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# MONOPOLY PROVISION OF PRODUCT WARRANTIES

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## ABSTRACT

This article considers the problem of monopoly provision of product warranties when consumers are heterogeneous and when the probability of product malfunction depends on both the quality of the product and on the consumers' care. The optimal warranty contract is characterized to maximize the expected profit for the monopolistic seller. The properties of the optimal contract depend on the nature of the product. If the quality of the product is more important as a determinant of reliability than consumer care then standard results are obtained; that is, a positive correlation between warranties and reliability and between price and reliability are observed, and higher type buyers buy more expensive versions of the product with higher warranties. On the other hand, if consumer care is more important in increasing reliability, the results are exactly opposite; for example, there is a negative correlation between warranty coverage and reliability. Also, when consumer care is important, higher type buyers buy versions of the product with lower warranty and lower quality. Other features of the optimal warranty contract are also characterized in this paper.

## MONOPOLY PROVISION OF PRODUCT WARRANTIES

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### 1. Introduction

In this paper, we are concerned with a monopolist's choice of prices and warranties when consumers are heterogeneous and when the probability of product malfunction depends on both the quality of the product and on the consumers' care. The consumers' valuations of the functioning product and care cannot be observed by the producer; the quality of the product cannot be observed by the consumers. A warranty is modeled as a payment to the consumer in the event the product fails. Because the cost of product failure to the firm is greater when the warranty is larger, products with higher warranties will be of higher quality. But when the warranty is larger, the consumer has less incentive to take care of the product and the product is more likely to fail. Because of these two opposite effects, the correlation between the warranties and observed product reliability may or may not be positive. As we shall show, if seller quality is a more important determinant of reliability for most products, a positive correlation between warranty coverage and reliability will be observed in market data and higher type consumers will buy more expensive products with higher warranties. If consumer care is more important in determining reliability, the results switch. This opens up the possibility that consumers who more highly value successful functioning of the product may nonetheless not buy versions of the product with higher warranties and higher quality. Their desire to have the product not fail may cause them to expend sufficient effort to make the warranty of relatively little value to them especially when effort is significant in increasing reliability or is not so costly. As a result, they buy a version of the product with lower warranty and lower quality. This can occur even though the probability of successful functioning is assumed separable in product quality and consumer effort.

A large literature exists about product reliability and warranties in commodity markets. Four theories have been proposed to explain the role of product warranties in sales transactions: exploitation theory, signal theory, insurance theory and investment theory.[1] Exploitation theory views warranties as devices used by monopolistic manufacturers to exploit consumers by unilateraly limiting legal obligations (Kessler [1943]). Signal theory treats warranties as messages signaling the reliability of the product (Akerlof [1970], Spence [1977]). Higher quality producers can afford more complete warranties since their products are less likely to break down. Thus, consumers may infer the reliability of a product by inspecting the warranty. In contrast to theses theories, several empirical studies have found evidence that the correlations between warranty coverage and product quality, and price and product quality are weak and not necessarily positive (Priest [1981], Gerner and Bryant [1981], Garvin [1983] and Gerstner [1985]). Insurance theory (Heal [1977]) suggests that warranties provide insurance by the seller against variations in product performance. So the warranties have an optimal risk-sharing property. But insurance theory does not explain why the warranty is not provided by any insurance company. Investment theory (Priest [1981]) views a warranty as a contract that optimizes the productive services of a good by dividing responsibility between the manufacturer and the consumer to prolong the useful life of a product. This theory predicts that disclaimers of liability and exclusions of coverage will be observed in consumer product warranties for those specific allocative or insurance investments that the consumer can provide more cheaply than the manufacturer.

Recent papers have built formal models along the lines of investment theory to characterize the optimal warranty contract by emphasizing the importance of bilateral moral hazard (Cooper and Ross [1985], Mann and Wissink [1986]).[2] Observed product reliability is viewed as the joint outcome of two actions, one taken by the firm in making a product and another by the consumer in using a product. In Cooper and Ross's model, only a single consumer type is considered, and the analysis indicates that, owing to incentive problems resulting from moral hazard, warranties offer only partial insurance. Mann and Wissink [1986] investigate warranties in a competitive market structure with several different consumers. Their analysis shows that equilibrium contracts involve a positive correlation between price and warranty coverage, but that observed product reliability may vary directly or inversely with both price and warranty coverage. In a similar model, Mann [1986] considers only unilateral moral hazard on the buyer side of the market. He shows that a single warranty contract is offered in a competitive equilibrium, and that the contract offered by a monopolist separates consumer types whenever the competitive equilibrium involves a pooling equilibrium.

Mussa and Rosen [1978] have discussed a model of monopoly provision of product quality. In their model, a monopolist uses both price and product quality to screen heterogeneous consumers when the consumer's evaluation of the product cannot be observed by the monopolist. Consumers are risk-neutral and can observe the quality of any product they purchase. The single-crossing property of the utility curves allows a characterization of the optimal monopoly solution in which the monopolist offers a broader range of qualities of goods and in which consumers of higher type buy a more expensive good with a higher warranty coverage. Matthews and Moore [1987] extend the model of Mussa and Rosen by assuming that consumers are risk-averse. One of the contracts they consider includes a price, a quality level, and warranty coverage. They use a non-local approach to characterize optimal allocations that may not be monotonic. Their analysis shows that although the more eager types of buyers do pay higher prices and yield higher profit to the monopolist, they may receive lower quality or lower warranty coverage.

To generalize both Cooper and Ross [1985] and Matthews and Moore [1987], we consider a problem of monopoly provision of product warranties in the face of both bilateral moral hazard and adverse selection problems. In our model, buyers are assumed to be risk-neutral, but vary with the willingness to pay for a single unit of the product. Product reliability depends on both the seller's precaution and the buyers' care. The monopolist seller can observe neither the buyers' care or their willingness to pay for the product, and buyers cannot observe the seller's product quality either. These result in a bilateral moral hazard and asymmetric information between the monopoly seller and the buyers. We will emphasize the interaction of the bilateral moral hazard and the adverse selection problems. As we shall show, this interaction will yield different predictions on price and warranty policies from those in the literature we discussed above.

In the next section, we will specify our formal model of the monopoly design problem. The problems of bilateral moral hazard are analyzed in Section 3. Section 4 considers incentive compatibility due to the unobservability of the buyers' evaluation of the good by the seller and characterizes the optimal monopoly solution of product warranties. We also derive some properties of the monopoly solution and compare these with the work of Cooper and Ross [1985], Mann and Wissink [1986], Mann [1986], and Matthews and Moore [1987]. A simple example is provided to illustrate our model and findings in Section 5. Section 6 concludes the paper.

