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BINARY RELATIONS, NUMERICAL COMPARISONS WITH ERRORS AND RATIONALITY CONDITIONS FOR CHOICE

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1 Introduction

Since the famous work by P. Samuelson (1938) many papers have been published dealing with the problem of a description of human behavior in terms of numerical representation (e.g. utility functions), revealed preferences (or more generally, binary relations) and different conditions of rationality. K. Arrow (1959) generalizing the idea of rationality suggested to consider them as a properties of corresponding choice functions, and showed that choice according to maximization of some criteria is equivalent to the choice of non-dominated options on some weak order, and that corresponding choice function satisfies to the condition K of constancy[1]

R.D. Luce (1956) introduced some other numerical representation — that of criterial estimates with constant error. He found the equivalent representation of such numerical estimation —- the binary relation which he called semiorder. This fruitful idea was developed in several papers for the case when error value is not constant but depends of the option to which it is prescribed.

In this paper different cases for a numerical comparisons with errors are systematically developed, the corresponding rationality conditions are established and the corresponding binary relations are investigated. The main generalization considered in this paper is that

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[1]All necessary definitions are given in Section 2.

the error function depends not only on one option x but on the other option y or on the feasible set X given for choice.

Section 2 of the paper counting of all preliminary notions. In Section 3 the different models of unicriterial choice with error function are given. In Sections 4 and 5 a brief survey for some cases considered before by different authors is given.

Sections 6 to 10 containing the new results on unicriterial choice with error functions. Section 11 describes the open problems in the field. In Appendix the theorems given in Sections 6 to 10 are proved.

2 The General Notions and Classic Models

The finite set A of options is considered; any arbitrary non-empty subset X of A can be presented for choice. The set of all non-empty subsets of A is hereafter denoted as $\mathcal{A}^o$, i.e. $\mathcal{A}^o = 2^A \setminus \{\phi\}$.

The notion of the mechanism of choice is exploited in the following sense: it assumed that some information is given about options, e.g. numerical estimates, binary relations, etc. This information is informally called a structure on the set A, the choice rule π prescirbes how to use this information for choice of the best options. Both a structure and a choice rule are called a mechanism of choice and denoted as M with corresponding indices.

Choice function is denoted as $C(\cdot)$; point in brackets stands each time some set $X \in \mathcal{A}^o$.

For the classic model of uni-criterial choice the structure is given in the form of numerical function $\phi(x)$ for each $x \in A$, and the choice rule determines the set of chosen options for each $X \in \mathcal{A}^o$ as

$$C(X) = \{y \in X | \overline{\exists} x \in X \text{ s.t. } \phi(x) > \phi(y)\}. \quad (1)$$

On the other hand, in the classic model of pair-dominant choice the information about options is given in the form of binary relation β and the choice rule π_β prescribes to choose the undominated options, i.e. for $X \in \mathcal{A}^o$

$$C(X) = \{y \in X | \overline{\exists} x \in X \text{ s.t. } x \beta y\}. \quad (2)$$

This choice mechanism $M = <\beta, (2)>$ is called a pair-dominant one and denoted as M_{PD}.

Two choice mechanisms $M_1 =< \sigma_1, \pi_1 >$ and $M_2 =< \sigma_2, \pi_2 >$ are called equivalent if the choice functions generated by them coincide. If not concrete choice mechanism $M =< \sigma, \pi >$ is considered but the class $\mathcal{M}$ of mechanisms arising when σ varies in some class Σ, e.g. all scalar criteria, all binary relations, etc., the notion of equivalency between two classes $\mathcal{M}_1$ and $\mathcal{M}_2$ means only the existence of one-to-one mapping between them.

The class of mechanisms on all one scalar criteria with the rule (1) is denoted as M_ϕ. The classic generalization of the idea of uni-criterial choice with strict values had led to the notion of multi-criteria choice model, i.e. to each option $x \in A$ the vector of criterial values $\vec{\phi}(x) = \{\phi_1(x), \ldots, \phi_n(x)\}$, is prescribed, the choice rule for this case can be defined in different ways, e.g.

$$C(X) = \{y \in X | \overline{\exists} x \in X \text{ s.t. } \forall i \ \phi_i(x) > \phi_i(y)\} \tag{3}$$

or

$$C(X) = \{y \in X | \overline{\exists} x \in X \text{ s.t. } (\forall i \ \phi_i(x) \geq \phi_i(y) \& \exists i_o \text{ s.t. } \phi_{i_o}(x) > \phi_{i_o}(y)\}. \tag{4}$$

The rule (3) is called Sleuter (or weak Pareto) rule, the rule (4) is called Pareto one.

It turns out that the class of multicriteria mechanisms with the rule (3) is equivalent to that one which determined by the rule (4), i.e. for each multicriteria choice mechanism defined on the vector of criteria $\{\phi_i\}_1^n$ with the rule (3) it is possible to find some mechanism on the vector $\{\psi_i\}_1^m$ with rule (4) (and vice versa) such that the choice function generating by the first mechanism coincides with the choice function generating by the second one. So hereafter the multicriteria choice mechanism $M_{\vec{\phi}}$ with the rule (3) will be considered.

Let us study now some particular cases of pair-dominant mechanisms. These cases arise according to some special restrictions to a binary relation β in the definition of pair-dominant mechanism $M_{PD} =< \beta, (2) >$. Below the definition of different binary relations are given which will be widely used in the next sections.

Definition 1. The binary relation β is called to be

a) irreflexive iff $\forall x \in A \quad (x,x) \notin \beta$;

b) aciclic iff there is no such $r (1 \leq r \leq |A|)$ and options $x_1, \ldots, x_r \in A$ that $x_1 \beta x_2 \beta \ldots \beta x_r \beta x_1$;

c) transitive iff $x \beta y$, $y \beta z$ implies $x \beta z$;

d) negatively transitive iff $x \overline{\beta} y$, $y \overline{\beta} z$ implies $x \overline{\beta} z$;

e) complete iff $\forall x, y$ either $x \beta y$ or $y \beta x$ hold;

f) strict partial order iff β satisfies to a) and c);

g) weak order iff β obeys to the conditions a), c) and d);

h) linear order iff β obeys to the conditions a), c) and e).

The class of pair-dominant mechanisms on a) aciclic binary relations, b) strict partial orders, c) weak orders, d) linear orders will be denoted as a) $\mathcal{M}_{ac}$, b) $\mathcal{M}_{s.p.o.}$ c) $\mathcal{M}_{w.o.}$, and d) $\mathcal{M}_{\ell.o.}$ correspondingly.

Theorem 1 *The following classes of choice mechanisms are equivalent*

$$\mathcal{M}_{\phi} \sim \mathcal{M}_{w.o.},$$
$$\mathcal{M}_{\vec{\phi}} \sim \mathcal{M}_{s.p.o.}.$$

The following definition 2 gives different rationality conditions.

Definition 2. A choice function is called to satisfy to the condition of

a) Heritage (H) iff $\forall X, X' \in \mathcal{A}^o, X' \subseteq X \Rightarrow C(X') \supseteq C(X) \bigcap X'$;

b) Concordance (C) iff $\forall X', X'' \in \mathcal{A}^\bullet \rightarrow C(X') \bigcap C(X'') \subseteq C(X' \bigcup X'')$;

c) Independence of Outcast options (O) iff $\forall X, X' \in \mathcal{A}^0, X' \subseteq X \setminus C(X) \Rightarrow C(X \setminus X') = C(X)$;

d) Constancy (K) iff $\forall X, X' \in \mathcal{A}^o, X' \subseteq X, C(X') \bigcap X \neq \phi \Rightarrow C(X') = X \bigcap C(X)$;

These conditions were introduced and investigated by different authors, e.g. conditions H and C coincide correspondingly with conditions α and γ introduced by A. Sen (1974), conditon O is more strong than that one called δ-condition which was also introduced by A. Sen. Condition K was introduced by H. Chernoff (1954), and in other form used by K. Arrow (1959). We follow here the conditions and their notations used by M. Aizerman and A. Malishevski (1981) and elaborated by M. Aizerman and F. Aleskerov (1990). The following mutual relations shown in the form of Euler-Vienn diagram on the Fig. 1 between the classes of choice functions isolated by above conditions takes place. The choice functions which satisfy to these conditions create the classes in the set (space) $\mathcal{C}$ of all choice functions on A; these classes will be denoted by the same letters as the corresponding conditions. Let us note here that conditions H, C, O and K has been introduced for arbitrary choice functions (admitted also empty choice on some $X \in \mathcal{A}^o$); the subspace of non-empty choice functions will be denoted as $\widehat{\mathcal{C}}$, and correspondingly these conditions will be denoted as $\widehat{H}, \widehat{C}$, etc. A class $\mathcal{M}$ of choice mechanisms obviously generates some class of choice functions, and this class will be denoted as $\mathcal{C}(\mathcal{M})$ with a corresponding indicies.

