows there are incentives not to share for the case we address. Li [1985] extends these results to a linear information

structure, and n firms and also shows that any information sharing is not an equilibrium.) Second, our information structure is more general and includes a wide class of conjugate pairs which satisfy certain linearity conditions.

2. The Model

Consider an industry with n identical firms, facing a stochastic inverse demand function of the form $D^{-1}(Q, \theta)$, where $Q = \sum_{i=1}^n q_i$ is the total quantity produced, and θ is the true state of the world. Each firm has an identical cost function, $C(q_i, t_i)$, which is a function of its own production, q_i , and of the amount of research, t_i , that it undertakes. Both D^{-1} and C are linear, so that

$$D^{-1}(Q, \theta) = a_0 + a\theta - bQ$$

with $a_0, b > 0$, and

$$C(q_i, t_i) = c_0 + c_1 q_i + c t_i$$

with $c, c_0, c_1 > 0$. While we assume the uncertainty arises in the intercept of the demand function, the analysis is identical if the uncertainty arises from the coefficient c_1 in the cost function. Also, while we assume the cost function is linear in t_i , our results could be extended to deal with the case when cost is convex in t_i .

Given choices $q = (q_1, \dots, q_n)$ and $t = (t_1, \dots, t_n)$, by the n firms, the profit of firm 1 is then of the form

$$\begin{aligned} \pi_1(t, q, \theta) &= q_1 D^{-1}(\sum_{i=1}^n q_i, \theta) - C(q_1, t_1) \\ &= q_1 (A + a\theta - b \sum_{j=1}^n q_j) - c_0 - c t_1 \end{aligned} \quad (2.1)$$

where $a_0 - c_1 = A$

The model is a two stage game, whose extensive form is drawn in Figure 1. In the first stage, the true value of θ is generated, then firms have a simultaneous move in which they select research levels $t = (t_1, \dots, t_n)$. In the second stage, (conditional on the research choice), there is first a chance move that generates the private data, $y = (y_1, \dots, y_n)$ for each firm. Each firm observes the research levels selected by each firm, but only observes its own private data. The firms then have a simultaneous move in which each firm determines its output level. Thus a strategy for firm i in the second stage is a function $q_i(y_i)$ which specifies an output for each possible observed value, y_i , of the private signal.

The equilibrium definition we use is that of a subgame perfect Nash Equilibrium. Thus, we proceed by solving the second stage first, for any given $t = (t_1, \dots, t_n)$. The first stage is then solved by assuming that payoffs from the first stage are determined by equilibrium behavior in the second stage.

To solve the second stage, we note that this game is in the form of a game of incomplete information, where firm types are determined by their private information, y_i . Thus, the appropriate equilibrium is that of Bayesian Nash (see, e.g., Harsanyi [1967-8]). Here we find there is a unique linear Bayesian Nash equilibrium. Assuming that firms adopt this equilibrium in the second stage, it follows that in the first stage, and hence to the overall game, there is a unique symmetric Nash equilibrium in mixed strategies.

In order to complete the description of the extensive form, we must specify the probability distributions for the chance moves. We assume that θ is generated according to a distribution $g(\theta)$, and then that y_i is generated according to $h(y_i|\theta, t_i)$, where

$$t_i = \frac{1}{E[\text{var}(y_i|\theta)]} \quad (2.2)$$

Thus, the research level of each firm is a measure of the expected precision of the data it is to receive. The higher its research level, t_i , the lower the expected variance of its data. Both $g(\cdot)$ and $h(\cdot|\theta, t_i)$ are assumed to have finite variances. Also, for any fixed t_i, t_j and θ , the conditional distribution of y_i , given t_i and θ , is assumed independent of y_j , given t_j and θ . These probability distributions are common knowledge. Hence, $g(\theta)$ is the common prior distribution (before they see any data) that firms have for θ . Each firm only sees its own data, y_i , and on the basis of this, can compute a posterior $f(\theta|y_i, t_i)$ for θ .

We make the following assumptions about the information structure. For all $t = (t_1, \dots, t_n)$

$$A1: E(y_i|\theta) = \theta$$

$$A2: E(\theta|y_i) = \gamma_i + \delta_i y_i$$

for some $\gamma_i, \delta_i \in \mathbb{R}$. Hence, the firm's signal is an unbiased estimator of θ , and the expectation of θ given y_i is linear in y_i . Under assumptions A1 and A2, it follows, from a result of Ericson [1968], that

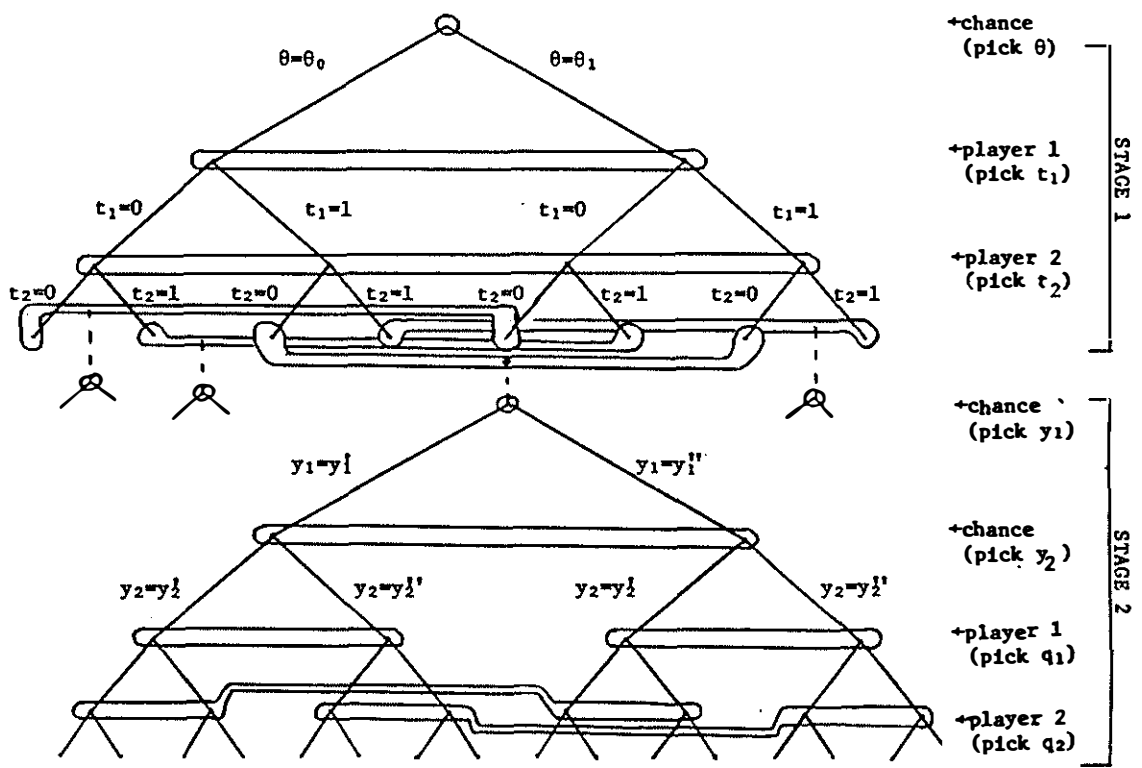


Figure 1
Extensive Form of Game

$$\delta_1 = \frac{\text{var}(\theta)}{\text{var}(\theta) + E[\text{var}(y_1|\theta)]} = \frac{\text{var}(\theta)}{\text{var}(y_1)}, \quad (2.3)$$

and, writing $\theta_0 = E(\theta)$,

$$\gamma_1 = (1 - \delta_1)\theta_0. \quad (2.4)$$

Substituting (2.2) in (2.3), and writing $R = \frac{1}{\text{var}(\theta)}$, for the precision of the prior, we get an expression for (A2) in terms of t_1 :

$$E(\theta|y_1) = \frac{R}{t_1 + R} \theta_0 + \frac{t_1}{t_1 + R} y_1, \quad (2.5)$$

with

$$\delta_1 = \frac{t_1}{t_1 + R}. \quad (2.6)$$

Since R is the precision of the prior, and t_1 is a measure of the expected precision of the data, this says that the posterior estimate of θ is a convex combination of the prior mean and the observed data. Each is weighted in proportion to its measure of precision. Also, using (A1) and (A2), we get

$$E(y_j|y_1) = E(\theta|y_1) = \gamma_1 + \delta_1 y_1, \quad (2.7)$$

$$\text{cov}(y_1, y_j) = \text{cov}(y_1, \theta) = \text{var} \theta. \quad (2.8)$$

$$\text{var}(y_1) = \frac{\text{var} \theta}{\delta_1} = \frac{R + t_1}{R t_1} \quad (2.9)$$

Before proceeding, we give several examples of information structures that satisfy our conditions. See e.g., Degroot [1970].

Example 1: Let the prior of θ be given by $g(\theta) \sim \text{Beta}(\alpha, \beta)$ and the data be described by $h(y_1|\theta, t_1) \sim \frac{1}{n_1} \text{Bin}(n_1, \theta)$ where $\text{Bin}(n_1, \theta)$ is a

binomial distribution with parameters n_1 and θ , and where $n_1 = t_1(\alpha + \beta)$ $\text{var}(\theta) = t_1 \frac{(\alpha + \beta)}{R}$. Then the posterior of θ is $f(\theta|y_1, t_1) \sim \text{Beta}(\alpha + n_1 y_1, \beta + n_1 - n_1 y_1)$, and

$$\begin{aligned} E(\theta|y_1) &= \frac{\alpha + y_1 n_1}{\alpha + \beta + n_1} = \frac{\alpha + \beta}{\alpha + \beta + n_1} \left(\frac{\alpha}{\alpha + \beta} \right) + \frac{n_1}{\alpha + \beta + n_1} y_1 \\ &= (1 - \delta) \theta_0 + \delta_1 y_1 \end{aligned}$$

where

$$\delta_1 = \frac{n_1}{\alpha + \beta + n_1} = \frac{t_1}{R + t_1}$$

In this example, $t_1 = \frac{R n_1}{\alpha + \beta}$ is a function of the number of observations.