## 2. The Basic Model

We consider contracts for a commodity which will be traded between a monopoly seller and many potential consumers. The product may or may not work after purchase. The "perfect" product is dependent both on the producer's specific investment and on the consumer's specific investment in a "production process". The probability that the product works is represented by $\pi$, called product reliability, $0 \leq \pi \leq 1$. $\pi$ is a function of two inputs: the quality level $q$ provided by the producer in making the product, and the level of care or effort $e$ expended by the consumer in using the product. Both $q$ and $e$ are one dimensional variables, where larger values of $q$ indicate higher quality and larger values of $e$ indicate higher level of care. We assume that the function $\pi(q, e)$ satisfies $\pi_q > 0$, $\pi_e > 0$, $\pi_{qq} \leq 0$ and $\pi_{ee} \leq 0$ for any $e$ and nonnegative $q$. This implies that the inputs are productive at a decreasing rate in the production of reliability. For simplicity of analysis, we rule out the "production process" in which $q$ and $e$ are complements or substitutes, that is, we assume $\pi_{qe} = 0$.[3]

A contract between the seller and the consumer specifies a pair $(p, w)$ where $p$ represents the price to be paid by the consumer for a single unit of the product and $w$ is the refund that the consumer receives from the seller if the product fails. We assume that there exists a third party who can costlessly determine whether or not the product works and enforce the terms of the contract without any dispute. In the mechanisms we consider here, the seller does not have to pay a fine to the third party if the product fails and the price paid by the consumer equals the price received by the seller.[4]

The monopolist sells a durable good for which each consumer is interested in buying one unit of the goods. Consumers are assumed to be risk neutral, but vary according to their willingness to pay for a single unit of the product. A consumer of type $\theta$ has a valuation of $\theta$ dollars for a functioning product. Income effects are ruled out. Each consumer's type $\theta$ is a private characteristic, unobservable to the monopoly seller, but the seller has a belief that $\theta$ is drawn from a random distribution $F(\theta)$ with support $[\underline{\theta}, \overline{\theta}]$, where $0 \leq \underline{\theta} < \overline{\theta}$. The density function $f(\theta)$ is assumed to be positive and continuous on $[\underline{\theta}, \overline{\theta}]$. Given $q$ and a contract $(p, w)$, the expected utility for a consumer of type $\theta$ with care level $e$ is

$$U(q, e, p, w; \theta) = \pi(q, e)\theta + [1 - \pi(q, e)]w - p - g(e), \tag{1}$$

where $g(e)$ is the consumer's disutility function for effort which is assumed to be the same for all consumers. We assume that $e$ is nonnegative, and $g(0) = 0$, $g_e > 0$ for $e \geq 0$, $g_e \leq 0$ for $e < 0$.

The monopolist can produce any number of products with a quality level $q$ at a unit cost $c(q)$. We assume that $c_q > 0$, and $c_{qq} > 0$. Then, the monopolist's expected profit obtained from each consumer who chooses the contract $(p, w)$ and effort level $e$ is

$$V(q, e, p, w) = p - c(q) - [1 - \pi(q, e)]w. \tag{2}$$

In this paper, we assume that functions $\pi(q, e)$, $g(e)$, $c(q)$ and $F(\theta)$ are twice differentiable and are common knowledge.

Under complete information, $q$, $e$ and $\theta$ are all costlessly observable by both parties. The Pareto-efficient contract is an allocation $(p, w, q, e)$ that maximizes the sum of the utilities since both parties are assumed to be risk neutral. The contract $(p, w)$ is determined by *bargaining between two parties. The Nash equilibrium $(q, e)$ satisfies the following equations

$$\pi_q \theta = c_q \tag{3}$$

$$\pi_e \theta = g_e \tag{4}$$

It is easy to check that both $q$ and $e$ are increasing in $\theta$. Under the complete information, the consumer with a higher evaluation of the functioning product will buy a higher quality product and take better care.

## 3. Bilateral Moral Hazard

When the input levels are not costlessly observable by the parties, a bilateral moral hazard problem arises: the monopolist cannot observe the consumer's effort and the consumer cannot observe the quality level provided by the monopolist. This results in an incentive problem in enforcing a contract. In this situation contracts frequently leave the input decisions by the parties unspecified. Rationally designed contracts anticipate the effect of contractual elements $(p, w)$ on the ex post input decisions made by both parties. Input decisions are then self-enforcing. Generally, the monopolist offers a set of contracts $(p, w)$ to consumers and then selects levels of quality $q$. Consumers then choose the contract they most prefer as well as a level of effort $e$. The product is then observed to work or fail; if it fails the warranty is exercised. This process can be modeled as a two-stage game played between the monopolist and the consumers. In the first stage, they sign a contract $(p, w)$ under rational expectations, where $p \geq 0$, $w \geq 0$. The second stage takes $(p, w)$ as given, and the players choose their inputs $q$ and $e$ noncooperatively. Payoffs are then realized after the condition of the product is determined.

We first look at the second stage of the game for arbitrary $(p, w)$. The first stage of the game will be discussed in the next section.

From the utility function (1) and (2), we obtain the first order conditions for Nash noncooperative equilibrium as follows:

$$\pi_q w = c_q \tag{5}$$

$$\pi_e(\theta - w) = g_e \tag{6}$$

Let $\hat{q}$ and $\hat{e}$ be the solution to (5) and (6). At equilibrium, the second order conditions are satisfied. Since $\pi_{qe} = 0$, the monopolist does not have to know the consumer's type $\theta$ when choosing $q$. From (5) we get $\hat{q} = \hat{q}(w)$ if $w > 0$, and $\hat{q} = 0$ if $w = 0$. Therefore, the assumption that $\pi_{qe} = 0$ is very important in this paper. From (6), we can get $\hat{e} = \hat{e}(\theta - w)$ if $w < \theta$, and $\hat{e} = 0$ if $w \geq \theta$. When $0 < w < \theta$, we can easily compute the partial effects of changes in $w$ and $\theta$ on $\hat{q}$ and $\hat{e}$:

$$\hat{q}_w = \frac{\pi_q / c_q}{c_{qq} / c_q - \pi_{qq} / \pi_q}$$

$$\hat{e}_w = -\hat{e}_\theta = -\frac{\pi_e / g_e}{g_{ee} / g_e - \pi_{ee} / \pi_e}$$

Since $\hat{q}_w > 0$ at equilibrium, higher warranty coverage results in a higher quality level for the monopolist. That is, when a contract specifies a high level of warranty coverage, product failure is relative costly to the monopolist, and the monopolist is willing to spend more to reduce the chance of this outcome. Therefore, from the warranty of the particular contract offered by the monopolist a consumer can infer the quality level of the product. This is consistent with signal theory. But on the other hand, higher warranty coverage results in a lower level of effort by all consumers. That is, product failure is more costly to a consumer when the level of warranty coverage is low. Full warranty coverage may result in consumers breaking the product deliberately. Because of these two opposite effects, observed product reliability may or may not be positively correlated with the warranty coverage. Let

$$S(q) = \frac{\pi_q^2 / c_q}{c_{qq} / c_q - \pi_{qq} / \pi_q},$$

$$B(e) = \frac{\pi_e^2 / g_e}{g_{ee} / g_e - \pi_{ee} / \pi_e}.$$

At equilibrium, product reliability $\pi$ is a function of $w$ and $\theta$. Given $\theta$, taking the derivative of $\pi$ with respect to $w$, we get

$$\frac{\partial \pi}{\partial w} = \pi_q \hat{q}_w + \pi_e \hat{e}_w = S[\hat{q}(w)] - B[\hat{e}(\theta - w)]. \tag{7}$$

Let $g(\theta, w) = S[\hat{q}(w)] - B[\hat{e}(\theta - w)]$. Consider a product such that $g(\theta, w) > 0$ for all $\theta$ and $w$. That is, product quality is the relatively more important determinant of product reliability or is less costly than consumer effort in the production of reliability. In this case, a high product reliability will be observed if the contract specifies a high warranty coverage. In contrast, when $g(\theta, w) < 0$, (consumer effort is less costly or more important to product reliability), a high warranty coverage will result in a lower observed product reliability. This result may explain part of the empirical evidence found by Gerner and Bryant [1978] and Priest [1981].

