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NOTES ABOUT THEORY OF PSEUDO–CRITERIA AND BINARY PSEUDO–RELATIONS AND THEIR APPLICATION TO THE THEORY OF CHOICE AND VOTING

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SOCIAL SCIENCE WORKING PAPER 766

June 1991

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Abstract

The pages of this working paper are copies of transparencies used in a lecture on the general theory of choice given at the California Institute of Technology June 1991.

Notes about Theory of Pseudo–Criteria and Binary Pseudo–Relations and Their Application to the Theory of Choice and Voting

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This working paper includes copies of transparencies of a lecture[1] given at the California Institute of Technology in June 1991. The lectures were a summary of a line of investigation that has taken place in the Soviet Union over the past several years. Hopefully the material is enough to provide an understanding of the ideas and results.

The group lead by Professor Mark Aizerman at the Institute of Control Sciences of the USSR Academy of Science during the last ten to twelve years has developed many new ideas concerning the theory of choice and voting. The main line of this group's investigations is to find methods for describing individual and collective choice. In particular, the focus is upon circumstances in which the evaluation of alternatives depends not only on properties of the alternatives but on the full context of a given set of alternatives as well.

The investigation began with studying not an internal description of choice (algorithms) but external (input–output) description of choice by use of the language of choice functions. Later the internal description was also generalized from binary relations (oriented graphs) to group relations (hyper–graphs) [Aizerman, 1985].

During the past several years a new approach has been developed to describe choice algorithms. This approach uses a direct generalization of the notion of a criterion function (or utility function) and the associated "binary relation." These generalizations have been named pseudo–criterion function and binary pseudo–relations, correspondingly.

In order to select the best alternative, we typically construct a numerical scale for evaluating alternatives. Classical choice theory assumes that the numerical evaluation of each alternative depends only upon its properties and remains invariant across all opportunity sets (all representations which include this alternative). Contrary to this classical approach, however, we assume that such numerical evaluations can depend upon context (or the presence or absence of other alternatives in the opportunity set). This feature of choice problems is typical of evaluations generated by Borda counts, "sum of

[1] The financial support of the Laboratory for Experimental Economics and Political Science at the California Institute of Technology is gratefully acknowledged.

points" in tournaments, application of the minmax regret criterion, and for many other well known methods.

The numerical function which evaluates alternatives, taking into account the context, is called a pseudo–criterion (pseudo–value function). We develop the language for describing classes of pseudo–criteria and for investigating choices that use pseudo–criteria.

Natural generalizations of this idea brought us to consideration of binary relations that depend upon context, which we labeled a binary pseudo–relation (a pseudo–graph). These new ideas give us the opportunity to discover new properties of some well known voting procedures such as the Borda count, procedures that use the scale "number of voices for," and Fishburn's procedure, and some others.

Transparencies 1–6 describe the main features of the classical approach to the subjects. Such results are necessary for an understanding of the remaining transparencies and for introducing necessary notations.

Transparencies 7 and 8 explain why further generalizations of the classical approach are necessary.

Transparencies 9–11 provide the main results about the structure of the space of all choice functions and transparency 12 contains the main results using hyper–graphs for the internal description of choice.

Transparencies 13–27 provide the main ideas and results about

pseudo–criteria and their applications. Transparencies 29–38 contain results about binary pseudo–relations and their application to the choice theory.

The reference in transparencies 39–41 includes the main publications in the field.

The authors understand that the notes are not enough to go deeply into this new field and invite any contacts by those who want more material. Correspondence should be directed to

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Notion of "Criterion"

(synonims: "goal function", "utility function", "winning function", "scale estimation" etc.)

$A = \{ x_1, x_2, \dots \quad \}; \; |A| \leq \infty$

objects for estimations
alternatives

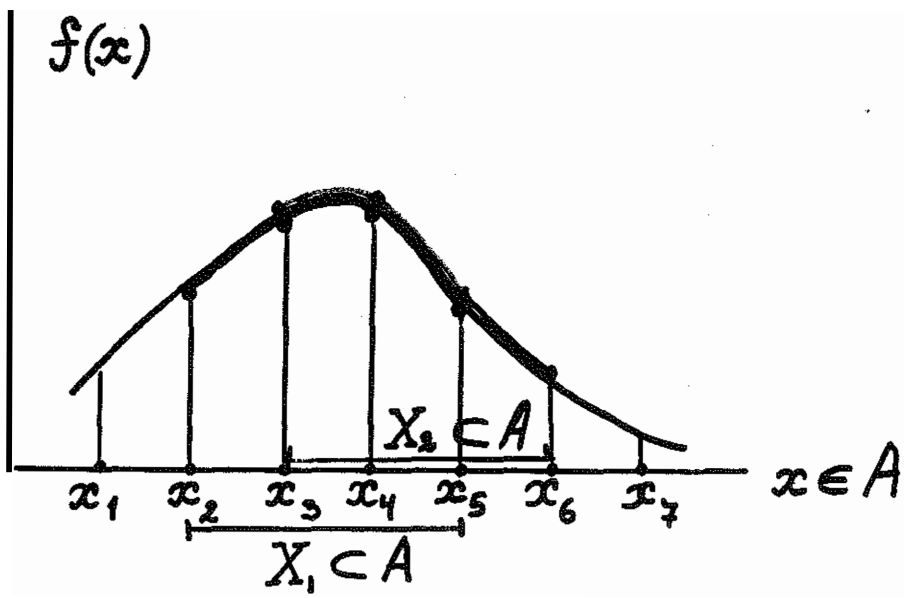
$$\boxed{\begin{array}{ll} A \longrightarrow R & z = \underbrace{f(x)}_{\text{is given } \forall x \in A} \\ \text{numerical or range scale} & \text{numbers} \end{array}}$$

Property of mass character of criterion

$X \subseteq A; \; X \in 2^A$

$$\boxed{\forall X \text{ such that } \; x^* \in X : z = f(x^*) = const}$$

i.e. $f(x^*)$ - criterion estimation of x^* does not depend on $X \ni x^*$ (on "context")

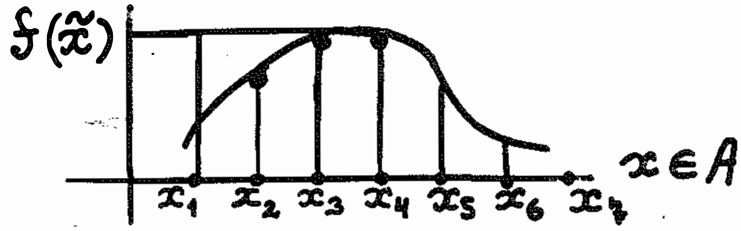


Rules which use criterion

For $\forall X : Y \subseteq X$ is constructed so that

1°. $Y = \{\tilde{x}\}$, where $\tilde{x} = \arg\max\limits_{x \in X} f(x)$

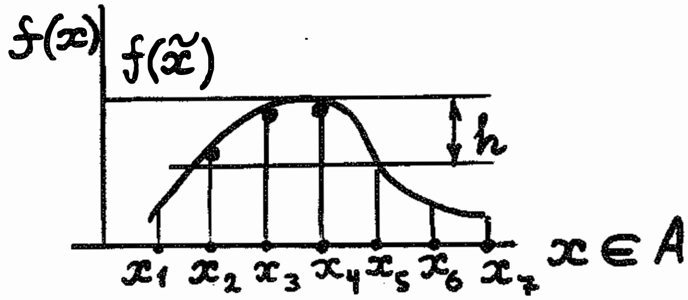
or $\boxed{Y = \{x^* \in X \mid \overline{\exists} x \in X : f(x) > f(x^*)\}}$



$\tilde{x} = x_3$ и x_4

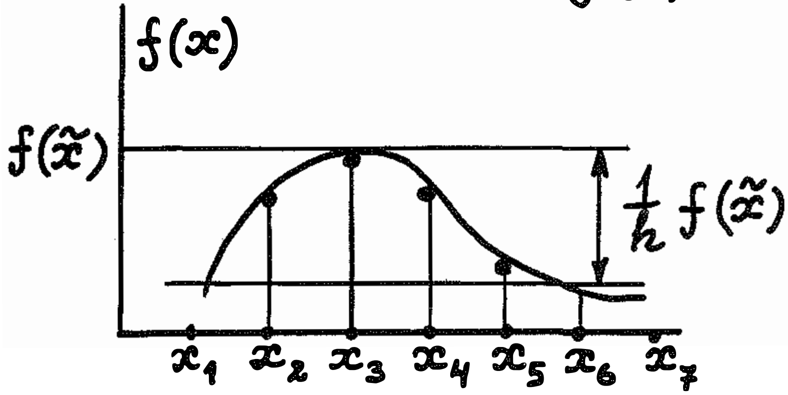
$Y = \{x_3, x_4\}$

2°. $Y = \{x^* \in X \mid \overline{\exists} x \in X : f(x) > f(\tilde{x}) - h\}$



$Y = \{x_2, x_3, x_4\}$

3°. $Y = \{x^* \in X \mid \overline{\exists} x \in X : f(x) > \frac{1}{h} f(\tilde{x})\}$



$Y = \{x_2, x_3, x_4, x_5\}$

etc.

Classical generalizations

First generalization

Set ("vector") of criteria

$$
\begin{array}{lll}
 & \nearrow R_1 & z_1 = f_1(x) \\
A & \rightarrow R_2 & z_2 = f_2(x) \\
 & & \vdots \\
 & \searrow R_n & z_n = f_n(x)
\end{array}
$$

$$\boxed{z = \underbrace{\{f_i(x)\}}_{n \text{ functions}} \quad \forall x \in A}$$

Maintenance of mass character property

$$\boxed{\forall X \subseteq A, \text{ so that } x^* \in X, \text{ for each } i \in [1, 2, \dots, n] : f_i(x^*) = Const_i}$$

i.e. $z = \{f_i(x^*)\}$ – all criterial estimations – do not depend on $X \ni x^*$ (on context)

Rule of obtaining $Y \subseteq X$ is maintained, if in rules 1°, 2° and 3° inequalities $f(x) > \dots$ are true for each component $f_i(x) > \dots$, $\forall i \in [1, 2, \dots, n]$. For example,

$$1^\circ \quad Y = \{x^* \in X \mid \overline{\exists} x \in X \text{ and } \overline{\exists} i : f_i(x) > f_i(x^*)\}$$

$$\downarrow$$

$$\boxed{Y = Par\, X}$$

Second classical generalization

Transition to arbitrary binary relation

Let's recollect rule 1°:

1°_{ext} $$\boxed{Y=\{x^*\in X \mid \bar{\exists} x\in X: \underbrace{f(x)>f(x^*)}_{\text{to be in the relationship "f(x) is greater"}}\}}$$

Arbitrary binary relation $\mathcal{D}$:

$$\boxed{x\mathcal{D}x^*}$$

Rule 1°_{ext} turns into

1°_{dom} $$\boxed{Y=\{x^*\in X \mid \bar{\exists} x\in X: x\mathcal{D}x^*\}}$$

Binary relaton $\mathcal{D}$ and the rule 1°_{dom} pick out from X variants which are named <u>dominants of X</u>.

The idea of mass character generates the notion of <u>equivalence between two procedures of choice</u>: they are equivalent if for each $X \subseteq A$ both procedures of choice produce the same $Y \subseteq X$.