Theorem 2

$$\begin{aligned} \mathcal{C}(\mathcal{M}_{\phi}) &= \mathcal{C}(\mathcal{M}_{w.\bullet.}) = \widehat{K} \\ \mathcal{C}(\mathcal{M}_{\vec{\phi}}) &= \mathcal{C}(\mathcal{M}_{s.p.\bullet.}) = \widehat{H} \bigcap \widehat{C} \bigcap \widehat{O}, \\ \mathcal{C}(\mathcal{M}_{ac}) &= \widehat{H} \bigcap \widehat{C}, \end{aligned}$$

i.e. the class of choice functions generating by the class of choice mechanisms on one scalar criteria coincides with the class generating by choice mechanisms on weak orders and coincides with the class of choice functions isolated by condition $\widehat{K}$, etc.

3 Uni-Criterial Choice with Insensitivity

Let us now following to R.D. Luce (1956) extend the idea of uni-criterial choice.

Let a criterial scale $\phi(a)$ be defined over the set A; below, its strictness or non-strictness will be specified if necessary. In contrast to the classic uni-criterial extremizational choice mechanism, consider another rule for choosing along this scale the best options with allowance for insensitivity (tolerance).

Assume that there exists an ϵ-wide ($\epsilon > 0$) "insensitivity zone" ("tolerance") for comparison of the estimates $\phi(x)$ and $\phi(y)$ of options $x, y \epsilon A$, and y is regarded as preferable to x only if $\phi(y) - \phi(x) > \epsilon$. Here, the rule for choosing "best option" can be written as follows:

$$y \in C(X) \Leftrightarrow (y \in X \& \bar{\exists} x \in X : \phi(x) - \phi(y) > \epsilon) \tag{5}$$

The choice rule (1) used in the uni-criterial extremizational choice mechanism is a special case of the rule (5) if $\epsilon \equiv 0$. That means that we choose the option y such that there is no other option x with the interval left side of which is disposed on the scale strictly on the left than the right side of interval corresponding to y. For brevity hereafter this kind of choice models will be called interval choice. In Fig. 2 it is shown the situation of choice in such case. For this example if the set $A = \{x, y, z\}$ is given for choice, then $C(A) = \{x, y\}$, and z is not chosen because $\phi(y) - \phi(z) > \epsilon$.

Generalizing this idea of uni-criterial choice with insensitivity let us consider different definitions of error function ϵ. Below the different cases of error function ϵ are listed which will be studied in the next sections:

1. $\epsilon =$ const $\neq 0$; 2. $\epsilon = \epsilon(x)$; 3. $\epsilon = \epsilon(x, y)$; 4. $\epsilon = \epsilon(y, X)$; 5. $\epsilon = \epsilon(x, X)$; 6. $\epsilon = \epsilon(X)$.

Using these different error functions in (5) we can obtain different mechanisms of choice and using scalar functions ϕ and error functions ϵ, different classes of choice mechanisms, which will be denoted as $\mathcal{M}_{\epsilon}$, $\mathcal{M}_{\epsilon(x)}$, $\mathcal{M}_{\epsilon(x,y)}$, $\mathcal{M}_{\epsilon(y,X)}$, $\mathcal{M}_{\epsilon(x,X)}$, $\mathcal{M}_{\epsilon(X)}$, correspondingly.

As before the notation $\mathcal{C}(\mathcal{M})$ is used, $\mathcal{M}$ with some subindex in each case, to denote the choice functions class generating by this class of mechanisms.

Let us discuss these different mechanisms of choice with insensitivity. In cases 1) and 2) the error function is either constant, or depends on the option x (or y). In case 3) this function depends not only on one option x, but on the option y as well with which x is compared; cases 4) to 6) are more general, i.e. the error function depends not only of options but also on the set given for choice. There is some asymmetry between cases 2) on one hand, and 4) and 5) on the other, because we did not mention the different case for $\epsilon = \epsilon(y)$, but we did it for cases 4) and 5). The cause of such asymmetry will be shown below.

4 Interval Choice with $\epsilon = \epsilon(x) \geq 0$

These interval choice models deal with the error function which can be either constant or non-negative. These models were investigated by R.D. Luce (1956), B Mirkin (1974), and P. Fishburn (1974, 1985).

The term "interval choice rule" is due to the following construction. To each $x \in X$ assign on the numerical axis ϕ an interval of the form of $[\phi^{\bullet}(x) - \epsilon^{-}(x), \phi(x) + \epsilon^{+}(x)]$ where $\phi^{\bullet}(x)$ is the "true" estimate of x, and $\epsilon^{-}(x)$ and $\epsilon^{+}(x)$ characterize the estimate "scatter" with respect to $\phi^{\bullet}(x)$. The option y will be regarded as better than x if $\phi^{o}(y) - \epsilon^{-}(y) > \phi^{\bullet}(x) + \epsilon^{+}(x)$. Let y be chosen from X if no option $x \in X$ exceeds it, that is

$$y \in C(X) \Leftrightarrow (y \in X \& \bar{\exists} x \in X : \phi^{\bullet}(x) - \epsilon^{-}(x) > \phi^{\bullet}(y) + \epsilon^{+}(y)) \qquad (6)$$

Assuming that

$$\phi(y) = \phi^{o}(y) - \epsilon^{-}(y), \phi(x) = \phi^{o}(x) - \epsilon^{-}(x), \epsilon(y) = \epsilon^{-}(y) + \epsilon^{+}(y)$$

obtain (5) from (6). In virtue of this remark, introduction into the choice rule of the "tolerance" ϵ amounts to considering interval in each scale point (its length may be different for different y); the estimates of options under consideration are compared with due regard for the length of interval characterizing the measurement error.

Return to (5) and introduce for $\epsilon = \epsilon(y)$ (in particular, for ϵ=const) the following relation β:

$$x\beta y \leftrightarrow \phi(x) - \phi(y) > \epsilon(y). \quad (7)$$

One can see directly that (5) can be rewritten on this "β- structure" as follows:

$$y \in C(X) \Leftrightarrow (y \in X \& \overline{\exists} x \in X : x\beta y),$$

i.e. any mechanism for choosing best options over a scale with insensitivity that makes use of the choice rule (5) is pair-dominant representable for both ϵ=const and $\epsilon = \epsilon(y)$.

The relation β in (7) is irreflexive and transitive, but, generally, not negatively transitive. Irreflexivity and transitivity of β are evident, and the fact that the negative transitivity condition is not satisfied is demonstrated via an example of Figure 2 where $y\overline{\beta}x$, $x\overline{\beta}z$, but $y\beta z$.

As follows from the above, the choice mechanism under consideration is pair-dominant representable by strict partial orders, but not weak orders, and, thus, any choice mechanism of the class under review is reducible to the multi-criterial extremizational choice mechanism,[2] but not to the extremizational choice by one scale, be it even different from the original scale ϕ. As follows then from Theorem 2, the choice functions generated by this mechanism are not empty and belong to the classical domain $\frown{H} \bigcap \frown{C} \bigcap \frown{O}$ in the subspace $\mathcal{C}$.

The inverse assertion is incorrect because there are functions belonging to $\frown{H} \bigcap \frown{C} \bigcap \frown{O}$, that are non-empty and generated by the multi-criterial extremizational mechanism, that therewith cannot be generated for any ϵ=const or $\epsilon = \epsilon(y)$ by any mechanism of choice by a single scale with insensitivity.