The above partial analysis shows that observed product reliability $\pi$ and warranty coverage $w$ may be negatively correlated for a given consumer type $\theta$. What we actually observe in the data is the different warranties and product reliabilities across heterogeneous consumers. Warranties vary with the consumer types, let us say, $w = w(\theta)$. Observed product reliability is then $\pi(\theta) = \pi\{\hat{q}[w(\theta)], \hat{e}[\theta - w(\theta)]\}$. The question is whether a higher product reliability will be observed for a consumer with a higher evaluation of the product and whether $\pi$ and $w$ are positively correlated. If $w(\theta)$ is differentiable, then $\pi'(\theta) = g(\theta, w(\theta))w'(\theta) + \frac{\partial \pi}{\partial \theta}$. Since $\frac{\partial \pi}{\partial \theta} > 0$, if $g(\theta, w(\theta))w'(\theta) \geq 0$, then $\pi'(\theta) > 0$. That is, a higher product reliability will be observed for a higher type consumer. In the next section, we will see that because of the unobservability of consumer types the monopolist will design warranty contracts such that $g(\theta, w(\theta))w'(\theta) \geq 0$ for all $\theta$, which leads each consumer to reveal his type truthfully.

## 4. Incentive Compatibility and Optimal Warranty Contracts

In this section, we analyze the first stage of the game. Using the input decisions by both parties in the second stage, the induced expected utility for a consumer of type $\theta$ who chooses a contract $(p, w)$ can be written as

$$U(p, w; \theta) \equiv \pi[\hat{q}(w), \hat{e}(\theta - w)](\theta - w) + w - p - g[\hat{e}(\theta - w)]$$

Similarly, the induced monopolist's profit from a consumer of type $\theta$ who chooses a contract $(p, w)$ is

$$V(p, w; \theta) \equiv p - c[\hat{q}(w)] - \left(1 - \pi[\hat{q}(w), \hat{e}(\theta - w)]\right) w$$

At the first stage, the monopolist and the consumer behave as principal and agent, yielding a principal-agent model with adverse selection. The monopolist will offer a contract for each consumer to maximize his induced expected profits subject to consumer's incentive and individual rationality constraints. By the revelation principle, we can consider the truth-telling mechanism $[p(\theta), w(\theta)]$ without loss of generality. Then, the expected utility for each consumer of type $\theta$, which is

$$U(\tilde{\theta}; \theta) = \pi\{\hat{q}[w(\tilde{\theta})], \hat{e}[\theta - w(\tilde{\theta})]\}[\theta - w(\tilde{\theta})] + w(\tilde{\theta}) - p(\tilde{\theta}) - g\{\hat{e}[\theta - w(\tilde{\theta})]\},$$

will be maximized by reporting truth $(\tilde{\theta} = \theta)$. The first order condition is

$$\frac{\partial U(\tilde{\theta}; \theta)}{\partial \tilde{\theta}} \Big|_{\tilde{\theta} = \theta} = -p'(\theta) + \left(\pi_q \hat{q}_w[\theta - w(\theta)] + 1 - \pi\right) w'(\theta) = 0 \qquad (8)$$

for all $\theta \in (\underline{\theta}, \overline{\theta})$. The second order condition is

$$\frac{\partial^2 U(\tilde{\theta};\theta)}{\partial \tilde{\theta}^2}\Big|_{\tilde{\theta}=\theta} = -p''(\theta) + \left[\pi_{qq}\hat{q}_w^2[\theta - w(\theta)] + \pi_q \hat{q}_{ww}[\theta - w(\theta)] - 2\pi_q \hat{q}_w - \pi_e \hat{e}_w\right] w'^2(\theta) \tag{9}$$

$$+ \left[\pi_q \hat{q}_w [\theta - w(\theta)] + 1 - \pi\right] w''(\theta) \le 0.$$

Taking the derivative of (8) with respect to $\theta$ and using the fact that $\hat{e}_w = -\hat{e}_\theta$, we can rewrite (9) as

$$\frac{\partial^2 U(\tilde{\theta};\theta)}{\partial \tilde{\theta}^2}\Big|_{\tilde{\theta}=\theta} = -(\pi_q \hat{q}_w + \pi_e \hat{e}_w) w'(\theta) \le 0.$$

Using (7), this can be written as

$$\frac{\partial \pi}{\partial w} w'(\theta) = g(\theta, w(\theta)) w'(\theta) \ge 0. \tag{10}$$

Therefore, we do have the restriction on $w(\theta)$ that we discussed at the end of the last section. In order to interpret this condition, notice that

$$\frac{\partial U}{\partial w} = \pi_q \hat{q}_w (\theta - w) + 1 - \pi, \qquad \frac{\partial U}{\partial p} = -1,$$

and

$$\frac{\partial^2 U}{\partial w \partial \theta} = \pi_q \hat{q}_w - \pi_e \hat{e}_\theta = g(\theta, w).$$

The incentive compatibility condition (10) implies

$$\frac{\partial}{\partial \theta}\left(-\frac{\partial U}{\partial w} \Big/ \frac{\partial U}{\partial p}\right) w'(\theta) \ge 0.$$

That is, when $\theta$ changes, the warranty and the marginal rate of substitution between the warranty and price have the same sign. Condition (10) is an extension of the usual monotonicity condition and refers to compatibility between the consumer's preferences and the contract $[p(\theta), w(\theta)]$ offered by the monopolist.[5]

Let $U(\theta) \equiv U(\theta; \theta)$ be the maximum expected utility for the consumer of type $\theta$. From the envelope theorem, the necessary condition (8) is equivalent to

$$U'(\theta) = \pi\{\hat{q}[w(\theta)], \hat{e}[\theta - w(\theta)]\} \equiv \pi(\theta). \tag{11}$$

Then $\pi'(\theta) = g(\theta, w(\theta)) w'(\theta) + \frac{\partial \pi}{\partial \theta} > 0$. We have shown the following

***Proposition 1***: Local incentive compatibility implies: i) $g(\theta, w(\theta))w'(\theta) \geq 0$; ii) $\pi(\theta)$ is strictly increasing in $\theta$; and iii) $U(\theta)$ is a strictly increasing and convex function.

Since $g(\theta, w(\theta))$ may change its sign over the range of $\theta$, the necessary condition of incentive compatibility does not necessarily imply monotonicity of $w(\theta)$. Especially, $w(\theta)$ and also $p(\theta)$ may not be increasing in $\theta$. This is different from what Mussa and Rosen [1978] and Matthews and Moore [1987] have found. But observed product reliability $\pi(\theta)$ is always strictly increasing in $\theta$ under the incentive requirements.

