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# STATIONARITY AND CHAOS IN INFINITELY REPEATED GAMES OF INCOMPLETE INFORMATION

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## ABSTRACT

Consider an incomplete information game in which the players first learn their own types, and then infinitely often play the same normal form game with the same opponents. After each play, the players observe their own payoff and the action of their opponents. The payoff for a strategy $n$-tuple in the infinitely repeated game is the discounted present value of the infinite stream of payoffs generated by the strategy. This paper studies Bayesian learning in such a setting. Kalai and Lehrer [1991] and Jordan [1991] have shown that Bayesian equilibria to such games exist and eventually look like Nash equilibria to the infinitely repeated full information game with the correct types. However, due to folk theorems for complete information games, this still leaves the class of equilibria for such games to be quite large.

In order to refine the set of equilibria, we impose a restriction on the equilibrium strategies of the players which requires stationarity with respect to the profile of current beliefs: if the same profile of beliefs is reached at two different points in time, the players must choose the same behavioral strategy at both points in time. This set, called the *belief stationary equilibria*, is a subset of the Bayesian Nash equilibria. We compute a belief stationary equilibrium in an example. The equilibria that result can have elements of chaotic behavior. The equilibrium path of beliefs when types are not revealed can be chaotic, and small changes in initial beliefs can result in large changes in equilibrium actions.

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Stationarity and Chaos in
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## 1. Introduction

We consider an infinitely repeated $n$-person game of incomplete information, where the game starts with each player privately observing its type, and then there is an infinite sequence of moves in which players simultaneously choose an action in a normal form game. After each move, all players observe their own payoff and the choice of the other player. The payoff for a strategy $n$-tuple in the infinitely repeated game is the present value of the stream of payoffs generated by the strategy $n$-tuple. We consider Bayesian learning in such a setting. Previous work by Kalai and Lehrer [1991] and Jordan [1991] has shown that equilibria exist in such games, and that in the limit, as time goes to infinity, players eventually play a Nash equilibrium of the infinitely repeated game of complete information.[2]

However, it is well known from the "folk theorem" (See, eg., Fudenberg and Maskin [1986]) that in repeated games of complete information, for every individually rational payoff in the one stage game, there is a high enough discount factor so that one can find a subgame perfect equilibrium to the infinitely repeated game whose equilibrium path yields that payoff every period. The folk theorem has also been extended to games of incomplete information. The above results of Kalai and Lehrer and Jordan connecting equilibria in incomplete information games to those in complete information games thus leave open the possibility that the set of limiting equilibria in an incomplete information game is as big as the union of the set of equilibria in all the complete information games that are in the support of the original incomplete information game. This is potentially a very large set.

In this paper, we investigate a refinement of equilibria for the infinitely repeated game of incomplete information that may allow for more specific predictions of the outcomes of such games. Specifically, we impose a stationarity assumption on the types of strategies that are adopted by the players in equilibrium: we require that equilibrium

[2]This result does not hold in common value settings. See McKelvey and Palfrey (1992a) for a counterexample.

strategies are only a function of current beliefs. If a player ever holds the same beliefs at two different points in time, it is assumed that it will take the same action at both points in time. We call this condition ***belief stationarity.***

In this paper, we are unable to answer the question of general existence of belief stationary equilibria. However, the conditions of belief stationarity are sufficiently restrictive that we are able to compute unique equilibria for simple examples (see also McKelvey and Palfrey, 1992a and 1992b). In this paper, we give an example in which a unique equilibrium can be computed over a dense subset of the belief space. The example has certain features that are characteristic of chaotic dynamical systems: The path of beliefs when types are not revealed is chaotic, and small changes in prior beliefs can yield large and unpredictable changes action probabilities.

## 2. The Basic Setup

Consider an $n$-person, one stage Bayesian game of the following form: Set $T = \prod_{i \in N} T_i$, and $A = \prod_{i \in N} A_i$, where $N$ is a finite set of ***n players***, $T_i$ is a finite set of possible ***types*** for player $i$, and $A_i$ is a finite set of ***actions*** for player $i$. We let $u\colon A \times T \rightarrow \mathbb{R}^n$ be the vector of ***utility functions***, and $\rho \in \mathcal{M}(T)$ be the ***common prior*** over $T$. Here $\mathcal{M}(S)$ denotes the set of probability measures over the set $S$. The collection $\Gamma = \{N, T, A, u, \rho\}$ is called a ***Bayesian game.***

We define an equilibrium for a Bayesian game in the standard way: Let $S_i = \{s_i\colon T_i \rightarrow \mathcal{M}(A_i)\}$ be the set of ***(type contingent) strategies*** for player $i$, and $S = \prod_{i \in N} S_i$ be the set of ***strategy profiles.*** Elements of $S$ are written in the form $s(t) = (s_1(t_1), \ldots, s_n(t_n))$. Write $s_i(t_i)(a_i) = s_i(t_i)(\{a_i\})$, and define $s\colon T \rightarrow \mathcal{M}(A)$ by $s(t)(a) = \prod_{i \in N} s_i(t_i)(a_i)$. Write $u_i(s(t), t) = \sum_{a \in A} u_i(a, t)s(t)(a)$. Define $M_i\colon S \rightarrow \mathbb{R}$ by

$$M_i(s) = \sum_{t \in T} u_i(s(t), t)\rho(t) = \sum_{t \in T} \sum_{a \in A} u_i(a, t)s(t)(a)\rho(t) \qquad (1)$$

A ***Bayesian Nash equilibruim*** is a strategy profile, $s$, satisfying $M_i(s) \geq M_i(s_i', s_{-i})$ for all $s_i' \in S_i$. In other words, a Bayesian Nash equilibruim is Nash equilibrium to the normal form game $M\colon S \rightarrow \mathbb{R}^n$.

