DIVISION OF THE HUMANITIES AND SOCIAL SCIENCES
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA 91125

# THE OPTIMAL PRODUCT-MIX FOR A MONOPOLIST IN THE PRESENCE OF CONGESTION EFFECT: A MODEL AND SOME RESULTS

Parkash Chander
California Institute of Technology
Indian Statistical Institute

Luc Leruth
CEME, Université Libre de Bruxelles

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ABSTRACT

The paper develops a model of product differentiation in which the quality of a product may be negatively affected by the number of consumers buying it, as it is the case for any good affected by congestion. It is shown that for any positive degree of heterogeneity among the consumers, a monopolist will always find it more profitable to differentiate, i.e., to sell more than one quality of the product at different prices.

# THE OPTIMAL PRODUCT–MIX FOR A MONOPOLIST IN THE PRESENCE OF CONGESTION EFFECT: A MODEL AND SOME RESULTS

Parkash Chander * and Luc Leruth **

## I. INTRODUCTION

The pricing problem of a monopolist producing several qualities of a good has received some theoretical attention in recent years. Mussa and Rosen (1978) analyze the optimal product–mix for a monopolist offering several qualities of a product to heterogeneous consumers differentiated by their preference for quality. Itoh (1983) studies the welfare aspects of such monopolistic behavior. Gabszewicz and Thisse (1980) and Shaked and Sutton (1982) show that the degree of heterogeneity among the consumers determines the number of firms which can make positive profits on the market. More recently, Gabszewicz, et.al. (1986) derives a condition on the degree of heterogeneity leading to the existence of a "natural" monopolist whose profit maximizing policy is to sell the highest quality of his product only. Product differentiation thus disappears.

Though these papers differ from each other in several respects, there is one assumption common to them, viz. the qualities of the product are exogenously given: They do not depend either on the prices being charged or on the number of consumers buying them.

In this paper, we study the best strategy for a monopolist selling a commodity characterized by the fact that the utility a consumer derives from it is negatively affected by the number of other consumers who also buy it. For example, in Paris, each train in the subway has one coach painted in a special color (but otherwise identical to any other coach) to which only those people who bought a more expensive ticket have access. As a result, these coaches are less crowded, which is why some consumers are willing to pay the extra fee.

Congestion effects have been studied in the theory of clubs or in a general equilibrium framework (see Marchand (1968) and Levy–Lambert (1968)), but less so in industrial organization, if one excepts Wilson (1986) for the case of electricity supply. In this paper, we show that a monopolist selling a good the quality of which is affected by the number of consumers who buy it, will *always* choose to differentiate as long as there is a positive degree of heterogeneity among the consumers.

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* California Institute of Technology and Indian Statistical Institute
** CEME, Université Libre de Bruxelles
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In Section II we introduce the model. The main result is derived in Section III. Section IV is devoted to some conclusions.

## II. THE MODEL

We consider a single firm selling at most $m$ distinct, substitute, goods labelled by an index $i = 1, 2, \ldots, m$. It is assumed that the quality level of each good can be represented by a real number. The firm sets the prices for these goods and each consumer buys none or just one unit of one of the goods.

We assume that each consumer can be characterized by a parameter $\theta$ which represents his preference for quality and that the consumers are distributed according to a function $f(\theta)$ which is nonnegative, differentiable, and integrable over a certain interval $[\underline{\theta}, \overline{\theta}], \underline{\theta} < \overline{\theta}, \underline{\theta} \geq 0$, such that $f(\theta) = 0$ for all $\theta \notin [\underline{\theta}, \overline{\theta}]$.

As in Mussa and Rosen (1978) and Itoh (1982), we assume that a consumer $\theta$ who buys one unit of a good of quality $k$ at a price $p$ derives a surplus given by

$$S(\theta; k, p) = \theta k - p \; . \tag{1}$$

Let $p_1, p_2, \ldots, p_m$ and $k_1, k_2, \ldots, k_m$ be some prices and qualities of these goods. Suppose $p_1 > p_2 > \cdots > p_m$ and $k_1 > k_2 > \cdots > k_m$. Suppose each consumer chooses among the goods so as to maximize its surplus as defined by (1).