Definition 3. A binary relation β is called to satisfy

a) the strong intervality condition

b) the semitransitivity condition iff $\forall\ x, y, z, t \in\ A$

[2]This gives rise to the following problem: knowing the scale $\phi(\cdot)$ and the tolerance function $\epsilon(\cdot)$, construct a scale set such that vector extremization (identification of the Pareto set) coincides for all X with the initial choice by single scale with insensitivity. A possible construction of this criterial set may be found in Aleskerov (1980)

a) $x\beta y \& z\beta t \Rightarrow x\beta t$ or $z\beta y$;

b) $x\beta y\beta z \Rightarrow x\beta t$ or $t\beta z$,

respectively.

Definition 4. A binary relation β which satisfies irreflexivity and strong intervality conditions is called to be an interval order. A binary relation β obeying irreflexivity and semitransitivity conditions is called to be a semiorder.

For the case when $\epsilon = \epsilon(x) \geq 0$ this binary relation β satisfies to the conditions of irreflexivity and intervality. If $\epsilon = \text{const} > 0$, then β satisfies in addition to the semitransitivity condition. So, according to Definition 4, in the first case β is an interval order, in the second $-a$ semiorder. The classes of choice mechanisms on a set of interval orders and on a set of semiorders will be denoted as $\mathcal{M}_{i.o.}$ and $\mathcal{M}_{s\bullet}$ correspondingly. The following theorem holds:

Theorem 3 *The classes of choice mechanisms $\mathcal{M}_{i.\bullet.}$ and $\mathcal{M}_{\epsilon(x)}$ are equivalent; i.e. $M_{i.\bullet} \sim M_{\epsilon(x)}$. The classes of mechanisms $\mathcal{M}_{so}$ and $\mathcal{M}_{\epsilon}$ are also equivalent, i.e. $\mathcal{M}_{so} \sim \mathcal{M}_{\epsilon}$.*

These statements were proved in the papers cited above. Let us study now the conditions of rationality for such kind of interval choice.

These conditions were given independently by P. Fishburn (1975) and T. Schwartz (1975). Below the Fishburn's conditions are given however rewritten in terms of this paper:

$$\forall X', X'' \in \mathcal{A}^o \; C(X')\bigcap(X'' \setminus C(X'')) \neq \phi \Rightarrow C(X'')\bigcap(X' \setminus C(X')) = \phi.$$

This condition can be called functional asymmetry condition. The next Fishburn's condition (axiom 5) can be written as follows

$$\forall X, X', X'' \in \mathcal{A}^o \; X \subseteq X' \setminus C(X') \text{ and } C(X')\bigcap X'' \neq \phi \Rightarrow (X \setminus C(X))\bigcap C(X'') = \phi.$$

The following theorem holds.

Theorem 4 *1. Let choice function $C(\cdot)$ satisfies to the conditions $\widehat{H} \bigcap \widehat{C} \bigcap \widehat{O}$. This choice function is a pair-dominant one on some interval order iff it satisfies functional asymmetry condition; 2. let choice function $C(\cdot)$ generated by pair-dominant mechanism on some interval order. This choice function is a pair-dominant one on some semi-order iff it satisfies to the Fishburn's axiom 5.*

Theorems 3 and 4 give a complete description of the choice models of this kind.

The other necessary and sufficient conditions for the choice functions generating by pair-dominant mechanism on interval orders are given in M. Aizerman and F. Aleskerov (1990).

5 Interval Choice with Arbitrary Error Functions $\epsilon = \epsilon(x)$

In this case and it is easily be shown the binary relation β satisfies the strong intervality condition but not irreflexive.

Definition 5. A binary relation β which satisfy the strong intervality condition is called to be a bi-order. So, according to Definitions 4 and 5, an interval order is a bi-order which obey also irreflexivity condition.

It seems that Riguet (1951) was the first who introduced bi-orders for additive decomposition of integer numbers and called them Ferrers' relations. Later they were discussed in many publications among which complete studies made by Ducamp and Falmagne (1969) and Doignon et al. (1986) deserve special mentioning.

The bi-orders can be characterized in other terms, namely β is a bi-order iff $\beta\overline{\beta}^{-1}\beta \subseteq \beta$, where $\beta^{-1} = \{(x,y) | (y,x) \in \beta\}$ and $\overline{\beta} = A \times A \setminus (\beta \bigcup \beta^{-1})$.

In the case when ϵ=const and can be negative, then β obeys also to the condition of semitransitivity. Such binary relations were called by Doignon et al. (1986) as coherent biorders. The equivalent formulation for coherent biorders is as follows:

$$\beta\overline{\beta}^{-1}\beta \bigcup \overline{\beta}^{-1}\beta\beta \subseteq \beta.$$

Let us show that there are acyclic (and even transitive) binary relations which are not bi-orders. Let $\beta = \{(x,y),(z,t)\}$. This binary relation is not bi-order, but is strict partial order.

Let us note that negative value of error function corresponds to the situation, when the right boundary of the interval for some x can be displaced on the left side of the left boundary of this interval.

An abstract approach to this situation admits to introduce for each $x \in A$ two scalar functions $f(x)$ and $g(x)$ and rewrite rules (5) and (7) in the following way (see Ore (1962)).

$$y \in C(X) \Leftrightarrow (y \in X \& \exists x \in X \text{ s.t. } f(x) > g(y))$$

and

$$x\beta y \Leftrightarrow f(x) > g(y). \tag{8}$$

In the case when $\epsilon \geq 0$, this functions satisfy the condition $\forall x \in A \; f(x) \leq g(x)$ and correspond respectively to the left boundary of interval for x and to the right boundary of it.

The corresponding result can be formulated as follows

Theorem 5 *A binary relation β is a bi-order iff there exist two functions $f(x)$ and $g(x)$ defined on A such that (8) holds.*

Definition 6. Binary relation β will be called an equivalent according to choosing options extention of a given binary relation $\tilde{\beta}$ (or briefly an equivalent extention of $\tilde{\beta}$) if

$$\beta = \tilde{\beta} \bigcup \left\{ (x, y) | (y, y) \in \tilde{\beta} \right\},$$

i.e. β can be obtained from $\tilde{\beta}$ if we complete $\tilde{\beta}$ with all pairs from element x to y where the pair (y, y) belongs to $\tilde{\beta}$.

Lemma 1. $C_{\beta}(\cdot) = C_{\tilde{\beta}}(\cdot)$, where $C_{\beta}(\cdot)$ is the pair-dominant choice function on β, and $C_{\tilde{\beta}}(\cdot)$ is that one on $\tilde{\beta}$.

Theorem 6 *Let some binary relation $\tilde{\beta}$ is given, and β — its equivalent extension. Then the function $C_{\beta}(\cdot)$ is a pair-dominant one on a biorder iff it satisfies the condition of functional asymmetry.*

Remark. Let us give the example which shows that for a given $\tilde{\beta}$, the binary relation β can be a biorder even if $\tilde{\beta}$ is not. Let $\tilde{\beta} = \{(x, y), (y, z), (z, z)\}$, and β is a biorder, but $\tilde{\beta}$ is not because $x\tilde{\beta}y \& y\tilde{\beta}z$, but $(x, z) \notin \tilde{\beta}$ and $(y, y) \notin \tilde{\beta}$.

The choice mechanism generated with scalar function $\phi(\cdot)$ and constant error function ϵ can be equivalently represented by binary relations which were called by Doignon et al. (1986) coherent biorders; these relations satisfy the conditions of intervality and seitransitivity, but can not obey the irreflexivity condition.

The functional asymmetry condition and axiom 5 by Fishburn are necessary and sufficient conditions for rationality of corresponding choice functions provided that the equivalent extention of a given coherent biorder is considered.

6 Choice Mechanism with $\epsilon = \epsilon(x, y)$

Consider now another definition of the "error" function: $\epsilon = \epsilon(x, y)$. Introduction of this function implies that measurement insensitivity may be dependent on both compared options x and y.

