DIVISION OF THE HUMANITIES AND SOCIAL SCIENCES

CALIFORNIA INSTITUTE OF TECHNOLOGY

PASADENA, CALIFORNIA 91125

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Quang H. Vuong
California Institute of Technology

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ABSTRACT

We consider Wald tests based on consistent estimators of g-inverses of the asymptotic covariance matrix $\sum$ of a statistic that is $n^{1/2}$-asymptotically normal distributed under the null hypothesis. Under the null hypothesis and under any sequence of local alternatives in the column space of $\sum$, these tests are asymptotically equivalent for any choice of g-inverses. For sequences of local alternatives not in the column space of $\sum$ and for a suitable choice of g-inverse, the asymptotic power of the corresponding Wald test can be made equal to zero or arbitrarily large. In particular, the test based on a consistent estimator of the Moore-Penrose inverse of $\sum$ has zero asymptotic power against sequences of local alternatives in the orthogonal complement to the column space of $\sum$.

KEY WORDS: Generalized Inverses, Wald Tests, Asymptotic Power

GENERALIZED INVERSES AND ASYMPTOTIC PROPERTIES OF WALD TESTS

Quang H. Vuong*
California Institute of Technology

1. INTRODUCTION

Wald Tests based on statistics having singular asymptotic normal distributions are more and more frequent in econometrics (see Hausman and Taylor (1981), Holly (1982), Szroeter (1983), Vuong (1983), Andrews (1985a, b), Gourieroux and Monfort (1985), Holly and Monfort (1986) among others). Specifically, if τ_n is a p-dimensional statistic that is $n^{1/2}$-asymptotically normally distributed $N_p(0,\sum)$ with some unknown covariance matrix $\sum$, one considers the quadratic form $n\tau_n'Q_n\tau_n$ where Q_n is a suitably chosen p X p matrix. As noted by Moore (1977), when $\sum$ is singular, the matrix Q_n can be obtained in two different ways.

A first method consists in defining Q_n as a generalized (g-) inverse of $\sum_n$ where $\sum_n$ is a consistent estimator of $\sum$. A major but often overlooked difficulty with this method arises from the fact that the sequence $n\tau_n'Q_n\tau_n$ is not well-defined without further qualifications on Q_n.[1] Indeed, for any given n, there exists many g-inverses of $\sum_n$ so that $n\tau_n'Q_n\tau_n$ is not necessarily invariant with respect to the choice of Q_n (see Rao and Mitra (1971)). A frequent choice for Q_n is the Moore-Penrose inverse $(\sum_n)^+$ of $\sum_n$. However, as noted by Stewart (1969) and Andrews (1985c), the Moore-Penrose inverse of a matrix is not continuous in the elements of the matrix. As a

matter of fact, these authors showed that $(\sum_n)^+ \to \sum^+$ if and only if rank $\sum_n$ = rank $\sum$ with probability one. Thus, the consistent estimator $\sum_n$ of $\sum$ must satisfy this additional condition in order to ensure that the statistic $n\tau_n'(\sum_n)^+\tau_n$ converge to the usual central chi-square distribution with r degrees of freedom, $\chi^2(r,0)$, where r = rank $\sum$. When Q_n is any other g-inverse of $\sum_n$, one imposes on $\sum_n$ a second additional requirement, namely that τ_n belongs with probability one to the space $M(\sum_n)$ generated by the columns of $\sum_n$ (see Moore (1977), Andrews (1985c)). This ensures that the sequence of statistics $n\tau_n'Q_n\tau_n$ is, with probability one, invariant with respect to the choice of g-inverses and hence well-defined.