Define $m$ numbers $\theta_1, \ldots, \theta_m$ as

$$\theta_j = \frac{p_j - p_{j+1}}{k_j - k_{j+1}} \;, j = 1, \ldots, m-1,$$

and

$$\theta_m = \max \{\underline{\theta}, p_m / k_m\} \; .$$

If the prices and qualities are related in such a way that $\overline{\theta} \geq \theta_1 > \theta_2 > \cdots > \theta_m \geq \underline{\theta}$, then the consumers belonging to $[\theta_j, \theta_{j-1}]$ will choose to buy $k_j$ at price $p_j$. Their number is given by

$$n_j = \int_{\theta_j}^{\theta_{j-1}} f(\theta)\, d\theta \; .$$

However, if $\overline{\theta} \geq \theta_1 > \theta_2 > \cdots > \theta_m \geq \underline{\theta}$ is not true, then some of the qualities will not be bought. A representation is given in Figure 1 for the case $m = 3$. The three lines $S_1, S_2$ and $S_3$ give the surplus which would be enjoyed by each consumer in each case and the thick line represents the surplus derived from the actual choice. Note that if $\theta$ prefers $k_i$ to $k_{i+1}$, then $\theta' > \theta$ will also prefer $k_i$ to $k_{i+1}$.

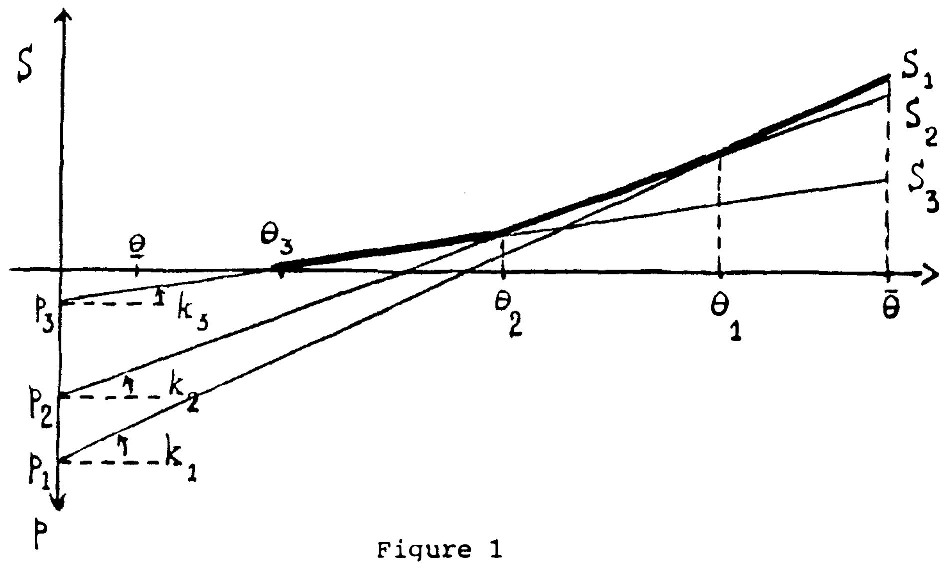


Figure 1

Let $k$ be a real valued function having the following properties:

(i) $k$ is at least twice differentiable;

(ii) $k(r) > 0$ and $dk(r)/dr < 0$ for all $r \geq 0$; and

(iii) $k(0) < +\infty$.

We shall assume that if $n_j$ is the number of consumers buying good $j$, then $k_j = k(n_j)$ is its quality.

A good example is provided by the transport network in New Delhi. There are two sorts of buses differentiated *only* by the price of the ticket, which is announced on the window ( 2 Rs. or 1 Re. ). The people at the stand consider the levels of congestion before getting in or not. The system is working quite well with the 2 Rs. buses obviously less crowded than the 1 Re. ones.

At this stage, it is worth noticing that if quality were not affected by congestion, consumers would simply not buy a good $i$ which is of lower quality and higher price than another one. But in our model, by buying another good in place of good $i$ , the consumers decrease the congestion in good $i$ , which increases its quality $k_i$. Thus, after a certain number of consumers withdraw from buying good $i$ , it might be the case that $k_i$ will rise enough to prevent the remaining consumers from withdrawing. Notice that a good which is not bought at all must have the highest quality and price.