<u>Definition</u>:

$\mathcal{D}$ is transitive $\left.\begin{matrix} x_1 \mathcal{D} x_2 \\ x_2 \mathcal{D} x_3 \end{matrix}\right\} \longrightarrow x_1 \mathcal{D} x_3$

$\mathcal{D}$ is nego-transitive $\left.\begin{matrix} x_1 \overline{\mathcal{D}} x_2 \\ x_2 \overline{\mathcal{D}} x_3 \end{matrix}\right\} \longrightarrow x_1 \overline{\mathcal{D}} x_3$

<u>Basic theorem</u>

The choice procedure, using binary relations $x_i \mathcal{D} x_j$ and the rule 1^o_{dom} of picking out dominants, is equivalent to procedure, using some set of criteria $\{f_i(x)\}$ and rule of extremization 1^o_{ext} if and only if $\mathcal{D}$ is transitive. What is more this procedure is equivalent to extremization of one criterion, if and only if it is transitive and nego-transitive.

Graph interpretation of classical generalizations

One criterion $f(x)$	~	one graph of weak order
Many criteria $\{f_i(x)\}$	~	one graph of partial order
?	~	one acyclic graph

Operation $\max\limits_{x \in X}$ ~ Operation of choosing of non-dominated vertices of subgraph for X

Why further generalizations (non-classical) are necessary?

new generalization generated → "by mass character approach"
→ by notion "equivalence"

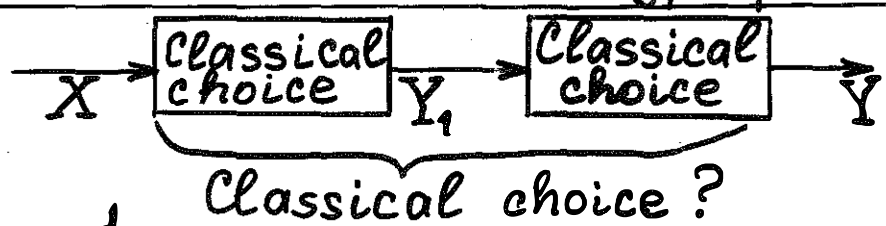
① There exist logical examples of scales which are not included by classical approach, for instance:

1. Tournament and scale "sum of scores"
2. Scale of estimations with "error"
3. Almost all scales produced of rules for choosing a part of the Pareto set

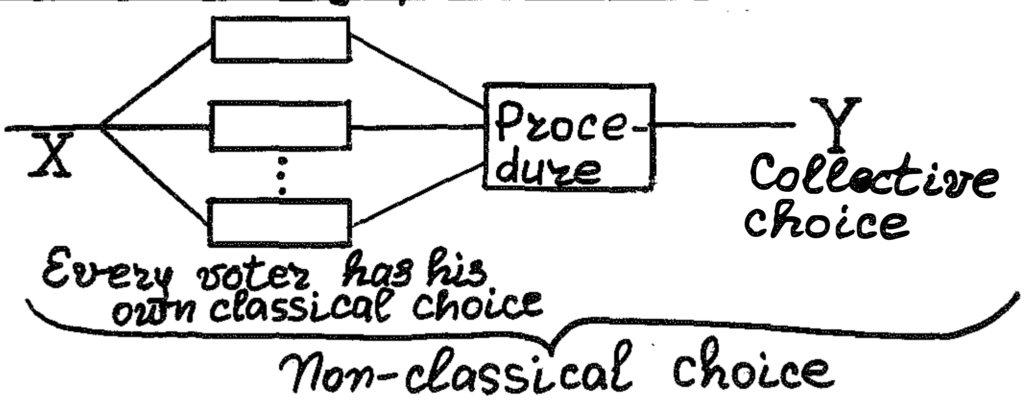
and many others

② Non-closedness of criteria and of their classical generalizations (almost for any rule of choice) relative to operations of union ($\cup$), intersection ($\cap$) and their combinations ($\cup\cap$)

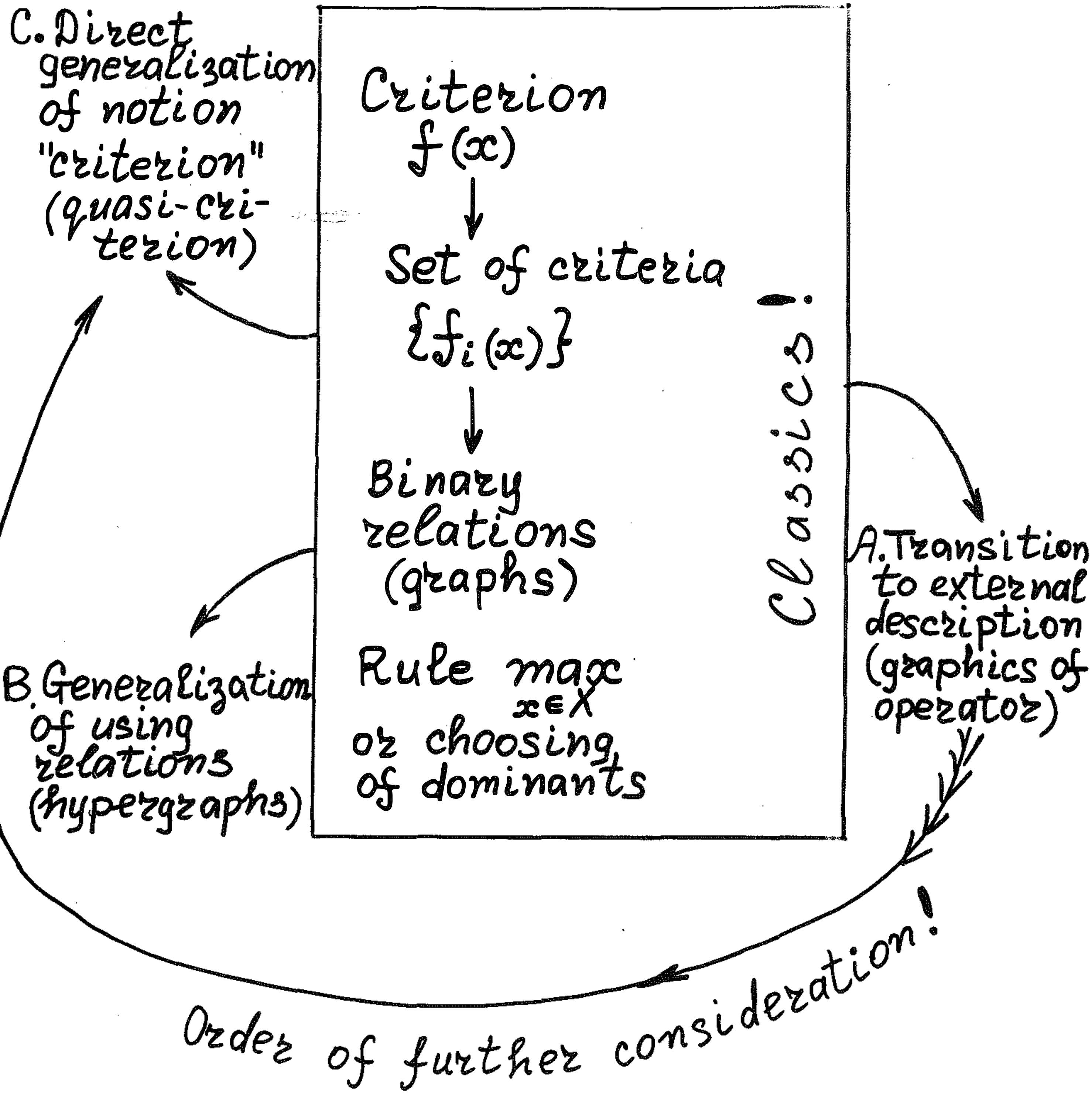
③ Non-closedness relative to syperposition



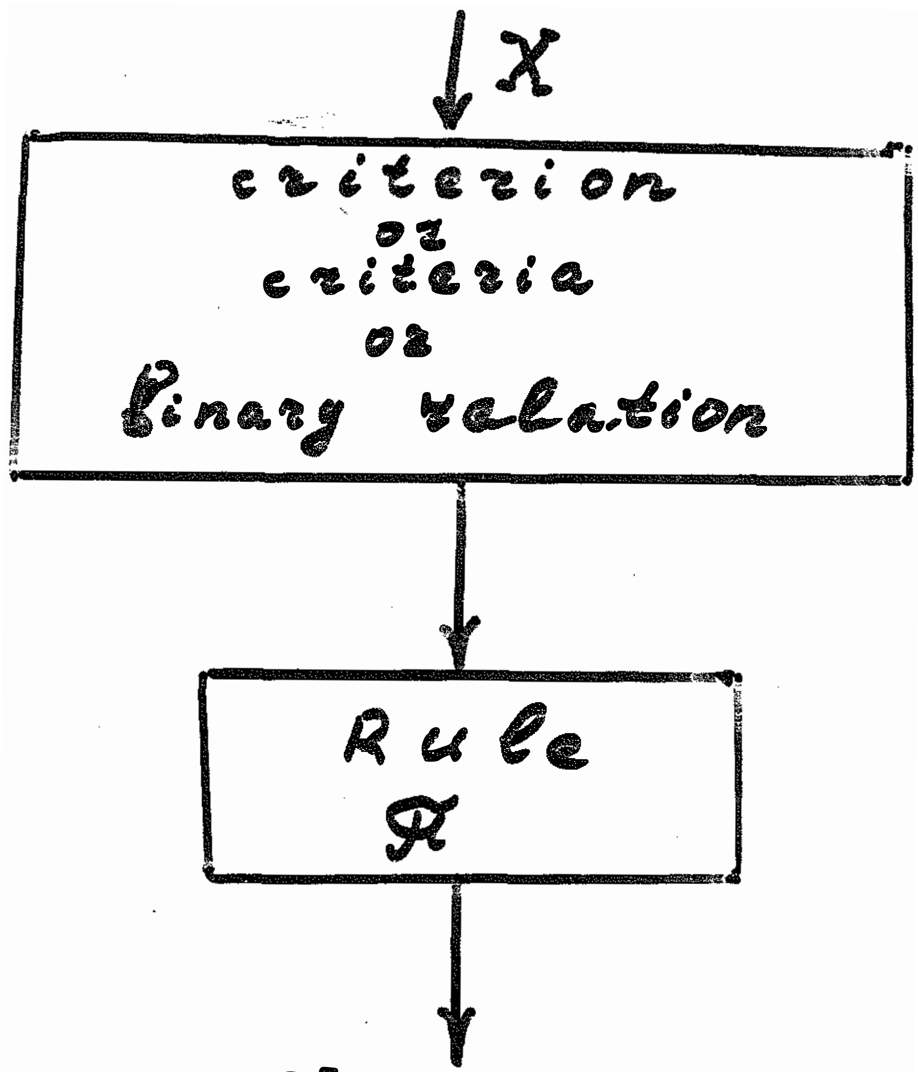
④ Non-closedness relative to almost all known voting procedures