Condition i) in Proposition 1 may not be sufficient for a contract $[w(\theta), p(\theta)]$ to be incentive compatible. We will consider two extreme cases each of which gives the consumer's utility curves the single-crossing property. In the first case $g(\theta, w) > 0$ for all $\theta$ and $w$. That is, product quality is more important than consumer care in determining product reliability. Since $\frac{\partial}{\partial \theta}(-\frac{\partial U}{\partial w} / \frac{\partial U}{\partial p}) > 0$, higher type consumers have steeper indifference curves in the $(p, w)$ space. The marginal rate of substitutions are ordered by type. Since the contracts we consider here include only two attributes, this ordering property is equivalent to the well known single-crossing property under which any distinct pair of the utility curves $U(p, w; \theta)$ and $U(\tilde{p}, \tilde{w}; \theta)$ intersect only once on the interval $[\underline{\theta}, \bar{\theta}]$.[6] In this case, the necessary condition of incentive compatibility is simply $w'(\theta) \geq 0$ and, as we will show, it is also sufficient. For the second case, $g(\theta, w) < 0$ for all $\theta$ and $w$. That is, consumer effort is the more important determinant of product reliability and higher type consumers have flatter indifference curves. The incentive condition will now be $w'(\theta) \leq 0$.

***Proposition 2***: Assume $p(\theta)$ satisfies (8). If $g(\theta, w) > 0$ for any $(\theta, w)$ then $w'(\theta) \geq 0$ for all $\theta$ is sufficient for the contract $[p(\theta), w(\theta)]$ to be incentive compatible. If $g(\theta, w) < 0$ for all $(\theta, w)$ then the sufficient condition is $w'(\theta) \leq 0$ for all $\theta$.

Proof: We only prove the result for the first case. The second case is proven similarly. Since we only consider continuous and piecewise differentiable contracts $[p(\theta), w(\theta)]$, the consumer's expected utility $U(\tilde{\theta}; \theta)$ is also continuous and piecewise differentiable. Let

$$\phi(\tilde{\theta}; \theta) = \pi_q \hat{q}_w [\theta - w(\tilde{\theta})] + 1 - \pi\{\hat{q}[w(\tilde{\theta})], \hat{e}[\theta - w(\tilde{\theta})]\},$$

then $\frac{\partial \phi(\tilde{\theta}; \theta)}{\partial \theta} = g[\theta, w(\tilde{\theta})] > 0$ for any $\theta$ and $\tilde{\theta}$, and $\frac{\partial U(\tilde{\theta}; \theta)}{\partial \tilde{\theta}} = -p'(\tilde{\theta}) + \phi(\tilde{\theta}; \theta)w'(\tilde{\theta})$ for all $\theta$ and $\tilde{\theta}$. If $w'(\tilde{\theta}) = 0$, then $\frac{\partial U(\tilde{\theta}; \theta)}{\partial \tilde{\theta}} = 0$. For any $\tilde{\theta} < \theta$ and $w'(\tilde{\theta}) > 0$, we get

$$\frac{\partial U(\tilde{\theta}; \theta)}{\partial \tilde{\theta}} > -p'(\tilde{\theta}) + \phi(\tilde{\theta}; \tilde{\theta})w'(\tilde{\theta}) = 0$$

since $\phi(\tilde{\theta}; \theta) > \phi(\tilde{\theta}; \tilde{\theta})$ and $[p(\tilde{\theta}), w(\tilde{\theta})]$ satisfies (8). Similarly, for any $\tilde{\theta} > \theta$ and $w'(\tilde{\theta}) > 0$, we obtain

$$\frac{\partial U(\tilde{\theta};\theta)}{\partial \tilde{\theta}} < -p'(\tilde{\theta}) + \phi(\tilde{\theta},\tilde{\theta})w'(\tilde{\theta}) = 0.$$

These results together with the continuity of $U(\tilde{\theta};\theta)$ imply that $U(\tilde{\theta};\theta)$ is maximized at $\tilde{\theta} = \theta$.

Q.E.D.

Since the monopolist is uncertain about the consumer's taste for the product he sells, he faces a screening problem. Incentive compatibility requires that the price $p(\theta)$, warranty $w(\theta)$ and quality level $\hat{q}[w(\theta)]$ be nondecreasing in the consumer's type $\theta$ when the monopolist's input is the less costly and/or more important determinant of product reliability. But when the consumer's effort is less costly and/or more important, the results switch. This switching result can be explained as the joint outcome of adverse selection and bilateral moral hazard. The presence of bilateral moral hazard really changes the behavior of the consumer's utility function.

Recognizing the incentive of the consumers, the monopolist maximizes its expected profit subject to the incentive and individual rationality constraints. Since $U(\theta)$ is increasing in $\theta$ from Proposition 1, the individual rationality constraints $U(\theta) \geq 0$ for all $\theta$ can be replaced by $U(\underline{\theta}) = 0$. From (11), integrating by parts, we obtain

$$\int_{\underline{\theta}}^{\bar{\theta}} U(\theta)f(\theta)d\theta = \int_{\underline{\theta}}^{\bar{\theta}} \pi\{\hat{q}[w(\theta)], \hat{e}[\theta - w(\theta)]\}[1 - F(\theta)]d\theta$$

From the definition of $U(\theta)$, we know that $p(\theta) = \pi(\theta)[\theta - w(\theta)] + w(\theta) - g[\hat{e}(\theta - w(\theta))] - U(\theta)$. Combining these two expressions with the monopolist's profit function, we can write the monopolist's expected profit as $\int_{\underline{\theta}}^{\bar{\theta}} H[\theta, w(\theta)]f(\theta)d\theta$, where

$$H(\theta, w) = \pi[\hat{q}(w), \hat{e}(\theta - w)][\theta - Z(\theta)] - g[\hat{e}(\theta - w)] - c[\hat{q}(w)]$$

and $Z(\theta) = [1 - F(\theta)]/f(\theta)$ is the hazard rate. Thus, the optimization problem for the monopolist is

$$(P) \quad \underset{w}{Max} \int_{\underline{\theta}}^{\bar{\theta}} H[\theta, w(\theta)]f(\theta)d\theta$$

subject to $w(\theta) \geq 0$ and incentive constraint: (I) $w'(\theta) \geq 0$ for all $\theta$. For the second case, the optimization problem is to maximize $\int_{\underline{\theta}}^{\bar{\theta}} H[\theta, w(\theta)]f(\theta)d\theta$ subject to $w(\theta) \geq 0$ and incentive constraint; (II) $w'(\theta) \leq 0$ for all $\theta$. Let $\hat{w}(\theta)$ to be the solution to the equation $H_w(\theta, w) = 0$, that is, $\hat{w}(\theta)$ satisfies

$$\left([\theta - Z(\theta)]\pi_q - c_q\right)\hat{q}_w + \left([\theta - Z(\theta)]\pi_e - g_e\right)\hat{e}_w = 0 \tag{12}$$

Substituting equation (5) and (6) into (12) and rearranging this equation, we get

$$\hat{w}(\theta) = Z(\theta) + [\theta - 2Z(\theta)]r[\theta, \hat{w}(\theta)] \tag{13}$$

where $r(\theta, w) \equiv r[\hat{q}(w), \hat{e}(\theta - w)]$ and $r(q, e) \equiv \dfrac{S(q)}{S(q) + B(e)}$. From (13), the derivative of $\hat{w}$ with respect to $\theta$ is

$$\hat{w}'(\theta) = \frac{Z'(\theta)(1 - 2r) + r + [\theta - 2Z(\theta)]r_\theta}{1 - [\theta - 2Z(\theta)]r_w}$$

In the special case where $S$ and $B$ are constants, which implies $r$ is a constant as well, $\hat{w}'(\theta) = Z'(\theta)(1 - 2r) + r$. Since we assume that $Z'(\theta) \le 0$, then $\hat{w}'(\theta) > 0$ when $r \ge 1/2$. That is, $\hat{w}(\theta)$ satisfies the incentive constraints $w'(\theta) \ge 0$. Therefore, $\hat{w}(\theta)$ is a solution to $(P)$ when $r \ge 1/2$. That is, we obtain the optimal solution $w^*(\theta) = \hat{w}(\theta)$. But when $r$ , 1/2, the incentive constraints $w'(\theta) \le 0$ may not be satisfied by $\hat{w}(\theta)$. Therefore, $\hat{w}(\theta)$ is not optimal solution. In order to get a complete solution to the optimization problems for the monopolist in general, we make the following assumption:

Assumption A: $H(\theta, w)$ is twice continuously differentiable and strictly concave in $w$, and $\hat{w}'(\theta)$ changes sign a finite number of times.