Let $p_1, p_2, \ldots, p_m$ be some given prices of the $m$ goods. Suppose an equilibrium is achieved in the sense that everybody has made a choice and does not want to change it, given what the other consumers have chosen. Let $k_1, \ldots, k_m$ be the resultant qualities.

*Proposition 1*: If $p_1 \geq p_2 \geq \cdots \geq p_m$, then $k_1 \geq k_2 \geq \cdots \geq k_m$.

*Proof*: Suppose it is not true. Then there is some $i$ and $j$ for which $p_i \leq p_j$ and $k_i > k_j$. But this cannot happen because the consumers who had chosen $j$ would prefer $i$ . Thus they will shift until $k_i \leq k_j$.

Hereafter we call a set of prices $p_1, \ldots, p_m$ such that $p_1 \geq p_2 \geq \cdots \geq p_m$ a *price vector* $\vec{p}$ and a set of $\theta's$, $\theta_1, \ldots, \theta_m$ such that $\bar{\theta} \geq \theta_1 \geq \cdots \geq \theta_m \geq \underline{\theta}$, a *partition vector* $\vec{\theta}$. The existence of an equilibrium is determined by the ability of a price vector $\vec{p}$ to induce a partition (possibly unique) vector $\vec{\theta}$.

Let us now consider all the partition vectors $\vec{\theta}$ which are such that

(a) $\theta_1 \leq \bar{\theta}$

(b) $\int_{\theta_1}^{\bar{\theta}} f(\theta)\, d\theta \leq \int_{\theta_2}^{\theta_1} f(\theta)\, d\theta \leq \cdots \leq \int_{\theta_m}^{\theta_{m-1}} f(\theta)\, d\theta$

(c) $\theta_m \geq \underline{\theta}$.

The first two conditions ensure that the vector $\vec{\theta}$ is relevant to our analysis, but the third one may appear to be restrictive. However, we shall later show why condition (c) is justified. Also notice that the possibility of $\theta_i = \theta_{i-1} = \cdots = \theta_1 = \bar{\theta}$ for some $i \leq m$ is not being ruled out.

Let $\Theta$ be the set of all vectors $\vec{\theta}$ verifying the above conditions. Then $\Theta$ is clearly compact and simply connected.

*Proposition 2*: For any $\vec{\theta}$ in $\Theta$, there is one and only one price vector $\vec{p}$ sustaining it, i.e., such that given $\vec{p}$ the consumers maximize their surplus by making their choices according to $\vec{\theta}$.

*Proof*: Define $\vec{p} = (p_1, \ldots, p_m)$ as

$$\left.\begin{aligned}
p_1 - p_2 &= \theta_1[k(\int_{\theta_1}^{\bar{\theta}} f(\theta)\, d\theta) - k(\int_{\theta_2}^{\theta_1} f(\theta)\, d\theta)] \\
p_2 - p_3 &= \theta_2[k(\int_{\theta_2}^{\theta_1} f(\theta)\, d\theta) - k(\int_{\theta_3}^{\theta_2} f(\theta)\, d\theta)] \\
p_{m-1} - p_m &= \theta_{m-1}[k(\int_{\theta_{m-1}}^{\theta_{m-2}} f(\theta)\, d\theta) - k(\int_{\theta_m}^{\theta_{m-1}} f(\theta)\, d\theta)] \\
p_m &= \theta_m\, k(\int_{\theta_m}^{\theta_{m-1}} f(\theta)\, d\theta)
\end{aligned}\right\} \qquad (2)$$

Then $\vec{p}$ is the unique price vector that sustains $\vec{\theta}$. Notice that $p_1 \geq p_2 \geq \cdots \geq p_m$ and $k_1 \geq k_2 \geq \cdots \geq k_m$, where

$$k_i = k(\int_{\theta_i}^{\theta_{i-1}} f(\theta)\, d\theta) .$$

Given $\Theta$, let $P$ be the set of all price vectors which correspond to some $\vec{\theta}$ in $\Theta$. Then $P$ includes all the price vectors which are relevant to a monopolist's strategy.