A second and less frequent method consists in defining Q_n as a consistent estimator of a g-inverse of $\sum$, i.e., $Q_n = (\sum^-)_n$ (see Moore (1977), Szroeter (1983)). This second method does not have the problems associated with the first one since the statistic $n\tau_n'Q_n\tau_n$ converges to the usual chi-square distribution with r degrees of freedom for any choice of g-inverses (see Moore (1977)). One purpose of this note is to show that the Wald statistics constructed in this manner are all asymptotically equivalent for any choice of g-inverses. In addition, we show that the associated Wald tests have identical Pitman asymptotic power for any sequence of local alternatives in the column space $M(\sum)$. However, when the sequence of local alternatives is <u>not</u> in $M(\sum)$, the asymptotic power of these tests differs, and it can be made arbitrarily large or small by a suitable choice of g-inverse. In particular, we show that, if $\sum^+$ is the Moore-Penrose

inverse of $\sum$, then the corresponding Wald test based on $n\tau_n'(\sum^+)_n\tau_n$ does not have any power against any sequence of local alternatives in the orthogonal complement of $M(\sum)$.

Finally, our results are applied to the case where one considers equivalent versions of the null hypothesis. We show that Wald tests based on equivalent versions are asymptotically equivalent for any choice of g-inverses, and that they have the same asymptotic power for any sequence of local alternatives in $M(\sum)$. This latter property no longer holds if the sequence of local alternatives is not in $M(\sum)$.

The note is organized as follows. Section 2 establishes the asymptotic equivalence of the Wald tests. Section 3 studies their asymptotic local power. Section 4 considers Wald tests based on equivalent versions of the null hypothesis.

2. ASYMPTOTIC EQUIVALENCE

Let θ_o be the parameter vector of interest which is supposed to belong to Θ, an open subset of $\mathbb{R}^p$. Let a be a given vector in Θ. Throughout, we shall be interested in testing.

$$H_o: \theta_o = a \quad \text{vs.} \quad H_A: \theta_o \in \Theta - \{a\}.$$

To do so, we consider an estimator $\hat{\theta}_n$ of θ_o that satisfies the following assumption.

Assumption A1: Under θ_o, $n^{1/2}(\hat{\theta}_n - \theta_o) \xrightarrow{D} N_p(0,\sum)$ where $\sum$ is unknown and of rank $r \leq p$.

As mentioned in Section 1, we shall consider Wald statistics based on consistent estimators of generalized inverses of $\sum$. Thus, let Q be a g-inverse of $\sum$, i.e., $\sum Q \sum = \sum$, and let $\{Q_n\}$ be a sequence of $p \times p$ matrices satisfying:

Assumption A2: Q_n converges in probability to Q, i.e., $Q_n = Q + o_p(1)$.[2]

The Wald statistic based on the g-inverse Q is defined by:

$$W_n^Q = n(\hat{\theta}_n - a)'Q_n(\hat{\theta}_n - a). \tag{2.1}$$

It is clear that different choices of g-inverses Q lead to different Wald statistics W_n^Q. In particular, W_n^Q is not in general invariant with respect to Q. The next result gives the asymptotic properties of the Wald statistics W_n^Q under the null hypothesis H_o. Part (i) is well-known and gives the null asymptotic distribution of any W_n^Q (see, e.g., Moore (1977)). Part (ii) establishes the frequently quoted asymptotic equivalence of all these Wald statistics.

Theorem 1: Given A1 - A2, under H_o:

(i) $W_n^Q \xrightarrow{D} \chi^2(r,0)$ for any g-inverse Q of $\sum$.

(ii) $W_n^{Q_1} - W_n^{Q_2} = o_p(1)$ for any g-inverses Q_1 and Q_2 of $\sum$.