Example 2: Let the prior of θ be given by $g(\theta) \sim N(\theta_0, \sigma_0^2)$ and $h(y_1|\theta, t_1) \sim N(\theta, \frac{1}{t_1})$, where $N(\theta, \sigma^2)$ is a normal distribution with mean θ , and variance σ^2 . So $R = \frac{1}{\sigma_0^2}$, and $f(\theta|y_1, t_1) \sim N(\theta_1, \sigma_1^2)$ where

$$\theta_1 = \frac{R \theta_0 + t_1 y_1}{R + t_1},$$

and

$$\frac{1}{\sigma_1^2} = R + t_1.$$

Hence,

$$E(\theta|y_1) = \theta_1 = (1 - \delta_1) \theta_0 + \delta_1 y_1,$$

where

$$\delta_1 = \frac{t_1}{R + t_1}.$$

In this example, we can let y_1 be a sample of n_1 independent

observations from $N(\theta, \sigma^2)$, in which case $t_1 = \frac{n_1}{\sigma^2}$, or we could let y_1 be a sample of a fixed number of observations from $N(\theta, \sigma(t_1)^2)$, in which case $t_1 = \frac{n}{\sigma(t_1)^2}$.

Example 3: Let the prior of θ be given by $g(\theta) \sim \Gamma(\alpha, \beta)$ and the data be described by $h(y_1 | \theta, t_1) \sim \frac{1}{n_1} P(n_1 \theta)$ where $\Gamma(\alpha, \beta)$ is a Gamma distribution, with parameters α, β , where $P(\lambda)$ is a Poisson with parameter λ , and where $n_1 = \frac{\alpha t_1}{\beta}$. Then the posterior of θ has $f(\theta | y_1, t_1) \sim \Gamma(\alpha + n_1 y_1, \beta + n_1)$. So

$$\begin{aligned} E(\theta | y_1) &= \frac{\alpha + n_1 y_1}{\beta + n_1} = \frac{\beta}{\beta + n_1} \left[\frac{\alpha}{\beta} \right] + \frac{n_1}{\beta + n_1} y_1 \\ &= (1 - \delta_1) \theta_0 + \delta_1 y_1 \end{aligned}$$

where

$$\delta_1 = \frac{n_1}{\beta + n_1} = \frac{t_1}{\frac{\beta^2}{\alpha} + t_1} = \frac{t_1}{R + t_1}$$

In this example, $t_1 = \frac{\beta}{\alpha} n_1$ is a function of the number of observations

We emphasize that since our information structure need not be normal, this leaves some freedom to model the uncertainty in ways which constrain the set of possible states θ to be reasonable for the specific problem. For example the Gamma-Poisson is reasonable if one wishes to impose nonnegativity constraints on θ (to keep the intercept of the demand function positive). On the other hand, if one views the uncertainty as arising from the cost function, then it is sometimes

useful to let the parameter, θ , represent the probability of an event. (See, e.g., McKelvey and Page [1985] who model an industry producing potentially toxic chemicals in such a fashion). In this case the Beta-Binomial is an obvious choice. Ericson [1968] points out several other distributions satisfying the linearity assumption.

3. The Second Stage Game

In this section, we fix $t = (t_1, \dots, t_n)$, and derive the Bayesian equilibrium strategy functions $q_1 = q_1(y_1)$.

Proposition 1 For any fixed $t = (t_1, \dots, t_n)$, there is a unique Bayesian equilibrium to the second stage game. The equilibrium strategy for each player is linear in his information.

Proof: For any n tuple of strategies, $q = (q_1, \dots, q_n)$ = $(q_1(y_1), \dots, q_n(y_n))$, we can write the expected profit for firm 1 as

$$\begin{aligned} E(\pi_1 | y_1) &= E(\pi_1(t, q, \theta) | y_1) = q_1(A - bq_1 - b \sum_{j \neq 1} E(q_j | y_1) \\ &\quad + aE(\theta | y_1)) - c_0 - ct_1. \end{aligned} \quad (3.1)$$

Any equilibrium point must satisfy the first order conditions:

$$2q_1 = \frac{A}{b} + (\gamma_1 + \delta_1 y_1) \frac{a}{b} - \sum_{j \neq 1} E(q_j | y_1) \text{ for all } i. \quad (3.2)$$

This can be rewritten as

$$2[q_1 - (\alpha_1 + \beta_1 y_1)] = \frac{A}{b} + (\gamma_1 + \delta_1 y_1) \frac{a}{b} - \sum_{j \neq 1} E(q_j | y_1)$$

$$-2\alpha_1 - 2\beta_1 y_1. \quad (3.3)$$

where we define α_1 and β_1 by

$$\beta_1 = \frac{a}{b} \left[\frac{\lambda_1}{1 + \sum_j \lambda_j} \right], \text{ where } \lambda_1 = \frac{\delta_1}{2 - \delta_1} = \frac{t_1}{t_1 + 2R}, \quad (3.4)$$

and

$$\alpha_1 = \frac{A}{(1+n)b} + \frac{2R\theta_0\beta_1}{t_1} - \frac{2}{1+n} \sum_j \frac{R\theta_0\beta_1}{t_j}. \quad (3.5)$$

Note that $\beta_1 \lambda_j = \beta_j \lambda_1$, so, from (3.4), we have

$$\frac{a}{b} \lambda_1 = \beta_1 + \sum_j \beta_1 \lambda_j = \beta_1 + \lambda_1 \sum_j \beta_j.$$

Hence, using $\frac{\lambda_1}{\lambda_1 + 1} = \frac{\delta_1}{2}$, we get

$$\beta_1 = \frac{\delta_1}{2} \left[-\sum_{j \neq 1} \beta_j + \frac{a}{b} \right]. \quad (3.6)$$

Further, from (3.5),

$$\sum_j \alpha_j = \frac{n}{n+1} \frac{A}{b} + \frac{2}{n+1} \sum_j \frac{R\theta_0\beta_1}{t_j}$$

Solving for α_1 in terms of $\sum_{j \neq 1} \alpha_j$, and adding to (3.5), we get

$$\alpha_1 = \frac{1}{2} \left[\frac{A}{b} - \sum_{j \neq 1} \alpha_j + \frac{2R\theta_0\beta_1}{t_1} \right]. \quad (3.7)$$

Substituting (3.7) and then (3.6) in (3.3), we get

$$2[q_1 - (\alpha_1 + \beta_1 y_1)] = (\gamma_1 + \delta_1 y_1) \frac{a}{b} - \sum_{j \neq 1} E(q_j | y_1)$$

$$\begin{aligned} & + \sum_{j \neq 1} \alpha_j - 2\beta_1 \left(\frac{\gamma_1}{\delta_1} + y_1 \right) \\ & = - \sum_{j \neq 1} E(q_j | y_1) + \sum_{j \neq 1} \alpha_j + \sum_{j \neq 1} \beta_j (\gamma_1 + \delta_1 y_1) \\ & = - \sum_{j \neq 1} E(q_j - (\alpha_j + \beta_j y_j) | y_1). \end{aligned} \quad (3.8)$$

Writing $V_j(y) = q_j - (\alpha_j + \beta_j y_j)$, it follows from (3.8) that each i 's Bayesian strategy must satisfy, for any y_1 ,

$$V_1(y_1) = - \sum_j E[V_j(y_j) | y_1] \quad (3.9)$$

Multiplying (3.9) by $V_1(y_1)$, and taking expectations, we get $E[V_1(y_1)^2] = - \sum_j E[V_1(y_1) E[V_j(y_j) | y_1]] = - \sum_j E[V_1(y_1) V_j(y_j)]$. Then summing over all i , we get

$$\sum_{i=1}^n E[V_i(y_i)^2] = - \sum_{i=1}^n \sum_{j=1}^n E[V_i(y_i) V_j(y_j)] \quad (3.10)$$

But $E[V_i(y_i) V_j(y_j)]$ is just the variance covariance matrix of the random vector $V(y) = (V_1(y_1), V_2(y_2), \dots, V_n(y_n))$, hence is positive semi definite. So (3.10) implies $E[V_i(y_i)^2] = 0$, and hence $V_i(y_i) = 0$ almost surely, i.e., $q_i = \alpha + \beta_i y_i$ almost surely.

Q.E.D.

We next derive the expected total output at the Bayesian Nash Equilibrium.

Proposition 2 For any fixed research levels, $t = (t_1, \dots, t_n)$ the expected total output at the Bayesian equilibrium is

$$E\left(\sum_{i=1}^n q_i\right) = \frac{n}{(n+1)b} (A + a\theta_0).$$

Proof: For any given θ , if $q_i = q_i(y_i)$ is the Bayesian strategy,

$$E[q_i|\theta] = a_i + \beta_i E(y_i|\theta) = a_i + \beta_i \theta.$$

So

$$\begin{aligned} E[q_i] &= a_i + \beta_i \theta_0 \\ &= \frac{A}{(n+1)b} + \frac{2R\theta_0\beta_i}{t_i} - \frac{2}{n+1} \sum_j \frac{R\theta_0\beta_j}{t_j} + \beta_i \theta_0 \\ &= \frac{A}{(n+1)b} - \frac{2}{n+1} \sum_j \frac{\lambda_j}{\lambda_i} \frac{\beta_j R\theta_0\beta_j}{t_j} + \frac{\beta_i \theta_0}{\lambda_i} \\ &= \frac{A}{(n+1)b} + \frac{\beta_i}{(n+1)\lambda_i} \left[-2 \sum_j \frac{R\theta_0\beta_j \lambda_j}{t_j} + (n+1)\theta_0 \right] \\ &= \frac{A}{(n+1)b} + \frac{\beta_i}{(n+1)\lambda_i} [1 + \sum_j \lambda_j] \theta_0 \\ &= \frac{A}{(n+1)b} + \frac{a}{b} \frac{\theta_0}{(n+1)} \\ &= \frac{1}{(n+1)b} (A + a\theta_0). \end{aligned} \tag{3.11}$$

So

$$E\left[\sum_{j=1}^n q_j\right] = \frac{n}{(n+1)b} (A + a\theta_0). \tag{3.12}$$

Q.E.D.

