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Higher Derivative Chern–Simons Extensions

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We study the higher-derivative extensions of the D=3 Abelian Chern–Simons topological invariant that would appear in a perturbative effective action's momentum expansion. The leading, third-derivative, extension I_{ECS} turns out to be unique. It remains parity-odd but depends only on the field strength, hence no longer carries large gauge information, nor is it topological because metric dependence accompanies the additional covariant derivatives, whose positions are seen to be fixed by gauge invariance. Viewed as an independent action, I_{ECS} requires the field strength to obey the wave equation. The more interesting model, adjoining I_{ECS} to the Maxwell action, describes a pair of excitations. One is massless, the other a massive ghost, as we exhibit both via the propagator and by performing the Hamiltonian decomposition. We also present this model's total stress tensor and energy. Other actions involving I_{ECS} are also noted.

1. Introduction

The remarkable properties of the Chern–Simons (CS) topological invariant in D=3 gauge theories [1, 2] are by now well-appreciated. For Abelian vector fields, $I_{CS} = m/2 \int d^3x \, \epsilon^{\alpha\beta\gamma} A_\alpha \partial_\beta A_\gamma$ is parity violating, of first derivative order, metric-independent, and gauge invariant. Viewed as an action, I_{CS} leads to the locally "flat" field equation $F_{\alpha\beta} \equiv \partial_\alpha A_\beta - \partial_\beta A_\alpha = 0$. Instead, when the Maxwell action I_{MAX} is adjoined, the resulting topologically massive electrodynamics (TME) describes a helicity ± 1 (depending on the sign of the mass m) mode [2]. The gravitational CS analog is of third derivative order ($I_{CS} \sim m^{-1} \int \Gamma R$), and its Euler–Lagrange equation $C^{\mu\nu} \equiv \epsilon^{\mu\alpha\beta} D_\alpha (R^\nu_\beta - \frac{1}{4}\delta^\nu_\beta R) = 0$ states that space is locally conformally flat: the Cotton tensor $C^{\mu\nu}$ is the 3-space conformal tensor. Adjoining the Einstein action leads to a dynamical system, topologically massive gravity (TMG); despite its higher order, the linearized limit of TMG describes a massive helicity ± 2 excitation [2].

The first derivative I_{CS} appears naturally in a perturbative effective action expansion of QED_3, because the electron's mass term is also P-violating in D=3, and higher derivative extensions of it (which we denote by I_{ECS}) should appear as well [3] in a (∂/m) power series,

$$I_{EFF}[A_\mu] = I_{CS} + I_{MAX} + I_{ECS} + \mathcal{O}(m^{-2}) \; .$$

It is therefore a natural question, both in connection with this expansion and for comparison with the third derivative TMG, to consider such I_{ECS}. Additional derivative powers must be even (if the parity violating $\epsilon^{\alpha\beta\gamma}$ is retained), the lowest extensions being the most illuminating. In actuality, there is only one such extension, as exhibited by the equalities

$$\begin{aligned} I_{ECS} &= (2m)^{-1}\int d^3x\, \epsilon^{\alpha\beta\gamma}\Box A_\alpha \partial_\beta A_\gamma \;=\; -(2m)^{-1}\int d^3x \epsilon^{\alpha\beta\gamma}\partial_\lambda A_\alpha \partial^\lambda \partial_\beta A_\gamma \\ &= -(2m)^{-1}\int d^3x\, \epsilon^{\alpha\beta\gamma} f_\alpha \partial_\beta f_\gamma \;, \quad f^\alpha \equiv \tfrac{1}{2}\,\epsilon^{\alpha\mu\nu}F_{\mu\nu}\;. \end{aligned} \tag{1}$$

Each term follows from its predecessor by an obvious integration by parts. Hence, unlike the original I_{CS}, I_{ECS} depends locally on the field strength and *not* on the potential, and so carries no "large gauge" information. Our signature conventions are (+,–,–), $\epsilon^{012} = +1 = \epsilon_{012}$.

We shall firstly exhibit the excitations described by actions containing I_{ECS}, including especially its sum with the Maxwell action (ETME). There, we find a massless particle plus a massive ghost. We then consider I_{ECS} in a gravitational background, where the higher derivatives necessarily engender metric dependence, in contrast to the topological character of I_{CS}, and give rise to a stress tensor that contributes explicitly to the energy of ETME, as we shall display in terms of the two degrees of freedom.