It is worth clarifying that if $\vec{\theta}$ were such that $\theta_m \leq \underline{\theta}$, then the sustaining price vector cannot be unique. Indeed the last equation of (2) disappears and we have a system of $(m - 1)$ equations to determine $m$ prices. Such a vector $\vec{\theta}$ could be sustained by a vector of prices $\vec{p}$ as well as by all price vectors $\hat{p} - \Delta\vec{p}$ where $\Delta\vec{p} = (\Delta p, \Delta p, \ldots, \Delta p)$, $\Delta p > 0$. This can be derived from (2) and is illustrated in Figure 2. Note that when $\theta_m < \underline{\theta}$, the consumer characterized by $\underline{\theta}$ has a positive surplus. It is therefore possible to increase all the prices in such a way that $\theta_m \geq \underline{\theta}$ and equalities (2) are satisfied. Since a monopolist, given the choice, will always charge higher prices, our assumption that $\vec{\theta}$ should be such that $\theta_m \geq \underline{\theta}$ is justified (see (c) above).

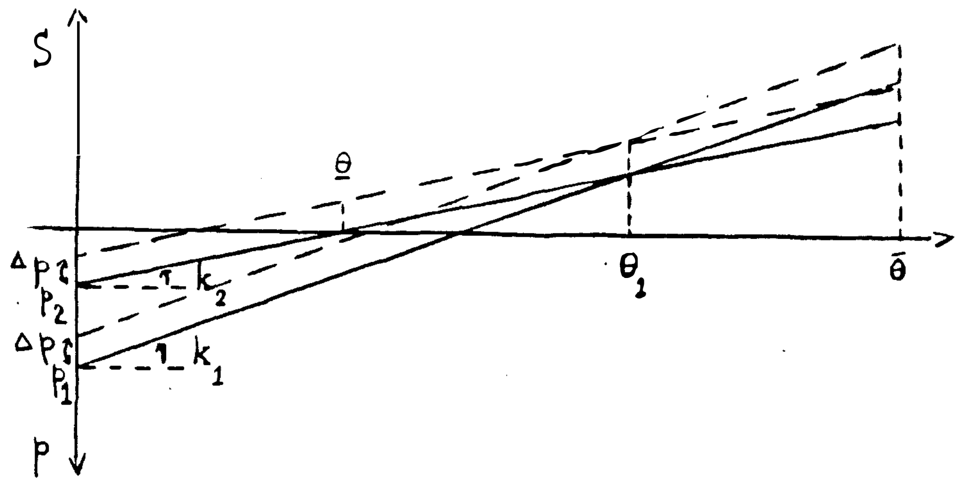


Figure 2

It may also be observed that two or more goods may have the same quality and prices. In such cases the consumers may choose between the goods either randomly or according to $\vec{\theta}$. From the point of view of profits of the firm, however, it does not matter how the consumers choose between such goods. Hence it is assumed that they choose according to $\vec{\theta}$.

*Proposition 3*: If $\vec{p} \in P$, then there is one and only one partition $\vec{\theta} \in \Theta$ which can be sustained by it.

*Proof*: Since $\vec{p} \in P$, there exists by definition a $\vec{\theta} \in \Theta$ such that equalities (2) above are satisfied. We show that $\vec{\theta}$ is unique. Suppose not. Suppose $\vec{\theta}' = (\theta_1', \ldots, \theta_m') \neq (\theta_1, \ldots, \theta_m)$ also satisfies (2). Let $(k_1, \ldots, k_m)$ and $(k_1', \ldots, k_m')$ be the corresponding qualities. Without loss of generality assume that $\theta_1' > \theta_1$. Then we must have

$$k_1' = k(\int_{\theta_1'}^{\bar{\theta}} f(\theta)\, d\theta) > k(\int_{\theta_1}^{\bar{\theta}} f(\theta)\, d\theta) = k_1 .$$

The first equation in (2) then implies $\theta_2' > \theta_2$. The process goes on until we reach $\theta_m' > \theta_m$ and

$$k_m' = k(\int_{\theta_m'}^{\theta_{m-1}'} f(\theta)\, d\theta) > k(\int_{\theta_m}^{\theta_{m-1}} f(\theta)\, d\theta) = k_m$$

But this cannot be true, since we must have

$$\theta_m' k_m' = p_m = \theta_m k_m \ . \quad \text{(by (2))} .$$

If $\theta_1' = \theta_1$ , the process can start from the first $\theta_j$ such that $\theta_j \neq \theta_j'$ . And we shall again get a contradiction unless $\vec{\theta}' = \vec{\theta}$.