4. The First Stage Game

In this section, we characterize the Nash equilibrium to the full game. Thus, we allow $t = (t_1, \dots, t_n)$ to vary, and search for a Nash equilibrium in t given that Bayesian strategies, $q_i = q_i(y_i)$ (as described in Proposition 1), are adopted in the second stage game. For any fixed research levels, $t = (t_1, \dots, t_n)$ if firm i follows its Bayesian equilibrium strategy, then substituting (3.2) into (3.1), we find its expected profit conditioned on y_i , is

$$E(\pi_i|y_i) = bq_i^2 - c_0 - ct_i. \tag{4.1}$$

We write firm i 's expected profit as a function of the research level, t_j , as $\Pi_i(t)$. Then,

$$\Pi_i(t) = E(\pi_i(t, q, \theta)) = E(E(\pi_i(t, q, \theta)|y_i))$$

$$= bE[q_i^2|y_i] - c_0 - ct_i$$

$$= bE(q_i^2) - c_0 - ct_i.$$

But

$$E(q_i^2) = [E(q_i)]^2 + \text{var}(q_i) = [E(q_i)]^2 + \beta_i^2 \text{var}(y_i).$$

So, using (3.11), and (2.9),

$$\Pi_1(t) = \frac{1}{b(n+1)^2} (A + a\theta_0)^2 + \frac{a^2}{b} D_1(t) - c_0 - ct_1 \quad (4.2)$$

where

$$D_1(t) = \frac{b^2 \beta_1^2}{a^2 \delta_1 R} = \frac{t_1(t_1 + R)}{R(t_1 + 2R)^2 \left(1 + \sum_j \frac{t_1}{t_1 + 2R}\right)^2} \quad (4.3)$$

Thus, $\Pi_1(t)$ is the payoff function for the first stage game, given Bayesian strategies are followed in the second stage. Our main results establish existence and properties of the equilibrium for this game. We consider two cases, depending on what constraints are placed on the strategy space. The most natural choice for the strategy space is to let player 1's strategy space be \mathbb{R}_+ , where \mathbb{R}_+ is the non negative reals. Then a choice of $t_1 \in \mathbb{R}_+$ simply selects a signal of a given expected precision, t_1 . However some of the information structures (such as those in Examples 1 and 3 of Section 2) restrict the precision of the signal, since the precision is only a function of the signal through the number of observations. Hence it is also of interest to consider the case when player 1's strategy space is S , where S is a set (possibly infinite) of isolated points of \mathbb{R}_+ . We then define the two games

$$\Gamma = \langle \Pi, \mathbb{R}_+^n \rangle$$

$$\Gamma' = \langle \Pi, S^n \rangle$$

and refer to them, respectively as the continuous and discrete research games.

Proposition 3. For the continuous research game, Γ ,

- (a) There is a unique Nash equilibrium, $t = (t_1, \dots, t_n)$ which is symmetric (i.e. $t_i = t_j$ for all i, j).
- (b) The equilibrium, t_1 , is a continuous, decreasing, and convex function of c and b ; a continuous, increasing function of a ; a continuous decreasing function of R , and a decreasing function of n .

Proposition 4. For the discrete research game, Γ' ,

- (a) There is a unique symmetric equilibrium in mixed strategies.
- (b) The expected research at the symmetric mixed strategy equilibrium is a continuous decreasing function of c , b , and R , and a continuous increasing function of a .
- (c) Let s'_1 and s''_1 be the smallest and largest elements of S with positive support in the symmetric mixed strategy equilibrium for player 1, and let \underline{t}_1 be the Nash equilibrium for player 1 in the corresponding continuous game Γ (with the same parameters, a , a_0 , n , etc.). Then

$$\underline{t}_1 \leq s'_1 \leq s''_1 \leq \bar{t}_1$$

where \underline{t}_1 is the greatest element of S less than or equal to t_1 , and \bar{t}_1 is the smallest element of S greater than or equal to t_1 .

Propositions 3 and 4 follow from a more general lemma which states conditions for monotonicity of the reaction functions and the existence of a unique symmetric equilibrium. The lemma (Lemma 1) and

its application are in the appendix.

Propositions 3 and 4 completely characterize the equilibrium of the continuous research game and approximately characterize that of the discrete game. From (A.2) the equilibrium t_1 for the continuous game can be easily calculated for any parameters. Proposition 4 says that the symmetric equilibrium for the discrete research game will be close to the equilibrium in the corresponding continuous game, in the sense that the support of the symmetric mixed strategy equilibrium consists of at most 2 adjacent elements of S , which are "straddled" by the best approximations in S to the equilibrium of the corresponding continuous game. It follows from (c) that if the continuous game has an equilibrium which happens to be in S , then the equilibrium of the discrete game will coincide with it. In the discrete research game there can be additional asymmetric equilibria (due to the discrete nature of the game); however, in what follows we will focus on the symmetric equilibrium which is approximated by the unique equilibrium of the continuous game.

We next consider the competitive situation, by letting the number of firms get large. We write t_1^n for the equilibrium research for firm 1 in an n firm oligopoly, and $T_n = \sum_{i=1}^n t_1^n = nt_1^n$. Then write

$$t_1^c = \lim_{n \rightarrow \infty} t_1^n$$

and

$$T_c = \lim_{n \rightarrow \infty} T_n$$

for the competitive individual and total research.

Proposition 5: In the continuous research game, Γ ,

(a) For $0 < c < \frac{a^2}{4bR}$,

$$t_1^c = 0,$$

$$T_c = \left[\frac{a^2}{cb} \right]^{1/2} - 2R.$$

(b) For $\frac{a^2}{4bR^2} \leq c$, $t_1^c = T_c = 0$

Proof: In the continuous game, when n increases the equilibrium total $T = \sum t_1 = nt_1$ approaches, by (A.5) and (A.8), as a function of c , the envelope

$$\frac{1}{(T + 2R)^2} = \frac{cb}{a^2} \quad (4.4)$$

By substituting $t = 0$ in (A.8), we see that, for any n , when $c > \frac{a^2}{4bR^2}$, the equilibrium T is zero in either game.

Q.E.D.

5. Efficiency

In this section, we solve for the research and output which would result if a "center" attempted to maximize social welfare (consumer plus producer surplus). We then compare the oligopoly solution to this efficiency standard. We assume that for any given research, T , undertaken by the center, it obtains a signal, Y , having the same properties as those of the firms (in (A1) and (A2)). Namely

$$E(Y|\theta) = \theta \text{ and } E(\theta|Y) = \frac{R\theta_0 + TY}{R + T} \quad (5.1)$$

where T is the expected precision of the data obtained by the center.

To avoid problems of multiple equilibria and non-differentiability, we assume throughout this section a continuous information structure. For this continuous case we find that if equilibrium oligopoly research T_n is positive but finite, the expected net benefits in the oligopoly equilibrium are less than those of the efficiency standard, even for the limiting competitive case, when $n \rightarrow \infty$.

To characterize the sources of inefficiency we introduce an intermediate game where information is pooled, and derive conditions where in the efficiencies arise. The intermediate game is an artificial situation in which firms decide on research levels $t = (t_1, \dots, t_n)$ as in the oligopoly game, but then, after having done the research, the results of research output of all firms is pooled by the "center", and made publicly available (as common knowledge) to each of the n firms. The firms do not know that the information will be pooled when they make the research decision. Once the research is pooled, since each firm has the same information, the second stage of this game is essentially the same as the classical oligopoly game.