Proof of Theorem 1: From A1 - A2, we have under H_o:

$$W_n^Q = n(\hat{\theta}_n - \theta_o)'Q(\hat{\theta}_n - \theta_o) + o_p(1) \qquad (2.2)$$

for any g-inverse Q. Part (i) follows from A1 and Rao and Mitra (1971, Theorem 9.2.2). To prove Part (ii), it is necessary and sufficient to show that

$$n(\hat{\theta}_n - \theta_o)'(Q_1 - Q_2)(\hat{\theta}_n - \theta_o) = o_p(1). \qquad (2.3)$$

We have $\sum(Q_1 - Q_2)\sum = 0$ since $\sum Q_i \sum = \sum$ by definition of a g-inverse of $\sum$. Hence $\sum(Q_1 - Q_2)\sum(Q_1 - Q_2)\sum = \sum(Q_1 - Q_2)\sum$. Thus, from A1 and Rao and Mitra (1971, Theorem 9.2.1), it follows that

$$n(\hat{\theta}_n - \theta_o)'(Q_1 - Q_2)(\hat{\theta}_n - \theta_o) \xrightarrow{D} \chi^2(k,0) \qquad (2.4)$$

where $k = tr[(Q_1 - Q_2)\sum] = tr(Q_1\sum) - tr(Q_2\sum)$. But $k = 0$ since $tr(Q\sum) = rank(Q\sum) = rank(\sum)$ for any g-inverse Q of $\sum$ (see Rao and Mitra (1971, Definition 3, p.21)). Thus, the left-hand side of (2.4) converges in distribution to the degenerated distribution with mass point at 0. Since convergence in distribution to a constant implies convergence in probability to that constant, (2.3) follows.

Q.E.D.

3. ASYMPTOTIC POWER

In the previous sections, we have established the asymptotic equivalence of all tests based on Wald statistics of the form (2.1) where Q is any g-inverse of $\sum$. It is important to know if the choice of a g-inverse of $\sum$ matters for the local asymptotic power of these

Wald tests, i.e., if these Walds tests have identical asymptotic power for any given sequence of local alternatives. If not, it would be interesting to know against which sequence of local alternatives, a given choice of g-inverse leads to a test with least or most local asymptotic power. Alternatively, it would be interesting to know if there exists a choice of a g-inverse that leads to a test which is uniformly most locally powerful.

Since we consider Pitman (1979) definition of local asymptotic power, we make the following assumption on the statistic $\hat{\theta}_n$.

Assumption A3: Under any θ_o in a neighborhood of a, $n^{1/2}(\hat{\theta}_n - \theta_o) \xrightarrow{D} N_p(0, \sum(\theta_o))$ uniformly in θ_o where $\sum(\theta_o)$ is a continuous function of θ_o such that $\sum(a) = \sum$.

Note that A3 is stronger than A1. Let $\{\theta_n\}$ be a sequence of local alternatives around a of the form $\theta_n = a + n^{-1/2}b$ with $b \in \mathbb{R}^p - \{0\}$. It is well-known that, under A3, we have:

$$n^{1/2}(\hat{\theta}_n - \theta_n) \xrightarrow{D} N_p(0, \sum) \qquad (3.1)$$

under the sequence of local alternatives $\{\theta_n\}$ so that:

$$n^{1/2}(\hat{\theta}_n - a) \xrightarrow{D} N_p(b, \sum) \qquad (3.2)$$

(see, e.g., Pitman (1979)).[3]

The next result gives the asymptotic distribution of the Wald statistic (2.1) for any sequence of local alternatives such that b belongs to $M(\sum)$ the column space of $\sum$. This result is known (see,

e.g., Moore (1977)) and its proof is given for completeness.

Theorem 2: Given A2 - A3, under the sequence of local alternatives $\theta_n = a + n^{-1/2}b$ where $b \in M(\sum)$, $W_n^Q \xrightarrow{D} \chi^2(r, b'Qb)$ for any choice of g-inverse Q of $\sum$. Moreover the non-centrality parameter $b'Qb$ is independent of Q.

Proof of Theorem 2: From (2.1), (3.2), and A2 we have:

$$W_n^Q = n(\hat{\theta}_n - a)'Q(\hat{\theta}_n - a) + o_p(1). \qquad (3.3)$$

Since Q is a g-inverse of $\sum$ and since $b \in M(\sum)$, the first part follows from (3.2) and Rao and Mitra (1971, Theorem 9.2.3). The second part follows from Rao and Mitra (1971, Lemma 2.2.4 - (iii)).