*Proposition 4*: (a) The set of all suitable partition vectors $\Theta$ and the set of all sustaining price vectors $P$ are homeomorphic and (b) $P$ is simply connected.

*Proof*: (a) There is a continuous bijection between $\Theta$ and $P$ as a consequence of equation (2) (continuity) and propositions (2) and (3) (bijection). Since $\Theta$ is a compact subspace of $R^m$ and $P$ is Hausdorff, the continuous bijection is a homeomorphism. (b) Since $\Theta$ is simply connected and simple connectedness is a topological property, $P$ is simply connected.

For $m = 2$, the simple connectedness of $P$ implies that the set does not have any holes and is perfectly defined by its bounds. As a result, we can define $P$ as

$$P = \{\vec{p} = (p_1, p_2) \mid p_1 \geq p_2, p_1 \leq p_{1,\max}(p_2),$$

$$\text{and } p_2 \geq p_{2,\min}(p_1), \text{ where } p_{1,\max} \text{ solves}$$

$$\theta_1(p_{1,\max}, p_2) = \overline{\theta} \text{ and } p_{2,\min} \text{ solves}$$

$$\theta_2(p_1, p_{2,\min}) = \underline{\theta}\} .$$

An example is provided in Figure 3.

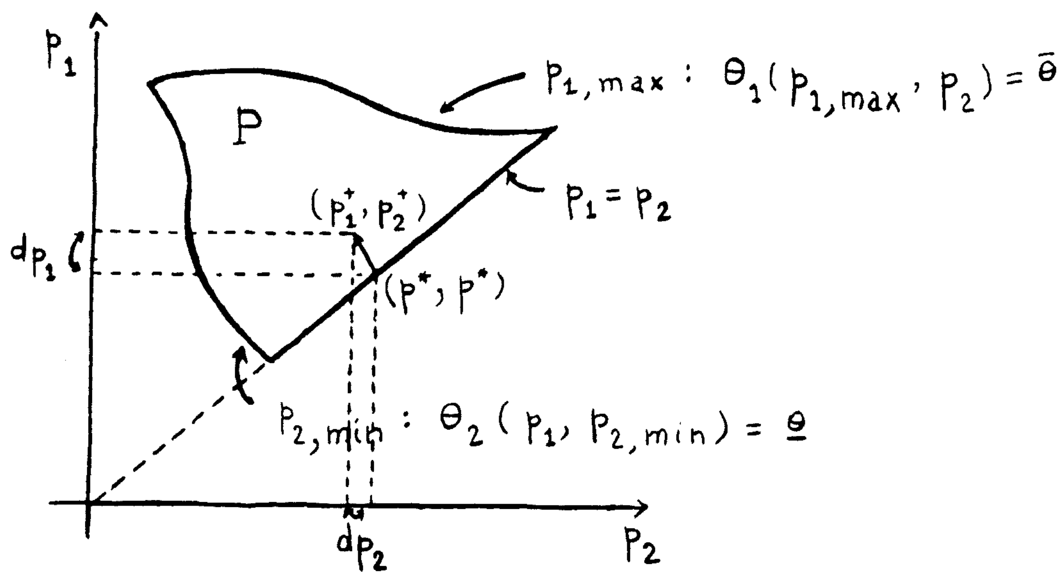


Figure 3

## III. THE MONOPOLIST

We now show that, in order to maximize his profit, a monopolist will always set differentiated prices and thus, differentiated qualities. Without loss of generality, we consider the case of two goods only.

*Proposition 5*: $p_1 = p_2$ is never a profit–maximizing price policy if there is a positive degree of heterogeneity among consumers, i.e., $\underline{\theta} < \bar{\theta}$.