To ensure that the aggregate information has the same content whether it is gathered on a decentralized basis and then pooled, or gathered on a centralized basis, we introduce an additional assumption. We assume that for any $t = (t_1, \dots, t_n)$, and $y = (y_1, \dots, y_n)$,

A3: $E(\theta|y)$ is affine in y

This assumption (together with A1) can be shown to imply assumption A2. It also follows (see Li [1985]) that for $\bar{T} = \sum t_i$, the aggregate signal, $\bar{Y} = \frac{\sum t_i y_i}{\bar{T}}$ is a sufficient statistic of y with

$$E(\bar{Y}|\theta) = \theta \text{ and } E(\theta|\bar{Y}) = \frac{R\theta_0 + \bar{T}\bar{Y}}{R + \bar{T}} \quad (5.2)$$

We define the consumer plus producer surplus ("social benefits" for short) for T , Q , and θ by

$$\begin{aligned} W &= \int_0^Q D^{-1}(Q, \theta) dQ - C(Q, T) \\ &= AQ + aQ\theta - \frac{b}{2}Q^2 - c_0 - c_2T \end{aligned} \quad (5.3)$$

So

$$E(W|Y, T) = AQ + aQE(\theta|Y) - \frac{b}{2}Q^2 - c_0 - c_2T \quad (5.4)$$

and for fixed T the efficient output Q_e which maximizes (5.4) over Q is given by the first order condition

$$Q_e = Q_e(Y) = \frac{1}{b}[A + aE(\theta|Y)] \quad (5.5)$$

Define q_1^n as the equilibrium output for firm 1 in the continuous game of Section 4, and total output $Q_n = \sum_{i=1}^n q_1^n$. Define q_1^p as the equilibrium output for firm 1 in the intermediate, pooled information game, and total output $Q_n^p = \sum q_1^p$. We now show that

$$Q_e = Q_e(Y) = \frac{1}{b} \left[A + a \frac{R\theta_0 + TY}{R + T} \right] \quad (5.6)$$

$$Q_n^P = Q_n^P(Y) = \frac{n}{(n+1)b} \left[A + a \frac{R\theta_0 + T_n Y}{R + T_n} \right] \quad (5.7)$$

$$Q_n = Q_n(Y) = \frac{n}{(n+1)b} \left[A + a \frac{2nR\theta_0 + (n+1)T_n Y}{2nR + (n+1)T_n} \right] \quad (5.8)$$

Equation (5.6) follows from applying (5.1) to (5.5). To establish (5.7) we write down the first order conditions corresponding to (3.2)

$$2q_1^P = \frac{A}{b} + \frac{a}{b} E(\theta|\bar{Y}) - \sum_{j \neq 1} q_j^P$$

$$\text{So } q_1^P = q_j^P.$$

$$Q_n^P(\bar{Y}) = \frac{n}{(n+1)b} \left[A + aE(\theta|Y) \right]$$

and applying (5.2) with $\bar{T} = T_n$ being the equilibrium research, equation (5.7) follows directly.

To establish (5.8) note that in the unique oligopoly equilibrium of the continuous case $t_i = t_j$, and we can write $t_i = t$, $\alpha_i = \alpha$, $\beta_i = \beta$, $T_n = nt$, $nY = \sum y_i$, and $Q_n = nq_1^n$. By Proposition 1 $Q_n = n\alpha + \beta \sum y_i = n\alpha + n\beta Y$.

$$\beta = \frac{a}{b} \frac{\frac{t}{2R+t}}{1 + \frac{nt}{2R+t}} = \frac{a}{b} \left[\frac{T_n}{2nR + (n+1)T_n} \right]$$

$$\alpha = \frac{A}{(n+1)b} + \frac{2R\theta_0\beta}{(n+1)t} = \frac{A}{(n+1)b} + \frac{2aRn\theta_0 T_n}{(n+1)bT_n(2nR + (n+1)T_n)}$$

$$Q_n = n\alpha + n\beta Y = \frac{n}{(n+1)b} \left[A + a \frac{2nR\theta_0 + (n+1)T_n Y}{2nR + (n+1)T_n} \right]$$

From (5.3) the expected social benefits for $Q = Q(Y)$ and T is

$$\begin{aligned} W(T) &= E[E[W|Q,T]] = AE(Q) + aE(Q\theta) - \frac{b}{2}E(Q^2) - c_0 - c_2 T \\ &= AE(Q) + a[\theta_0 E(Q) + \text{cov}(Q,\theta)] - \frac{b}{2}[(E(Q))^2 + \text{var } Q] \\ &\quad - c_0 - cT \\ &= (A + a\theta_0 - \frac{b}{2}E(Q))E(Q) + a \text{cov}(Q,\theta) \\ &\quad - \frac{b}{2}\text{var}(Q) - c_0 - cT \end{aligned} \quad (5.9)$$

To derive the efficient level of research, note first

$$E(Q_e) = \frac{1}{b}(A + a\theta_0)$$

$$\text{var}(Q_e) = \text{var} \left[\frac{aTY}{b(R+T)} \right] = \frac{a^2 T}{b^2 R(R+T)}$$

$$\text{cov}(Q_e, \theta) = \frac{aT}{b(R+T)} \text{cov}(Y, \theta) = \frac{aT}{b(R+T)} \text{var}(\theta) = \frac{aT}{bR(R+T)}$$

Thus for any given T , the expected social benefits from the optimal Q_e are

$$W_e(T) = \frac{1}{2b}(A + a\theta_0)^2 + \frac{a^2 T}{2bR(R+T)} - c_0 - cT \quad (5.10)$$

The efficient research is the research, T_e , satisfying the first order conditions

$$\frac{1}{2(T+R)^2} = \frac{bc}{a^2} \text{ for } 0 < c < \frac{a^2}{2R^2 b}, \text{ and } T = 0 \text{ for } c \geq \frac{a^2}{2R^2 b} \quad (5.11)$$

so

$$T_e = \left[\frac{a^2}{2bc} \right]^{1/2} - R \text{ for } 0 < c < \frac{a^2}{2R^2 b} \quad (5.12)$$

The second order condition yields

$$\frac{\partial^2 W_e(T)}{\partial T^2} = - \frac{a^2}{b(T+R)^2} < 0 \quad (5.13)$$

To derive the expected social benefits in the pooled information game, note

$$\begin{aligned} E(Q_n^P) &= \frac{n}{(n+1)b}(A + a\theta_0) \\ \text{var}(Q_n^P) &= \frac{n^2 a^2 T_n}{(n+1)^2 b^2 R(R + T_n)} \\ \text{cov}(Q_n^P, \theta) &= \frac{naT_n}{bR(R + T_n)} \end{aligned} \quad (5.14)$$

So the expectation of W in the pooled information equilibrium is

$$W_n^P(T_n) = \frac{(n+2)n}{(n+1)^2 b} (A + a\theta_0)^2 + \frac{(n+2)na^2 T_n}{(n+1)^2 bR(R + T_n)} - c_0 - cT_n \quad (5.15)$$

To derive the expected social benefits for the oligopoly game, note from (5.6)

$$\begin{aligned} E(Q_n) &= \frac{n}{(n+1)b} (A + a\theta_0) \\ \text{var } Q_n &= \frac{\frac{2}{n} a^2 T_n (R + T_n)}{b^2 R(2nR + (n+1)T_n)^2} \\ \text{cov}(Q_n, \theta) &= \frac{\frac{na}{b} T_n}{R(2nR + (n+1)T_n)} \end{aligned} \quad (5.16)$$

So the expectation of W in this oligopoly game, for equilibrium T_n is

$$W_n(T_n) = \frac{n(n+2)}{2(n+1)^2 b} (A + a\theta_0)^2 + \frac{a^2 n T_n ((n+2)T_n + 3nR)}{2bR((n+1)T_n + 2nR)^2} - c_0 - cT_n \quad (5.17)$$

To identify three sources of inefficiency we derive the main welfare result (proposition 9) in terms of the following lemma.

Lemma 2.

- (i) $W_e(T_e) - W_e(T_n) > 0$ for $T_n \neq T_e$.
- (ii) $W_e(T_n) - W_n^P(T_n) > 0$, for $n \geq 1$ and $T_n \geq 0$, and the difference converges to zero as $n \rightarrow \infty$.
- (iii) $W_n^P(T_n) - W_n(T_n) > 0$, for $n \geq 2$ and $T_n > 0$ and the difference does not converge to zero except when $T_n = 0$ or $T_n \rightarrow \infty$.

Proof. (i) From the definition of T_e , and the second order condition (5.13), (i) is obvious.

(ii) From (5.10) and (5.15)

$$W_e(T_n) - W_n^P(T_n) = \frac{(A + a\theta_0)^2}{2(n+1)^2 b} + \frac{aT_n}{2(n+1)^2 bR(T_n + R)} > 0,$$

(iii) From (5.15) and (5.17),

$$W_n^P(T_n) - W_n(T_n) = \frac{n(n-1)a^2 T_n (n(n+3)R + 2(n+1)T_n)}{2(n+1)^2 bR(R + T_n)(2nR + (n+1)T_n)^2} > 0,$$

for $n \geq 2$ and $T_n > 0$.

Asymptotic properties can be seen by examining the above differences.

Q.E.D.

Proposition 6. In the continuous game

- (a) If $0 < c < \frac{a^2}{2R^2 b}$, the expected social benefits in the oligopoly equilibrium converge as n increases, to a limit below the

efficient level.

- (b) If $c > \frac{a^2}{2R^2b}$ is fixed and $n \rightarrow \infty$, or if $c \rightarrow 0$ and $n \rightarrow \infty$, the expected social benefits in the oligopoly equilibrium converge to the efficient level.

Proof.

$$W_e(T_e) - W_n(T_n) = [W_e(T_e) - W_e(T_n)] + [W_e(T_n) - W_n^P(T_n)] \\ + [W_n^P(T_n) - W_n(T_n)]$$

where we know from Lemma 2 that each of the bracketed terms is non-negative.

Case (a). If $0 < c < \frac{a^2}{4R^2b}$, by Proposition 5 as $n \rightarrow \infty$, T_n converges to T_c , where $0 < T_c < \infty$. By Lemma 2(iii), $[W_n^P(T_n) - W_n(T_n)]$ does not converge to zero and hence $[W_e(T_e) - W_n(T_n)]$ does not converge to zero. If $\frac{a^2}{4R^2b} \leq c \leq \frac{a^2}{2R^2b}$, by Proposition 5, equilibrium research is zero for all n . By (5.11) $T_e > 0$. Hence by Lemma 2(i), $[W_e(T_e) - W_e(T_c)] > 0$ and $[W_e(T_e) - W_n(T_n)]$ does not converge to zero.

Case (b). If $c > \frac{a^2}{2R^2b}$, then $T_n = T_e = 0$. By Lemma 2(i) $W_e(T_e) - W_e(T_n) = 0$ and by Lemma 2(iii), $W_n^P(T_n) - W_n(T_n) = 0$. Since, by Lemma 2(ii), for any T_n , $W_e(T_n) - W_n^P(T_n)$ converges to zero, $W_e(T_e) - W_n^P(T_n)$ converges to zero. If $c \rightarrow 0$ and $n \rightarrow \infty$, a similar argument applies.