For this case, (5) is representable as

$$y \in C(X) \Leftrightarrow (y \in X \& \overline{\exists} x \in X : \phi(x) - \phi(y) > \epsilon(x, y)) \tag{9}$$

Denote the choice mechanism $< \phi(\cdot), \epsilon(\cdot, \cdot); (9) >$ by $M_{\epsilon(x,y)}$, and the class of mechanisms generated for different $\phi(x)$ and $\epsilon(x, y)$ by $\mathcal{M}_{\epsilon(x,y)}$. The corresponding binary relation β can be constructed as

$$x \beta y \Leftrightarrow \phi(x) - \phi(y) > \epsilon(x, y) \tag{10}$$

Theorem 7 *The class $\mathcal{M}_{\epsilon(x,y)}$ is equivalent to the class of pair-dominant choice mechanisms with arbitrary binary relation β and, under the constraint $\epsilon(x, y) \geq 0$ for all $x, y \in A$, to the class of pair- dominant mechanisms with acyclic binary relation β.*

Theorem 8 *For the binary relation β in (10) to be a)transitive, b)negatively transitive it is sufficient that the condition $\forall x, y, z \in A$ a) $\epsilon(x, z) \leq \epsilon(x, y) + \epsilon(y, z)$, b) $\epsilon(x, z) \geq \epsilon(x, y) + \epsilon(y, z)$ holds respectively.*

Let us note that the conditions used in Theorem 7 are sufficient, but not necessary to represent transitive and negatively-transitive binary relations. Author could not obtain the necessary conditions which would be satisfied for an arbitrary $\phi(\cdot)$. Let us note also that the constraints on the function $\epsilon(x, y)$ providing the transitivity of β are analogues (or at least look like) to those of stochastic transitivity used in the models of paired comparisons (see David (1968)).

Consider now the case when the function $\epsilon(x,y)$ can be represented as $\epsilon(x,y) = \epsilon(x) + \epsilon(y)$, i.e. joint error function depends on "independent errors" $\epsilon(x)$ and $\epsilon(y)$ additively. Then the following theorem holds.

Theorem 9 *The mechanism* $< \phi(x), \epsilon(x,y), (9) >$ *with additive error function* $\epsilon(x,y) = \epsilon(x) + \epsilon(y)$ *is equivalent to the mechanism* $< \phi(x), \epsilon(x), (5) >$ *of interval choice.*

The proof of this theorem is literally coincide with considerations used above in studying of different types of definitions of options estimates intervals, and hence is omitted.

The example of the existence of a joint error function of a type $\epsilon(x,y)$ are given by experiments made by Fechner (1860), in which ϕ was the real value of stimulus (irritation), and $\epsilon(x,y)$ was the error value in the comparison of options (irritations). G.T. Fechner showed that the error value depends logarithmically on the values of stimula. This regilarity is one of the fundamental laws of psychophisic called Fechner-Veber law.

7 General Case: Error Function Depends on Feasible Set X

Let us consider the choice mechanisms with error function in the forms $\epsilon = \epsilon(y,X)$, $\epsilon = \epsilon(x,X)$ and $\epsilon = \epsilon(X)$.[3]

The choice mechanism for these cases will be called below generalized interval choice mechanisms.

Let the function $\phi(\cdot)$ be defined as well as the error values $\epsilon(y,X)$ for each y in X. Then, (5) becomes

$$y \in C(X) \Leftrightarrow (y \in X \& \overline{\exists} x \in X : \phi(x) - \phi(y) > \epsilon(y,X)) \tag{11}$$

Table 1 presents an example proving that the mechanism of generalized interval choice may generate a choice function $C(\cdot)$ not satisfying the Heritage and Concordance conditions. In this example, y belongs to the choice from $\{x,y,z\}$ but does not belong to that from $\{x,y\}$, that is condition H is violated. Besides, z belongs to the choice from $\{x,z\}$

[3]The results about these cases also were considered in Agaev and Aleskerov (1993).

Table 1

$u \in A$	$\phi(u)$	X	$\epsilon(x,X)$	$\epsilon(y,X)$	$\epsilon(z,X)$	$C(X)$
x	3	$\{x,y,z\}$	0	1	0	$\{x,y\}$
y	2	$\{x,y\}$	0	0	-	$\{x\}$
z	1	$\{x,y\}$	0	-	2	$\{x,z\}$
		$\{y,z\}$	-	0	1	$\{y,z\}$

and from $\{y,z\}$ but does not belong to the choice from $\{x,y,z\}$, that is condition C is violated.[4]

If one assumes in this example that $\epsilon(y,\{x,y\}) = \epsilon(z,\{x,z\}) = 1$, the function defined by Table 1 will satisfy both conditions H and C.

Thus, the generalized interval choice mechanism can generate a function either satisfying both the Heritage and Concordance conditions, or none.

For the case when $\epsilon = \epsilon(x,X)$ the formula (5) will be rewritten as follows:

$$y \in C(X) \Leftrightarrow (y \in X \& \exists x \in X : \phi(x) - \phi(y) > \epsilon(x,X)) \tag{11}$$

The choice mechanisms $< \phi(y), \epsilon(y,X), (10) >$ and $< \phi(x), \epsilon(x,X), (11) >$ will be denoted by $M_{\epsilon(y,X)}$ and $M_{\epsilon(x,X)}$, respectively.

Theorem 10 *The class of choice mechanisms generating by the class of mechanisms $\mathcal{M}_{\epsilon(y,X)}$ coincides with the space of all choice functions, i.e. $\mathcal{C}(\mathcal{M}_{\epsilon(y,X)}) = \mathcal{C}$.*

Otherwise speaking for an arbitrary choice function $C(\cdot)$ one can find the mechanism $M_{\epsilon(y,X)}$ generating this function. It turns out that the class of mechanisms $M_{\epsilon(x,X)}$ is narrower than that of $M_{\epsilon(y,X)}$. Below the example is given showing that not an arbitrary choice function can be represented making use of mechanism $M_{\epsilon(x,X)}$.

Example. Let $\beta = \{(x,y),(z,t)\}$ is given, and consider the pair-dominant choice function $C(\cdot)$ on such β. Let us consider the sets $X_1 = \{x,y,t\}$ and $X_2 = \{z,y,t\}$. Obviously, $C(X_1) = \{x,t\}$ and $C(X_2) = \{z,y\}$. Hence according to the rule (11) for y not to be chosen there exist some u s.t. $\phi(u) - \epsilon(u,X_1) > \phi(y)$, and because t is chosen

$$\forall u \ \phi(u) - \epsilon(u,X_1) \le \phi(t), \text{ then } \phi(t) > \phi(y).$$

[4]Table 1 does not list the values of $\epsilon(y,\{y\})$ that are assumed to be 1.

Considering now the set X_2 obtain that $\phi(t) < \phi(y)$. This contradiction shows that such function $C(\cdot)$ can be represented by no mechanism of the form $M_{\epsilon(x,X)}$.

We investigate now the possibility of representing a choice function by means of the generalized interval mechanism $M_{\epsilon(x,X)}$.

Definition 7. The function $C(\cdot)$ will be said to satisfy the functional acyclicity condition if there exist no r sets $X_1, X_2, \ldots, X_r \in \mathcal{A}^o$ such that $(X_1 \setminus C(X_1)) \bigcap C(X_2) \neq \phi$, $(X_2 \setminus C(X_2)) \bigcap C(X_3) \neq \phi, \ldots$, $(X_r \setminus C(X_r)) \bigcap C(X_1) \neq \phi$.

We cite an example illustrating the meaning of the functional acyclicity condition.

Example. Let $A = \{x_1, x_2, x_3\}$, $X_1 = \{x_1, x_2\}$, $X_2 = \{x_2, x_3\}$ and $X_3 = \{x_1, x_3\}$, and let the function $C(\cdot)$ be such that $C(X_1) = \{x_1\}$, $C(X_2) = \{x_2\}$ and $C(X_3) = \{x_3\}$. By Definition 7, the sets X_1, X_2, and X_3 make up in this case a "functional cycle" or, stated differently, violate the functional acyclicity condition.

This functional cycle of length 3 is depicted in Figure 3. Notably, if the relation β is constructed through $C(\cdot)$ so that $x\beta y \Leftrightarrow C(\{x,y\}) = \{x\}$, it will contain the cycle $x_1 \beta x_2 \beta x_3 \beta x_1$.