*Proof*: Profits are given by

$$\pi = p_1 \int_{\theta_1}^{\bar{\theta}} f(\theta)\,d\theta + p_2 \int_{\theta_2}^{\theta_1} f(\theta)\,d\theta \ , \tag{3}$$

where

$$p_1 - p_2 = \theta_1 [k(\int_{\theta_1}^{\bar{\theta}} f(\theta)\,d\theta) - k(\int_{\theta_2}^{\theta_1} f(\theta)\,d\theta)] \tag{4}$$

and

$$p_2 = \theta_2 k(\int_{\theta_2}^{\theta_1} f(\theta)\, d\theta) \ . \tag{5}$$

Therefore, $\pi$ is a continuous function defined over the compact set $\Theta$. It must therefore attain its maximum over some $\vec{\theta} \in \Theta$. We prove that $\vec{\theta}$ must be such that $p_1 \neq p_2$. We have

$$d\pi = \frac{\partial \pi}{\partial p_1} dp_1 + \frac{\partial \pi}{\partial p_2} dp_2 \ ,$$

$$\frac{\partial \pi}{\partial p_1} = \int_{\theta_1}^{\bar{\theta}} f(\theta) d\theta - p_1 f(\theta_1) \frac{\partial \theta_1}{\partial p_1} + p_2 \left[ f(\theta_1) \frac{\partial \theta_1}{\partial p_1} - f(\theta_2) \frac{\partial \theta_2}{\partial p_1} \right]$$

$$\frac{\partial \pi}{\partial p_2} = -p_1 f(\theta_1) \frac{\partial \theta_1}{\partial p_2} + \int_{\theta_2}^{\theta_1} f(\theta)\, d\theta + p_2 \left[ f(\theta_1) \frac{\partial \theta_1}{\partial p_2} - f(\theta_2) \frac{\partial \theta_2}{\partial p_2} \right] \ .$$

Therefore, at $p_1 = p_2$ and $dp_1, dp_2$ such that

$$d\theta_2 = \frac{\partial \theta_2}{\partial p_1} dp_1 + \frac{\partial \theta_2}{\partial p_2} dp_2 = 0 \ , \tag{6}$$

we get

$$d\pi \Big|_{\theta_2 = \text{constant}} = dp_1 \int_{\theta_1}^{\bar{\theta}} f(\theta)\, d\theta + dp_2 \int_{\theta_2}^{\theta_1} f(\theta)\, d\theta$$

$$= \int_{\theta_1}^{\bar{\theta}} f(\theta)\, d\theta\, (dp_1 + dp_2) \ ,$$

since $p_1 = p_2$. Totally differentiating (4) and (5) and using (6), we get

$$dp_1 - dp_2 = -2\theta_1 \frac{dk}{dn} f(\theta_1) \left( \frac{\partial \theta_1}{\partial p_1} dp_1 + \frac{\partial \theta_1}{\partial p_2} dp_2 \right]$$

and

$$dp_2 = \theta_2 \frac{dk}{dn} f(\theta_1) \left( \frac{\partial \theta_1}{\partial p_1} dp_1 + \frac{\partial \theta_1}{\partial p_2} dp_2 \right] \ ,$$

where

$$\frac{dk}{dn} = \frac{dk_1}{dn_1} \Big|_{p_1 = p_2} = \frac{dk_2}{dn_2} \Big|_{p_1 = p_2}$$

Therefore,

$$dp_1 + dp_2 = \frac{2(\theta_2 - \theta_1)}{\theta_2} dp_2 \ .$$

Since $\theta_1 > \theta_2$ , for $dp_2 < 0$ we get $dp_1 + dp_2 > 0$ and hence $d\pi > 0$. This shows that if $p_1 = p_2$, then the prices can be adjusted in such a way that $d\pi > 0$. This completes the proof.

*Intuition*: We start with a unique price $p^* (= p_1 = p_2)$ for both goods and show that, for any $p^*$ , the monopolist can always increase his profit by decreasing $p_2$ and increasing $p_1$ in such a way that $\theta_2$ remains constant. By doing so, we make sure that the result will hold for any distribution of the $\theta$'s such that $\underline{\theta} < \overline{\theta}$ , no matter how narrow it is. Indeed, we know that the ratio $p_2/k_2$ must be higher or equal to $\underline{\theta}$ and thus, the prices and qualities must be changed in such a way that $p_2/k_2$ would not go below $\underline{\theta}$ .

From $(p^*,p^*)$ , we go to $(p_1^+, p_2^+) = (p^*,p^*) + (dp_1, dp_2)$ with $(dp_1 + dp_2)$ such that $\theta_2$ is constant as shown in Figure 3. By decreasing $p_2$ and increasing $p_1$ , we change the qualities to $k_1^+, k_2^+$ compared to the situation in which $p_1 = p_2 = p^*$ and $k_1^* = k_2^* = k^*$ . See Figure 4.