We now consider an infinitely repeated game of incomplete information, where types are drawn once, at the beginning of the game, and then the same stage game is played repeatedly, with discounting, with players observing the strategy choices of the other players between rounds. We define, for any $\tau \geq 0$, $H^\tau = \prod_{1 \leq j \leq \tau} A$ to be the set of ***histories of length*** $\tau$, and $H = \cup_{\tau=1}^{\infty} H^\tau$ to be the set of all ***histories***. Then the set $\Sigma_i$ of ***strategies*** for player $i$ is the set of all functions, $\sigma_i$: $H \times T_i \to \mathcal{M}(A_i)$, and we write $\Sigma = \prod_{i \in N} \Sigma_i$ for the set of all strategy profiles. Elements of $\Sigma$ are written on the form $\sigma: H \times T \to \mathcal{M}(A)$, where $\sigma(h,t) = (\sigma_1(h,t_1),\ldots,\sigma_n(h,t_n))$. For any $\eta \in \mathcal{M}(A)$, and $t \in T$, define $u_i(\eta,\ t) = \int_A u_i(a,t)d\eta(a)$ to be the expected utility of $\eta$. We define $v$: $\mathcal{M}(T) \times \Sigma \to \mathbb{R}^n$ for any $\sigma \in \Sigma$ and $i \in N$ by

$$v_i(\rho,\sigma) = \sum_{\tau=0}^{\infty} \delta^\tau \sum_{t \in T} \sum_{h \in H^\tau} u_i(\sigma(h,t),t)\pi(h,t)\rho(t) \qquad (2)$$

where $\pi(h,t)$ is the probability of observing history $h$ given type $t$, and is defined inductively on $\tau$ by $\pi(\emptyset,t) = 1$, and for $h = (h',\ a) \in H^{\tau-1} \times A = H^\tau$,

$$\pi(h,t) = \pi(h',t)\sigma(h',t)(a).$$

Given an initial prior $\rho$, a strategy $n$-tuple, $\sigma \in \Sigma$ is said to be a ***Bayesian Nash equilibrium*** to the infinitely repeated game if $v_i(\rho,\sigma) \geq v_i(\rho,(\sigma'_i,\ \sigma_{-i}))$ for all $\sigma'_i \in \Sigma_i$. For any history $h \in H$, define the strategy $\sigma_h \in \Sigma$ to be the strategy generated on the subgame starting at $h$: Thus, for any $t \in T$, and $h' \in H$, $\sigma_h(h',t) = \sigma((h,h'),t)$. A strategy $n$-tuple $\sigma$ is said to be a ***subgame perfect Nash equilibrium*** if $\sigma_h$ is a Nash equilibrium for any $h \in H$.

We know that there are typically a multiplicity of equilibria in discounted repeated games. In fact, the folk theorems tell us that one can generally construct a subgame perfect equilibrim to support any individually rational outcome to the one shot game. These results also extend to games of incomplete information similar to the class considered above.

In this paper, we propose an equilibrium refinement to infinitely repeated games of incomplete information intended to isolate a reasonable subset of equilibria in such games. The critereon that we propose is that the equilibrium must be stationary in the players' current beliefs. Thus, we require that the strategy should be a function only of the current posterior beliefs, and not of the history that led to those beliefs. Further, small perturbations in beliefs should not lead to large perturbations in actions. We define these ideas more precisely.

Given any history, $h \in H^{\tau}$, we can compute the posterior probability distribution over types at time $\tau$, $\rho(t \mid h) \in \mathcal{M}(T)$ using Bayes rule:

$$\rho(t \mid h) = \frac{\pi(t,h)\rho(t)}{\sum_{t' \in T}\pi(t',h)\rho(t')}. \tag{3}$$

Thus, in this paper, we will be concerned with equilibria in which the strategies satisfy the following condition:

DEFINITION 1: A strategy $\sigma \in \Sigma$ is ***stationary in beliefs*** if for any $h, h' \in H$,

$$\rho(\cdot \mid h) = \rho(\cdot \mid h') \Rightarrow \sigma(h, \cdot) = \sigma(h, \cdot)$$

A subgame perfect Nash equilibrium which is stationary in beliefs is called a ***belief dependent Bayesian Nash equilibrium.***