Q.E.D.

In case (b) each firm has essentially the same information and the resulting efficiency is the same as in the classical model. The extension of the classical model is in case (a), where there is (with positive probability) asymmetric private information and resulting inefficiency. Lemma 2 traces three sources of inefficiency.

The term $[W_e(T_e) - W_e(T_n)]$ represents a research inefficiency which arises when oligopoly equilibrium research T_n does not equal the efficient T_e . The term $[W_e(T_n) - W_n^P(T_n)]$ represents the classical oligopoly inefficiency which arises from strategic exploitation of the demand curve and goes away as n becomes large. The term $[W_n^P(T_n) - W_n(T_n)]$ represents (as we will later discuss) an inefficiency from incomplete pooling. It is especially noteworthy that the research inefficiency and the incomplete pooling inefficiency do not vanish when n gets large.

The following proposition states when the research inefficiency arises (when $T_n \neq T_e$).

Proposition 7. In the continuous oligopoly game, for each $n \geq 2$,

there is a positive constant k_n such that

(a) if $0 < c < k_n$, then $T_n > T_e$

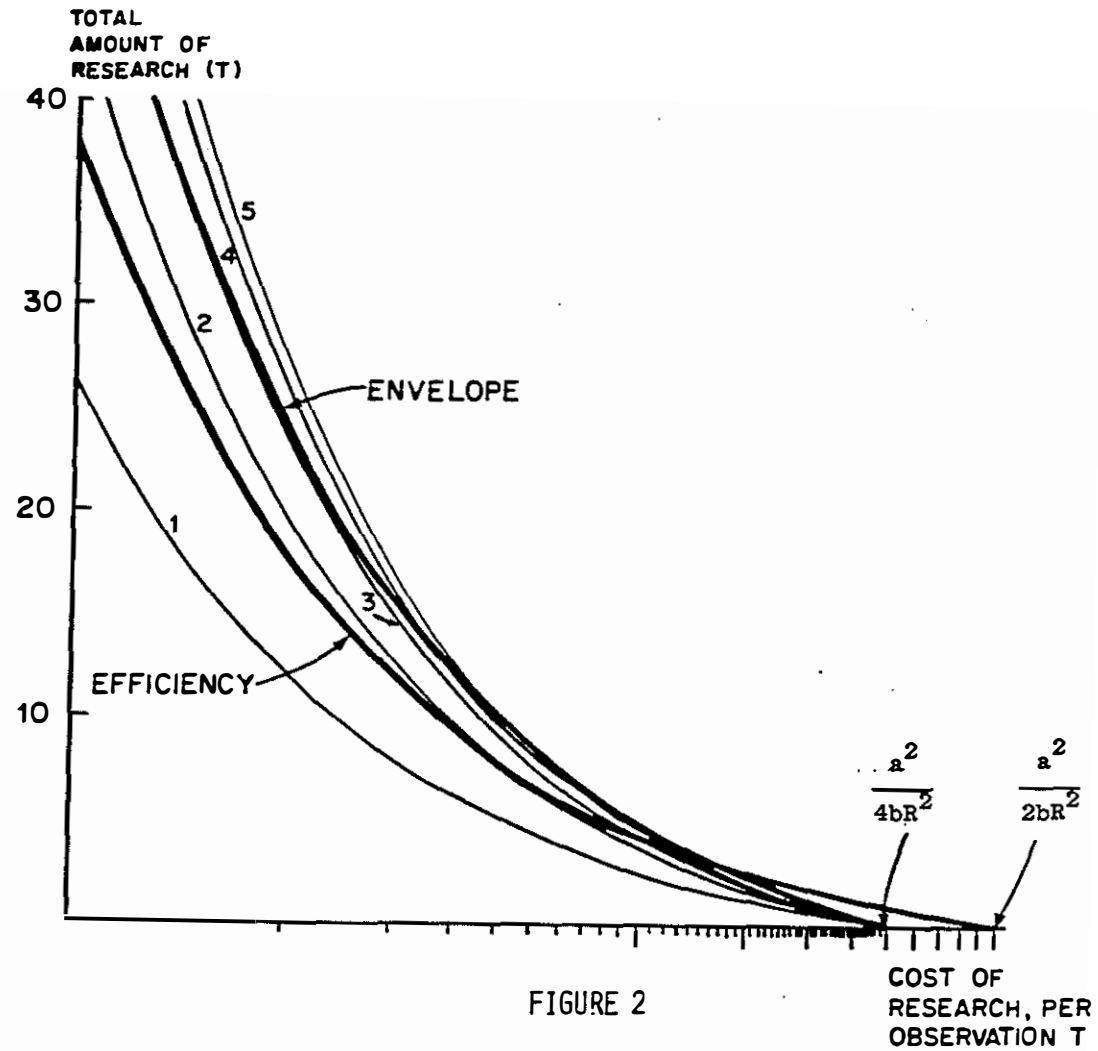
(b) if $c = k_n$, then $T_n = T_e$

(c) if $k_n < c < \frac{a^2}{2bR^2}$, then $T_n < T_e$.

Moreover, in the monopoly case, $n = 1$, if $0 > c > \frac{a^2}{2bR^2}$, then $T_1 < T_e$.

And for all n , if $c > \frac{a^2}{2bR^2}$, then $T_n = T_e = 0$.

AMOUNT OF RESEARCH AS A FUNCTION OF COST



Proof: (in appendix).

The efficient T_e is plotted as a function of c in Figure 2. The equilibrium T_n for the continuous game is also plotted for $n = 1, 2, 3, 4, 5$, and for the competitive envelope of Proposition 5. Proposition 7 makes it easy to construct examples where there is a unique equilibrium to the discrete research game and where there is either more or less than the efficient amount of research.

To investigate the question of incomplete pooling, we first show that Q_n converges to Q_n^P only in the classical case when $c \rightarrow 0$ (and $T \rightarrow \infty$).

Proposition 8.

- (a) If $0 < c < \frac{a^2}{4bR^2}$, then as $n \rightarrow \infty$ $\text{var } Q_n > \text{var } Q_p$.
- (b) If $c \rightarrow 0$ and $n \rightarrow \infty$, then $Q_n \xrightarrow{\text{a.s.}} Q_p$.

Proof. (a) From (5.14) and (5.16),

$$\text{var } Q_n^P = \frac{n^2 a^2 T_n}{(n+1)^2 b^2 R(R+T_n)}, \quad \text{and} \quad \text{var } Q_n^P \rightarrow \frac{a^2 T_c}{b^2 R(R+T_c)}.$$

$$\text{var } Q_n = \frac{n^2 a^2 T_n (R+T_n)}{b^2 R(2nR + (n+1)T_n)^2}, \quad \text{and} \quad \text{var } Q_n \rightarrow \frac{a^2 T_c (R+T_c)}{b^2 R(2R+T_c)^2}.$$

Hence, $\lim_{n \rightarrow \infty} (\text{var } Q_n - \text{var } Q_n^P) = \frac{a^2 T_c (3R + 2T_c)}{b^2 (2R + T_c)^2 (R + T_c)} > 0$.

(b) From (5.7) and (5.8)

$$Q_n - Q_n^P = \frac{n(n-1)aRT_n}{(n+1)b(R+T_n)(2nR+(n+1)T_n)} (Y - \theta_0)$$

As $c \rightarrow 0$ and $n \rightarrow \infty$, $T_n \rightarrow \infty$ and $Q_n - Q_n^P \rightarrow 0$ for any $y = (y_1, \dots, y_n)$.

Q.E.D.

By (5.6) and (5.7), as $n \rightarrow \infty$, $Q_n^P \xrightarrow{\text{a.s.}} Q_e$. Since demand is linear, convergence of Q_n^P to Q_e implies convergence of equilibrium price in the pooled information game to the efficient price. So the nonconvergence of Q_n to Q_n^P implies that the equilibrium price in the oligopoly game does not converge to the efficiency price. This last conclusion follows for any $0 < T_n < \infty$, including the special case when $c = k_m$ and when the competitive research level is efficient.

Palfrey [1984] concludes that in large markets, even if no firm knows the true market demand curve and firms are not price-takers and do not use price as a signal to improve their information, the efficient price will prevail with certainty. The above proposition points out that this is not true if the information is costly to acquire.

6. Conclusion

We conclude that when research acquisition is endogenously determined, and when the research amount is positive but finite, the symmetric equilibrium for an oligopoly is inefficient. This negative result includes the limiting case of a competitive market. For the competitive case, the source of inefficiency is traced to a lack of information pooling as well as an inefficient total amount of research.

In terms of the modeling, there are two main contributions of this paper. First, we model the research decision as endogenous. Most other models of oligopolies take the research levels as fixed. The second contribution is the proof (Theorem 1) that existence of a unique linear Bayesian equilibrium to the second stage game depends only on a linear information structure. This seems to be a useful generalization of existing results, which depend on a normal information structure.

APPENDIX

The proofs of Theorems 3 and 4 follow from the monotonicity of the best response correspondences. Monotonicity of the reaction functions for the continuous game follows from the fact that $\partial^2 \pi_1 / \partial t_1 \partial t_j < 0$. In the discrete game the corresponding condition is in terms of differences. We state the condition in a way which applies to both games. Monotonicity is then used to establish the existence of a unique symmetric equilibrium in each game.