General scheme of further generalizations



A. Transition to external description (choice function)

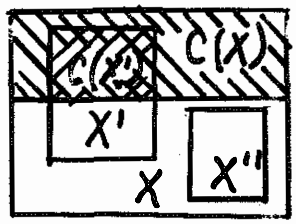
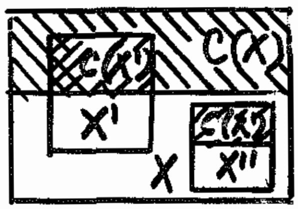
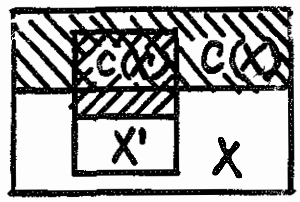
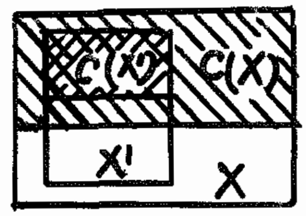
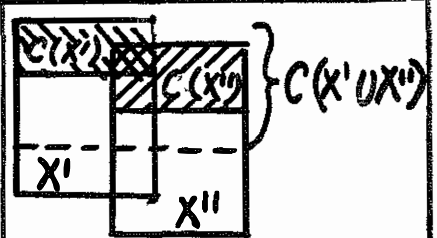
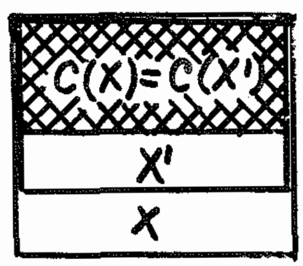


$$\underline{Y} \subseteq X$$

$$\{X, \underline{Y}\}, \quad \forall X \subseteq A$$

$$\underline{Y} = C(X) \text{ - choice function}$$

Domains in C

Nomination of condition	Designation	Determination	Illustration
Summation (trivial choice)	S	$X' \subset X \Rightarrow C(X') = C(X) \cap X'$	
Constancy	K	$X' \subset X,\ C(X) \cap X' \neq \emptyset \Rightarrow C(X') = C(X) \cap X'$	
Heritance	H	$X' \subset X \Rightarrow C(X') \supseteq C(X) \cap X'$	
Monotonicity	M	$X' \subset X \Rightarrow C(X') \subseteq C(X)$	
Concordance	C	$X = X' \cup X'' \Rightarrow C(X') \cap C(X'') \subseteq C(X)$	
Rejection	O	$C(X) \subseteq X' \subset X \Rightarrow C(X') = C(X)$	

A. Transition to external description (choice function)

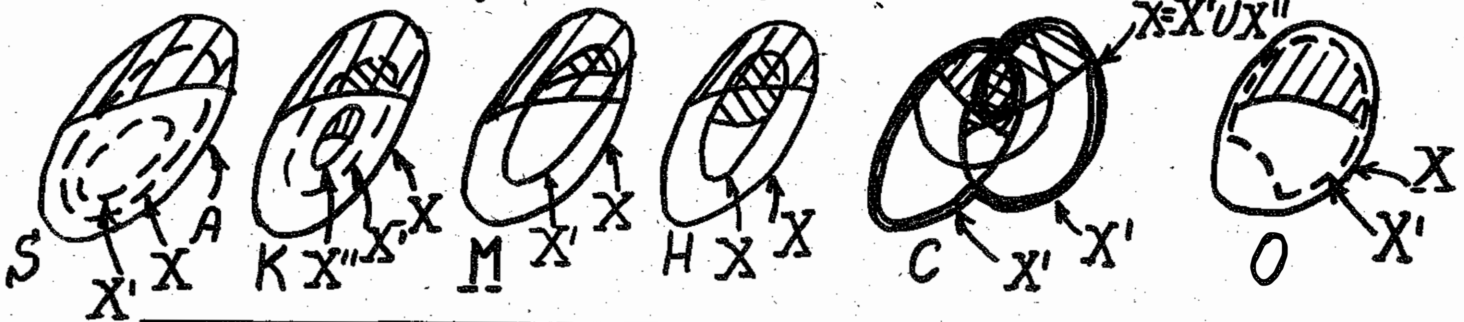
$$X \longrightarrow \boxed{\text{Criterion or criteria or Binary relation}} \longrightarrow \boxed{\text{Rule } \pi} \longrightarrow Y \subseteq X$$

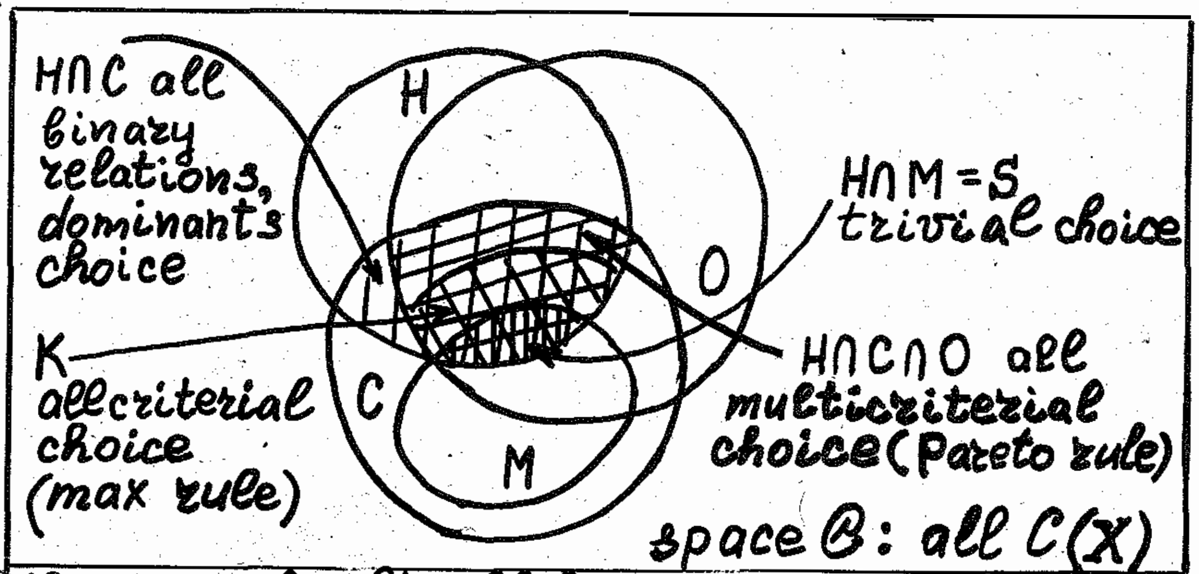
Instead of direct task

$$\{X, Y\}, \quad \forall X \subseteq A$$

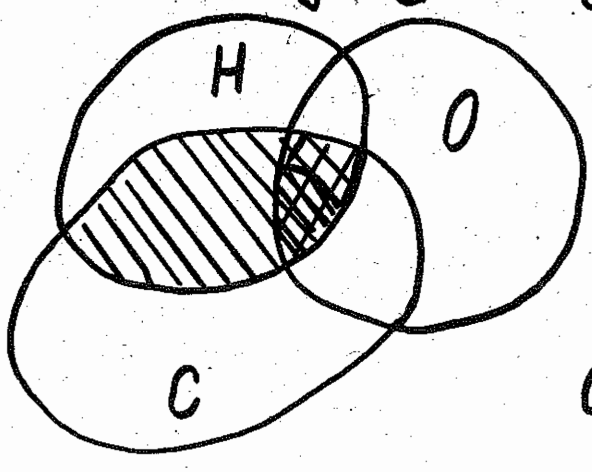
$Y = C(X)$ - choice function

Language for classification of $C(X)$





the same for $\overset{\circ}{B}$: all $\overset{\circ}{C}(X)$ (non-empty choice)



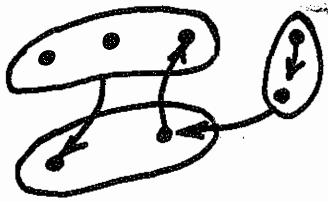
H is closed relative to $\cap, \cup, \cap\cup$

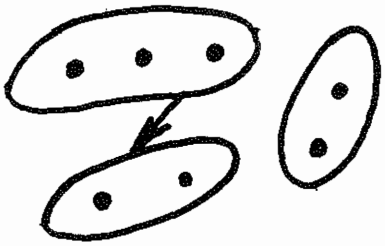
O is closed relative to $\cup$

C is closed relative to $\cap$

Hypergraphs

Structure	Rules	Domain of C
$\boxed{V\beta x}$ "One-side hyperrelations"	Strong dominant $Y=\{x\in X \mid \bar{\exists} V\subseteq X : V\beta x\}$	all H
	weak dominant $Y=\{x\in X \mid \bar{\exists} y\in X$ that $\forall V\subseteq X : [y\in V]\rightarrow[V\beta x]\}$	all C
$\boxed{V\beta M}$ "two-sided hyperrelations"	Hyperdominant $Y\subseteq X \mid \bar{\exists} V : V\beta Y$	all O





Extension of notion "criterion"

Extension of notion $\max\limits_{x\in X}$ and $\operatorname{Par}\limits_{x\in X}$

C. Direct generalization of notion "criterion"

"Criterion": $f(x)$; $\{f_i(x)\}$
$\forall x \in A$

"pseudocriterion": $\psi(x,X)$; $\{\psi_i(x,X)\}$
$\forall x \in A$, $\forall X \subseteq A$

For operation $\max\limits_{x\in X}$

$\max\limits_{x\in X} f(x)$; $\qquad \operatorname{Par}\limits_{x\in X}\{f_i(x)\}$

result Y depends on X, because X determines restrictions.

$\max\limits_{x\in X} \psi(x,X)$; $\operatorname{Par}\limits_{x\in X}\{\psi_i(x,X)\}$

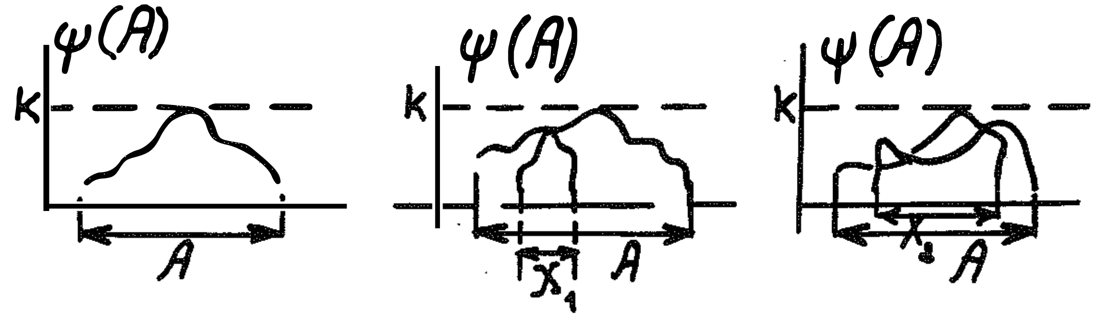
result Y depends on X because X determines restrictions and because criterial estimations depend on X.