Since we are limiting our analysis to strategies that are stationary in beliefs, we reformulate the above game. We write $\mathcal{B} = \mathcal{M}(T)$ for the set of possible common knowledge beliefs. Define $\underline{\Sigma} = \prod_i \underline{\Sigma}_i$, where $\underline{\Sigma}_i$ is the set of measurable functions of the form $\sigma_i: \mathcal{B} \times T_i \to \mathcal{M}(A_i)$. Elements of $\underline{\Sigma}$ are written on the form $\sigma: \mathcal{B} \times T \to \mathcal{M}(A)$, where $\sigma(\rho, t) = (\sigma_1(\rho, t_1), \ldots, \sigma_n(\rho, t_n))$. Thus, $\underline{\Sigma}$ represents the set of strategies that are stationary in beliefs, and can be thought of as a subset of $\Sigma$. We can now rewrite the value function $v: \mathcal{B} \times \underline{\Sigma} \to \mathbb{R}^n$ of equation (2) as follows: For each $i \in N$, $\sigma \in \underline{\Sigma}$, and $\rho \in \mathcal{B}$,

$$v_i(\rho, \sigma) = \sum_{\tau=0}^{\infty} \delta^{\tau} \sum_{h \in H^{\tau}} \sum_{t \in T} u_i(\sigma(\rho(\cdot \mid h), t))\pi(h,t)\rho(t) \tag{4}$$

Now for each $\sigma \in \underline{\Sigma}$, $\rho \in \mathcal{B}$, and $a \in A$, define $\rho'(\rho, \sigma, a) \in \mathcal{B}$ by:

$$\rho'(\rho, \sigma, a)(t) = \frac{\sigma(\rho, t)(a)\rho(t)}{\sum_{t'}\sigma(\rho, t')(a)\rho(t')}. \tag{5}$$

when the denominator in non-zero. Thus, $\rho'(\rho, \sigma, a) \in \mathcal{B}$ is the result of Bayesian updating for one period under the strategy $\sigma$ if one starts at $\rho$ and observes $a$. Then write $\pi^{\rho}(\sigma, \cdot)$ for the probability distribution over next period beliefs under $\sigma$ given that one starts at $\rho$. It follows that $\pi^{\rho}(\sigma, \rho'(\rho, \sigma, a)) = \mathrm{q}(\rho, \sigma, \mathrm{a})$, where

$$q(\rho, \sigma, a) = \sum_{t' \in T} \sigma(\rho, t')(a)\rho(t'). \tag{6}$$

Now, using equations (5) and (6), we can rewrite

$$v_i(\rho,\sigma) = \sum_{t \in T} u_i(\sigma(\rho,t),t)\rho(t) +$$
$$\delta_i \sum_{a \in A} \sigma(\rho,\sigma,a) \cdot \Big\{\sum_{\tau=0}^{\infty} \delta^{\tau} \sum_{h \in H^{\tau}} \sum_{t \in T} u_i(\sigma(\rho'(\rho,\sigma,a)(\cdot \mid h),t),t)\pi(h,t)\rho'(\rho,\sigma,a)(t)\Big\}$$
$$= \sum_{t \in T} u_i(\sigma(\rho,t),t)\rho(t) +$$
$$\delta_i \sum_{a \in A} \Big\{\sum_{\tau=0}^{\infty} \delta^{\tau} \sum_{h \in H^{\tau}} \sum_{t \in T} u_i(\sigma(\rho'(\rho,\sigma,a)(\cdot \mid h),t),t)\pi(t,h)\sigma(t,\rho)(a)\rho(t)\Big\}$$
$$= \sum_{t \in T}\Big\{u_i(\sigma(\rho,t),t) +$$
$$\delta_i \sum_{a \in A} v_i(\rho'(\rho,\sigma,a),\sigma)\sigma(\rho,t)(a)\Big\}\rho(t)$$
$$= \sum_{t \in T}\sum_{a \in A}\Big\{u_i(a,t) + \delta_i v_i(\rho'(\rho,\sigma,a),\sigma)\Big\}\sigma(\rho,t)(a)\rho(t) \tag{7}$$

(Note that whenever $\rho'(\rho,\sigma,a)$ is undefined in the above expression, the term it is in occurs with zero probability.) In a similar fashion, we can define the value of a one period unilateral deviation by player $i$ to $\sigma_i$ in the first period to be

$$v_i(\rho,\sigma;\sigma_i') = \sum_{t \in T}\sum_{a \in A}\Big\{u_i(a,t) +$$
$$\delta_i v_i(\rho'(\rho,\sigma,a),\sigma)\Big\}\sigma_{-i}(\rho,t_{-i})(a_{-i})\sigma_i'(\rho,t_i)(a_i)\rho(t). \tag{8}$$

Note that it is not generally the case that $v_i(\rho,\sigma;\sigma_i') = v_i(\rho,(\sigma_{-i},\sigma_i'))$.

The payoff function $v_i(\rho,\sigma)$ can be thought of as the value function arising from a stochastic game with state space $\mathfrak{B}$: For any $\rho \in \mathfrak{B}$, define the game element $\Gamma^{\rho} = (S^{\rho}, M^{\rho}, \pi^{\rho})$, where $S^{\rho}$ and $M^{\rho}$ are the strategy sets and payoff function for the Bayesian game $\{N,\ T,\ A,\ u,\ \rho\}$, and $\pi^{\rho}$ is a transition function as defined above. So for any $\rho\ \in \mathfrak{B}$, $S^{\rho} = S = \prod_{i \in N} S_i$, where $S_i = \{s_i\colon T_i \to \mathcal{M}(A_i)\}$, and $M_i^{\rho}\colon S^{\rho} \to \mathbb{R}$ is defined by

$$M_i^{\rho}(s) = \sum_{t \in T} u_i(s(t),t)\rho(t). \tag{9}$$

If we were to treat the game as a stochastic game, the natural definition of equilibrium would be that $\sigma$ is an equilibrium if for all $i \in N$, $\rho \in \mathfrak{B}$, and $\sigma_i' \in \Sigma_i$, $v_i(\rho,\sigma) \geq v_i(\rho,(\sigma,\sigma_i'))$. This definition is unsatisfactory for our purposes, since it implies that when one player contemplates a deviation from equilibrium, the deviation is common knowledge to the other players. Instead we define a different equilibrium condition:

DEFINITION: If, for all $i \in N$, $\rho \in \mathfrak{B}$, and $\sigma_i' \in \underline{\Sigma}_i$, $v_i(\rho,\sigma) \geq v_i(\rho,\sigma;\sigma_i')$, then we say that $\sigma$ is a **Belief Stationary Equilibrium** (BSE).

It should be emphasized that while a belief stationary equilibrium is defined only to be Nash over $\underline{\Sigma}$, it is also a Bayesian Nash equilibrium over $\Sigma$. In other words, if $\sigma$ is a belief stationary equilibrium, it is also a belief dependent equilibrium. This follows since for any $\sigma \in \underline{\Sigma}$, player $i$ faces a stationary problem, any optimal response can be achieved by a stationary strategy (See, eg., Denardo [1967], Sobel [1971], or Blackwell [1962]). Thus a belief stationary equilibrium is in fact a refinement of the set of Bayesian Nash equilibria over of the original infinitely repeated game.

## 3. The Two Player Case

In the case that there are two players, with two strategies for each player, Table 1 gives the game matrix for type $t=(j,\ k)$, i. e., where player 1 is of type $j$, and player 2 is of type $k$. Let $r_{ij}$ be the prior probability that player $i$ is of type $j$, $p_{j\ell}$ be the probability that player 1, of type $j$, plays strategy $\ell$, and $q_{km}$ be the probability that player 2, of type $k$, plays strategy $m$. Note that $p_{j1} = 1 - p_{j2}$, so we write $p_j = p_{j1}$. Similarly, we write $q_k = q_{k1}$. A strategy for player 1 is then just a vector $p=(p_1,\ p_2)$, and a strategy for player 2 is a vector $q=(q_1,\ q_2)$.

| | | | | $t_2 = k$, $r_{2k}$ | |
|---|---|---|---|---|---|
| | | | | L | R |
| | | | | $q_{k1}$ | $q_{k2}$ |
| $t_1 = j$ | $r_{1j}$ | U | $p_{j1}$ | $(a_{11}^{jk},\ b_{11}^{jk})$ | $(a_{12}^{jk},\ b_{12}^{jk})$ |
| | | D | $p_{j2}$ | $(a_{21}^{jk},\ b_{21}^{jk})$ | $(a_{22}^{jk},\ b_{22}^{jk})$ |

Table 1

Payoff matrix for type $t=(j,\ k)$

We consider the case where there are just two types for each player. So $T_i = \{1,2\}$. Note that if there are just two types for each player, then $r_{i1} = 1 - r_{i2}$. So we write $r_i = r_{i1}$. An initial prior is then just a vector $r = (r_1, r_2)$.

## 4. An Example

We now consider an example. The example is a game of one sided incomplete information, where player 1's type is common knowledge, and player 2 can have one of two possible types. The stage game is $2\times 2$, and has the following payoff function:

$r$: $t_2 = 1$

| | L | R |
|---|---|---|
| U | 0, 2 | 4, 0 |
| D | 1, 0 | 0, 4 |

$1-r$: $t_2 = 2$

| | L | R |
|---|---|---|
| U | 0, 4 | 4, 0 |
| D | 1, 0 | 0, 2 |

To solve for a belief stationary equilibrium, we define the following notation:

| | |
|---|---|
| $r$ | prior probability Player 2 is a type 1, |
| $p(r)$ | probability player 1 chooses U, |
| $q_i(r)$ | probability player 2 of type $i$ chooses $L$, |
| $w(r)$ | value of the game to player 1, |
| $w(r \mid a)$ | value of the game to player 1 if it chooses $a$, |
| $v_i(r)$ | value of the game to player 2 of type $i$, |
| $v_i(r \mid a)$ | value of the game to player 2 of type $i$ if it chooses $a$, |
| $\hat{r}_a(r)$ | posterior belief that player 2 is type 1 if action $a$ is observed (abbreviated $\hat{r}_a$). |

We also define

$$s(r) = Eq_i(r) = rq_1(r) + (1-r)q_2(r)$$

to be the expected probability that player 2 chooses $L$. We then have the following conditions that must be met at a belief stationary equilibrium.

For player 1:

$$w(r \mid \mathrm{U}) = 4(1-s(r)) + \delta Ew(r)$$

$$w(r \mid D) = s(r) + \delta Ew(r),$$

where $Ew(r) = s(r)w(\hat{r}_L) + (1 - s(r))w(\hat{r}_R)$.

For player 2:

Type 1

$$v_1(r \mid \mathrm{L}) = 2p(r) + \delta v_1(\hat{r}_L)$$

$$v_1(r \mid \mathrm{R}) = 4(1 - p(r)) + \delta v_1(\hat{r}_R)$$

Type 2

$$v_2(r \mid \mathrm{L}) = 4p(r) + \delta v_2(\hat{r}_L)$$

$$v_2(r \mid \mathrm{R}) = 2(1 - p(r)) + \delta v_2(\hat{r}_R)$$

We now consider some basic properties of the solution to the infinitely repeated game, when $\delta$ is small.