Let $M_1: T^n \rightarrow \mathbb{R}$ be 1's payoff function for the game $\langle M, T^n \rangle$, where $M = (M_1, \dots, M_n)$. Write $\Phi: T^n \rightarrow T^n$ as the best response correspondence for $\langle M, T^n \rangle$. We define Φ as monotonic if, $t' \leq s'$ whenever $t > s$, $t' \in \Phi(t)$, and $s' \in \Phi(s)$. Denote t_{-1} for $(t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n)$ and $M_1(t'_{-1}, t_{-1}) = M_1(t_1, \dots, t'_{i-1}, \dots, t_n)$.

The condition corresponding to $\partial^2 \pi_1 / \partial t_1 \partial t_j < 0$ is stated as follows. For each $t''_1 > t'_1$,

$$M_1(t''_1, t_{-1}) - M_1(t'_1, t_{-1}) \text{ is strictly decreasing in } t_{-1}. \quad (\text{A.1})$$

Lemma 1. If $T \subseteq \mathbb{R}$ is compact and convex, $\langle M, T^n \rangle$ is a concave, continuous, and symmetric game, and each M_i satisfies condition (A.1), then

- (i) Φ is symmetric, convex valued, upper hemi continuous, and monotonic;
- (ii) There is a unique symmetric equilibrium for $\langle M, T^n \rangle$.

Proof:

(i) The facts that Φ is symmetric, convex valued, and upper hemi continuous follow in standard fashion from the hypothesis that $\langle M, T^n \rangle$ is a continuous, concave, and symmetric game. To show monotonicity, choose $t > s$ and any $t' \in \Phi(t)$ and any $s' \in \Phi(s)$, and suppose $t'_i > s'_i$ for some i . By condition (A.1)

$$M_i(t'_i, t_{-i}) - M_i(s'_i, t_{-i}) < M_i(t'_i, s_{-i}) - M_i(s'_i, s_{-i}).$$

But since $s'_i \in \Phi_i(s_i)$, $M_i(t'_i, s_{-i}) - M_i(s'_i, s_{-i}) \leq 0$. So $M_i(t'_i, t_{-i}) < M_i(s'_i, t_{-i})$, contradicting $t'_i \in \Phi_i(t)$. Hence $t_i \leq s_i$ for all i , or $t \leq s$.

(ii) Let $\Delta = \{t \in T^n \mid t_j = t, \text{ all } j\}$ be the diagonal, and let $F(t) = \Delta \cap \Phi(t)$. Note that for each $t \in \Delta$, $\Phi_i(t_{-i}) = \Phi_j(t_{-j})$ (all i, j), and there is at least one scalar $s_k \in \Phi_i(t_{-i})$ for all i . So $s = (s_k, \dots, s_k) \in \Phi(t)$ and $F(t)$ is non empty for any $t \in \Delta$. Since Φ is convex and upper hemi continuous, it follows that $F: \Delta \rightarrow \Delta$ is also convex and upper hemi continuous. Hence F has a fixed point.

Let t^* be a fixed point of F and choose any $t \in \Delta$ with $t > t^*$. Since $t^* \in \Phi(t^*)$, by monotonicity $t' \leq t^*$ for all $t' \in F(t) \subseteq \Phi(t)$. So $t' < t$ and $t \notin \Phi(t)$. Similarly $t < t^*$ implies $t \notin \Phi(t)$. Therefore t^* must be a unique symmetric equilibrium.

Q.E.D.

Proof of Proposition 3:

(a) To apply Lemma 1, we set $M_1(t) = \Pi_1(t)$, and calculate

$$\frac{\partial M_1}{\partial t_1}, \frac{\partial^2 M_1}{\partial t_1^2}, \text{ and } \frac{\partial^2 M_1}{\partial t_1 \partial t_j}$$

$$M_1(t) = \Pi_1(t) = C_0 + \frac{a^2}{b} D_1(t) - ct_1,$$

where

$$C_0 = \frac{1}{b(n+1)^2} (A - a\theta_0)^2 - c_0,$$

and

$$D_1(t) = \frac{\delta_1}{R(2 - \delta_1)^2 \left(1 + \sum_j \left(\frac{\delta_1}{2 - \delta_j}\right)^2\right)}.$$

Note that

$$\begin{aligned} (2 - \delta_1) \left(1 + \sum_j \frac{\delta_1}{2 - \delta_j}\right) &= (2 - \delta_1) \left(\frac{2}{2 - \delta_1} + \sum_{j \neq 1} \frac{\delta_1}{2 - \delta_j}\right) \\ &= 2 + (2 - \delta_1)\Delta_1. \end{aligned}$$

where

$$\Delta_1 = \sum_{j \neq 1} \frac{\delta_1}{2 - \delta_j}.$$

Rewriting,

$$D_1 = \frac{\delta_1}{R(2(1 + \Delta_1) - \delta_1\Delta_1)^2},$$

we find

$$\frac{\partial D_1}{\partial \delta_1} = \frac{2(1 + \Delta_1) - \delta_1\Delta_1 + 2\delta_1\Delta_1}{R(2(1 + \Delta_1) - \delta_1\Delta_1)^3}$$

$$= \frac{2(1 + \Delta_1) + \delta_1 \Delta_1}{R(2(1 + \Delta_1) - \delta_1 \Delta_1)^3} > 0$$

$$\frac{\partial^2 D_1}{\partial \delta_1^2} = \frac{\Delta_1(2(1 + \Delta_1) - \delta_1 \Delta_1) + 3\Delta_1(2(1 + \Delta_1) + \delta_1 \Delta_1)}{R(2(1 + \Delta_1) - \delta_1 \Delta_1)^4}$$

$$= \frac{2\Delta_1(4(1 + \Delta_1) + \delta_1 \Delta_1)}{R(2(1 + \Delta_1) - \delta_1 \Delta_1)^4}$$

Also note that

$$\frac{\partial \delta_1}{\partial t_1} = \frac{R}{(t_1 + R)^2} > 0.$$

$$\frac{\partial^2 \delta_1}{\partial t_1^2} = -\frac{2R}{(t_1 + R)^3} < 0.$$

Then

$$\frac{\partial D_1}{\partial t_1} = \frac{\partial D_1}{\partial \delta_1} \cdot \frac{\partial \delta_1}{\partial t_1} = \frac{2(1 + \Delta_1) + \delta_1 \Delta_1}{(2(1 + \Delta_1) - \delta_1 \Delta_1)^3} \cdot \frac{1}{(t_1 + R)^2} > 0$$

$$\frac{\partial^2 D_1}{\partial t_1^2} = \frac{\partial^2 D_1}{\partial \delta_1^2} \left[\frac{\partial \delta_1}{\partial t_1} \right]^2 + \frac{\partial D_1}{\partial \delta_1} \frac{\partial^2 \delta_1}{\partial t_1^2}$$

$$= \frac{\partial^2 D_1}{\partial \delta_1^2} \cdot \frac{R^2}{(t_1 + R)^4} + \frac{\partial D_1}{\partial \delta_1} \frac{2R}{(t_1 + R)^3}$$

$$= \frac{1}{(t_1 + R)^3} \left[\frac{\partial^2 D_1}{\partial \delta_1^2} \cdot (1 - \delta_1) - 2 \frac{\partial D_1}{\partial \delta_1} \right]$$

$$= \frac{1}{(t_1 + R)^3} \left[\frac{2\Delta_1(4(1 + \Delta_1) + \delta_1 \Delta_1)(1 - \delta_1) - 2(4(1 + \Delta_1)^2 - \delta_1^2 - \delta_1^2 \Delta_1^2)}{(2(1 + \Delta_1) - \delta_1 \Delta_1)^4} \right]$$

$$= \frac{2\Delta_1(4 + 4\Delta_1 + 4\Delta_1\delta_1 + 3\Delta_1^2\delta_1)}{(t_1 + R)^3 (2(1 + \Delta_1) - \delta_1 \Delta_1)^4} < 0 \quad (\text{A.2})$$

Rewrite

$$\frac{\partial D_1}{\partial t_1} = \frac{2 + (2 + \delta_1)\Delta_1}{(2 + (2 - \delta_1)\Delta_1)^3} \cdot \frac{(1 - \delta_1)^2}{R^2} \quad (\text{A.3})$$

Note

$$\frac{\partial \Delta_1}{\partial \delta_k} = \frac{2 + \delta_k}{(2 - \delta_k)^2}, \quad k \neq 1, \text{ so}$$

$$\frac{\partial^2 D_1}{\partial t_1 \partial \delta_k} = \frac{\partial^2 D_1}{\partial t_1 \partial \Delta_1} \frac{\partial \Delta_1}{\partial \delta_k}$$

$$= -\frac{(1 - \delta_1)^2}{R^2} \cdot \frac{2 + \delta_k}{(2 - \delta_k)^2} \cdot \frac{8(1 - \delta_1) + 2(4 - \delta_1^2)\Delta_1}{(2 + (2 - \delta_1)\Delta_1)^4} < 0 \quad (\text{A.4})$$

$$\frac{\partial^2 D_1}{\partial t_1 \partial t_k} = \frac{\partial^2 D_1}{\partial t_1 \partial \delta_k} \cdot \frac{\partial \delta_k}{\partial t_k}$$

$$= -\frac{(1 - \delta_i)^2(1 - \delta_k)^2}{R^3} \cdot \frac{2 + \delta_k}{(2 - \delta_k)^2} \cdot \frac{8(1 - \delta_i) + 2(4 - \delta_i^2)\Delta_i}{(2 + (2 - \delta_i)\Delta_i)^4} < 0 \quad (\text{A.5})$$

By (A.2) and (A.3) the continuous research game is concave, continuous and symmetric. Property (A.1) follows directly from (A.5). Hence Lemma 1 applies and the continuous research game has a unique symmetric Nash equilibrium.