2. Maxwell–ECS Dynamics

We shall work in source-free flat space in this section, our aim being to characterize the excitations described by actions that include I_{ECS}. It is clear from (1) that, taken alone, I_{ECS} yields the unconstrained massless propagation of the field strength,

$$\Box f^\mu = 0\;. \tag{2}$$

Next, define the extended system by adjoining to I_{ECS} the Maxwell action

$$I_{ETME} = -\tfrac{1}{2}\int d^3x[f_\mu^2 + m^{-1}\epsilon^{\alpha\beta\gamma} f_\alpha \partial_\beta f_\gamma] \tag{3a}$$

resulting in the field equations

$$m\;\delta I_{ETME}/\delta A_\mu = \Box f^\mu - m\epsilon^{\mu\alpha\beta}\partial_\alpha f_\beta = -\epsilon^{\mu\alpha\beta}\partial_\alpha(m f_\beta + \epsilon_\beta{}^{\gamma\delta}\;\partial_\gamma f_\delta) = 0\;. \tag{3b}$$

Here m is seen to have dimensions of inverse length or mass. [We remark that (3a) is known [4] to be precisely equivalent to TME, the Maxwell–CS action, if f^μ is taken to be the fundamental variable rather than, as for us, the curl of an underlying vector potential. Our equations are correspondingly the curl of those of TME, as shown by the last equality in (3b).] Note that ETME can be formally obtained from (the appropriate helicity branch of) TME by the replacement $m \to \Box/m$. Hence, we can immediately write the form of our propagator, using that of TME. There [2],

$$G^{\mu\nu}_{TME} = (\Box + m^2)^{-1}(g^{\mu\nu} - (m/\Box)\epsilon^{\mu\alpha\nu}\partial_\alpha) \tag{4}$$

when acting on conserved sources; it clearly described a massive excitation. Hence, the ETME propagator becomes

$$G^{\mu\nu}_{ETME} = (\Box + \Box^2/m^2)^{-1}(g^{\mu\nu} - m^{-1}\epsilon^{\mu\alpha\nu}\partial_\alpha)\;, \tag{5}$$

as of course follows directly from (3a). This time, the denominator describes two excitations,

$$m^2\Box^{-1}(\Box + m^2)^{-1} = \Box^{-1} - (\Box + m^2)^{-1}\ . \tag{6}$$

One is massless, the other is massive, with a relative ghost sign. The limit $m \to \infty$ of G_{ETME} correctly reproduces the Maxwell propagator $g^{\mu\nu}/\Box$. Its small m limit should correspond to pure I_{ECS} and indeed we find $G_{ETME} \to -m/\Box^2\ \epsilon^{\mu\alpha\nu}\partial_\alpha$, the propagator corresponding to (2). Note that, irrespective of the signs in (3), there is never a tachyon: there cannot be a $(\Box - m^2)$ pole.

We next perform a detailed canonical analysis, decomposing f^μ in terms of the vector potential $(A_i,\ A_0)$, and writing

$$A_i \equiv \epsilon_i{}^j\hat{\partial}_j a + \partial_i\Lambda\ , \qquad \hat{\partial}_i \equiv \partial_i/\sqrt{-\nabla^2}\ , \qquad (\epsilon_i{}^j \equiv -\epsilon^{ij})\ , \tag{7}$$

to yield

$$f^0 = \epsilon^{ij}\partial_i A_j = -\sqrt{-\nabla^2}\ a\ , \quad f_i = \hat{\partial}_i\dot{a} + \epsilon_{ij}\hat{\partial}_j\ E\ , \qquad E \equiv -\sqrt{-\nabla^2}(A_0 - \dot{\Lambda})\ . \tag{8}$$

The action I_{ETME} then reduces to

$$I_{ETME} = \tfrac{1}{2}\int d^3x(-a\Box a + E^2) + m^{-1}\int d^3x E\Box a\ . \tag{9}$$