Q.E.D.

Theorem 2 implies that the local asymptotic power, as measured by $b'Qb$, of tests based on the Wald statistics W_n^Q is independent of the choice of g-inverses Q of $\sum$ if the sequence of local alternatives belongs to $M(\sum)$.[4]

Let us now consider sequences of local alternatives not in $M(\sum)$, i.e., for which b does not belong to $M(\sum)$. In this latter case, however, the statistic W_n^Q is not necessarily asymptotically chi-squared distributed as shown in the next lemma.

Lemma 1: Given A2 - A3, let Q be a g-inverse of $\sum$, then W_n^Q is asymptotically chi-square distributed under the sequence of local alternatives $\theta_n = a + n^{1/2}b$ if and only if:

$$b'Q'\sum Qb = b'Qb. \qquad (3.4)$$

Proof of Lemma 1: This immediately follows from (2.1), (3.2), Q being a g-inverse of $\sum$, and Rao and Mitra (1971, Theorem 9.2.1).

Q.E.D.

Thus, there may exist g-inverses Q of $\sum$ for which W_n^Q is not asymptotically chi-square distributed under some sequence of local alternatives. If, however, we consider g-inverses for which W_n^Q is asymptotically chi-square distributed for any direction b, then we have the following result of which the "if" part is known (see Rao and Mitra (1971, Theorem 9.2.3)).

Lemma 2: Given A2 - A3, let Q be a g-inverse of $\sum$, then W_n^Q is asymptotically chi-square distributed under any sequence of local alternatives of the form $\theta_n = a + n^{1/2}b$, $b \in \mathbb{R}^p$, if and only if Q is a symmetric reflexive g-inverse of $\sum$, in which case $W_n^Q \xrightarrow{D} \chi^2(r, b'Qb)$.

Proof of Lemma 2: From Lemma 1, Equation (3.4) must hold for any b. Thus $Q'\sum Q = Q$ which implies that $Q = Q'$ and $Q\sum Q = Q$. Hence Q is a symmetric reflexive g-inverse of $\sum$. The converse follows from Rao and Mitra (1971, Theorem 9.2.3). Finally, the number of degrees of freedom and the non-centrality parameters follow from Rao and Mitra (1971, Theorem 9.2.1).

Q.E.D.

In view of Lemma 2, we now restrict our attention to symmetric

reflexive g-inverses of $\sum$. It is useful to characterize these g-inverses. Since $\sum$ is a p.s.d. matrix, there exists a matrix P such that $PP' = P'P = I_p$ and

$$P'\sum P = D \tag{3.5}$$

where

$$D = \begin{bmatrix} D_r & 0 \\ 0 & 0 \end{bmatrix} \tag{3.6}$$

and D_r is a diagonal matrix of dimension r = rank $\sum$ of which all the diagonal elements are strictly positive.

Lemma 3: A matrix Q is a symmetric reflexive g-inverse of $\sum$ if and only if it is of the form

$$Q = P\begin{bmatrix} D_r^{-1} & B \\ B' & B'D_rB \end{bmatrix}P' \tag{3.7}$$

for some $r \times (p - r)$ matrix B.

Proof of Lemma 3: Using $\sum = PDP'$, it can readily be shown that Q is a symmetric reflexive g-inverse of $\sum$ if and only if it is of the form

$$Q = PHP' \tag{3.8}$$

where H is a symmetric reflexive g-inverse of D. It now suffices to show that any symmetric reflexive g-inverse of D is of the form of the second matrix in the right-hand side of (3.7).

Let H be a symmetric reflexive g-inverse of D. By symmetry, we must have

$$H = \begin{bmatrix} A & B \\ B' & C \end{bmatrix} \tag{3.9}$$

But, since DHD = D, it follows from (3.6) and (3.9), that we must have $D_rAD_r = D_r$, i.e., $A = D_r^{-1}$. Finally, since HDH = H, it follows from (3.6), (3.9), and $A = D_r^{-1}$ that $C = B'D_rB$. Hence any symmetric reflexive g- inverse of D is of the form (3.9) with $A = D_r^{-1}$ and $C = B'D_rB$. The converse is easily established.