**P1:** If $\delta < \frac{1}{3}$, then any equilibrium must have $0 < p(r) < 1$ and $s(r) = \frac{4}{5}$ for all $r$. It follows that $\hat{r}_L = \frac{5}{4} r q_1(r)$, and $\hat{r}_R = 5r(1 - q_1(r))$.

***Proof:*** We first prove that $0 < p(r) < 1$. Assume that $p(r) = 1$ for some $r$. First note that all one period payoffs must lie between 0 and 4. Therefore, $0 < v_i(r) < \frac{4}{1-\delta} = 6$ for all $r$, or $v_i(\hat{r}_L) - v_i(\hat{r}_R) > -6$. So for player 2 of type 1

$$v_1(r \mid \mathrm{L}) - v_1(r \mid \mathrm{R}) = 2 + \delta(v_1(\hat{r}_L) - v_1(\hat{r}_R)) > 2 - 2 = 0,$$

and for player 2 of type 2,

$$v_2(r \mid \mathrm{L}) - v_2(r \mid \mathrm{R}) = 4 + \delta(v_2(\hat{r}_L) - v_2(\hat{r}_R)) > 4 - 2 > 0.$$

Hence, $q_1(r) = q_2(r) = 1$. But then $s(r) = 1$, so

$$w(r \mid \mathrm{U}) = \delta Ew(r) < 1 + \delta Ew(r) = w(r \mid \mathrm{D}),$$

contradicting the fact that $p(r) = 1$. A similar contradiction arises if $p(r) = 0$. Hence $0 < p(r) < 1$.

Since player 1 must mix for all $r$, it follows that $w(r \mid \mathrm{U}) = w(r \mid D) \Rightarrow 4(1 - s(r)) = s(r) \Rightarrow s(r) = \frac{4}{5}$.

Finally, from Bayes rule, it follows that

$$\hat{r}_L = \frac{rp_1(r)}{s(r)} = \tfrac{5}{4} r q_1(r),$$

and

$$\hat{r}_R = \frac{r(1-p_1(r))}{1-s(r)} = 5r(1-q_1(r)).$$

*Q.E.D.*

**P2:** If $\delta < \frac{1}{5}$, then any equilbium must satisfy:

For $r \le \frac{1}{5}$, $q_1(r) = 0$, $q_2(r) = \frac{4}{5(1-r)}$, $\hat{r}_L = 0$, and $\hat{r}_R = 5r$.

For $r \ge \frac{1}{5}$, $q_1(r) = 1 - \frac{1}{5r}$, $q_2(r) = 1$, $\hat{r}_L = \frac{5}{4}(r - \frac{1}{5})$, and $\hat{r}_R = 1$.

***Proof:*** We first show that if $\delta < \frac{1}{5}$, then for all $r$, either

$$q_1(r) = 0 \text{ or } q_2(r) = 1.$$

Assume the result is false. Then there is an $r$ where it is simultaneously the case that $q_1(r) > 0$ and $q_2(r) < 1$. But $q_1(r) > 0$ implies

$$\begin{aligned} v_1(r \mid \mathrm{L}) \ge v_1(r \mid \mathrm{R}) &\Rightarrow 2p(r) + \delta v_1(\hat{r}_L) \ge 4(1-p(r)) + \delta v_1(\hat{r}_R) \\ &\Rightarrow p(r) \ge \tfrac{2}{3} + \tfrac{\delta}{6}(v_1(\hat{r}_R) - v_1(\hat{r}_L)) > \tfrac{2}{3} - \tfrac{\delta}{6}(\tfrac{4}{1-\delta}) \\ &= \tfrac{2}{3}(1 - \tfrac{\delta}{1-\delta}) = \tfrac{2}{3}(\tfrac{1-2\delta}{1-\delta}) > \tfrac{2}{3}(\tfrac{3}{4}) = \tfrac{1}{2}, \end{aligned}$$

and $q_2(r) < 1$ implies

$$\begin{aligned} v_2(r \mid \mathrm{L}) \le v_2(r \mid \mathrm{R}) &\Rightarrow 4p(r) + \delta v_2(\hat{r}_L) \le 2(1-p(r)) + \delta v_2(\hat{r}_R) \\ &\Rightarrow p(r) \le \tfrac{1}{3} + \tfrac{\delta}{6}(v_2(\hat{r}_R) - v_2(\hat{r}_L)) < \tfrac{1}{3} + \tfrac{\delta}{6}(\tfrac{4}{1-\delta}) < \tfrac{1}{2}. \end{aligned}$$

This yields a contradiction.