The Maxwell contribution ($m = \infty$) is that of the transverse a-mode, together with a nonpropagating longitudinal electric field term. Note the absence of dangerous explicit third time derivative terms. This is unsurprising: they can only come from the $\int \epsilon^{ij} f_i\partial_0 f_j$ part of I_{ECS}. Due to the explicit ϵ^{ij}, all "diagonal" terms (aa) and (EE) vanish by antisymmetry (after spatial partial integration) since these scalars would have to carry ∂_i, ∂_j to saturate ϵ^{ij}. The aE cross-term has no ϵ^{ij} and so can and does contain the third derivative $a\,\partial_0^3\,\Lambda$, but (since Λ is a gauge parameter) one ∂_0 is harmlessly buried as part of the gauge-invariant field variable E. The field equations from (9) are obviously

$$\Box(a - \bar{E}) = 0\ , \quad \Box a + m^2\bar{E} = 0\ , \qquad \bar{E} \equiv m^{-1}E\ . \tag{10}$$

The appropriate diagonalization is also clear from (9):

$$I_{ETME} = -\tfrac{1}{2}\int d^3x\ \bar{a}\Box\bar{a} + \tfrac{1}{2}\int d^3x\ \bar{E}(\Box + m^2)\bar{E} \tag{11}$$

in terms of $\bar{a} \equiv a - \bar{E}$. Here we have the normal transverse massless photon $\bar{a}$, together with a massive ghost $\bar{E}$ (longitudinal electric field), as evidenced by the relative minus sign between the two modes. We again recover the Maxwell theory in the $m \to \infty$ limit, while as $m \to 0$, we find $\Box f^\mu = 0$ in terms of the two invariant parts (a, E) of f^μ. All this agrees with the analysis above of the propagator (5,6). We have not studied the spin character of our excitations; any massless particle must be spinless in D=3 [5], while the massive mode presumably has helicity ± 1 according to the sign of m. We also omit details of coupling ETME to sources, which is a straightforward exercise in the propagator's properties for minimally coupled $\sim A_\mu j^\mu$ currents. Non-minimal interactions $\sim f_\mu k^\mu$ are also permitted here, and would presumably be equivalent to minimal couplings in TME, in view of the equivalence of (3) to TME, in terms of f_μ alone.

The most general gauge invariant action involving the terms discussed so far (apart from obvious $\Box^2$ or higher insertions in I_{CS}) would be the linear combination

$$I_{TOT} = I_{ECS} + I_{MAX} + I_{CS}\ . \tag{12}$$

The only unexplored 2-term action, $I_{ECS} + I_{CS}$, obviously just corresponds to massive propagation, $(\Box + m^2) f^\mu = 0$, of the field strength. If we keep all three terms, the propagator can once again be read off from $G^{\mu\nu}_{TME}$ by the replacement $m \to m + c\Box/m$, where we have allowed for different mass parameters m, m/c in the two CS variants. The G_{TOT} denominator has the form $m^{-2}[c^2\Box^2 + m^2\Box(1+2c) + m^4]$ with roots $m^2[-(1+2c) \pm (1+4c)^{1/2}]/2c^2$. The degenerate root at $c = -\frac{1}{4}$ corresponds to a double (massive) pole. Clearly there is a range of special cases, both physical and not, that can be explored, but no massless excitation remains.

3. Curved Space Gauge Invariance

When the geometry is nonflat, even the Maxwell action must be written properly to preserve gauge invariance. For example if instead of writing the manifestly invariant form $F_{\mu\nu}F^{\mu\nu}$, we had continued from its equally correct flat space expansion $(\partial_\mu A_\nu)^2 - (\partial^\mu A_\mu)^2$ to the covariant expression $(D_\mu A_\nu)^2 - (D^\mu A_\mu)^2$, gauge invariance would be lost; the only correct order in the last term is $(D_\mu A_\nu)(D^\nu A^\mu)$, which differs [6] from $(D^\mu A_\mu)^2$ by a gauge-variant, curvature-dependent term $R^{\mu\nu} A_\mu A_\nu$. It is similarly easy to see that only the last, manifestly gauge invariant derivative ordering of (1) preserves curved space gauge invariance

$$I_{ECS} = -(2m)^{-1} \int d^3x\, \epsilon^{\alpha\beta\gamma} f_\alpha \partial_\beta f_\gamma \;, \quad f_\alpha \equiv g^{-1/2} g_{\alpha\beta} \epsilon^{\beta\mu\nu} \partial_\mu A_\nu \;. \tag{13}$$