Q.E.D.

From Lemma 2, we know that under any sequence of local alternatives, the asymptotic power of a Wald test based on W_n^Q where Q is a symmetric reflexive g-inverse of $\sum$ can be measured by the non-centrality parameter $b'Qb$. The non-centrality parameter $b'Qb$ is not, however, invariant with respect to Q when b does not belong to $M(\sum)$. The next result characterizes the asymptotic power of the tests based on W_n^Q when Q is any symmetric reflexive g-inverse of $\sum$ under sequences of local alternatives not in $M(\sum)$.

Theorem 3: Given A2 - A3,

(i) Under any given sequence of local alternatives $\theta_n = a + n^{1/2}b$ such that $b \notin M(\sum)$, and for any $c \in [0,\infty)$, there exists a symmetric reflexive g-inverse Q of $\sum$ such that the test based on W_n^Q has asymptotic power equal to c.

(ii) For any symmetric reflexive g-inverse Q of $\sum$, the test based on W_n^Q has zero asymptotic power under the sequence of local alternatives $\theta_n = a + n^{1/2}b$ if and only if b belongs to a p - r dimensional complement of $M(\sum)$.

(iii) The test based on W_n^Q where Q is the Moore-Penrose inverse of $\sum$ has zero asymptotic power under the sequence of local alternatives $\theta_n = a + n^{1/2}b$ if and only if b belongs to the orthogonal complement of $M(\sum)$.

Proof of Theorem 3: (i) Partition P into P = [M,N] where M is made of the first r column vectors of P. Since $\sum = PDP'$, then from (3.6) it follows that $\sum = MD_rM'$ so that the column space of $\sum$ is equal to the column space of M. Let $[\alpha',\beta']'$ be the vector of coordinates of b in the orthonormal basis P, i.e., $b = M\alpha + N\beta$. Then, using Lemma 3, it is easy to show that the non-centrality parameter satisfies:

$$b'Qb = [D_r^{-1}\alpha + B\beta]'D_r[D_r^{-1}\alpha + B\beta]. \tag{3.10}$$

Let $\tilde{c}$ be a r-dimensional vector such that $\tilde{c}'D_r\tilde{c} = c$. For instance $\tilde{c}$ can be the vector of which the i-th component is $(c/d_i)^{1/2}$ where d_i is the (i,i) element of D_r. Then $b'Qb = c$ if and only if

$$B\beta = \tilde{c} - D_r^{-1}\alpha. \tag{3.11}$$

Since $b \notin M(\sum)$, then the euclidean norm of β, denoted $\| \beta \|$, is not zero. Let $\beta^- = \beta'/\| \beta \|^2$. Then, it is clear that the necessary and sufficient condition of Theorem 2.3.2 in Rao and Mitra (1971) is

satisfied so that (3.11) has a solution in B. As a matter of fact all the solutions are of the form:

$$B = (\tilde{c} - D_r^{-1}\alpha)\beta'/\| \beta \|^2 + Z - Z\beta\beta'/\| \beta \|^2 \tag{3.12}$$

for some r X (p - r) matrix Z.

(ii) $b'Qb = 0$ if and only if b belongs to the kernel of Q. Since Q is a reflexive g-inverse of $\sum$, it follows from Rao and Mitra (1971, Lemma 2.5.1) that rank Q = rank $\sum$ = r so that the kernel of Q has dimension p - r. Finally, $\text{Ker}(Q) \cap M(\sum) = \{0\}$. Indeed, if $b \in M(\sum)$, then β is zero in the decomposition $b = M\alpha + N\beta$ which implies from (3.10) that $b'Qb = \alpha'D_r^{-1}\alpha$. Hence, if $b \in \text{Ker}(Q)$, then $Qb = 0$ so that α must also be equal to zero.