$\boxed{\Psi \text{- space of all pseudo-criteria}}$

Examples of pseudo-criteria:

1. Scale "number of voices" obtaining by voting;
2. Sum of points in tournament;
3. Borda scale - sum of ranges; etc.

Pseudocriterion

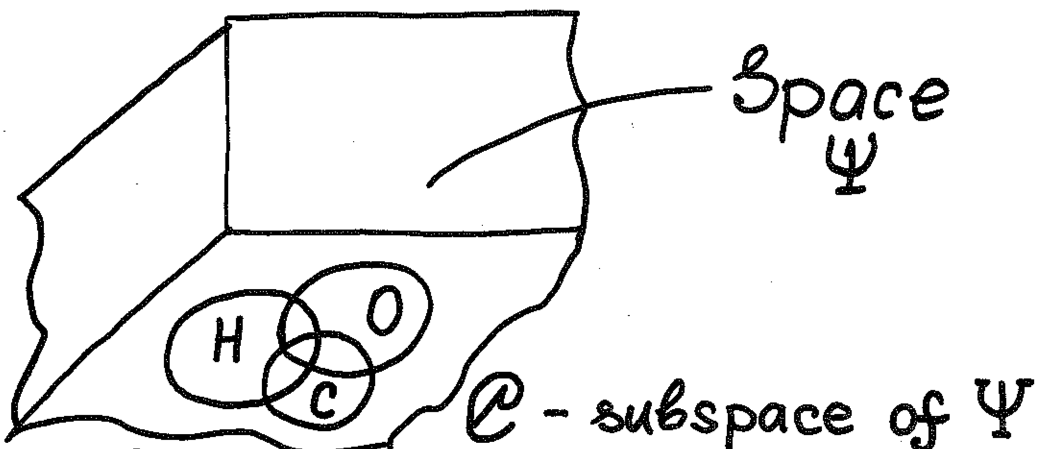


and so on for $\forall X \subseteq A$

$$0 \leq \psi(x, X) \leq k \quad \text{for} \quad \forall X \subseteq A, \forall x \in X$$

The space $\mathcal{C}$ of all choice functions $C(X)$ is the subspace of the space Ψ pseudocriteria (then $k=1$ and $\psi(x,X)=1$ or 0!)

$$\boxed{C(X) = \begin{cases} 1, & \text{if } x \in Y \\ 0, & \text{if } x \in X \setminus Y \end{cases}}$$



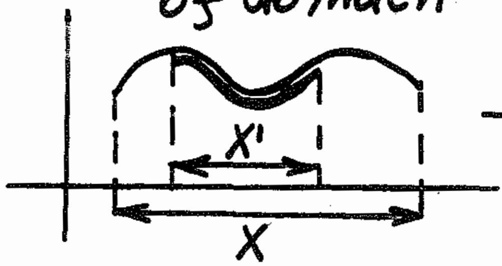
In this sense "pseudocriterion" is the generalization both of notion of "criterion" and notion of "choice function."

Because of that:
Notion of pseudocriterion is too wide.
It is necessary to introduce additional restrictions on $\psi(x, X)$.

Narrowing of notion pseudocriterion (domains in space Ψ)

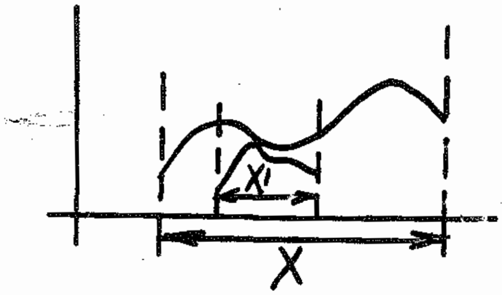
Name of domain | Definition of domain (for $\forall X, X' \subseteq A$)

S_ψ

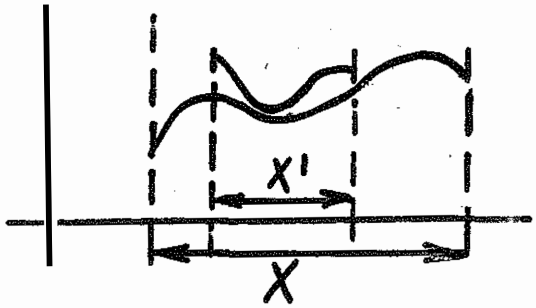


$\rightarrow$ $\psi \in S_\psi$ – usual criterion $f(x)$

M_ψ

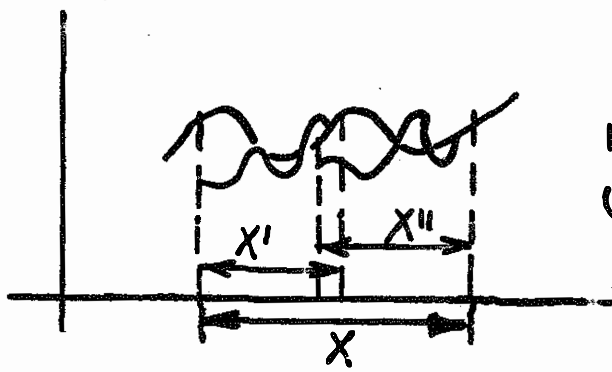


H_ψ



$S_\psi = M_\psi \cap H_\psi$

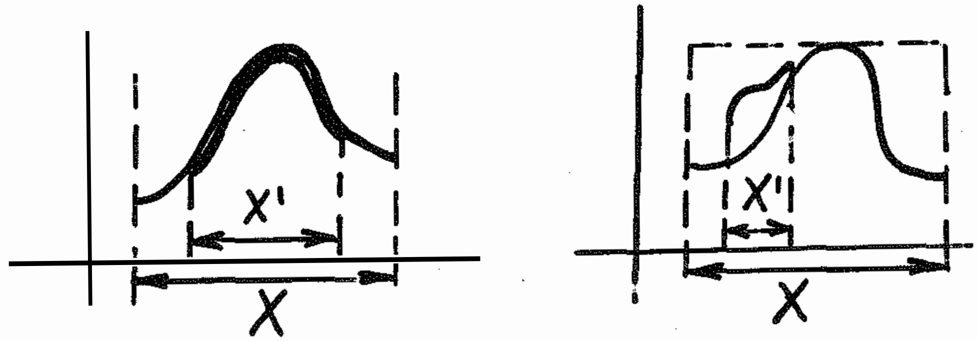
C_ψ



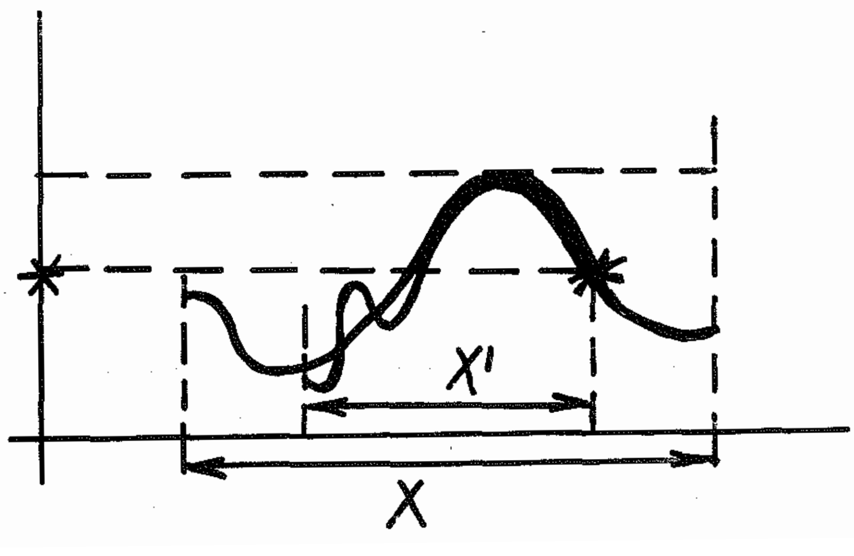
$\min[\psi(x,X'), \psi(x,X'')] \leq \psi(x,X)$; $X', X'' \subset X$

$(\psi(x,X)=0$ for $x \in X)$

$$K_\psi \begin{cases} \psi\in H_\psi\cap C_\psi\cap O_\psi \text{ and } \\ X'\subset X,\ \max\limits_{x\in X'}\psi(x,X) = \\ = \max\limits_{x\in X}\psi(x,X) \Rightarrow \psi(x,X') = \\ = \psi(x,X)\quad \forall x\in X' \end{cases}$$



O_ψ



$*$ - is $\max \psi(x,X)$ in $X\setminus X'$!

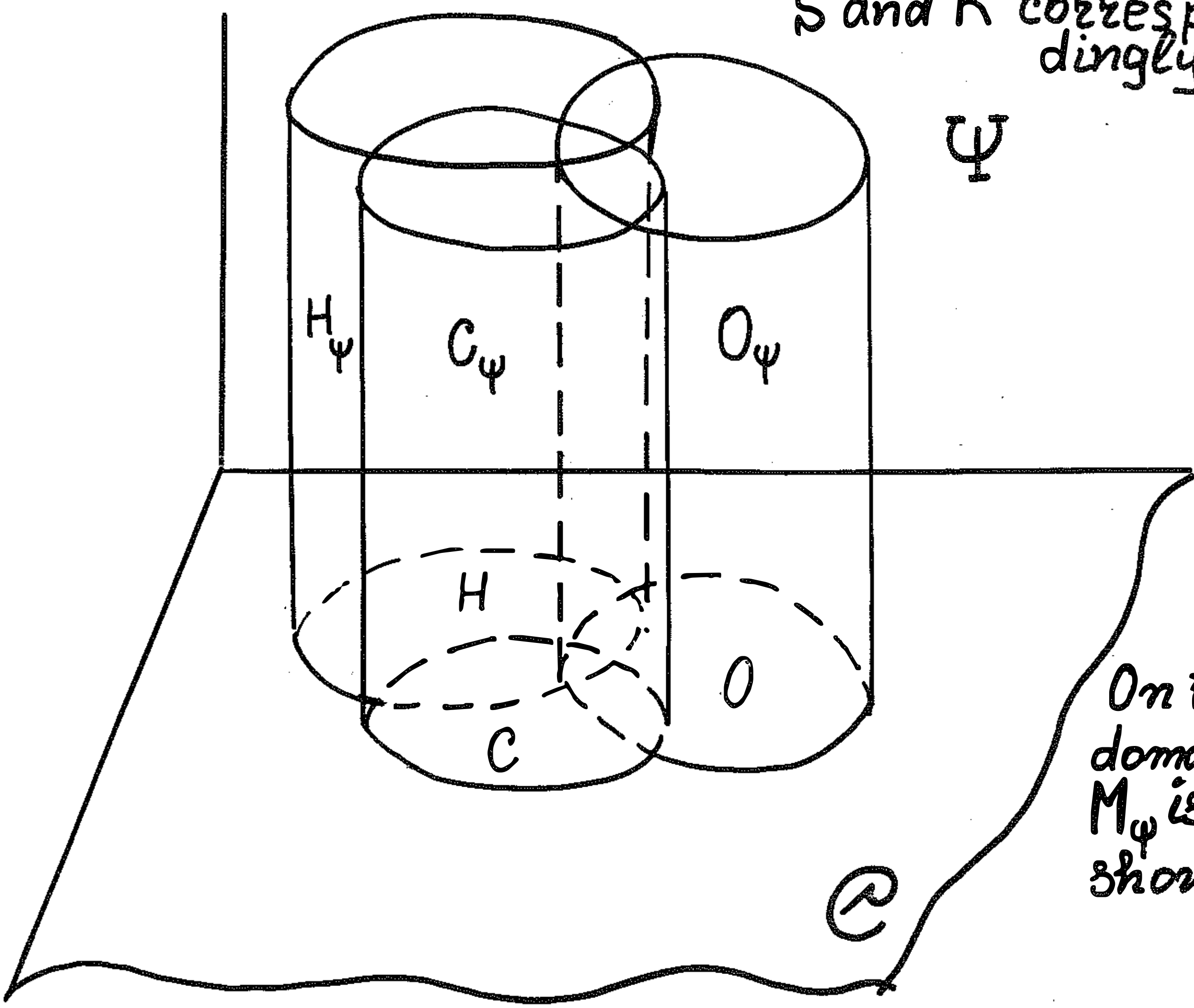
Basic theorem

1. In space Ψ conditions H_ψ, C_ψ and O_ψ are independent in total (all 8 domains are non-empty: $H_\psi \cap C_\psi \cap O_\psi, \bar{H}_\psi, C_\psi \cap O_\psi, \ldots, \bar{H}_\psi \cap \bar{C}_\psi \cap \bar{O}_\psi$)