If $\pi(q, e)$, $g(e)$ and $c(q)$ are three times continuously differentiable and $f(\theta)$ is twice continuously differentiable, then $H(\theta, w)$ is twice continuously differentiable. We can calculate $H_{ww} = -\left[S[\hat{q}(w)] + B[\hat{e}(\theta - w)]\right]\left[1 - [\theta - 2Z(\theta)]r_w\right]$. When $\pi_{qqq} \le 0$, $\pi_{eee} \le 0$, $g_{eee} \ge 0$, and $c_{qqq} \ge 0$, then $r_w \le 0$. When $[\underline{\theta} f(\underline{\theta}) - 2]r_w \le f(\underline{\theta})$, we obtain $1 - [\theta - 2Z(\theta)]r_w \ge 0$ and thus $H$ is concave in $w$.

In the first case where $g(\theta, w) > 0$ for all $(\theta, w)$, the optimization problem is

$$(P_I) \qquad \underset{w}{Max} \int_{\underline{\theta}}^{\bar{\theta}} H[\theta, w(\theta)]f(\theta)d\theta$$

$$s.t. \qquad w'(\theta) \ge 0$$

$$w(\theta) \ge 0$$

Under Assumption A, Guesnerie and Laffont [1984] have shown the existence and uniqueness of a solution to a problem similar to $(P_I)$ without the restriction $w(\theta) \ge 0$. The results are similar with this restriction. We can state the results as the following:

***Proposition 3:*** Under Assumption A, there exists a unique piecewise differentiable solution $w^*(\theta)$ to $(P_I)$. When $w^*(\theta)$ is increasing, $w^*(\theta)$ is the same as $\hat{w}(\theta)$ which is determined by the equation (13). When $w^*(\theta)$ equals a constant $\tilde{w}$ on some subinterval $[\theta_1, \theta_2]$, then $\tilde{w} = \hat{w}(\theta_1) = \hat{w}(\theta_2)$, unless $\theta_i \in \{\underline{\theta}, \bar{\theta}\}$ for some $i = 1, 2$ in which case $\tilde{w} = \hat{w}(\theta_j)$, where $j \ne i$. When $w^*(\theta)$ is constant on $[\underline{\theta}, \bar{\theta}]$, then $w^*(\theta) = \tilde{w}$ if there exists a $\tilde{w} > 0$ such that $\int_{\underline{\theta}}^{\bar{\theta}} H_w(\theta, \tilde{w})d\theta = 0$, and $w^*(\theta) = 0$ otherwise.

The proof is similar to the proof of Theorem 4 in Guesnerie and Laffont [1984]. In an example we present in the next section, we are able to find an analytical solution $w^*(\theta)$. It is difficult to get a solution to $(P_I)$ explicitly in general. Guesnerie and Laffont offer a constructive algorithm for the optimal solution $w^*(\theta)$.

Given the optimal warranty $w^*(\theta)$, we can calculate the consumer's maximum expected utility $U^*(\theta)$ from (11) and the optimal price offer $p^*(\theta)$ as well. The price $p^*(\theta)$ is also piecewise differentiable and nondecreasing in $\theta$. From (8), when $w^*(\theta)$ is strictly increasing in $\theta$, $p^*(\theta)$ is strictly increasing in $\theta$ also. $w^*(\theta)$ and $p^*(\theta)$ are constants at the same range of $\theta$. Thus, we have an optimal contract $[p^*(\theta), w^*(\theta)]$ which is pooling for a set of consumers and separating for the other consumers. That is, the monopolist offers a menu of contracts $p = P(w)$, from which a set of consumers choose the same price and warranty coverage, but the others choose different pricies and warranty coverages.

In the second case, we need to solve the following problem:

$$(P_{II}) \quad \underset{w}{Max} \int_{\underline{\theta}}^{\bar{\theta}} H[\theta, w(\theta)]f(\theta)d\theta$$

$$s.t. \quad w'(\theta) \leq 0$$

$$w(\theta) \geq 0$$

The same technique in Guesnerie and Laffont [1984] can be applied here. Under Assumption A, there exists a unique solution $w^*(\theta)$ to $(P_{II})$ which is nonincreasing in $\theta$. When $w^*(\theta)$ is decreasing (separating) within a subset of $[\underline{\theta}, \bar{\theta}]$, it equals $\hat{w}(\theta)$. When $w^*(\theta)$ is pooling, it is determined in the same way as that in Proposition 3. Similarly, the optimal price $p^*(\theta)$ can be calculated.

In summary, we have determined the monopoly solution $p^*(\theta)$ and $w^*(\theta)$ in two extreme cases. In the first case we discussed above, price and warranty are nondecreasing in the consumer's type. That is, higher type consumers purchase the more expensive product and get higher warranty coverage. The quality of product is higher as well. In the second case where consumer effort is the less costly and/or more important determinant of the product reliability, the results switch. That is, price, warranty, and quality are nonincreasing in the consumer's type. This switching result is different from Matthews and Moore's [1987] finding that the price is always nondecreasing in the consumer's type. The difference comes from the fact that no moral hazard is considered in Matthews and Moore's model. In our model, consumer effort which is not observable to the seller influences the probability of product malfunction. When this influence is very significant, consumer's desires to have the product not fail may cause them to extend sufficient effort to make the warranty of relative little value to them. As a result, consumers who more highly value successful functioning of the product may purchase a version of the product with lower warranty and lower price and that is therefore of lower quality.

In Mann and Wissink's [1986] model of the competitive case with the finite type of consumers, they get similar switching results which relate to the single-crossing-property. But in their model, the correlation between the product reliability and warranty coverage is ambiguous. In our model, this correlation is positive or negative dependent on whether product quality or consumer effort is the less costly and/or the more important determinant of product reliability, respectively. From i) in Proposition 1, our model predicts that a higher product reliability is always observed for a consumer with higher evaluation of the product.

Because of the interactions of asymmetric information and bilateral moral hazard, the monopolist may not be able to separate consumer types. Pooling for different consumers happens at equilibrium in our model. This is different from Mann's [1986] result that the contracts offered by a monopolist separate consumer types whenever the competitive equilibrium involves a pooling equilibrium. When we get a pooling equilibrium, the optimal warranty may be bigger than the consumer's evaluation of the product, that is, a full warranty coverage or even over-coverage is offered for low type consumers.