Note that as defined here, f_α is a covariant vector; f^a will denote the contravariant vector (not the usual density) $g^{\alpha\beta} f_\beta$. The metric dependence I_{ECS} is now entirely contained in f_α, so that the stress tensor is

$$\sqrt{g}\, T^{\mu\nu}_{ECS} = 2\delta I_{ECS}/\delta g_{\mu\nu} = -m^{-1} \left\{ (\epsilon^{\mu\alpha\beta} f^\nu + \epsilon^{\nu\alpha\beta} f^\mu) \partial_\alpha f_\beta - g^{\mu\nu} \epsilon^{\alpha\beta\gamma} f_\alpha \partial_\beta f_\gamma \right\} \;, \tag{14}$$

in contrast to $T^{\mu\nu}_{CS} \equiv 0$. The (covariant) conservation of $T^{\mu\nu}$ on ECS shell is easily checked, using the Bianchi identities $\partial_\mu(\sqrt{g}\, f^\mu) \equiv 0$, the (covariant) field equations

$$\epsilon_\mu{}^{\alpha\lambda}\, \partial_\lambda (g^{-1/2} \epsilon_\alpha{}^{\beta\gamma} \partial_\beta f_\gamma) \equiv D^\nu(-\partial_\mu f_\nu + \partial_\nu f_\mu) \equiv [g_{\mu\nu}\, D^2 - D_\nu D_\mu] f^\nu = D^2 f^\mu - R^{\mu\nu} f_\nu = 0 \tag{15}$$

and the identity $\epsilon^{\alpha\beta\gamma} \partial_\beta f_\gamma (\partial_\alpha f_\mu - \partial_\mu f_\alpha) \equiv 0$.

The total ECSE stress tensor includes the usual Maxwell contribution,

$$T^{\mu\nu}_{ECSE} = -\tfrac{1}{2}(f^\mu f^\nu + f^\nu f^\mu - g^{\mu\nu} f^\lambda f^\sigma g_{\lambda\sigma}) + T^{\mu\nu}_{ECS} \tag{16a}$$

and is likewise conserved on combined shell, where is can be written very simply: By virtue of the middle term in (3b), $T^{\mu\nu}_{ECSE}$ reduces to its Maxwell part but (in flat space) with the operator $(1 + 2m^{-2}\Box)$ rather than just unity, between the f's:

$$-2m^2\, T^{\mu\nu}_{ECSE} = f^\mu (m^2 + 2\Box) f^\nu + f^\nu (m^2 + 2\Box) f^\mu - \eta^{\mu\nu} f^\alpha (m^2 + 2\Box) f_\alpha \;. \tag{16b}$$

[In curved space, $g_{\alpha\beta}\Box$ is replaced by the operator $(D^2 g_{\alpha\beta} - R_{\alpha\beta})$ of (15).] Inserting the canonical decomposition (7,8) into T^{00}_{ECSE} directly gives the total energy as the expected difference between photon and ghost mode contributions; in our signature,

$$\begin{aligned} P_0 \;&=\; -\int d^2r\, T^{00} = \tfrac{1}{2} m^{-2} \textstyle\int d^2r [f_i (m^2 + 2\Box) f_i + f_0 (m^2 + 2\Box) f_0] \\ &= \tfrac{1}{2} \textstyle\int d^2r \left\{ \left[(\partial_0 \bar{a})^2 + (\boldsymbol{\nabla} \bar{a})^2 \right] - \left[(\partial_0 \bar{E})^2 + (\boldsymbol{\nabla} \bar{E})^2 + m^2 \bar{E}^2 \right] \right\} \;. \end{aligned} \tag{17}$$

4. Summary

We have studied the first higher derivative analog of the CS topological invariant, which would arise in the effective QED_3 action's expansion in powers of ∂/m. This I_{ECS} invariant turns out to be unique, and while formally similar to I_{CS}, differs profoundly from it in two respects: first, I_{ECS} is a local function of the field strength, insensitive to the "large gauge" aspects captured by I_{CS}; second, it is no longer topological but depends explicitly on the background geometry. When I_{ECS} is added to the Maxwell action, the resulting ETME system describes two degrees of freedom, one massless, the other a massive ghost. This is in contrast with the otherwise similar gravitational TMG model: while both are of overall third (but of second time) derivative order, TMG represents a single massive excitation. The reasons for this difference can be traced to the roles played by the respective component actions: First, the Maxwell term, unlike the Einstein one in TMG, already describes a (massless) degree of freedom. Second, the triple derivative CS term in gravity is conformally invariant and this higher symmetry, absent in I_{ESC}, eliminates one candidate mode.

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