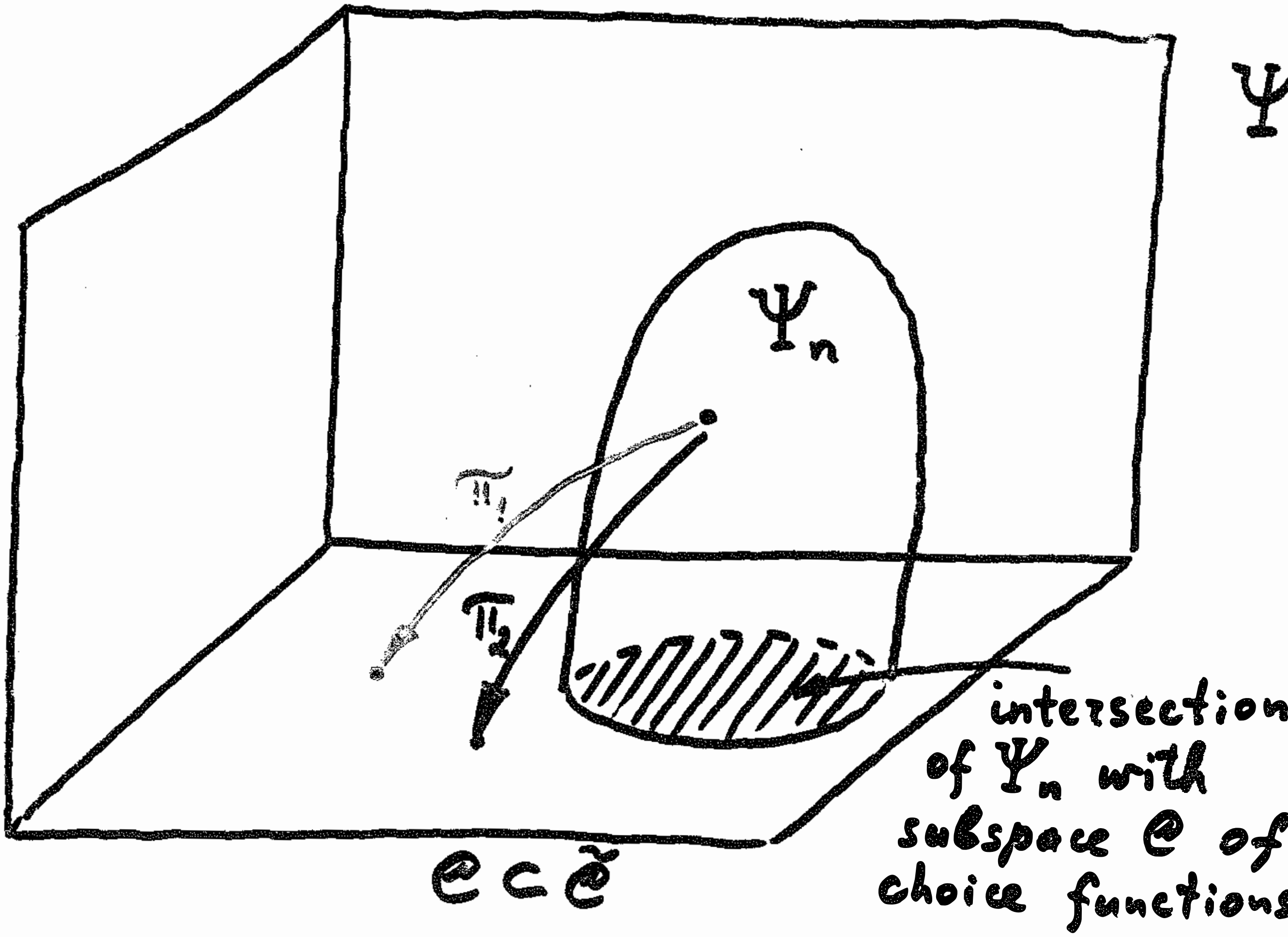
2. $M_\psi \subset C_\psi$, and $H_\psi \cap M_\psi = S_\psi$.

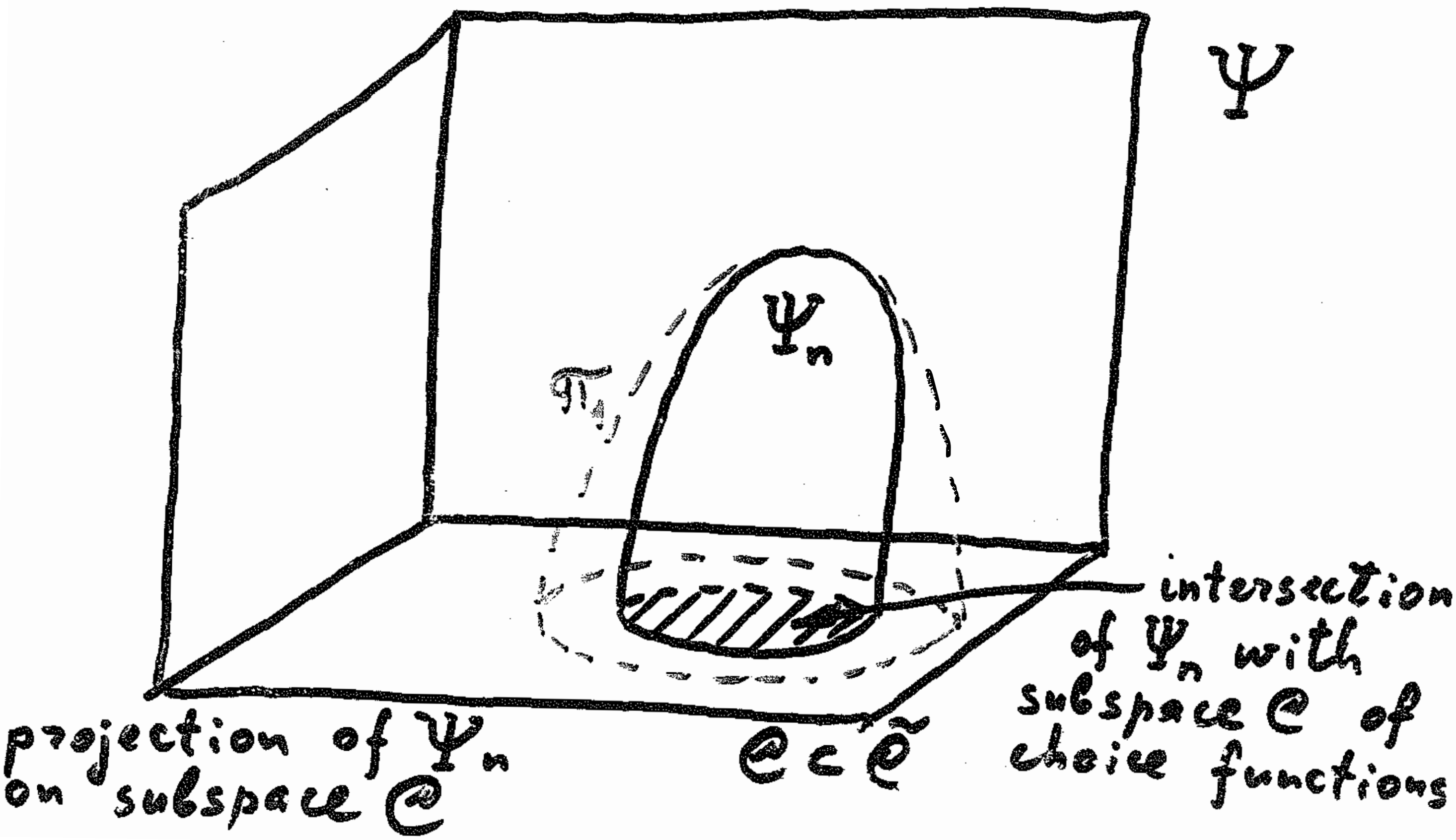
3. $S_\psi \subset K_\psi \subset H_\psi \cap C_\psi \cap O_\psi$.

4. In intersection with subspace @ of choice functions $Y = C(X)$ domains H_ψ, $M_\psi, C_\psi, O_\psi, S_\psi$ and K_ψ turn into H, M, C, O, S and K correspondingly



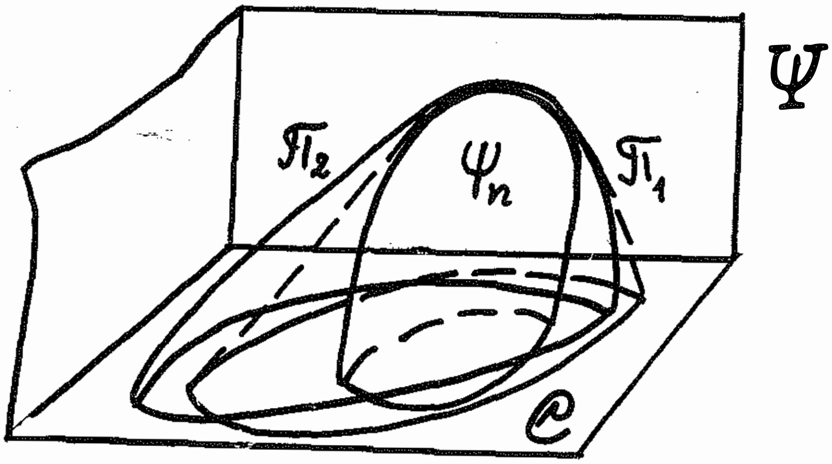
On the fig. domain M_ψ is not shown.


Ψ
Ψ_n
π_1
π_2
intersection of Ψ_n with subspace @ of choice functions
$@ \subset \tilde{@}$


Ψ
Ψ_n
π_1
intersection of Ψ_n with subspace @ of choice functions
projection of Ψ_n on subspace @
$@ \subset \tilde{@}$

Rule of choice which use pseudocriterion for mapping from Ψ in @

$$X^* \longrightarrow \boxed{\psi(x,X)} \xrightarrow{\psi(x,X^*)} \underset{\text{Rule}}{\boxed{\pi}} \longrightarrow Y \subseteq X^*$$



Example of rule π:

$$\pi_{max}: Y = \arg\max_{x \in X^*} \psi(x, X^*)$$

Theorem

Operator π_{max} maps domains of Ψ in $\tilde{@}$ as it is shown in the table

Domain in ψ	S_ψ	K_ψ	M_ψ	H_ψ	C_ψ	O_ψ	$H_\psi \cap O_\psi$	$H_\psi \cap C_\psi$	$C_\psi \cap O_\psi$	$H_\psi \cap C_\psi \cap O_\psi$
π_{max}	$\tilde{K}$	$\tilde{K}$	all $\tilde{@}$	all $\tilde{@}$	all $\tilde{@}$	$\tilde{O}$	$\widetilde{H \cap O}$	$Q \supset \widetilde{H \cap C}$	$\widetilde{C \cap O} \subset Q_1 \subset \tilde{O}$	$\widetilde{H \cap C \cap O}$
π_d	$\tilde{S}$	?	M	H	C	O	$H \cap O$	$H \cap C$	$C \cap O$	$H \cap C \cap O$

Note: $\tilde{@}$ – space of non-empty choice functions, $\tilde{H}, \tilde{C}, \tilde{O}, \widetilde{H \cap C}$ etc. – domains in it.

The fact that estimation of variants x depends on X gives a possibility to suggest further generalizations of the notions max and min.