We now consider the effects of incomplete information on the separating solution. When the monopoly solution $w^*(\theta)$ is separating, it is the same as $\hat{w}(\theta)$ which satisfies (13). As $Z(\theta)$ approaches zero, i.e. the complete information case, (13) becomes $\hat{w}(\theta) = \theta r[\theta, \hat{w}(\theta)]$. Because of bilateral moral hazard, $r$ is strictly between 0 and 1 when the consumers expend positive efforts, and thus the monopolist offers partial warranty coverage. This is the argument in Cooper and Ross's [1985] paper. When $Z(\theta)$ is positive, i.e. the incomplete information case, we have a similar result for high type consumers at a separating equilibrium.

***Proposition 4***: When the optimal warranty $w^*(\theta)$ is separating, it has the following properties: i) if $\underline{\theta} f(\underline{\theta}) > 2$, then $Z(\theta) < w^*(\theta) < \theta - Z(\theta)$ for all $\theta \in [\underline{\theta}, \bar{\theta}]$; ii) if $\underline{\theta} f(\underline{\theta}) \le 2$, then there exists a type $\theta^* \in [\underline{\theta}, \bar{\theta}]$ such that $w^*(\theta^*) = \theta^*/2$, $Z(\theta) < w^*(\theta) < \theta - Z(\theta)$ for $\theta > \theta^*$, and $\theta - Z(\theta) < w^*(\theta) < Z(\theta)$ for $\theta < \theta^*$.

*Proof*: i) If $\underline{\theta} f(\underline{\theta}) > 2$, then $\theta - 2Z(\theta) \ge \underline{\theta} - 2Z(\underline{\theta}) = \underline{\theta} - 2/f(\underline{\theta}) > 0$ for all $\theta \in [\underline{\theta}, \bar{\theta}]$. From (13), we obtain $Z(\theta) < w^*(\theta) < \theta - Z(\theta)$ for all $\theta$.

ii) The condition $\underline{\theta} f(\underline{\theta}) \le 2$ is the same as $\underline{\theta} - 2Z(\underline{\theta}) \le 0$. Since $\bar{\theta} - 2Z(\bar{\theta}) = \bar{\theta} > 0$ and $\theta - 2Z(\theta)$ is increasing, then there exists a type $\theta^* \in [\underline{\theta}, \bar{\theta}]$ such that $\theta^* - 2Z(\theta^*) = 0$. Thus, from (13), $w^*(\theta^*) = Z(\theta^*) = \theta^*/2$. When $\theta > \theta^*$, $Z(\theta) < w^*(\theta) < \theta - Z(\theta)$. When $\theta < \theta^*$, $\theta - Z(\theta) < w^*(\theta) < Z(\theta)$.

Q.E.D.

From the above proof, we notice that the condition $\underline{\theta} f(\underline{\theta}) > 2$ implies $Z(\theta) < \theta/2$ which means the hazard rate due to unobservability of the consumer's willingness to pay of the product is relatively low. In this case, $0 < w^*(\theta) < \theta$ for all $\theta$; that is, there is a separating equilibrium with only a partial warranty coverage when the hazard rate is relatively low. However, when the hazard rate is high, consumers are more diverse. In this case, high type consumers ($\theta \ge \theta^*$) will also buy partial warranty coverage but the range of that coverage is more tightly bounded because $w^*(\theta)$ is now between $Z(\theta)$ and $\theta - Z(\theta)$. Low type consumers either do not buy the product or buy the product with full warranty coverage. Which they do, depends on whether consumer's care or producer's precaution is the more important determinant of reliability. We will show how this happens in the next section by considering a simple example.

Finally, we should notice that we have only solved the problem of monopoly provision of product warranties for two extreme cases. In the intermediate case where $g(\theta, w)$ may change its sign over $\theta$ and $w$, we are not able to find a necessary and sufficient condition for incentive compatibility. Even if the local incentive constraint (10) is sufficient for incentive compatibility, it is difficult to solve the monopolist's optimal control problem with this constraint. We need further study on this problem.

## 5. An Example

In this section, we consider a simple linear-quadratic case to illustrate the results we have shown in the above sections. We assume

$$\pi(q, e) = \alpha e + \beta q$$

and the cost of quality and disutility of effort are quadratic, that is,

$$c(q) = \gamma q^2, \qquad g(e) = \delta e^2$$

where $\alpha$, $\beta$, $\delta$ and $\gamma$ are all positive parameters, $e \in [0, 1/2\alpha]$, and $q \in [0, 1/2\beta]$. For convenience, we normalize the constants and assume that $\beta^2/2\gamma + \alpha^2/2\delta = 1$. Then $r$ defined in Section 3 is a constant, that is, $r = \beta^2/2\gamma$. We also assume that the consumer's type $\theta$ is uniformly distributed on $[0, 1]$, that is, $F(\theta) = \theta$ for all $\theta \in [0, 1]$.

As we discussed before, our model can be viewed as a two-stage game played between the monopolist and consumers. In the second stage of the game, each party chooses the optimal input level $\hat{q}$ and $\hat{e}$ given a price-warranty offer $(p, w)$. From (5) and (6), we get

$$\hat{q}(w) = \frac{\beta w}{2\gamma}$$

and

$$\hat{e}(\theta - w) = \frac{\alpha(\theta - w)}{2\delta}$$

Then $\hat{q}_w = \beta/2\gamma > 0$, $\hat{e}_w = -\hat{e}_\theta = -\alpha/2\delta < 0$ and $\pi = \alpha^2\theta/2\delta + (\beta^2/2\gamma - \alpha^2/2\delta)w$. Since $\frac{\partial \pi}{\partial w} = -\alpha^2/2\delta + \beta^2/2\gamma = 2r - 1$, for a given type of consumer the correlation between product reliability and warranty is not necessarily positive. When $r$ is small, that is, the consumer's care is more important for product reliability than the seller's contribution, observed product reliability will be negatively correlated with the warranty coverage. We can calculate the induced utility function for both the consumer and the monopolist as follows:

$$U(p, w; \theta) = -(3r - 1)w^2/2 + (2r - 1)\theta w + (1 - r)\theta^2/2 + w - p$$

$$V(p, w; \theta) = (3r - 2)w^2/2 + (1 - r)\theta w - w + p$$

Both are quadratic in the warranty coverage $w$. The local incentive compatibility constraints (10) will be $(2r - 1)w'(\theta) \geq 0$. As we showed in Proposition 1, when $r > 1/2$, $w'(\theta) \geq 0$ for all $\theta$ are necessary and sufficient for incentive compatibility. When $r < 1/2$, the incentive condition is that $w'(\theta)$ is nonincreasing in $\theta$. In the case where $r = 1/2$, incentive compatibility does not place any direct restriction on the warranty $w(\theta)$ except for equation (8) and (11). The monopolist's optimization problem is then the following:

$$(\hat{P}) \quad \underset{w}{Max} \int_0^1 \left[ -w^2(\theta)/2 + [(3r - 1)\theta + 1 - 2r]w(\theta)\right] d\theta$$

$$s.t. \quad (2r - 1)w'(\theta) \geq 0$$

$$w(\theta) \geq 0$$

***Proposition 5:*** There exists a unique piecewise differentiable solution $w^*(\theta)$ to $(\hat{P})$: i) if $r < 1/3$, then $w^*(\theta) = (3r - 1)\theta + 1 - 2r$ for all $\theta \in [0, 1]$; ii) if $1/3 \leq r < 1/2$, then $w^*(\theta) = (1 - r)/2$; iii) if $r \geq 1/2$, then

$$w^*(\theta) = \begin{cases} 0 & \text{if } \theta \leq (2r - 1)/(3r - 1) \\ (3r - 1)\theta + 1 - 2r & \text{if } \theta > (2r - 1)/(3r - 1) \end{cases}$$

Proof: See Appendix.