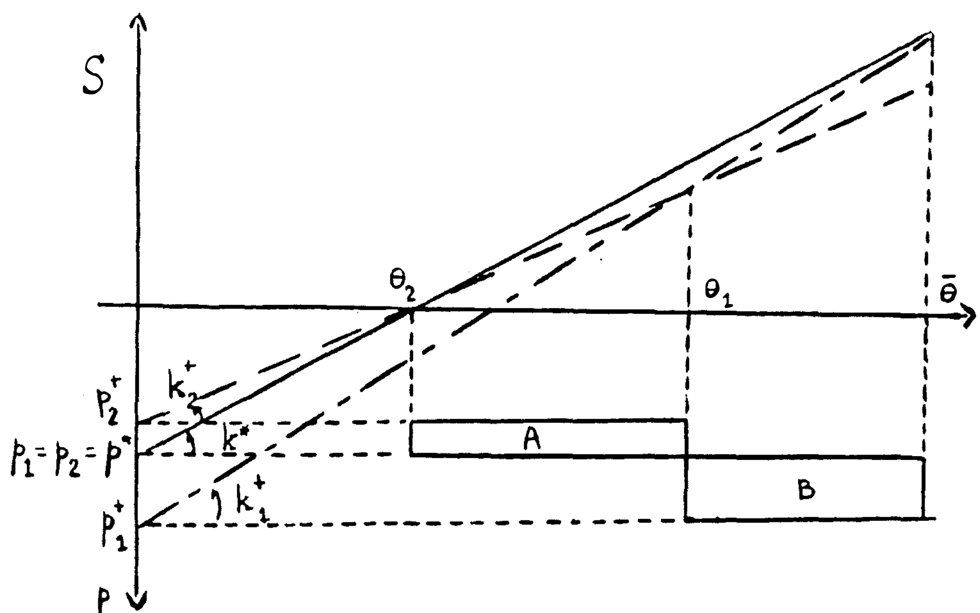


Figure 4

By decreasing $p_2$ and increasing $p_1$ , the monopolist gains B–A and the proof of proposition 5 shows that B–A is positive.

Finally, it may be interesting to note that we can also associate a planning problem with the above model. Suppose the firm is publicly owned and the Public Authority is interested in providing the good at as low a price as possible by subsidizing upto a maximum amount of S. Formally, the problem of the Public Authority is to

minimize $p_2$, subject to

$$\pi + S \geq 0 \ ,$$

where

$$\pi = p_2 \int_{\theta_2}^{\theta_1} f(\theta)\, d\theta + p_1 \int_{\theta_1}^{\bar{\theta}} f(\theta)\, d\theta$$

Proposition 5 shows that if $p_1 = p_2$, then the prices can be adjusted such that $dp_2 < 0$ and $d\pi > 0$ which implies that $p_1 = p_2$ cannot be a solution to the above problem. This shows that even from a social point of view price differentiation may be necessary.

## IV. COMPARISON WITH EXISTING MODELS AND CONCLUSIONS

It has been assumed above that each of the goods irrespective of its quality can be produced at a constant unit variable cost ( marginal cost ) which for simplicity has been taken to be zero everywhere. Our results, however, do not depend upon this assumption. The same results would obtain even if the unit variable cost were rising with quality.

In the analysis above we have also ignored fixed costs or taken them as sunk costs and the number of goods ( or qualities ) as exogenously given. If fixed costs were not sunk costs, then the number of goods offered by the monopolist will depend upon them. If fixed costs decline due to technological improvements, more goods of better quality will be offered.

In a recent paper, Gabszewicz et.al. (1986) show that when consumers are not "too" heterogeneous, the monopolist has an incentive to sell his top quality only. The reason why it is so is that prices can be so fixed that all prefer the top quality. Selling a lower one requires the monopolist to charge such a low price that it would not be profitable.

Above results show that this does not hold in a model in which quality is a function of the degree of congestion. The reason is that the top quality good can be sold very little or none at all at the prices at which it is the top quality. Thus, the monopolist can never do anything else than selling below top quality if he wants to maximize his profits. And that gives him an incentive to differentiate.

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