Let's designate:

$$F(Z) = \max_{z \in Z} \psi(z, Z),$$

$$\Phi(Z) = \min_{z \in Z} \psi(z, Z)$$

The first group of rules:

the rule maxmax: $Y = \{ y \in X \mid \psi(y, X) = \max_{X' \subseteq X} F(X') \}$

the rule maxmin: $Y = \{ y \in X \mid \psi(y, X) = \max_{X' \subseteq X} \Phi(X') \}$

the rule minmax: $Y = \{ y \in X \mid \psi(y, X) = \min_{X' \subseteq X} F(X') \}$

The second group of rules:

the rule MAXmax: $Y = \bigcup_i \{ Y_i \subseteq X \mid F(Y_i) = \max_{X' \subseteq X} F(X') \}$

the rule MAXmin: $Y = \bigcup_i \{ Y_i \subseteq X \mid \Phi(Y_i) = \max_{X' \subseteq X} \Phi(X') \}$

the rule MINmax: $Y = \bigcup_i \{ Y_i \subseteq X \mid F(Y_i) = \min_{X' \subseteq X} F(X') \}$

Each of these rules maps the domain $\Psi_D \subset \Psi$ into some domain in the subspace $\tilde{\mathbb{C}}$ - non-empty choice functions.

$$\xrightarrow{X}\boxed{\psi(x,X)}\longrightarrow\boxed{\pi}\longrightarrow Y\subseteq X$$

The first group of rules:

$$\boxed{\begin{array}{l}\text{the rule } \underline{\text{maxmax}}\text{:}\quad Y=\{y\in X\,|\,(y,X)=\max\limits_{X'\subseteq X}F(X')\}\\[2ex] \text{the rule } \underline{\text{maxmin}}\text{:}\quad Y=\{y\in X\,|\,\psi(y,X)=\max\limits_{X'\subseteq X}\Phi(X')\}\\[2ex] \text{the rule } \underline{\text{minmax}}\text{:}\quad Y=\{y\in X\,|\,\psi(y,X)=\min\limits_{X'\subseteq X}F(X')\}\end{array}}$$

$$F(Z)=\max_{z\in Z}\psi(z,Z),\ \Phi(Z)=\min_{z\in Z}\psi(z,Z)$$

<u>Theorem</u>

rule / domain of pseudocriteria	maxmax	maxmin	minmax
S_ψ	K	K	K
K_ψ	K	K	K
H_ψ	K	K	$\tilde{@}$
M_ψ	$\tilde{@}$	$\tilde{@}$	K
C_ψ	$\tilde{@}$	$\tilde{@}$	$\tilde{@}$
O_ψ	O	O	K
The whole space Ψ	$\tilde{@}$	$\tilde{@}$	$\tilde{@}$

There $K, O \subset \tilde{@}$

$$\xrightarrow{X} \boxed{\psi(x, X)} \longrightarrow \boxed{\pi} \longrightarrow Y \subseteq X$$

The second group of rules:

the rule <u>MAXmax</u>: $Y = \bigcup_i \{ Y_i \subseteq X \mid F(Y_i) = \max_{X' \subseteq X} F(X') \}$

the rule <u>MAXmin</u>: $Y = \bigcup_i \{ Y_i \subseteq X \mid \Phi(Y_i) = \max_{X' \subseteq X} \Phi(X') \}$

the rule <u>MINmax</u>: $Y = \bigcup_i \{ Y_i \subseteq X \mid F(X_i) = \min_{X' \subseteq X} F(X') \}$

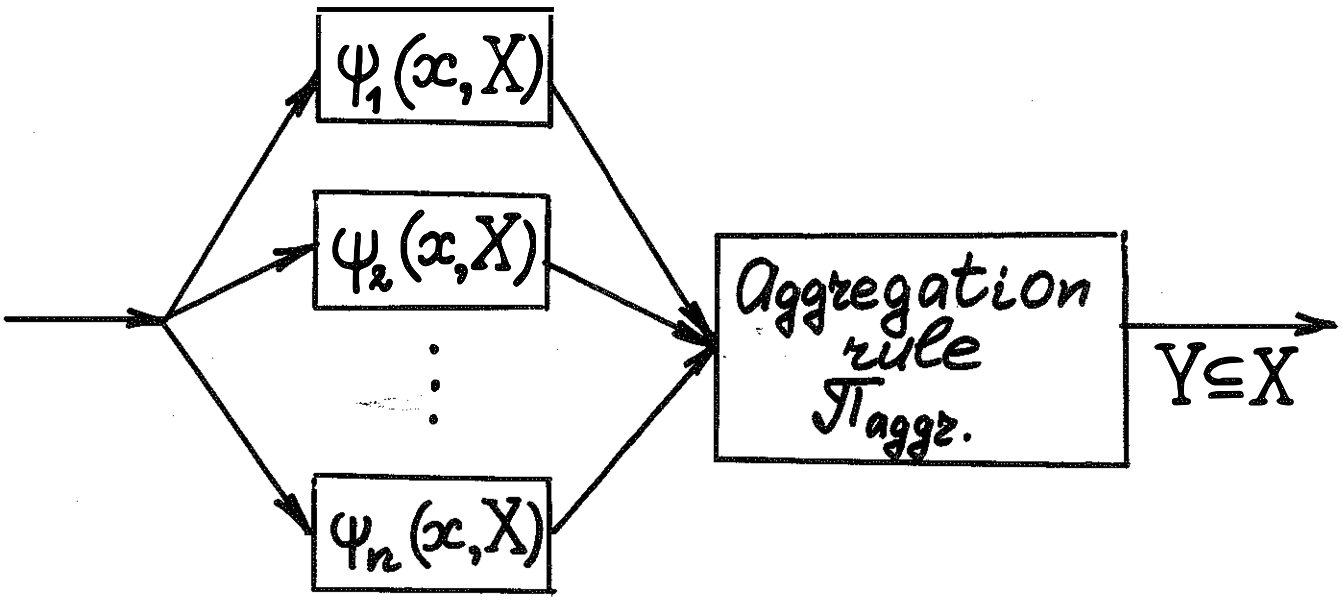
$$F(Z) = \max_{z \in Z} \psi(z, Z), \quad \Phi(Z) = \min_{z \in Z} \psi(z, Z)$$

<u>Theorem</u>

rule / domain of pseudo-criteria	MAXmax	MAXmin	MINmax
S_ψ	$C(X)=X \ \forall X \subseteq A$	K	K
K_ψ	$C(X)=X \ \forall X \subseteq A$	K	K
H_ψ	$C \cap O$	K	O
M_ψ	$C(X)=X \ \forall X \subseteq A$	O	K
C_ψ	O	O	O
O_ψ	$C(X)=X \ \forall X \subseteq A$	O	K
The whole space Ψ	O	O	O

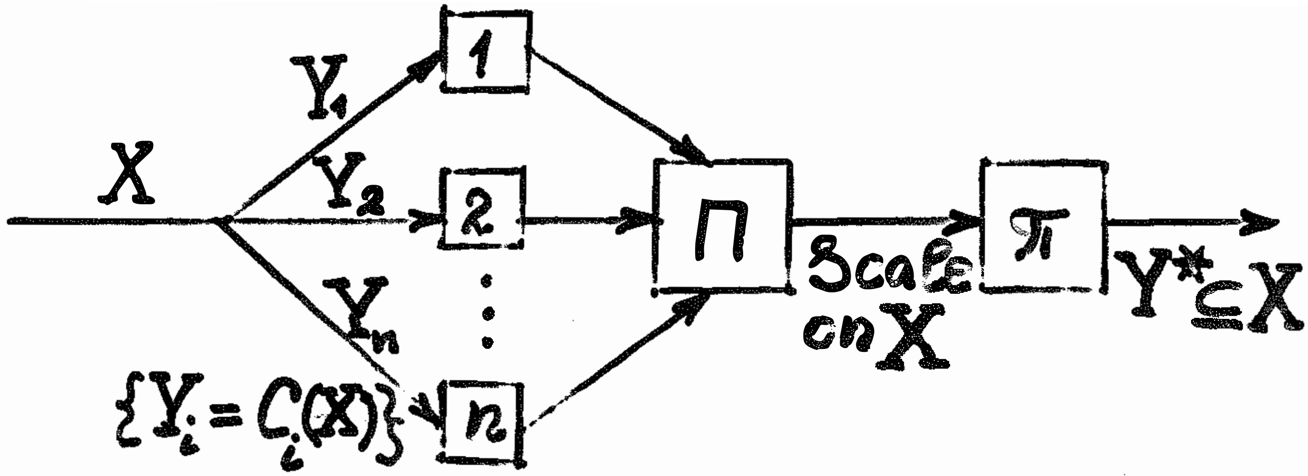
There $K, O \subset \widetilde{@}$

Collective pseudocriterial choice



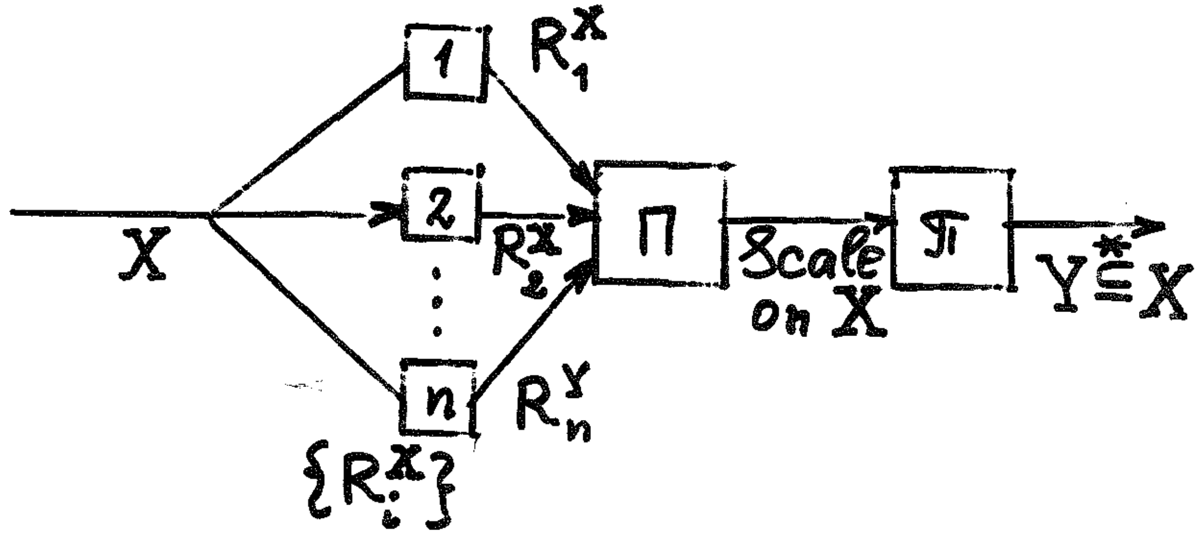
Theorem

Aggregation rule / Collection of pseudocriteria	Aggregate-extremal choice	Aggregate-overthreshold choice	Pareto
S_ψ	$\tilde{H} \cap \tilde{O}$	S	$\tilde{H} \cap \tilde{C} \cap \tilde{O}$
K_ψ	$\tilde{H} \cap \tilde{O}$	There is no invariant image	$Q \subseteq \tilde{O}$ $Q \nsubseteq \tilde{H}$ $Q \nsubseteq \tilde{C}$
M_ψ	$\tilde{@}$	M	$\tilde{@}$
H_ψ	$\tilde{@}$	H	$\tilde{@}$
C_ψ	$\tilde{@}$	$@$	$\tilde{@}$
O_ψ	$\tilde{O}$	O	$\tilde{@}$



The scale "number of votes for" – pseudocriterion (pseudoscale).