We have to check global incentive compatibility for the case $r = 1/2$. Since $w^*(\theta) = \theta/2$, we are able to calculate the maximum expected utility $U^*(\theta) = \theta^2/4$ and the optimal price $p^*(\theta) = \theta/2 - \theta^2/16$ for all $\theta$ from (11). Under the contract $[p^*(\theta), w^*(\theta)]$, the expected utility for the consumer of type $\theta$ who reports $\tilde{\theta}$ is $U(\theta; \tilde{\theta}) = U(\theta; p^*(\tilde{\theta}), w^*(\tilde{\theta})) = \theta^2/4$, which is independent of $\tilde{\theta}$. Each consumer of type $\theta$ gets the same utility no matter what he reports. Thus, consumers will choose any pair $(p, w)$ from the price-warranty schedule $p = w - w^2/4$.

From Proposition 5, when the producer's precaution in reliability is relatively important ($r$ is larger), high type consumers buy partial warranty policy, and low type consumers do not buy the product at all. When $r$ is very small, low type consumers buy too much coverage. The interesting case is when $1/3 \leq r < 1/2$ and the optimal contract involves pooling. The monopolist prefers not to discriminate among heterogeneous consumers and thus each consumer buys a version of the product with the same price-warranty coverage. In this case, optimal warranty policy makes low type consumers buy too much coverage.

Finally, we should point out that there is a discontinuity of our model. The optimal warranty policy $w^*(\theta)$ is not continuous at $r = 1/2$. When $r$ approaches, but is less than, 1/2, $w^*(\theta)$ approaches 1/4 for all $\theta$. When $r$ approaches, but larger than, 1/2, then $w^*(\theta)$ approaches $\theta/2$ for all $\theta$. We do not yet have a good explanation for this discontinuity.

## 6. Concluding Remarks

We have considered the problem of monopoly provision of product warranties in the face of both bilateral moral hazard and adverse selection problems. Buyers are heterogeneous according to their willingness to pay for the good in our model. The results indicate that, whether higher value buyers buy products with higher quality and fuller warranties or buy lower quality and lower warranties, depends upon the relative contributions made by seller quality and buyer care in enhancing product reliability. If seller quality is a more important determinant of reliability for most products, a positive correlation between warranty coverage and reliability will be observed in the data and higher type buyers will purchase more expensive products with higher warranties. If consumer care is more important in determining reliability, the results switch. This shows us that the correlation between the warranties and observed reliability may be negative for some products and that consumers who value the product more may still buy the version of the product with low warranty and low quality. Optimal pooling contracts may also exist. The monopoly seller may not be able to discriminate among. heterogeneous buyers because of the complex interactions between bilateral moral hazard and adverse selection problems. But for any product, our model predicts that higher product reliability is always observed to be correlated with higher types of buyers.

We have introduced moral hazard into a principal-agent problem with adverse selection. Both the principal and the agent face a moral hazard due to the unobservability of the other's action. As we have seen, bilateral moral hazard and adverse selection interact in very complex and interesting ways. We are only able to solve the problem for two extreme cases in this paper. For the more general case, in which the agent's utility curves for different contracts are not of the single-crossing type, it is difficult to identify the necessary and sufficient conditions for incentive compatibility by using the local (differential) approach. The non-local approach along the line taken by Matthews and Moore [1987] might be useful. Further study on this topic is needed.

## *Appendix*

***Proof of Proposition 5***: Let $\hat{w}(\theta) = (3r - 1)\theta + 1 - 2r$ and $a(\theta) = (2r - 1)w'(\theta)$. Then problem $(\hat{P})$ can be written equivalently as

$$(\hat{P}') \quad \underset{w,\, a}{Max} \int_0^1 \left[ -\frac{1}{2}w^2(\theta) + \hat{w}(\theta)w(\theta) \right] d\theta$$

$$s.t. \quad (2r - 1)w'(\theta) = a(\theta)$$

$$w(\theta) \geq 0, \quad a(\theta) \geq 0$$

If $r = \frac{1}{2}$, we can get a solution $w^*(\theta) = \hat{w}(\theta) = \frac{1}{2}\theta$ for all $\theta \in [0, 1]$. If $r \neq \frac{1}{2}$, let

$$\hat{H}(\theta, w, a, \lambda) \equiv -\frac{1}{2}w^2 + \hat{w}(\theta)w + \frac{\lambda a}{2r - 1}$$

is the Hamiltonian to $(\hat{P}')$ with the state variable $w$, the control variable $a$, and the multiplier $\lambda$. Then the necessary conditions for the optimal control problem $(\hat{P}')$ are the following (see Hadley and Kemp [1971], Theorem 5.4.1, p.291):

$$w'(\theta) = a(\theta)/(2r - 1)$$

$$\lambda'(\theta) = w(\theta) - \hat{w}(\theta) + \mu_2(\theta)$$

$$\mu_1(\theta) = \lambda(\theta)/(2r - 1) \leq 0, \quad \mu_2(\theta) \leq 0$$

$$\mu_1(\theta)a(\theta) = \mu_2(\theta)w(\theta) = 0$$

$$\lambda(0) = \lambda(1) = 0$$

where $\mu_i(\theta)$ is continuous in $\theta$. Since $\hat{H}$ is concave in $(w, a)$, and $R_1 = -a$, $R_2 = -w$ are all convex in $(w, a)$, the sufficient conditions are satisfied (see Hadley and Kemp [1971], Theorem 5.7.1, p.298).

We first consider the case in which $r > \frac{1}{2}$. When $\theta \leq \theta_0 \equiv (2r - 1)/(3r - 1)$, $\hat{w}(\theta) \leq 0$. Then, $w^*(\theta) = 0$ for all $\theta \leq \theta_0$. If not, there exists a $\theta' \leq \theta_1$ such that $w^*(\theta') > 0$, which implies

$\mu_2(\theta') = 0$ and $\lambda'(\theta') = w^*(\theta') - \hat{w}(\theta') > 0$. Since $\lambda(0) = 0$ and $\lambda(\theta) \le 0$, there exists a $\theta''(0 < \theta'' < \theta')$ such that $\lambda'(\theta'') < 0$. This implies $w^*(\theta'') = 0$. Since should be $w^*(\theta)$ is nondecreasing, $w^*(\theta) = 0$ for all $\theta \le \theta''$. For all $\theta$ such that $\lambda'(\theta) \le 0$, we also get $w^*(\theta) = 0$. For $\theta$ such that $\lambda'(\theta) > 0$, $\lambda(\theta) \ne 0$ and $w^{*'}(\theta) = 0$. By continuity of $w^*(\theta)$, $w^*(\theta) = 0$. It is also true for $\theta'$, that is, $w^*(\theta') = 0$. This contradicts $w^*(\theta') = 0$. Thus, $w^*(\theta) = 0$ for all $\theta \le \theta_0$.