Remark. The investigation of analogous conditions were done at first time by P. Samuelson (1938) in terms of consumer demand problem. When consumer is acting in a two-diensional space of commodities and the choice contains always only one option, P. Samuelson showed that the revealed preference relation P_C $(xP_Cy \Leftrightarrow (\exists X \in \mathcal{A}^o : \mathcal{C}(X) = \{x\}, y \in X, x \neq y)$ should be asymmetric for the choice function to be generated by in the terms of this paper uni-criterial mechanism. The corresponding condition was called in the literature as the Samuelson's axiom of revealed preferences.

In the case when the space of commodities has the dimension which is greater than 2, but under the singleton choice constraint, the corresponding result obtained by H.S. Houthakker (1950) already needs the acyclicity of the revealed preference relation.

The extension of this result on the situation when choice can contain not only one but several options was done by K. Arrow (1959) and the corresponding condition was that of K (constancy). We will not to analize different versions of abstract revealed preference axioms arising also in the case when the family of feasible sets is not complete, i.e. does not coincide with the family of all non-empty subsets of A. However it is necessary to mention that these axioms were oriented on the case when an arbitrary choice function can be equivalently described as a maximization of some utility function, or a choice of undominated options on some binary relation (Plott (1973), Suzumura (1974)).

Theorem 11 *The choice function $C(\cdot)$ is generated by the mechanism $M_{\epsilon(x,X)}$ if and only if $C(\cdot)$ satisfies the functional acyclicity condition.*

If now the error function has the form of $\epsilon = \epsilon(X)$, (5) becomes as follows:

$$y \in C(X) \Leftrightarrow (y \in X \& \overline{\exists} x \in X : \phi(x) - \phi(y) > \epsilon(X)) \tag{12}$$

The choice mechanism with choice rule (12) and structure $\phi(\cdot)$ will be symbolized by $M_{\epsilon(X)}$.

Theorem 12 *The mechanism classes $\mathcal{M}_{\epsilon(X)}$ and $\mathcal{M}_{\epsilon(x,X)}$ are equivalent, that is the classes of choice functions $\mathcal{C}(\mathcal{M}_{\epsilon(X)})$ and $\mathcal{C}(\mathcal{M}_{\epsilon(x,X)})$ coincide.*

8 Threshold Mechanism of Choice

Let us study now some mechanism of choice introduced by Aizerman and Malishevski (1982). Let the scalar criteria on options $\phi(y)$ and the threshold (scalar) function $V(X)$ on all X from $\mathcal{A}^o$ are given.

The choice rule is introduced as follows

$$y \in C(X) \Leftrightarrow (y \in X \& \phi(y) \geq V(X)), \tag{13}$$

i.e. the option y is included into the choice from the set X if and only if the estimation $\phi(y)$ is greater than the threshold value $V(X)$ on the given X. The threshold choice mechanism will be denoted as $M_{V(X)}$, and the class of such mechanisms arising with different functions $\phi(\cdot)$ and $V(\cdot)$ will be denoted as $\mathcal{M}_{V(X)}$.

Remark. The threshold function can be introduced in such a way to express the average criterial value on X, e.g.

$$V(X) = \frac{1}{|X|} \sum_{x \in X} \phi(x)$$

In this case the mechanism $M_{V(X)}$ choses the options with the estimation which is greater than the average value on X. The notion of average criterial value can be introduced in other way,

$$V(X) = \frac{1}{2}(\min_{x \in X} \phi(x) + \max_{x \in X} \phi(x)).$$

It turns out that

Theorem 13 *The class of threshold choice mechanisms* $\mathcal{M}_{V(X)}$ *is equivalent to the class of mechanisms* $\mathcal{M}_{\epsilon(x,X)}$, *i.e.* $\mathcal{M}_{V(X)} \sim \mathcal{M}_{\epsilon(x,X)}$.

From Theorems 11 and 12 obtain the following.

Corollary. The mechanism classes $\mathcal{M}_{\epsilon(X)}, \mathcal{M}_{V(X)}$ and $\mathcal{M}_{\epsilon(x,X)}$ are equivalent, i.e. $\mathcal{M}_{\epsilon(X)} \sim \mathcal{M}_{V(X)} \sim \mathcal{M}_{\epsilon(x,X)}$.

9 Generalized Interval Choice with Non-negative Error Functions

If the values of error function are not negative, the corresponding choice mechanisms will be denoted by $M_{\epsilon^+(y,X)}$ and $M_{\epsilon^+(x,X)}$. An example similar to that of Table 1 may be constructed for these mechanisms M in order to prove the possibility of violating the conditions H and C.

Consider first the choice functions generated by the mechanisms $M_{\epsilon^+(y,X)}$.

Definition 8. The choice function $C(\cdot)$ will be called to satisfy the functional non-dominance condition if for any $X \in \mathcal{A}^o$ there exists an option $x \in X$ such that $x \in X' \Rightarrow x \in C(X')$ holds for any $X' \subseteq X$.

The functional non-dominance condition requires that there exist in X an option x such that it is included into the choice from X as well as in the choice from all the subsets $X' \subseteq X$ involving this option. This option x according to the suggestion of T. Schwartz can be called a fixed point.

It might be well to note that choice non-emptiness follows from the functional non-dominance condition that is weaker for the non-empty choice function than that of heritage **H**. The latter requires that any option x is included into the choice from X must be also chosen from any subset $X' \subseteq X$ involving this variant. The functional non-dominance condition requires only the existence of such x. To take one example of a function satisfying the functional non-dominance condition and not satisfying H, we cite $A = \{x, y, z\}$, $C(X) = X$ if $|X| = 1$ or $|X| = 3$, $C(\{x, y\}) = \{x\}$, $C(\{x, z\}) = \{x, z\}$ and $C(\{y, z\}) = \{z\}$. Indeed, $C(A) = A$, but $y \notin C(\{x, y\})$.

The following theorem holds.

Theorem 14 *The choice function* $C(\cdot)$ *is generated by the mechanism* $M_{\epsilon^+(y,X)}$ *if and only if it satisfies the functional non-dominance condition.*

Remark. There is no special condition which isolates the choice functions generated by mechanisms $M_{\epsilon^+(x,X)}$ with non-negative error functions: the necessary and sufficient conditions are non-emptiness of choice and functional acyclicity. Let us note only that the algorithm used in the proof of theorem construct in the case of non-empty choice function the mechanism $M_{\epsilon^+(x,X)}$ with non-negative error function $\epsilon(x, X)$. As was mentioned above, in the general case the strict inclusion $\mathcal{C}(\mathcal{M}_{\epsilon(x,X)}) \subset \mathcal{C}(\mathcal{M}_{(y,X)}) = \mathcal{C}$ holds. Let us consider the relation between the classes $\mathcal{C}(\mathcal{M}_{\epsilon^+(x,X)})$ and $\mathcal{C}(\mathcal{M}_{\epsilon^+(y,X)})$.

The following theorem holds.

Theorem 15 *The choice function lies in the domain $\mathcal{C}(\mathcal{M}_{\epsilon^+(x,X)})$ if and only if it satisfies the functional acyclicity and non- dominance conditions, that is $\mathcal{C}(\mathcal{M}_{\epsilon^+(x,X)}) = \mathcal{C}(\mathcal{M}_{\epsilon(x,X)}) \bigcap \mathcal{C}(\mathcal{M}_{\epsilon^+(y,X)})$.*

Figure 4 shows for the choice function space $\mathcal{C}$ the mutual positions of domains $\mathcal{C}(\mathcal{M}_{\epsilon(x,X)}, \mathcal{C}(\mathcal{M}_{\epsilon(y,X)}), \mathcal{C}(\mathcal{M}_{\epsilon^+(x,X)}), \mathcal{C}(\mathcal{M}_{\epsilon^+(y,X)})$, and $\mathcal{C}(\mathcal{M}_{\epsilon(X)})$ comprising all the choice functions generated by the mechanisms from corresponding classes $\mathcal{M}_{\epsilon(x,X)}$, $\mathcal{M}_{\epsilon(y,X)}$, $\mathcal{M}_{\epsilon^+(x,X)}$, $\mathcal{M}_{\epsilon^+(y,X)}$, and $\mathcal{M}_{\epsilon(X)}$.