Theorem. In order to the scale "number of votes for" would belong in Ψ to domain H_φ, it is necessary and sufficient that all choice functions of voters C_i would belong in @ to domain H.



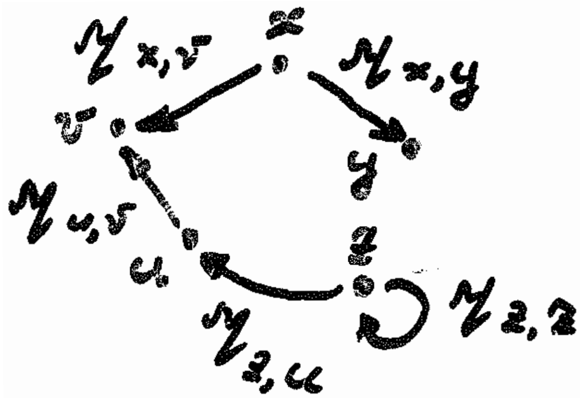
1°. The Borda scale - sum of rank points in $\{R_i\}$ (pseudoscale). A subclass of globally generated Borda scales is picked out (GGB).

Theorem. Pseudoscales GGB create in Ψ a domain strictly enclosed into M_Ψ.

2°. The pseudoscale "sum of first places" create in Ψ a domain, which is situated strictly within H_Ψ and has nonempty intersections with C_Ψ and O_Ψ.

Analogously the domains in Ψ generated by Coupland', Young' and the others procedures are studi-ed

Weighted-Graph mechanism for generating pseudo-criteria



G

$$0 \le \mu_{i,j} \le K$$

Graph on A; G_X - subgraph on $X \subseteq A$

Rule: $$f_X(x) = \min\Big[\min_{y \in X}(K - \mu_{y,x});\ F_X(x)\Big]$$

$$F_X(x) = \begin{cases} K & \text{for } x \in X; \\ 0 & \text{for } x \in A \setminus X; \end{cases}$$

Pair $\{G, \mathcal{H}_\psi\}$ - operator ψ to generate pseudo-criteria

Conditions:

"Transitivity": $\forall x, y, z \in A \quad \mu_{x,z} \ge \min(\mu_{x,y}, \mu_{y,z})$

"Regularity": $\forall x, y \in A \quad \mu_{y,y} \ge \min(\mu_{x,y}, \mu_{x,x})$

Theorem. For operator Ψ for all A and K conditions H_Ψ and C_Ψ are satisfied ($\Psi \in H_\Psi \cap C_\Psi$).

For each pseudo-criteria from $H_\Psi \cap C_\Psi$ there is an operator Ψ to generate it.

Theorem. Iff graph G is transitive and regular, pseudo-criteria generated by operator Ψ holds not only conditions H_Ψ and C_Ψ but also condition O_Ψ. ($\Psi \in H_\Psi \cap C_\Psi \cap O_\Psi$).

Remark. There exists an interval of graphs G which poduces such operator. Graph mentioned in Theorem is maximum of this interval.

Conditions for minimum graph of this interval is also obtained

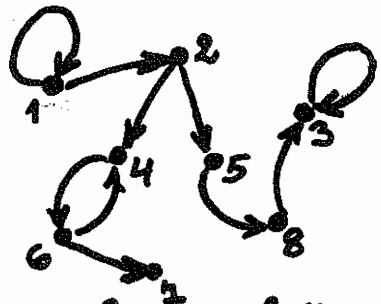
D. Generalization of notion "binary relation"

Binary relations

$\{yDx\}, \quad \forall x,y \in A$

Synonim: oriented graph

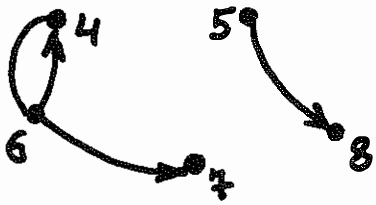
G_A :

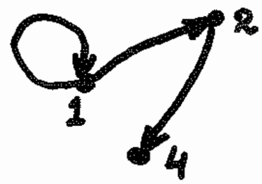


For $X \subset A$ "subgraph" is picked out

For example: for $X_1 = \{4,5,6,7,8\}$ | for $X_2 = \{1,2,4\}$

G_X :

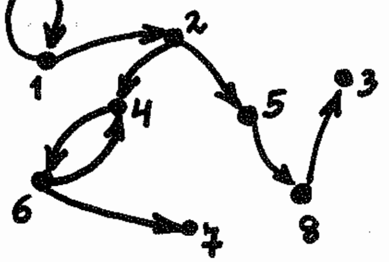


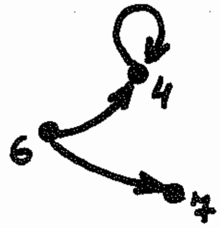


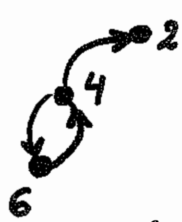
Binary pseudo-relations (pseudo-graph)

$\{yD(X)x\} \qquad \forall x,y \in X, \qquad \forall X \subseteq A$

for $A = \{1,2,3,4,5,6,7,8\}$ for $X_1 = \{6,4,7\}$ for $X_2 = \{2,4,6\}$



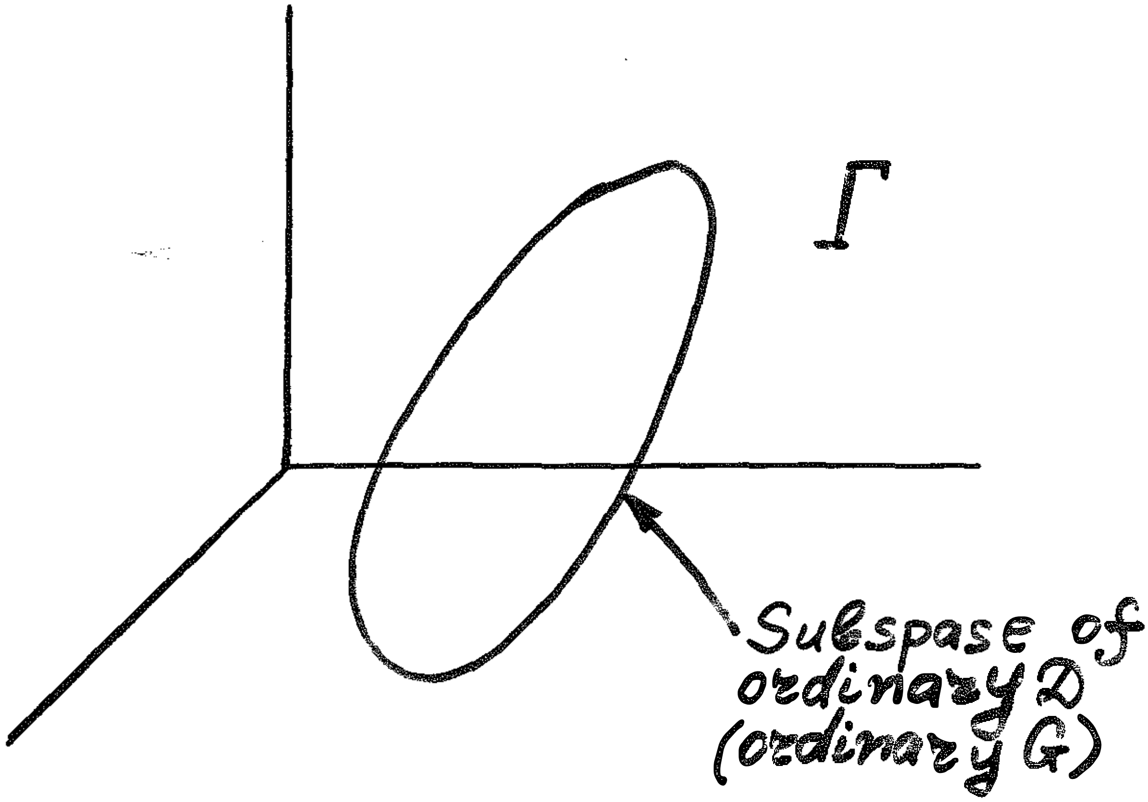




etc.

$\{G(X)\} = \{G(A), G(X_1), G(X_2), \dots\}$

Γ - space of all conseivable binary pseudorelations $\mathcal{D}(X)$ (all pseudographs $G(X)$)



It is necessary to introduce "language for description $\mathcal{D}(X)$," i.e. the other domains in Γ.
Let $X' \subset X$ and graph G_X is specified. Let's designate $\delta_X(X')$ - graph on X', picking out as subgraph from G_X.

Condition of Heritance H_Γ

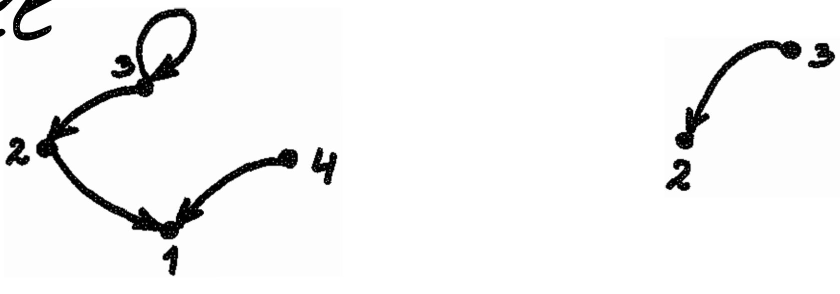
$$X' \subset X \Rightarrow G(X') \supseteq \delta_X(X')$$

i.e. all arcs from G(X) in G(X) remain, but it is possible that new ones will be added

Condition of Monotonicity M_Γ

$$X' \subset X \Rightarrow G(X') \subseteq \delta_X(X')$$

i.e. in G(X') may be only the arcs from G(X), but maybe not all



For $H_\Gamma \cap M_\Gamma$: $G(X') = \delta_X(X')$

i.e. in Γ domain of ordinary binary relations (ordinary graphs) is picked out.

Condition of Concordance C_Γ

$\forall X', X'' \subseteq A : G(X') \cap G(X'') \subseteq$
$\subseteq G(X' \cup X'')$

i.e. if $X' \cap X'' \neq \emptyset$, then the arcs, containing in $G(X')$ and in $G(X'')$, contain and in $G(X' \cup X'')$.

Let's designate $Z_{max}(X)$ maximal set of vertices of X, from which on $G(X)$ there are arcs to all vertices of X.

Condition Rejection O_Γ

$Z_{max}(X) \subseteq X' \subset X \Rightarrow G(X') = \delta_X(X')$

i.e. with rejecting from X alternatives, which are not contained in $Z_{max}(X)$, graph $G(X')$ is exactly extracted from graph $G(X)$, in other words, is its vertix-generated subgraph.