(iii) It is easy to show that the Moore-Penrose inverse of $\sum$ is given by (3.7) where B is the zero matrix. Then, using the partition P = [M,N], we have $\sum^+ = MD_r^{-1}M'$. Hence $b'\sum^+b = 0$ if and only if $M'b = 0$, i.e., if and only if b is in the orthogonal complement of $M(\sum)$ since the column space of M is equal to $M(\sum)$.

Q.E.D.

Part (i) of Theorem 3 implies that, for any given sequence of local alternatives not in $M(\sum)$, one can attain any level of asymptotic power for the test based on W_n^Q by appropriately choosing the g-inverse Q of $\sum$ (see Equation (3.12)). It also implies that there does not exist a g-inverse for which the corresponding Wald test is uniformly and asymptotically most powerful under all sequences of

local alternatives. Part (ii) says that there always exists a p - r dimensional space [with zero intersection with $M(\sum)$] for which a test based on W_n^Q has zero asymptotic local power. In particular, this p - r dimensional space is the orthogonal complement to $M(\sum)$ for the test based on W_n^Q when Q is the Moore-Penrose inverse of $\sum$.

Thus, contrary to Theorems 2 and 3 which establish the asymptotic equivalence of the tests based on the Wald statistics W_n^Q under the null hypothesis and under any sequence of local alternatives in $M(\sum)$, Theorem 3 shows that these tests no longer have identical asymptotic power under sequences of local alternatives not in $M(\sum)$.

4. GENERALIZATION

In the non-singular case, it is known that Wald tests of equivalent versions of the null hypothesis are asymptotically equivalent under the null hypothesis and have identical asymptotic power under any sequence of local alternatives. The purpose of this section is to investigate if these asymptotic properties still hold in the singular case.

Let $g(\cdot)$ be a vector function from $\mathbb{R}^p$ to $\mathbb{R}^q$ such that

$$g(\theta) = g(a) \quad \text{if and only if } \theta = a \tag{4.1}$$

for a and θ belonging to $\Theta \subset \mathbb{R}^p$. Then, to test H_o: $\theta_o = a$ against H_A: $\theta_o \in \Theta - \{a\}$, one can equivalently test

$$H_o^g\text{: } g(\theta_o) = g(a) \quad \text{vs.} \quad H_A^g\text{: } g(\theta_o) \neq g(a) \tag{4.2}$$

The hypotheses H_o^g and H_A^g are said to be <u>equivalent</u> to the hypotheses H_o and H_A. We assume that $g(\cdot)$ satisfies the following condition.

<u>Assumption A4</u>: $g(\cdot)$ is continuously differentiable on $\Theta \subset \mathbb{R}^p$, and a is a regular point of the $q \times p$ matrix $\partial g(\theta)/\partial\theta'$.[5]

It is easy to show that (4.1) and A4 imply that rank $[\partial g(a)/\partial\theta'] = p$. Thus $q \geq p$. In addition, A1 implies that

$$n^{1/2}[g(\hat{\theta}_n) - g(\theta_o)] \xrightarrow{D} N_q\left(0, \frac{\partial g(\theta_o)}{\partial\theta'} \sum \frac{\partial g'(\theta_o)}{\partial\theta}\right), \tag{4.3}$$

under H_o while A3 implies that under θ in a neighborhood of a:

$$n^{1/2}[g(\hat{\theta}_n) - g(\theta)] \xrightarrow{D} N_q\left(0, \frac{\partial g(\theta)}{\partial\theta'} \sum(\theta) \frac{\partial g'(\theta)}{\partial\theta}\right) \tag{4.4}$$

uniformly in θ.