10 Generalized Interval Choice – Binary Representation

Let us study now the special case of generalized interval choice - that one which can be equivalently represented as pair-dominant choice on some binary relation. As mentioned above, all choice functions from $H \bigcap C$ are generated by pair-dominant mechanisms. That is why the answer to a question about characteristic features of the mechanisms $M_{\epsilon(y,X)}$ and $M_{\epsilon(x,X)}$ generating functions from the domain $H \bigcap C$ may be formulated in terms of properties of relations β corresponding to pair-dominant mechanisms.

Consider first the mechanisms $M_{\epsilon(y,X)}$. Since the functional non-dominant condition implies non-emtpiness of choice and since non-emptyness in $H \bigcap C$ implies acyclicity of β, one can formulate the following.

Theorem 16 *For any choice function from the domain $\widehat{H} \bigcap \widehat{C}$, its generating mechanism $M_{\epsilon^+(y,X)}$ can be constructed.*

On the other hand, because of the fact that using error function in the form $\epsilon = \epsilon(y, X)$ it is possible to generate an arbitrary choice function, it is evident that an arbitrary pair-dominant choice function can be generated by mechanisms $< \phi(y), \epsilon(y, X), (11) >$ with arbitrary function $\epsilon(y, X)$. Let us study now the case of mechanisms $M_{\epsilon(x,X)}$.

Definition 9. The relation β will be said to satisfy the weak intervality condition if for any four distinct x_1, x_2, x_3 and x_4 satisfying $x_1 \beta x_2$ and $x_3 \beta x_4$ at least one of the following relations $x_1 \beta x_4$, $x_3 \beta x_2$, $x_2 \beta x_4$, $x_4 \beta x_2$, $x_2 \beta x_2$, or $x_4 \beta x_4$, holds.

Definition 10. The binary relation β will be identified as weak cyclic relation if with the availability of cycle $x_1 \beta x_2 \beta \ldots \beta x_r \beta x_1$ it has in any such cycle at least one symmetric pair of the form of $x_{i+1} \beta x_i$ $i \in \{1, \ldots, r\}$ (where $x_{r+1} = x_1$) or at least one pair of the form of $x_i \beta x_i$.

Definition 11. The binary relation β satisfying simultaneously the weak intervality and weak cyclicity conditions will be called weak bi-order relation.

As follows from Definitions 8 and 9, weak bi-order is "strictly weaker" than bi-order relation (stated differently, the set of bi-order relations is strictly embedded into the set of weak bi-orders).

Consider an example where the binary relation β has the form of $\beta = \{(x, y), (y, z), (z, x), (x, z)\}$. It is weak bi-order but does not satisfy the strong intervality condition since for the bi-order relation $x \beta z$ or $y \beta y$ must follow from $x \beta y$ and $y \beta z$.

Theorem 17 *Let a pair-dominant choice mechanism* $< \beta, (2) >$ *be defined. The function* $C(\cdot)$*generated by this mechanism satisfies the functional acyclicity condition (i.e., is generated in virtue of Theorem 10 by the mechanism* $M_{\epsilon(x,X)}$*) if and only if* β *is weak bi-order.*

As follows directly from Definition 9, if the irreflexivity and transitivity conditions are obeyed, weak bi-order relation becomes that of interval order, and, therefore, the mechanisms from the class $M_{\epsilon(x,X)}$ generate in the domain $H \bigcap C \bigcap O$ interval choice functions.

11 Open Problems

In Figure 5 an Euler-Vienn diagram depicts the domains of acyclic binary relations, strict partial orders, semitransitive relations, and bi-orders. According to the results given in previous sections the domain of bi-orders now has been completely studied, i.e. for all subdomains of this domain we have a complete description in terms of numerical representation and rationality conditions for choice functions. The situation is completely different for the domain of semitransitive relations — only for intersections of this domain with that of bi-orders the corresponding results have been obtained. This gives rise the following open problem — to describe somehow the domain of semitransitive relations in terms of numerical representation.

The second problem is connected with generalization of all cases on uni-criterial choice with errors considered above on a n-dimensional space of criteria. One result on this direction was obtained by F. Aleskerov (1983) (see also Aizerman and Aleskerov (1990). In that paper the direct generalization of Pareto rule was introduced for the case when $\epsilon_i = \epsilon_i(x)$, and it was shown that such class of choice mechanisms is equivalent to that of multicriterial choice procedures $\mathcal{M}_{\vec{\phi}}$.

The third open problem on my point of view, and I understand very obviously that it will be the object for strong criticism, because the models obtained above are principally algebraic but not statistical, is the description of human behavior in the experiments of psychophisical type. Using the mechanism $< \phi(y), \epsilon(y, X), (10) >$ according to Theorem 10, it is possible to explain any observed choice. But it will be much more interesting to find some type of experiments to explain the obtained choice making use the mechanism $M_{\epsilon(x,X)}$, or even more simple uni-criterial choice mechanisms with errors considered in this paper.

Appendix: Proofs

Proof of Lemma 1. For $\tilde{\beta} \subseteq \beta$, then $C_{\tilde{\beta}}(\cdot) \supseteq C_\beta(\cdot)$. Show that $C_{\tilde{\beta}}(\cdot) \subseteq C_\beta(\cdot)$. Let on the contrary there exist X and x s.t. $x \in C_{\tilde{\beta}}(X)$ and $x \notin C_\beta(X)$. It means that there is some y such that $y\beta x$ but $(y,x) \notin \tilde{\beta}$. According to the construction β, $y\beta x$ if $(x,x) \in \tilde{\beta}$, hence $x \notin C_{\tilde{\beta}}(X)$.

Proof of Theorem 6. Let β is a biorder. Show that $C_\beta(\cdot)$ satisfies to the condition of functional asymmetry. Suppose not, i.e. $\exists X_1,\ X_2 \in \mathcal{A}^o$ and x, y such that $x \in (X_1 \setminus C(X_1)) \bigcap C(X_2)$ and $y \in C(X_1) \bigcap (X_2 \setminus C(X_2)) \neq \phi$. For $x \in X_1 \setminus C(X_1)$ this implies that $\exists z \in C(X_1)$ s.t. $z\beta x$, $y \in X_2 \setminus C(X_2)$ implies that $\exists w \in C(X_2)$ s.t. $w\beta y$, and neither $z\beta y$ (because in that case $y \notin C(X_1)$) nor $w\beta x$ for the same reason.

Let now a function $C_\beta(\cdot)$ satisfy the condition of functional asymmetry. Show that β is a biorder. Let, on the contrary, $\exists x, y, z, w$ s.t. $x\beta y \& z\beta w$, but $x\overline{\beta}w$ and $z\overline{\beta}y$. According to the construction β $y\overline{\beta}y$ and $w\overline{\beta}w$. Let us consider two sets $X_1 = \{x, y, w\}$ and $X_2 = \{z, y, w\}$. $C(X_1) \ni w$, and $(X_1 \setminus C(X_1)) \ni y$, $C(X_2) \ni y, (X_2 \setminus C(X_2)) \ni w$, and we obtain the violation of functional asymmetry condition.

Proof of Theorem 7. Prove that if $M_{\epsilon(x,y)} \in \mathcal{M}_{\epsilon(x,y)}$ and $\forall x, y \in A\ \epsilon(x,y) \geq 0$, the constructed relation is acyclic. Let, on the contrary, there exist $x_1, x_2, \ldots, x_r$ such that $x_1\beta x_2\beta\ldots\beta x_n\beta x_1$. Obtain by the definition of β that

$$\begin{aligned}
\phi(x_1) - \phi(x_2) &> \epsilon(x_1, x_2) \\
\phi(x_2) - \phi(x_3) &> \epsilon(x_2, x_3) \\
&\cdots \\
\phi(x_r) - \phi(x_1) &> \epsilon(x_r, x_1)
\end{aligned}$$

By adding them obtain

$$\epsilon(x_1, x_2) + \epsilon(x_2, x_3) + \ldots + \epsilon(x_{r-1}, x_r) + \epsilon(x_r, x_1) < 0$$

Therefore, at least one of the addends is negative.