From P2 it follows that $s(r) = \frac{4}{5}$. From the above argument, it follows that for any $r$ we must have either $q_1(r) = 0$ or $q_2(r) = 1$. But

$$q_1(r) = 0 \Rightarrow s(r) = (1-r)q_2(r) = \tfrac{4}{5} \Rightarrow q_2(r) = \frac{4}{5(1-r)}.$$

But

$$1 \ge q_2(r) \Rightarrow (1-r) \ge \tfrac{4}{5} \Rightarrow r \le \tfrac{1}{5}.$$

Similarly,

$$q_2(r) = 1 \Rightarrow s(r) = rq_1(r) + (1-r) = \tfrac{4}{5} \Rightarrow q_1(r) = 1 - \tfrac{1}{5r}.$$

But

$$q_1(r) \geq 0 \Rightarrow 1 - \frac{1}{5r} \geq 0 \Rightarrow 1 \geq \frac{1}{5r} \Rightarrow r \geq \frac{1}{5}.$$

The equations for $\hat{r}_L$ and $\hat{r}_R$ in both cases follow directly from P1, by substitution of $q_1(r)$.

*Q.E.D.*

Note that player 2 can guarantee at least 4/3 each round by adopting the strategy of $q_1(r) = 2/3$, $q_2(r) = 1/3$. And player 1 can hold palyer 2 down to at most 8/3 by adopting any strategy satisfying $1/3 < p(r) < 2/3$ for all $r$. This suggests that $\frac{4}{3(1-\delta)} < v_i(r) < \frac{8}{3(1-\delta)}$. We suspect that it can be proven that in any solution this innequality must be satisfied. We have not yet been able to prove this, so for now, we just introduce it as an additional assumption. If we make this additional assumption, then the range of $\delta$ for which P1 and P2 are true expands:

**P1′**: Assume $\frac{4}{3(1-\delta)} < v_i(r) < \frac{8}{3(1-\delta)}$ for all $r$. Then if $\delta < \frac{3}{5}$, any equilibrium must have $0 < p(r) < 1$ and $s(r) = \frac{4}{5}$ for all $r$. It follows that $\hat{r}_L = \frac{5}{4} r q_1(r)$, and $\hat{r}_R = 5r(1 - q_1(r))$.

**P2′**: Assume $\frac{4}{3(1-\delta)} < v_i(r) < \frac{8}{3(1-\delta)}$ for all $r$. Then if $\delta < \frac{3}{7}$, then any equilbium must satisfy:

For $r \leq \frac{1}{5}$, $q_1(r) = 0$, $q_2(r) = \frac{4}{5(1-r)}$, $\hat{r}_L = 0$, and $\hat{r}_R = 5r$.

For $r \geq \frac{1}{5}$, $q_1(r) = 1 - \frac{1}{5r}$, $q_2(r) = 1$, $\hat{r}_L = \frac{5}{4}(r - \frac{1}{5})$, and $\hat{r}_R = 1$.

To prove the above two results, it is simply necessary to use the fact that $|\, v_i(\hat{r}_R) - v_i(\hat{r}_L)\,| < \frac{4}{3(1-\delta)}$ whenever this difference occurs in the proofs of P1 or P2.

We can now use the above properties to get equations on $v_1(r)$ and $v_2(r)$ which must be satisfied in any solution. There are two cases:

*Case 1:* $r < \frac{1}{5}$.

We have from P2 that $q_1(r) = 0$, and $0 < q_2(r) < 1$. Hence,

$$v_1(r) = v_1(r \mid \mathrm{R}).$$
$$v_2(r) = v_2(r \mid \mathrm{L}) = v_2(r \mid \mathrm{R}).$$

I. e.,

$$v_1(r) = 4(1 - p(r)) + \delta v_1(\hat{r}_R) \tag{1}$$

and

$$v_2(r) = 4p(r) + \delta v_2(\hat{r}_L) \tag{2}$$
$$v_2(r) = 2(1 - p(r)) + \delta v_2(\hat{r}_R). \tag{3}$$

Also, we have that $\hat{r}_L = 0$, and $\hat{r}_R = 5r$. Now since $\hat{r}_L = 0$, we can solve for $p(r)$ in terms of $v_2(r)$ using equation (2):

$$p(r) = \tfrac{1}{4}(v_2(r) - \delta v_2(0)),$$

So we need only solve for $v_1(r)$ and $v_2(r)$. Now, $v_1(0) = \frac{8}{3(1-\delta)}$, and $v_2(0) = \frac{4}{3(1-\delta)}$. So we get

$$v_2(r) = 4p(r) + \delta v_2(\hat{r}_L) = 4p(r) + \frac{4\delta}{3(1-\delta)} = 4(p(r) - \tfrac{1}{3}) + \frac{4}{3(1-\delta)}.$$

So

$$v_2(r) - v_2(0) = 4[p(r) - p(0)].$$

Equation (1) can be written

$$v_1(r) = 4(1 - p(r)) + \delta v_1(\hat{r}_R) = 4(\tfrac{1}{3} - p(r)) + \delta v_1(\hat{r}_R) + \tfrac{8}{3}$$
$$= -4(p(r) - p(0)) + \delta(v_1(\hat{r}_R) - v_1(1)) + \tfrac{8}{3} + \delta v_1(1)$$

So

$$v_1(r) - v_1(1) = \tfrac{8}{3} - (1-\delta)v_1(1) - 4[p(r) - p(0)] + \delta[v_1(\hat{r}_R) - v_1(1)]$$
$$= \tfrac{4}{3} - [v_2(r) - v_2(0)] + \delta[v_1(\hat{r}_R) - v_1(1)].$$