When $\theta > \theta_0$, $0 < \hat{w}(\theta) < \theta$. Then $\mu_2(\theta) = 0$ for all $\theta > \theta_0$. Otherwise, there is a $\theta'$ such that $w^*(\theta'') = 0$ and $\lambda'(\theta') < 0$. Since $\lambda(1) = 0$, there exists a $\theta'' > \theta_0$ such that $\lambda'(\theta'') > 0$. For all $\theta$ such that $\lambda'(\theta) > 0$, we get $w^*(\theta) > \hat{w}(\theta)$ and $w^{*'}(\theta) = 0$. For all $\theta$ such that $\lambda'(\theta) < 0$, which includes $\theta'$, $w^*(\theta) = 0$. This contradicts the continuity of $w^*(\theta)$. Thus, $\lambda'(\theta) = w^*(\theta) - \hat{w}(\theta)$. Then we are able to show $\lambda(\theta_0) = 0$. If not, because of the continuity of $\lambda(\theta)$, there exists an open interval $I \in [0, 1]$ such that $\theta_0 \in I$ and $\lambda(\theta) < 0$ for any $\theta \in I$ which implies $a(\theta) = 0$ and $w^{*'}(\theta) = 0$ for any $\theta \in I$. The continuity of $w^*(\theta)$ implies $w^*(\theta) = 0$ for all $\theta \in I$. For $\theta \in I$ and $\theta > \theta_0$, $\lambda'(\theta) = -\hat{w}(\theta) < 0$. Since $\lambda(1) = 0$, there exists a $\theta_1$ such that $\lambda'(\theta_1) > 0$ and $\lambda(\theta) < 0$ for all $\theta \in (\theta_0, \theta_1)$. Thus $w^*(\theta)$ is a constant and the continuity of $w^*(\theta)$ implies $w^*(\theta) = 0$ for any $\theta \in (\theta_0, \theta_1)$. Thus $\lambda'(\theta_1) = -\hat{w}(\theta_1) < 0$. This contradicts $\lambda'(\theta_1) > 0$. Therefore $\lambda(\theta_0) = 0$. This implies $\lambda(\theta) = 0$ for any $\theta \in (\theta_0, 1)$ by a similar argument. Thus, $w^*(\theta = \hat{w}(\theta)$.

In the case where $r < \frac{1}{3}$, similar to the above discussion, we get $w^*(\theta) = \hat{w}(\theta)$ for all $\theta \in [0, 1]$.

Finally, we consider the case in which $\frac{1}{3} \le r < \frac{1}{2}$. We claim $a(\theta) = 0$ for all $\theta \in [0, 1]$. If not, there exists an open interval $I \subseteq [0, 1]$ such that $a(\theta) > 0$ for any $\theta \in I$. This implies $\lambda(\theta) = 0$ and $\lambda'(\theta) = 0$ for any $\theta \in I$. We get $w^*(\theta) - \hat{w}(\theta) + \mu_2(\theta) = 0$ which implies $\mu_2(\theta) = 0$ since $\hat{w}(\theta) > 0$. Therefore, $w^*(\theta) = \hat{w}(\theta)$ for any $\theta \in I$. Then $a(\theta) = (2r - 1)w^{*'} = (2r - 1)(3r - 1) \le 0$ for $\theta \in I$. This contradicts $a(\theta) > 0$. Thus, $w^{*'}(\theta) = 0$. Since

$$0 = \lambda(1) = \int_0^1 [w^* - \hat{w}(\theta)]d\theta = w^* - \int_0^1 [(3r - 1)\theta + 1 - 2r]d\theta,$$

we get $w^* = \frac{1}{2}(1 - r)$.

Q.E.D.

## *Footnotes*

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1. Priest [1981] offers an excellent review of the exploitation and signal theories and their practical implications. He also discusses the investment theory in detail.

2. In Cooper and Ross's [1985] paper, they called the hidden actions faced by both the seller and buyer double moral hazard. We prefer to use bilateral moral hazard to emphasize the bilateral nature of each contract and the moral hazard which each party imposes upon the other.

3. In the next section, we will see the importance of this assumption in determining the Nash equilibrium $(\hat{q}, \hat{e})$. Without knowing the consumer's type the monopolist can choose a quality level which depends on the warranty coverage. When $\pi_{qe} \neq 0$, product quality and consumer care are complements or substitutes in the production of reliability and multiple equilibria may arise. Equilibrium selection and coordination become important issues. We do not consider these problems in this paper. See recent paper by Bigelow, Cooper, and Ross [1988] for discussions in this topic.

4. Kambhu [1982] considers both a balanced mechanism and an unbalanced mechanism. In his model of product quality, observed product quality is assumed to be a deterministic function of seller quality and buyer care and hence there is no warranty problem.

5. Guesnerie and Laffont [1984] have a similar condition for a more general principal-agent problem with no moral hazard, which they call a condition of minimal compatibility.

6. Matthews and Moore [1987] have a further discussion about the relationship between the single-crossing property and the ordering of marginal rate of substitutions by type.

*References*

Akerlof, G. A. [1970], "The Market for 'Lemons': Quality Uncertainty and the Market Mechanism," *Quarterly Journal of Economics*, 84, 488- 500.

Bigelow, J., Cooper, R. and Ross T. W. [1988], "Warranties Without Commitments to Market Participation," The Hoover Institution, E-88-3.

Cooper, R. and Ross, T. W. [1985], "Product Warranties and Double Moral Hazard," *The Rand Journal of Economics*, 16, 103-113.

Garvin, D. [1983], "Quality on the Line," *Harvard Business Review*, September / October, 64-75.

Gerner, J. and Bryant, W. K. [1978], "Appliance Warranties as a Market Signal?" *Journal of Consumer Affairs*, 75-83.

Gestner, E. [1985], "Do High Prices Signal Higher Quality?" *Journal of Marketing Research*, XXII, 209-215.

Guesnerie, R. and Laffont, J. J. [1984], "A Complete Solution to a Class of Principal-Agent Problems with an Application to the Control of a Self-Managed Firm," *Journal of Public Economics*, 25, 329- 369.

Hadley, G. and Kemp, M. C. [1971], *Variational Methods in Economics*, North-Holland / American Elsevier.

Heal, G. [1977], "Guarantees and Risk-Sharing," *Review of Economic Studies*, 44, 549-560.

Kambhu, J. [1982], "Optimal Product Quality under Asymmetric Information and Moral Hazard," *The Bell Journal of Economics*, 13 (Autumn), 483-492.

Kessler, [1943], "Contracts of Adhesion--Some Thoughts about Freedom of Contract," *Column. Rev.*, 43, 629.

Mann, D. P. and Wissink, J. P. [1986], "Hidden Actions and Hidden Characteristics in Warranty Markets," mimeo, Cornell University, Ithaca, NY.

Mann, D. P. [1986], "Product Warranties: Variegated Consumers and Buyer Care," mimeo, Rutgers University, New Brunswick, NJ.

Matthews, S. and Moore, J. [1987], "Monopoly Provision of Quality and Warranties: An

Exploration in the Theory of Multidimensional Screening," *Econometrica*, 55, 2, 441-467.

Mussa, M. and Rosen, S. [1978], "Monopoly and Product Quality," *Journal of Economic Theory*, 18, 301-317.

Priest, G. [1981], "A Theory of Consumer Product Warranty," *Yale Law Journal*, 90, 1297-1352.

Spence, A.M. [1977], "Consumer Misperceptions, Product Failure and Product Liability," *Review of Economic Studies*, 44, 561-572.