The $q \times q$ matrix $\Omega \equiv [\partial g(a)/\partial\theta'] \sum [\partial g'(a)/\partial\theta)$ has rank $r \leq p \leq q$. Thus, using the same method as in Section 2, we define Wald statistics for testing H_o^g against H_A^g as:

$$\tilde{W}_n^G = n[g(\hat{\theta}_n) - g(a)]' G_n [g(\hat{\theta}_n) - g(a)] \tag{4.5}$$

where $\{G_n\}$ is a sequence of $q \times q$ random matrices satisfying:

<u>Assumption A5</u>: $G_n = G + o_p(1)$ where G is a g-inverse of Ω.[6]

We have the following property:

<u>Theorem 4</u>: Suppose that A2, A4, and A5 hold.

(i) Given A1, under H_o, $W_n^Q - \tilde{W}_n^G = o_p(1)$ for any choice of g-inverses Q and G of $\sum$ and Ω.

(ii) Given A3, under any sequence of local alternatives $\theta_n = a + n^{1/2}b$ such that $b \in M(\sum)$, both W_n^Q and $\tilde{W}_n^G$ converge in distribution to the non-central chi-square $\chi^2(r,\delta)$ where

$$\delta = b'Qb = b'\frac{\partial g'(a)}{\partial \theta} G \frac{\partial g(a)}{\partial \theta'} b, \tag{4.6}$$

and δ is independent of the g-inverses Q and G.

Proof of Theorem 4: (i) In view of Theorem 1, it suffices to show that $W_n^Q - \tilde{W}_n^G = o_p(1)$ for a particular choice of g-inverses Q and G of $\sum$ and Ω. From (4.4), (4.5), and A5, we have:

$$\tilde{W}_n^G = n[g(\hat{\theta}_n) - g(a)]' G [g(\hat{\theta}_n) - g(a)] + o_p(1)$$

$$= n(\hat{\theta}_n - a)' \frac{\partial g'(a)}{\partial \theta} G \frac{\partial g(a)}{\partial \theta'} (\hat{\theta}_n - a) + o_p(1) \tag{4.7}$$

where we have used a Taylor expansion of $g(\cdot)$ around a. But G is a g-inverse of $\Omega \equiv [\partial g(a)/\partial \theta'] \sum [\partial g'(a)/\partial \theta]$, and rank Ω = rank $\sum$. Thus, if we let $Q = [\partial g'(a)/\partial \theta] G [\partial g(a)/\partial \theta']$, then Q is a g-inverse of $\sum$ from Rao and Mitra (1971, Lemma 2.2.5- c). Part (i) follows from (2.2) and (4.7).

(ii) Given (4.4), it follows that under $\{\theta_n\}$:

$$n^{1/2}[g(\hat{\theta}_n) - g(\theta_n)] \xrightarrow{D} N_q(0, \frac{\partial g(a)}{\partial \theta'} \sum \frac{\partial g'(a)}{\partial \theta})$$

so that, using a Taylor expansion of $g(\theta_n)$ around a:

$$n^{1/2}[g(\hat{\theta}_n) - g(a)] \xrightarrow{D} N_q(\frac{\partial g(a)}{\partial \theta'} b, \frac{\partial g(a)}{\partial \theta'} \sum \frac{\partial g'(a)}{\partial \theta}).^7 \tag{4.8}$$

Then, given A5, it follows from (4.5) that under this sequence of local alternatives:

$$\tilde{W}_n^G = n[g(\hat{\theta}_n) - g(a)]' G [g(\hat{\theta}_n) - g(a)] + o_p(1), \tag{4.9}$$

Since G is a g-inverse of Ω, this implies, using (4.8) and Rao and Mitra (1971, Theorem 9.2.3), that $\tilde{W}_n^G$ converges in distribution to a $\chi^2(r,\delta)$ with

$$\delta = b' \left[\frac{\partial g'(a)}{\partial \theta} G \frac{\partial g(a)}{\partial \theta'} \right] b \tag{4.10}$$

But from Lemma 2.2.4 in Rao and Mitra (1971), the non-centrality parameter δ is invariant with respect to G since $b \in M(\sum)$ and since the matrix in brackets in (4.10) is a g-inverse of $\sum$ as shown in Part (i). The other equality in (4.6) follows from Theorem 2.