We next show that there are no asymmetric equilibria. A Nash equilibrium $t = (t_1, \dots, t_n)$ must satisfy the Kuhn-Tucker conditions:

$$\frac{\partial D_i}{\partial t_1} = \frac{c_2 b}{a^2} \text{ and } t_1 \geq 0; \text{ or } \frac{\partial D_i}{\partial t_1} < \frac{c_2 b}{a^2} \text{ and } t_1 = 0 \quad (\text{A.6})$$

Now let $S = 1 + \sum_{j=1}^n \frac{\delta_j}{2 - \delta_j} = \Delta_i + \frac{2}{2 - \delta_i}$. Then we can rewrite

(A.5) as

$$\frac{\partial D_i}{\partial t_1} = \frac{(2 + (2 + \delta_i)(S - \frac{2}{2 - \delta_i}))}{(t_1 + R)^2 (2 - \delta_i)^3 S^3}$$

$$= \frac{2(t_1 + R) + (3t_1 + 2R)(S - \frac{2}{t_1 + 2R})}{(t_1 + 2R)^3 S^3}$$

Define

$$F(\tau, S) = \frac{[(3\tau + 2R)S - \frac{4\tau(\tau + R)}{\tau + 2R}]}{(\tau + 2R)^3 S^3} \quad (\text{A.7})$$

$$\text{So } F(\tau, S) = \frac{\partial D_i}{\partial t_1}.$$

$$\frac{\partial F}{\partial \tau} = \frac{1}{(\tau + 2R)^4 S^3} \cdot \left[(3S - \frac{4(\tau^2 + 4R\tau + 2R^2)}{(\tau + 2R)^2})(\tau + 2R) - 3((3\tau + 2R)S - \frac{4\tau(\tau + R)}{\tau + 2R}) \right]$$

$$= -\frac{1}{(\tau + 2R)^4 S^3} \cdot \left[6\tau S + \frac{-8\tau^2 + 4\tau R + 8R^2}{\tau + 2R} \right]$$

$$= -\frac{1}{(\tau + 2R)^4 S^3} \left[\frac{(6S - 8)\tau^2 + 4R(3S + 1)\tau + 8R^2}{\tau + 2R} \right].$$

Note that

$$\frac{\partial F(t_1, S)}{\partial \tau} = -\frac{1}{(t_1 + 2R)^4 S^3} \left[6t_1 \Delta_i + \frac{4t_1^2 + 16t_1 R + 8R^2}{t_1 + 2R} \right] < 0 \quad (\text{A.8})$$

and

$$\frac{\partial F(0, S)}{\partial \tau} = \frac{1}{4RS^3} < 0.$$

So $\frac{\partial F(\tau, S)}{\partial \tau} < 0$ for $0 \leq \tau \leq t_1$, and $i = 1, \dots, n$, since

$(6S - 8)\tau^2 + 4R(3S + 1)\tau + 8R^2 > 0$ if $6S - 8 \geq 0$ and it has only one positive root, which occurs for $\tau > \max_i(t_1)$ if $6S - 8 < 0$.

Now suppose (A.5) is satisfied by (t_1^0, \dots, t_n^0) for $i = 1, \dots, n$,

where some $t_1^0 < t_j^0$. Then $F(t_j^0, S^0) = \frac{c_2 b}{a_0^2} > 0$ and $F(t_1^0, S^0) \leq \frac{c_2 b}{a_0^2}$, so

$F(t_1^0, S^0) < F(t_j^0, S^0)$, a contradiction since F strictly declines in its first argument. We conclude that there exists a unique Nash

equilibrium and this equilibrium is symmetric.

(b) Let

$$K(t, R) = \frac{\partial D_i}{\partial t_i} \Big|_{t_1=t_j=t} = \frac{[(3n-1)t^2 + 2R(n+2)t + 4R^2]}{(t+2R)((n+1)t+2R)^3}$$

Direct calculation shows $\frac{\partial K}{\partial t} < 0$ and $\frac{\partial^2 K}{\partial t^2} > 0$. From (A.6), for any R , the condition which determines the symmetric equilibrium is

$$K(t, R) = \frac{cb}{a^2} \text{ and } t > 0, \text{ or } K(t, R) \leq \frac{cb}{a^2} \text{ and } t = 0. \quad (\text{A.9})$$

Since K is strictly decreasing, for $c < \frac{a^2}{4R^2b}$, the equilibrium t strictly declines as c increases, whereas for $c \geq \frac{a^2}{4R^2b}$, $t = 0$ and remains so as c increases. The convexity of t as a function of c simply follows from the convexity of K .

As b and $\frac{1}{2}$ play the same role as c in (A.9), it follows that the equilibrium t is a continuous, decreasing and convex function of b and a continuous, increasing function of a .

To show the equilibrium t is a decreasing function of n , simply note that K is an increasing function of n for any $t \geq 0$. As n increases, equilibrium t must decrease, unless of course equilibrium t is already on the boundary $t = 0$, in which case it remains zero.

Finally, to show that the equilibrium is decreasing in R , a direct calculation shows $\frac{\partial K}{\partial R} < 0$. The implicit function theorem then establishes that

$$\frac{\partial t}{\partial R} = - \frac{\partial K}{\partial R} / \frac{\partial K}{\partial t} < 0$$

Q.E.D.

Preliminary Observation for Proposition 4: Before proving Proposition 4, we observe the fact that the support of any mixed strategy equilibrium in the discrete research game consists of at most two adjacent elements of S . The argument is as follows. Note that $M_i(t)$ is a strictly concave function of t_i . For t restricted to be the set of n tuples of elements of S , the expected payoff function for i is again a strictly concave function of t_i since it is just a convex combination of $M_i(t_i, t_{-i})$ over t_{-i} . Therefore i 's best response is either a single element of S or indifference between two adjacent elements.

Proof of Proposition 4. First we define a game derived from the continuous research game. Let $S^n \subseteq \mathbb{R}_+^n$ be the restricted set of strategy n -tuples. For any $t \in \mathbb{R}_+^n$, define the measure $\lambda_t: S^n \rightarrow \mathbb{R}$ as follows. First, for $i \in N$, define $\lambda_t^i: I \rightarrow \mathbb{R}$ by

$$\lambda_t^i(s_i) = \begin{cases} \frac{t_i - \underline{t}_i}{\underline{t}_1^+ - \underline{t}_i} & \text{if } s_i = \underline{t}_1^+ \\ \frac{\underline{t}_1^+ - t_i}{\underline{t}_1^+ - \underline{t}_i} & \text{if } s_i = \underline{t}_i \\ 0 & \text{otherwise} \end{cases} \quad (\text{A.10})$$

Where \underline{t}_i is the greatest element of S less than or equal to t_i and \underline{t}_1^+ is the least element of S greater than \underline{t}_i . Then set

$$\lambda_t(s) = \prod_{i=1}^n \lambda_t^i(s_i).$$

Note that for any $t, t' \in \mathbb{R}_+$ with $t > t'$, $\lambda_t > \lambda_{t'}$ (i.e., λ_t strictly stochastically dominates $\lambda_{t'}$). That is, for all $s \in S^n$, $\lambda_{t'}\{s' \in S^n | s' \leq s\} \leq \lambda_t\{s' \in S^n | s' \leq s\}$, and strict inequality holds for some $s \in S^n$. We now define $M^*: \mathbb{R}_+^n \rightarrow \mathbb{R}^n$ by

$$M_i^*(t) = \int_{s \in S^n} \lambda_t(s) M_i(s) = \int M_i d\lambda_t. \quad (\text{A.11})$$

Clearly, the derived game is equivalent to the discrete research game based on the preceding observations. In the discrete research game, player i 's dominant strategy is to choose two adjacent elements s_i, s_i^+ and a probability weight on them, say $(1-p)$ on s_i and p on s_i^+ , while in the derived game, a choice of t_i means the choices of s_i and s_i^+ are \underline{t}_i and \underline{t}_1^+ and the choice of p is $\frac{t_i - \underline{t}_i}{\underline{t}_1^+ - \underline{t}_i}$.

Also notice that $t_i = (1-p)\underline{t}_i + p\underline{t}_1^+$ is i 's expected amount of research in the discrete research game.

It is easy to see that, for $t \in S^n$, $M_i^*(t) = M_i(t)$, and that M^* is a symmetric, continuous and concave game.

(a) To show there exists a unique symmetric Nash equilibrium, it is sufficient to verify condition (A.1) for M^* , in which case the Lemma applies. In the original game M , for any $t_1', t_1'' \in \mathbb{R}_+$ with $t_1'' > t_1'$, define

$$f_1(t_{-1}) = M_1(t_1'', t_{-1}) - M_1(t_1', t_{-1})$$

By (A.5), $f_1(t_{-1})$ is strictly monotone decreasing in t_{-1} . Thus, for any λ, λ' , if $\lambda > \lambda'$, then

$$\int f_1 d\lambda < \int f_1 d\lambda'.$$

By the stochastic dominance, if $t_{-1}, s_{-1} \in \mathbb{R}_+^{n-1}$ satisfy $t_{-1} > s_{-1}$, we have $\lambda_{t_{-1}} > \lambda_{s_{-1}}$. So for $t_1', t_1'' \in S$,

$$M_1^*(t_1'', t_{-1}) - M_1^*(t_1', t_{-1}) = \int f_1 d\lambda_{t_{-1}} \quad (\text{A.12})$$

$$< \int f_1 d\lambda_{s_{-1}} = M_1^*(t_1'', s_{-1}) - M_1^*(t_1', s_{-1})$$

For $t_1'' \in S$ and $t_1' \in \mathbb{R}^+$, set $p = \lambda_{t_1'}^i(\underline{t}_1^+)$. Then, using (A.12), we get

$$M_1^*(t_1'', t_{-1}) - M_1^*(t_1', t_{-1}) = (1-p) \left[M_1(t_1'', t_{-1}) - M_1(\underline{t}_i, t_{-1}) \right]$$

$$\begin{aligned}
& + p \left[M_1(t_1'', t_{-1}) - M_1(\underline{t}_1^+, t_{-1}) \right] \\
& < (1-p) \left[M_1(t_1'', s_{-1}) - M_1(\underline{t}_1, s_{-1}) \right] + p \left[M_1(t_1'', s_{-1}) - M_1(\underline{t}_1^+, s_{-1}) \right] \\
& = M^*(t_1'', s_{-1}) - M_1^*(t_1', s_{-1}) \quad (A.13)
\end{aligned}$$

In a similar fashion, extend the result (A.13) to the case when $t_1'' \in \mathbb{R}^+$. Thus we get that (A.12) holds for all $t_1', t_1'' \in \mathbb{R}^+$ with $t_1'' > t_1'$, and condition (A.1) is verified.