Eliminating $p(r)$ from equations (2) and (3) by adding twice the second equation to the first gives

$$3v_2(r) = 4 + \delta v_2(\hat{r}_L) + 2\delta v_2(\hat{r}_R)$$
$$= 4 + \delta v_2(0) + 2\delta v_2(5r)$$
$$\Rightarrow 3(v_2(r) - v_2(0)) = 4 - 3(1-\delta)v_2(0) + 2\delta(v_2(5r) - v_2(0))$$

Since the first and second terms cancel with each other, we get

$$v_2(r) - v_2(0) = \tfrac{2\delta}{3}(v_2(5r) - v_2(0))$$

Summarizing, we have

$$v_1(r) - v_1(1) = \tfrac{4}{3} - [v_2(r) - v_2(0)] + \delta[v_1(\hat{r}_R) - v_1(1)],$$
$$v_2(r) - v_2(0) = \tfrac{2\delta}{3}(v_2(5r) - v_2(0))$$

Or, defining $\underline{v}_1(r) = v_1(r) - v_1(1)$, and $\underline{v}_2(r) = v_2(r) - v_2(0)$,

$$\underline{v}_1(r) = \tfrac{4}{3} - \underline{v}_2(r) + \delta \underline{v}_1(\hat{r}_R),$$
$$\underline{v}_2(r) = \tfrac{2\delta}{3}\underline{v}_2(5r)$$

*Case 2:* $r > \frac{1}{5}$.

We have from P2 that $0 < q_1(r) < 1$, and $q_2(r) = 1$. Hence,

$$v_1(r) = v_1(r \mid \mathrm{L}) = v_1(r \mid \mathrm{R}).$$
$$v_2(r) = v_2(r \mid \mathrm{L}).$$

I. e.,

$$v_1(r) = 4(1 - p(r)) + \delta v_1(\hat{r}_R) \tag{1}$$
$$v_1(r) = 2p(r) + \delta v_1(\hat{r}_L). \tag{2}$$

and

$$v_2(r) = 4p(r) + \delta v_2(\hat{r}_L) \tag{3}$$

Also, we have that $\hat{r}_L = \frac{5}{4}(r - \frac{1}{5}) = \frac{5r-1}{4}$, and $\hat{r}_R = 1$. Now since $\hat{r}_R = 1$, we can solve for $p(r)$ in terms of $v_1(r)$ using equation (1):

$$1 - p(r) = \tfrac{1}{4}(v_1(r) - \delta v_1(1)),$$

So we need only solve for $v_1(r)$ and $v_2(r)$. Now, $v_1(1) = v_2(0) = \frac{4}{3(1-\delta)}$. So we get

$$v_1(r) = 4(1 - p(r)) + \delta v_1(\hat{r}_R) = 4(1 - p(r)) + \frac{4\delta}{3(1-\delta)} = 4(\tfrac{2}{3} - p(r)) + \frac{4}{3(1-\delta)}.$$

So

$$v_1(r) - v_1(1) = 4[p(1) - p(r)].$$

Equation (3) can be written

$$v_2(r) = 4p(r) + \delta v_2(\hat{r}_L) = 4(p(r) - \tfrac{2}{3}) + \delta v_2(\hat{r}_L) + \tfrac{8}{3}$$
$$= -4(p(1) - p(r)) + \delta(v_2(\hat{r}_L) - v_2(0)) + \tfrac{8}{3} + \delta v_2(0)$$

So

$$v_2(r) - v_2(0) = \tfrac{8}{3} - (1-\delta)v_2(0) - 4[p(1) - p(r)] + \delta[v_2(\hat{r}_L) - v_2(0)]$$

$$= \frac{4}{3} - [v_1(r) - v_1(1)] + \delta[v_2(1) - v_2(\hat{r}_L)].$$

Eliminating $p(r)$ from equations (1) and (2) by adding twice the second equation to the first gives

$$\begin{aligned} 3v_1(r) &= 4 + \delta v_1(\hat{r}_R) + 2\delta v_1(\hat{r}_L) \\ &= 4 + \delta v_1(1) + 2\delta v_1(\tfrac{5r-1}{4}) \\ \Rightarrow 3(v_1(r) - v_1(1)) &= 4 - 3(1-\delta)v_1(1) + 2\delta(v_1(\tfrac{5r-1}{4}) - v_1(1)) \end{aligned}$$

Since the first and second terms cancel with each other, we get

$$v_1(r) - v_1(1) = \frac{2\delta}{3}(v_1(\tfrac{5r-1}{4}) - v_1(1))$$

Summarizing, we have

$$\begin{aligned} v_2(r) - v_2(0) &= \frac{4}{3} - [v_1(r) - v_1(1)] + \delta[v_2(1) - v_2(\hat{r}_L)], \\ v_1(r) - v_1(1) &= \frac{2\delta}{3}(v_1(\tfrac{5r-1}{4}) - v_1(1)). \end{aligned}$$

Or, recalling $\underline{v}_1(r) = v_1(r) - v_1(1)$, and $\underline{v}_2(r) = v_2(r) - v_2(0)$,

$$\begin{aligned} \underline{v}_2(r) &= \frac{4}{3} - \underline{v}_1(r) + \delta\underline{v}_2(\tfrac{5r-1}{4}), \\ \underline{v}_1(r) &= \frac{2\delta}{3}\underline{v}_1(\tfrac{5r-1}{4}). \end{aligned}$$

***Solving on a dense subset:***

Define $\hat{r}(r)$ to be the updated value of $r$ when the type is not revealed. Thus,

$$\hat{r}(r) = \begin{cases} \hat{r}_R = 5r & \text{if } r < \frac{1}{5} \\ \hat{r}_L = \frac{5r-1}{4} & \text{if } r > \frac{1}{5}. \end{cases}$$

Also, we define

$$z(r) = \begin{cases} 0 & \text{if } r < \frac{1}{5} \\ 1 & \text{if } r > \frac{1}{5}. \end{cases}$$

Then we define a ***closed orbit*** to be a sequence $\{r_i\}_{i=0}^{n} \subseteq [0,1]$ satisfying $r_0 = r_n$, and $\hat{r}(r_i) = r_{i+1}$ for all $0 \leq i < n$.