Inversely, let a pair-dominant choice mechanism $< \beta, (2) >$ be defined with β being acyclic. Let us construct its equivalent mechanism $M_{\epsilon(x,y)}$.

Construct from β the system $\{Z_1, \ldots, Z_s\}$ as follows:

$$
\begin{aligned}
Z_1 &= \{x \in A | \overline{\exists} y \in A : y\beta x\} \\
Z_2 &= \{x \in A \setminus Z_1 | \overline{\exists} y \in A \setminus Z_1 : y\beta x\} \\
&\vdots \\
Z_s &= \{x \in A \setminus \bigcup_{j=1}^{s-1} Z_j | \overline{\exists} y \in A \setminus \bigcup_{j=1}^{s-1} Z_j : y\beta x\}
\end{aligned}
$$

The number s is defined here by $Z_s \neq \phi$ and $\bigcup_{j=1}^{s} Z_j = A$.

Obviously, the system of sets $\{Z_j\}_1^s$ is the partition of A. Assume for any $x \in Z_j$ that $\phi(x) = s - j$. Construct now the functions $\epsilon = \epsilon(x, y)$. For any x, y such that $x\beta y$ assume that $\epsilon(x, y) = \epsilon(y, x) = 0$. Let now $(x, y) \notin \beta, (y, x) \notin \beta, x \in Z_i, x \in Z_j$. For $i < j$, assume that $\epsilon(y, x) = 0$, $\epsilon(x, y) = \phi(x) - \phi(y)$ and for $i = j$ define $\epsilon(x, y) = \epsilon(y, x) = 0$.

With this definition of functions $\phi(x)$ and $\epsilon(x, y)$, the constructed choice mechanism $M_{\epsilon(x,y)}$ is equivalent to the initial mechanism $< \beta, (2) >$ because $x\beta y \Leftrightarrow \phi(x) - \phi(y) > \epsilon(x, y)$ for all $x, y \in A$.

If one does not impose the constraint requiring that $\forall x, y \in A \; \epsilon(x, y) \geq 0$, the class of choice mechanisms $M_{\epsilon(x,y)}$ will be equivalent to that of pair-dominant mechanisms with arbitrary β.

Really, assuming for all $x \in A \; \phi(x) = 0$ and for all $x, y \in A \; \epsilon(x, y) = -1$ for $x\beta y$ and $\epsilon(x, y) = 0$ for $x\bar{\beta} y$, obtain for any relation β a mechanism $M_{\epsilon(x,y)}$ equivalent to the original mechanism $< \beta, (2) >$.

Proof of Theorem 8: a) Let $\forall x, y, z \in A \; \epsilon(x, z) \leq \epsilon(x, y) + \epsilon(y, z)$ holds. Show that β is transitive. Indeed, let $x\beta y$ and $y\beta z$ hold, i.e. according to the formula (10) $\phi(x) - \epsilon(x, y) > \phi(y)$, $\phi(y) - \epsilon(y, z) > \phi(z)$. Adding these two inequalities obtain $\phi(x) - \epsilon(x, y) - \epsilon(y, z) > \phi(z)$ and making use of the inequality a) obtain $\phi(x) - \epsilon(x, z) > \phi(z)$, i.e. $x\beta z$.

The proof of part b) of the theorem is completely analogous.

Proof of Theorem 10: Let $\phi(x) = 0$ for every $x \in A$, and put

$$
\epsilon(y, X) = \begin{cases} -1, & \text{if } y \notin C(X); \\ 0, & \text{otherwise.} \end{cases}
$$

Then

$$
\phi(y) + \epsilon(y, X) = \begin{cases} -1, & \text{if } y \notin C(X) \\ 0, & \text{otherwise} \end{cases}
$$

and if $y \notin C(X)$, then for all $x \in X \phi(x) - \phi(y) = 0 > \epsilon(y, X) = -1$ holds, if $y \in C(X)$, then $\phi(x) - \phi(y) = 0 = \epsilon(y, X) = 0$.

Proof of Theorem 11. Let function $C(\cdot)$ a be generated by some mechanism $M_{\epsilon(x,X)}$. Prove that it satisfies the functional acyclicity condition. Let, on the contrary, there exist a number r, variants $x_1, \ldots, x_r$ and sets $X_1, \ldots X_r$ such that $x_i, x_{i+1} \in X_i$ ($i = \overline{1,r}, x_{r+1} = x_1$) and let the following be true:

$$x_1 \in C(X_1),\ x_2 \notin C(X_1),\ x_2 \in C(x_2),\ x_3 \notin C(X_3), \ldots,$$
$$x_r \in C(X_r),\ x_1 \notin C(X_r).$$

Since $C(\cdot)$ is generated by $M_{\epsilon(x,X)}$, it follows from $x_2 \notin C(X_1)$ that there exists $z \in X_1$ such that $\phi(z) - \epsilon(z, X_1) > \phi(x_2)$. Since $x_1 \in C(X_1)$ obtain $\phi(z) - \epsilon(z, X_1) \leq \phi(x_1)$.

These two inequalities lead to $\phi(x_2) < \phi(x_1)$. For each $i = \overline{1, r-1}$ from $x_i \in C(X_i)$ and $x_{i+1} \notin C(X_i)$ obtain similarly $\phi(x_{i+1}) < \phi(x_i)$ and, finally, $\phi(x_r) > \phi(x_1)$. The contradictory nature of this inequality system proves the need in the functional acyclicity condition.

Let $C(\cdot)$ satisfy the functional acyclicity condition. Construct a mechanism $M_{\epsilon(x,X)}$ such that the function $\tilde{C}(\cdot)$ it generates coincides with $C(\cdot)$.

Define a binary relation $\delta : x\delta y \Leftrightarrow \exists X \in \mathcal{A}^o : x \in C(X), y \in X \setminus C(X)$. Consider the following set system:

$$\begin{aligned}
Z_1 &= \{x \in A | \overline{\exists} y \in A : y\delta x\} \\
Z_2 &= \{x \in A \setminus Z_1 | \overline{\exists} y \in A \setminus Z_1 : y\delta x\} \\
\ldots \\
Z_r &= \{x \in A \setminus \bigcup_{j=1}^{n-1} Z_j | \overline{\exists} y \in A \setminus \bigcup_{j=1}^{n-1} Z_j : y\delta x\}
\end{aligned}$$

where n is defined by the condition $Z_{n+1} = \phi$. Obviously, $Z_i \bigcap Z_j = \phi$ for $i \neq j$. The acyclicity of the relation δ follows from the functional acyclicity of $C(\cdot)$. Therefore, $\bigcup_{j=1}^{n} Z_j = A$.

Define the estimates $\phi(\cdot)$. Assume for all $x \in Z_j$ that $\phi(x) = n - (j - 1)$.

Lemma. If $x\delta y$, $\phi(x) > \phi(y)$.

Proof. Let, on the contrary, $x\delta y$, but $\phi(x) \leq \phi(y)$. Let $x \in Z_i, y \in Z_j$. Then, $j < i$. Denote by B the set $A \setminus \bigcup_{k=1}^{i-1} Z_k$. From construction of Z_j $x, y \in B$. As follows from $y \in Z_j$ $\exists z \in B$ such that $z\delta y$, which contradicts the condition $x\delta y$.

The lemma is proved.

Consider an arbitrary set $X \in \mathcal{A}^o$. According to the proved lemma, $\phi(y) > \phi(z)$ if $y \in C(X), z \in X \setminus C(X)$. In virtue of the definition of $\phi(\cdot)$, obtain $\phi(y) \geq \phi(z) + 1$. For all $y \in C(X)$ assume $\epsilon(y, X) = \phi(y)$, for all $\mathbf{z} \in X \setminus C(X)$ assume $\epsilon(z, X) = -0.5$.