Q.E.D.

Contrary to Theorem 4, the Wald tests based on W_n^Q and $\tilde{W}_n^G$ no longer have identical asymptotic power under sequences of local alternatives $\theta_n = a + n^{1/2}b$ when $b \notin M(\sum)$. As in Section 3, we restrict our attention to symmetric reflexive g-inverses of $\sum$ and Ω. We have

Theorem 5: Suppose that A2 - A5 hold. Let Q and G be symmetric

reflexive g-inverses of $\sum$ and Q respectively. The Wald tests based on W_n^Q and $\tilde{W}_n^G$ have identical asymptotic power under any sequence of local alternatives $\theta_n = a + n^{1/2}b$, $b \in \mathbb{R}^p$, if and only if

$$Q = \frac{\partial g'(a)}{\partial \theta} G \frac{\partial g(a)}{\partial \theta'}, \tag{4.11}$$

where the right-hand side is a symmetric reflexive g-inverse of $\sum$.

Proof of Theorem 5: Since G is a symmetric reflexive g-inverse of Ω, it follows from (4.8), (4.9), and Rao and Mitra (1971, Theorem 9.2.3) that $\tilde{W}_n^G$ converges to a $\chi^2(r,\delta)$ with δ as given in (4.10). Then using Lemma 2, it follows that W_n^Q and $\tilde{W}_n^G$ have identical asymptotic power under any sequence of local alternatives $\theta_n = a + n^{1/2}b$ if and only if (4.11) holds. Moreover, let R be the matrix in the right-hand side of (4.11). Then, R is a symmetric reflexive g-inverse of $\sum$ since (i) $R = R'$, (ii) R is a g-inverse of $\sum$ (see proof of Theorem 4), and (iii)

$$R \sum R = \frac{\partial g'(a)}{\partial \theta} G \Omega G \frac{\partial g(a)}{\partial \theta'}$$
$$= R.$$

Q.E.D.

Equation (4.11) shows how the g-inverses Q and G must be related so that the Wald tests based on W_n^Q and $\tilde{W}_n^G$ have identical asymptotic power against all sequences of local alternatives. Theorem 5, however, implies that if Q and G do not satisfy (4.11), then there exist sequences of local alternatives such that the corresponding Wald

tests W_n^Q and $\tilde{W}_n^G$ do not have the same asymptotic power. In view of Theorem 4, these sequences of local alternatives are not in the column space of $\sum$.

FOOTNOTES

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1. Note that Q_n may not even have a limit. As a simple example, choose for any even n a g-inverse of minimum rank, i.e., of rank equal to that of $\sum_n$, and for any odd n a g-inverse of maximum rank, i.e., of rank equal to the dimension of τ_n.

2. When studying the null asymptotic properties of the Wald statistics, Assumption A2 need only hold under H_o. However, when studying local asymptotic power as in Sections 3 and 4, Assumption A2 should be understood as Q_n converges in probability to Q under the sequence of local alternatives $\theta_n = a + n^{1/2}b$, $b \neq 0$, i.e., $\forall \varepsilon > o$, $\lim_{n\to\infty} \Pr[\| Q_n - Q \| < \varepsilon | \theta_n] = 1$.

3. (3.1) and (3.2) means that for any $z \in \mathbb{R}^p$, $\lim_{n\to\infty} \Pr[\hat{\theta}_n - \theta_n \leq z | \theta_n] = \Phi_p(z;0,\sum)$ and $\lim_{n\to\infty} \Pr[\hat{\theta}_n - a \leq z | \theta_n] = \Phi_p(z;b,\sum)$.

4. To be rigorous, we should say for sequences of local alternatives in the affine space $a + M(\sum)$.

5. a is a regular point of $\partial g(\theta)/\partial \theta'$ if and only if the rank of $\partial g(\theta)/\partial \theta'$ is constant over a neighborhood of a.

6. See footnote 2.

7. See footnote 3.

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