(b) Let t be the unique symmetric equilibrium in mixed strategies.

Case 1: Suppose $\underline{t} < t < \underline{t}^+$ (i.e., the equilibrium is a mixed strategy). Then $M_1^*(\underline{t}_1^+, t_{-1}) = M_1^*(\underline{t}_1, t_{-1})$. Define $D_1^*(t) = \int D_1 d\lambda_t$ and write

$$M_1^*(\underline{t}_1^+, t_{-1}) - M_1^*(\underline{t}_1, t_{-1}) = \frac{2}{b} \left[D_1^*(\underline{t}_1^+, t_{-1}) - D_1^*(\underline{t}_1, t_{-1}) \right] - c = 0.$$

Thus,

$$D_1^*(\underline{t}_1^+, t_{-1}) - D_1^*(\underline{t}_1, t_{-1}) = \frac{cb}{a}.$$

But $D_1^*(\underline{t}_1^+, t_{-1}) - D_1^*(\underline{t}_1, t_{-1})$ is continuous and strictly decreasing in t , so as c increases, the equilibrium t decreases.

Case 2: Suppose $t = \underline{t}$ (the equilibrium is a pure strategy). A necessary condition for the equilibrium to be a pure strategy is

$$\begin{aligned}
M_1(\underline{t}_1^+, \underline{t}_{-1}) - M_1(\underline{t}_1, \underline{t}_{-1}) &\leq 0 \leq M_1(\underline{t}_1, \underline{t}_{-1}) - M_1(\underline{t}_1^-, \underline{t}_{-1}), \text{ or} \\
D_1(\underline{t}_1^+, \underline{t}_{-1}) - D_1(\underline{t}_1, \underline{t}_{-1}) &\leq \frac{cb}{a} \leq D_1(\underline{t}_1, \underline{t}_{-1}) - D_1(\underline{t}_1^-, \underline{t}_{-1}).
\end{aligned} \quad (A.14)$$

where \underline{t}_1^- is the greatest element of S less than \underline{t}_1 .

As long as c remains in the interval given by (A.14), equilibrium $t = \underline{t}$ is unchanged. Moreover as c decreases to

$$\frac{cb}{a^2} \left[D_1(\underline{t}_1^+, \underline{t}_{-1}) - D_1(\underline{t}_1, \underline{t}_{-1}) \right]$$

the equilibrium t is the same as the equilibrium t which is approached by c increasing to

$$\frac{cb}{a^2} \left[D_1^*(\underline{t}_1^+, \underline{t}_{-1}) - D_1^*(\underline{t}_1, \underline{t}_{-1}) \right]$$

in case (1).

We conclude that the symmetric equilibrium t is a continuous, decreasing function of c . Since b and $1/a^2$ play the same role as c , the symmetric equilibrium t is a continuous, decreasing function of b , and a continuous, increasing function of a . Also, t is a non increasing function of R , since $D_1(t_1^+, \underline{t}_{-1}) - D_1(\underline{t}_1, \underline{t}_{-1})$ is a decreasing function of R .

(c) Let t be the unique symmetric equilibrium for M , and t^* the unique symmetric equilibrium for M^* . Let Φ and Φ^* be the best response correspondences for M and M^* , respectively. Note that Φ is a function. In light of the preliminary comments to the proof of Proposition 4, it suffices to show $\underline{t} \leq t^* \leq \bar{t}$. By monotonicity,

$\Phi(\underline{t}) \geq \Phi(t) = t \geq \underline{t}$, which implies $\Phi^*(\underline{t}) \geq \underline{t}$. Similarly $\Phi^*(\bar{t}) \leq \bar{t}$. But then, it follows that the symmetric equilibrium t^* must satisfy $\underline{t} \leq t^* \leq \bar{t}$. Otherwise, if $t^* < \underline{t}$, for example, we have $\Phi^*(\underline{t}) \geq \underline{t} > t^*$. This is contrary to the monotonicity of Φ^* .

Q.E.D

In proving Proposition 7, we use the following useful fact.

Useful Fact: Let $f(x) = \sum_{n=0}^N d_n x^n$ be the Nth degree polynomial where $d_n > 0$, $d_0 < 0$, and the sequence of coefficients only switch once in sign (i.e., $d_i \geq 0$ for $i \geq k$ and $d_i \leq 0$ for $i < k$ for some $0 < k < N$). Then f has only one positive root.

Proof. For any $x > 0$

$$\frac{df}{dx} = \sum_{n=1}^N d_n n x^{n-1} > k \sum_{n=1}^N d_n x^{n-1} = k \left[\frac{f(x)}{x} - \frac{d_0}{x} \right] > \frac{kf(x)}{x} \text{ since } d_0 < 0$$

Since $d_n > 0$ and $d_0 < 0$, there is at least one positive root. Denote the smallest positive root x^0 . Then $f'(x^0) > kf(x^0)/x^0 = 0$, so f is increasing and turns positive at x^0 . But since $f'(x) > kf(x)/x > 0$ for $x > 0$, once f turns positive it must always remain positive. Hence there is only one positive root.

Q.E.D.

Proof of Proposition 7:

By (A.6), (A.9) and (5.9), it is easy to see that for

$$\frac{a^2}{4bR^2} < c < \frac{a^2}{2bR^2}, T_n = 0, \text{ and } T_e > 0, \text{ and for } c \geq \frac{a^2}{2bR^2}, T_n = T_e = 0 \text{ for}$$

any $n \geq 1$. So it is sufficient to deal with the case $0 < c \leq \frac{a^2}{4bR^2}$ in

which the interior solutions are obtained.

Define

$$G^n(T) \equiv \frac{n^2 [(3n-1)T^2 + 2Rn(n+1)T + 4R^2n^2]}{(T+2Rn)((n+1)T+2Rn)^3}$$

$$G^e(t) \equiv \frac{1}{2(T+R)^2}$$

Note that $G^n(T) = \frac{cb}{a^2}$ is the condition for an interior equilibrium,

given by (A.6) and (A.9) where $T = nt$, total research. Also note that

$G^e(T) = \frac{cb}{a^2}$ is the condition (5.9) for the continuous approximation

of the efficient level of research. Both $G^n(T)$ and $G^e(T)$ are

continuous and strictly decreasing in T with $G^n(\infty) = G^e(\infty) = 0$, and

$G^n(0) = 1/4R^2$ and $G^e(0) = 1/2R^2$. So for an industry of size n and for

$0 \leq c \leq \frac{a^2}{4R^2b}$, we can write the equilibrium research as the T_n

satisfying $G^n(T_n) = \frac{cb}{a^2}$, and the efficient level as the T_e satisfying

$G^e(T_e) = \frac{cb}{a^2}$. Define $h^n(T) = A(T)(G^n(T) - G^e(T))$ where

$$A(T) = 2(T+R)^2(T+2Rn)((n+1)T+2Rn)^3$$

Note $A(T) > 0$ for all $T \geq 0$ and $n \geq 1$, so the roots of $h^n(T)$ are the crossing points of $G^n(T)$ and $G^e(T)$.

It is easy to verify that

$$h^n(T) = d_4 T^4 + d_3 T^3 + d_2 T^2 + d_1 T + d_0$$

where

$$d_4 = 5n^3 - 5n^2 - 3n - 1$$

$$d_3 = 2Rn(n^3 + 2n - 11n - 4)$$

$$d_2 = 2R^2 n^2 (n + 1)(2n - 13)$$

$$d_1 = 4R^3 n^3 (n + 7)$$

$$d_0 = -8R^4 n^4$$

For

$$n = 1: d_4, d_3, d_2, d_1, d_0 < 0$$

$$n = 2: d_4 > 0; d_3, d_2, d_1, d_0 < 0$$

$$3 \leq n \leq 6: d_4, d_3 > 0; d_2, d_1, d_0 < 0$$

$$n \geq 7: d_4, d_3, d_2 > 0; d_1, d_0 < 0$$

From the useful fact we conclude that for each $n \geq 2$, $h^n(T)$ has only one positive root and $G^n(T)$ and $G^e(T)$ cross just once.

Denote the positive root of $h^n(T)$ as T_n^* and define

$$k_n = \frac{a^2}{b} G^e(T_n^*) = \frac{a^2}{b} G^n(T_n^*)$$

Since $G^n(0) = 1/4R^2$ and $G^e(0) = 1/2R^2$ the result follows for

$n \geq 2$.

When $n = 1$ (the monopoly case) it can be verified that

$d_4, d_3, d_2, d_1, d_0 < 0$, implying no positive root to $h^1(T)$. Since $G^1(0) = 1/4R^2$, the equilibrium research for a monopoly is always less than the efficiency level.

Q.E.D.

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