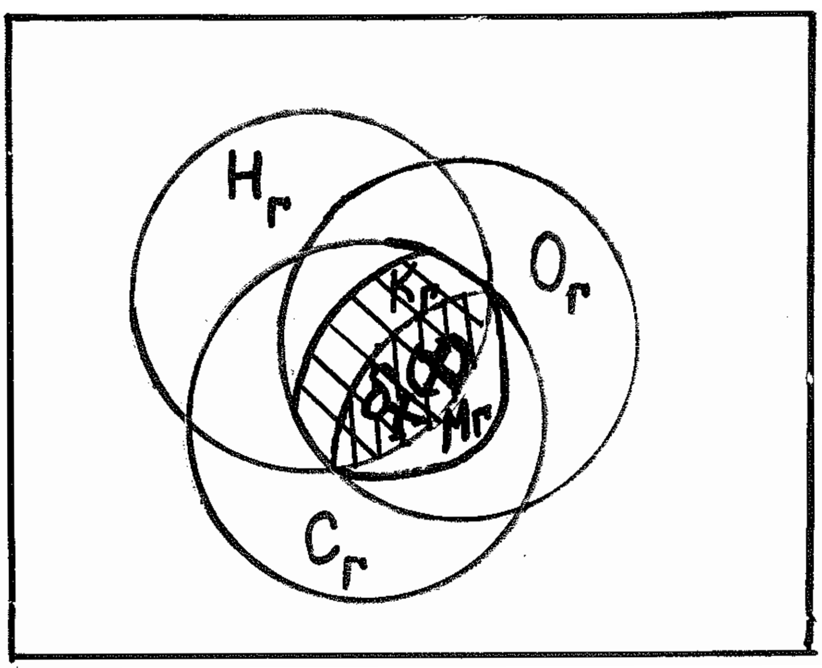
Condition of Constancy K_Γ:

if $X' \cap G_{max}(X) \neq \emptyset$, then

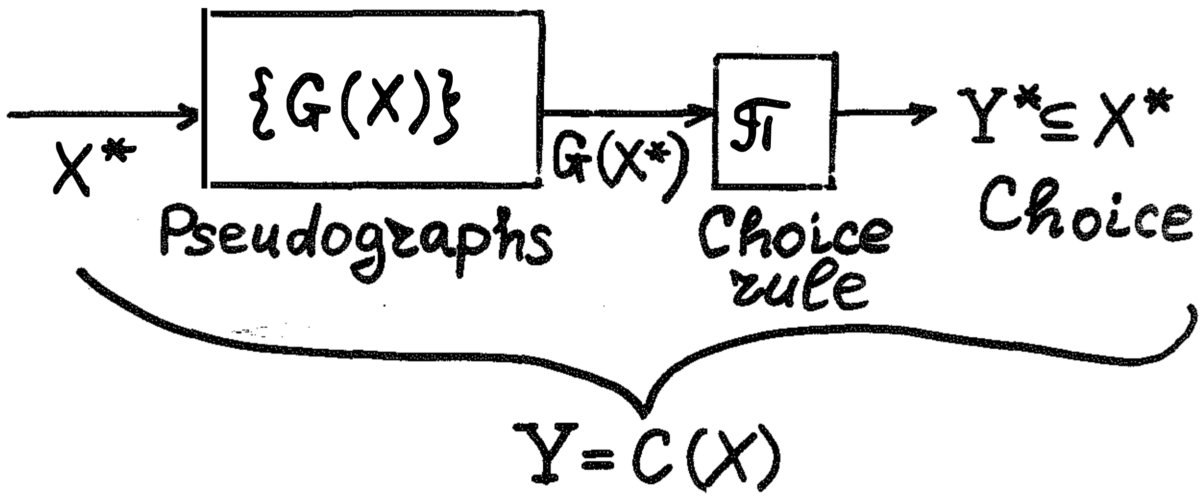
$$G(X') = \delta_X(X'),$$

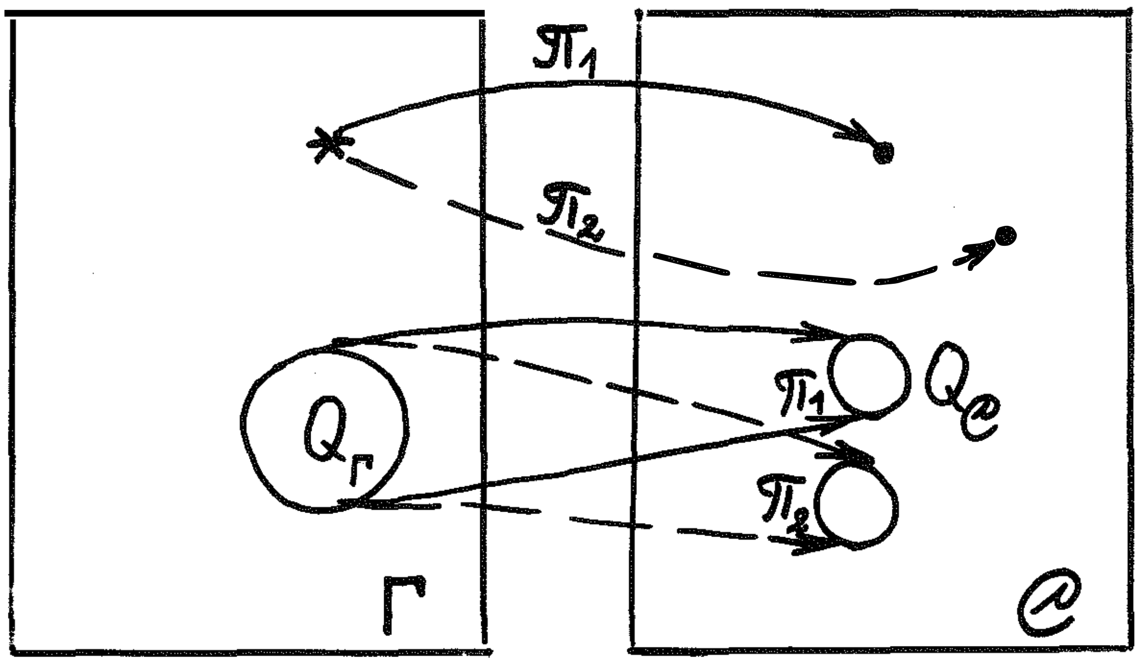
and if $X' \cap G_{max}(X) = \emptyset$ then conditions H_Γ, C_Γ and O_Γ are fulfilled.

Mutual disposition in Γ



Pseudograph choice





Space of pseudo-graphs

Space of choice functions

Rule π point by point maps domain $Q_\Gamma \subset \Gamma$ into domain $Q_{@} \subset @$.

Rules π:

π_D : with using $G(X)$ of X, vertices, from which go arcs to all vertices of $G(X)$, are picked out.

π_P : vertices, to which arcs from all vertices of $G(X)$ come up, are picked out.

π_Z : vertices, to which arcs of no vertices of $G(X)$ do not come up, are picked out.

Theorem about mapping

1°. Rule Π_D:

Domains in Γ

$$
\left\{
\begin{array}{l}
H_\Gamma \longrightarrow H \\
M_\Gamma \longrightarrow C \\
C_\Gamma \longrightarrow Q^1 \quad (C \subset Q^1 \subset @) \\
O_\Gamma \longrightarrow Q^2 \quad (O \subset Q^2 \subset @) \\
K_\Gamma \longrightarrow Q^3 \quad (H \cap C \subset Q^3 \subset H)
\end{array}
\right\}
$$

Domains in @

2°. Rule Π_p:

$H_\Gamma \longrightarrow H$

$M_\Gamma \longrightarrow C$

3°. Rule Π_z

$H_\Gamma \longrightarrow C$

$M_\Gamma \longrightarrow H$

Mappings are exact!

Connection of pseudo-criterial (rule $\max\limits_{X}$) and pseudo-graph (rule Π_D) choices

Let $Z_D(X)$ be set dominating vertices of graph $G(X)$.

Theorem

Pseudographs	Choice funct.	Pseudocriteria
Domain in Γ (condition $Z_D(X) \neq \emptyset$ for $\forall X \subseteq A$) Rule Π_D	Domain in $\mathcal{C}$	Domain in Ψ Rule $\max\limits_{X}$
K_Γ + condition of transitivity	$\tilde{K}$	K_Ψ
H_Γ	$\tilde{H}$	
M_Γ	$\tilde{C}$	L_Ψ
O_Γ + condition of transitivity	$\tilde{O}$	O_Ψ
$H_\Gamma \cap M_\Gamma$ + condition of transitivity	$\tilde{K}$	

Here S_Ψ and $\mathcal{L}_\Psi$ are new conditions for picking out domains in Ψ, determined on the following page.

Domains S_ψ and L_ψ used in previous theorem:

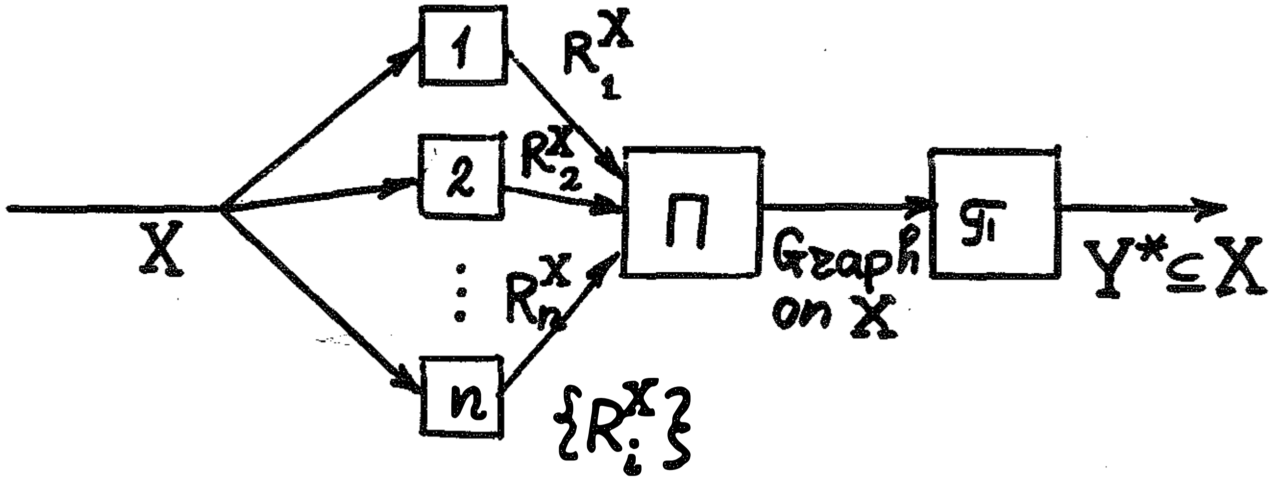
S_ψ, if

1°. $\psi(x, X) > \psi(z, X) \Rightarrow \psi(x, X') \geq \psi(z, X')$
$\forall X' \subset X$

2° if $\psi(x, X) \geq \psi(z, X)$ $\forall z \in X$ and
$\psi(y, X) \geq \psi(z, X)$ $\forall z \in X$, then
$\psi(x, X') = \psi(y, X')$ $\forall X' \subset X$,
$x, y \in X'$.

L_ψ Let $x \in X' \cap X''$ and
$\psi(x, X') \geq \psi(z, X'')$ $\forall z \in X'$,
$\psi(x, X'') \geq \psi(z, X'')$ $\forall z \in X''$.

Then $\psi(x, X' \cup X'') \geq \psi(z, X' \cup X'')$
$\forall z \in X' \cup X''$.



The Fishburn's procedure: first on X a majority graph is designed and by it - "Fishburn's graph". This is a pseudograph.

Theorem. The domain of Fishburn's pseudographs in Γ is strictly enclosed into domain C_Γ which has nonempty intersection with M_Γ.

Analogously the Miller's procedure and the others are studied.

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