Now for any $n > 1$, and sequence $\{z_i\}_{i=1}^{n} \subseteq \{0,1\}$, one can construct a closed orbit, $\{r_i\}_{i=1}^{n}$ satisfying $z(r_i) = z_i$ for all $i$ as follows: Construct the equations

$$r_{i+1} = \hat{r}(r_i) = \begin{cases} 5r_i & \text{if } r_i < \frac{1}{5} \\ \frac{5r_i - 1}{4} & \text{if } r_i > \frac{1}{5}. \end{cases}$$

for $0 \leq i < n$. Using the fact that $r_0 = r_n$, this gives a set of $n$ equations in $n$ unknowns, which can be solved for the $r_i$. It should be noted that the set of points that are members of a closed orbit is dense in $[0,1]$.

For any closed orbit, define

$$x_i = \begin{cases} \underline{v}_1(r_i) & \text{if } r_i < \frac{1}{5} \\ \underline{v}_2(r_i) & \text{if } r_i > \frac{1}{5}. \end{cases}$$

$$y_i = \begin{cases} \underline{v}_2(r_i) & \text{if } r_i < \frac{1}{5} \\ \underline{v}_1(r_i) & \text{if } r_i > \frac{1}{5}. \end{cases}$$

Now, a closed orbit, $\{r_i\}_{i=0}^{n}$, generates the following system of equations, for $0 \leq i < n$:

$$x_i = \tfrac{4}{3} - y_i + \delta x_{i+1},$$
$$y_i = \tfrac{2\delta}{3} y_{i+1}$$

if $z(r_i) = z(r_{i+1})$ and

$$x_i = \tfrac{4}{3} - y_i + \delta y_{i+1},$$
$$y_i = \tfrac{2\delta}{3} x_{i+1}$$

$z(r_i) \neq z(r_{i+1})$. Since $x_0 = x_n$, and $y_0 = y_n$, this gives a system of $2n$ equations in the $2n$ unknowns $\{x_i, y_i\}_{i=1}^{n}$, which can be solved for the $x_i$ and $y_i$. Since the points that are members of some closed orbit is a dense subset of $[0,1]$, this gives a method of solving for the functions $v_i(r)$ on a dense subset of closed orbits.

To get an idea of what the solution for our example looks like, we have generated all closed orbits of length less than or equal to $n = 10$, and solved the above system of equations for these closed orbits. The resulting functions are plotted in Figures 1 through 14. Figures 1 and 2 give the equilibrium mixed strategies $q_i(r)$ for player 2, and the updating rules $\hat{r}_L(r)$ and $\hat{r}_R(r)$. These parts of the solution are the same for all discount factors $0 \leq \delta \leq .6$. Figures 3 through 14 give the remainder of the solution for different discount factors $\delta = .2$, $\delta = .4$, $\delta = .6$. It follows from the results P1, P2, P1′ and P2′ that for $\delta < 1/5 = .2$ that this is a unique solution. For $\delta < 3/7 \simeq .429$, that this is a unique solution with the value function between $4/3(1-\delta)$ and $8/3(1-\delta)$. Also, it is verified by checking the equilibrium conditions, that this is a belief stationary equilibrium for any $\delta \leq 3/5 = .6$.

We see that for low values of the discount factor, the solution approximates what one would expect for the one stage bayesian game. (In the one stage game, the solution is that for $r < \frac{1}{5}$, player 1 chooses $p(r) = \frac{1}{3}$, and for $r > \frac{1}{5}$, player 1 chooses $p(r) = \frac{2}{3}$). However, as the discount factor increases, the solution becomes more and more discontinuous. It is at least monotonic up through about $\delta = .4$. However, for values of $\delta > .4$, the solution is no longer monotonic.

It should be pointed out that the method that we have used to compute the solution is only valid for points that are on some closed orbit. While these points are dense in the unit interval, nevertheless, the issue is still open as to what the solution looks like, or indeed if it even exists, for the "holes" in between points on the closed orbits. We do not have the answer to this question. Despite the fact that we cannot solve for the "holes," since beliefs cannot ever get off a closed orbit if they start there, the solution we have computed does give a complete characterization of the solution for any belief that starts on a closed orbit.

## 5. Conclusions

The example in this paper illustrates that the restriction of belief stationarity can place considerable structure on the solution of an infinitely repeated game of incomplete information. For low values of the discount factor we are able to find a unique solution. However, the resulting solutions have properties that are remeniscent of chaotic dynamical systems, and one wonders whether such a solution describes in either a normative or positive fashion the behavior individuals would or should adopt in such games.

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