Then, for any $y \in \tilde{C}(X)$ obtain $\exists x \in X : \phi(x) - \epsilon(x, X) > \phi(y)$. For any $z \in X \setminus \tilde{C}(X)$ obtain $\phi(z) < \phi(z) - \epsilon(z, X) = \phi(z) + 0.5$. Therefore, if $\tilde{C}(\cdot)$ is choice function generated by $M_{\epsilon(x,X)}$ under the introduced $\phi(\cdot)$ and $\epsilon(\cdot,\cdot)$, $C(\cdot) = \tilde{C}(\cdot)$.

Thus, for an arbitrary function satisfying the functional acyclicity condition its generating mechanism $M_{\epsilon(x,X)}$ has been constructed.

Proof of Theorem 12: Construct a mechanism $M_{\epsilon(X)}$ equivalent to the given one $M_{\epsilon(x,X)}$. To this end, define the error function $\epsilon(X)$ as follows:

$$\epsilon(X) = \max_{x \in X} \phi(x) - \left(\max_{x \in X} [\phi(x) - \epsilon(x, X)] \right) \tag{14}$$

and keep the same criterial estimates.

Denote by $C_{\epsilon(x,X)}(\cdot)$ the choice function generated by $M_{\epsilon(x,X)}$, and that generated by $M_{\epsilon(X)}$ denote as $C_{\epsilon(X)}(\cdot)$. Let $y \in C_{\epsilon(x,X)}$. Then $\phi(y) \geq \max_{x \in X} [\phi(x) - \epsilon(x, X)] = \max_{u \in X} \phi(u) - \epsilon(X)$, i.e. $y \in C_{\epsilon(X)}(X)$.

Let $y \notin C_{\epsilon(x,X)}(X)$. Then, there exists $x \in X$ such that $\phi(x) - \epsilon(x, X) > \phi(y)$. In this case, $\max_{x \in X} [\phi(x) - \epsilon(x, X)] > \phi(y)$; and $y \notin C_{\epsilon(X)}(X)$ follows from (12).

Inversely, let the mechanism $M_{\epsilon(X)}$ be defined. Assuming for all $x \in X$ that $\epsilon(x, X) = \epsilon(X)$, obtain its equivalent mechanism $M_{\epsilon(x,X)}$.

Proof of Theorem 13: Let us rewrite the rule (12) in the equivalent form

$$y \in C(X) \Leftrightarrow (y \in X \& \forall x \in X \quad \phi(x) - \epsilon(x, X) \leq \phi(y)) \tag{15}$$

and assume that $V(X) = \max_{x \in X} [\phi(x) - \epsilon(, X)]$. It is obvious that the rules (13) and (14) coincide.

On the other hand, the rule (14) is reduced to the rule (13) if $\epsilon(x, X)$ is assumed to be $\epsilon(x, X) = \phi(x) - V(X)$.

Proof of Theorem 14. Let the choice function $C(\cdot)$ has been generated by some mechanism $M_{\epsilon(y,X)}$. For an arbitrary $X \in \mathcal{A}^o$ we put $x = \arg\max_{u \in X} \phi(u)$. For an arbitrary $X' \subseteq X$ such that $x \in X'$ because of non-negativeness of ϵ the following inequality will be obtained $\forall u \in X \; \phi(x) + \epsilon(x, X') \geq \phi(u)$ i.e. $x \in C(X')$ and the condition of functional non-dominance is obeyed.

Consider now an arbitrary choice function for which the condition of functional non-dominance is satisfied, and construct the mechanism $M_{\epsilon(y,X)}$ generating this function. According to pre-assumption $\exists x \in A : \forall X \subseteq A$ with $x \in X$, $x \in C(X)$ holds. Through Z_1 we denote the set of options which satisfies to this condition. Let us construct the non-empty sets $\{Z_j\}_1^n$ (n is finite because of finiteness of A)

$$\begin{aligned} Z_1 &= \{x : \forall X \subseteq A, \; x \in X \Rightarrow x \in C(X)\}, \\ Z_2 &= \{x : \forall X \subseteq A \setminus Z_1, x \in X \Rightarrow x \in C(X)\}, \\ &\dots \\ Z_n &= \{x : \forall X \subseteq A \setminus \bigcup_{j=1}^{n-1} Z_j, x \in X \Rightarrow x \in C(X)\}. \end{aligned}$$

Apparently, the system $\{Z_j\}$ is a partition of the set A, i.e. $\bigcup_{j=1}^n Z_j = A$, $Z_i \bigcap Z_j = \phi$ when $i \neq j$. For each $x \in Z_j$ the criterial value $\phi(x) = n - (j-1)$ is prescribed.

The error functions $\epsilon(y, X)$ will be defined as follows: for an arbitrary $X \in \mathcal{A}^\bullet$ and $y \in X$ if $y \in C(X)$ we put $\epsilon(y, X) = \max_{u \in X} \phi(u) - \phi(y)$, and if $y \notin C(X)$ we put $\epsilon(y, X) = 0$.

Through $\tilde{C}(\cdot)$ is denoted the choice function which is generated by this mechanism. Let us prove that $\forall X \in \mathcal{A}^\bullet \; C(X) = \tilde{C}(X)$. It is obvious that if $y \in C(X)$ then according to the construction criterial values and error function

$$\phi(y) + \epsilon(y, X) = \max_{u \in X} \phi(u)$$

and $y \in \tilde{C}(X)$.

Let $y \notin C(X)$. Then $\exists x \in X : \phi(x) > \phi(y)$, because on the contrary putting $y \in Z_k$ we obtain $X \subseteq \bigcup_{j=1}^{n-1} Z_j$ and according to the construction $Z_k, y \in C(X)$. But from $\phi(x) > \phi(y)$ follows that $\phi(y) + \epsilon(y, X) = \phi(y) < \phi(x)$, i.e. $y \notin \tilde{C}(X)$.

Proof of Theorem 15. As follows from $C(\cdot) \in \mathcal{C}(\mathcal{M}_{\epsilon+(x,X)})$, it satisfies the functional acyclicity condition, and for each $X \in \mathcal{A}^o$, $x \in X$ $\epsilon(x,X) \geq 0$. Then, $\max\limits_{u \in X} [\phi(x) - \epsilon(x,X)] \leq \max\limits_{u \in X} \phi(y) = \phi(y_o)$, and $y_o \in C(X' \bigcup \{y_o\})$ for all $X' \subset X$. The functional non-dominance condition is obviously satisfied.

Let now $C(\cdot) \in \mathcal{C}(\mathcal{M}_{\epsilon+(x,X)}) \bigcap \mathcal{C}\mathcal{M}_{\epsilon(y,X)}$.

Redefine the error function as follows: $C(\cdot) \in \mathcal{C}(\mathcal{M}_{\epsilon+(x,X)})$ implies that for all $X \in \mathcal{A}^o$ there exists $y \in X$ such that $\forall X' \subseteq X, y \in X' \Rightarrow y \in C(X')$.

As follows from $y \in C(X')$, $\max\limits_{u \in X'} [\phi(u) - \epsilon(u, X'] \leq \phi(y)$. Assume that $\epsilon(y, X') = \phi(y) - \max\limits_{u \in X'} [\phi(u) - \epsilon(u, X')]$ and for the remaining $x \in X'$ assume that $\epsilon(x, X') = \phi(x) - \max\limits_{z \in X} \phi(z)$.

The choice function is generated by the determined values $\phi(x)$ and $\epsilon(x, X)$ for all $x \in \mathcal{A}^o$ and $x \in X$.

Proof of Theorem 16. Let $C(\cdot)$ be generated by the mechanism $< \beta, (2) >$ and satisfy the functional non-dominance condition. Let us demonstrate that β is an acyclic relation. Let, on the contrary, the variants $x_1, \ldots, x_p$ make up a cycle. Then, for the representation $X = \{x_1, \ldots, x_p\}$ obtain $C(X) = \phi$. The functional non-dominance condition is obviously violated.

Consider a mechanism $< \beta, (2) >$ where β is acyclic relation. For such a condition $\forall X \in \mathcal{A}^o$, $C(X) \neq \phi$, and for any X there exist x such that $x \in X' \subseteq X \Rightarrow x \in C(X')$ that is the functional non-dominance condition is satisfied.

Proof of Theorem 17. Consider a mechanism $< \beta, (2) >$ with weak bi-order relation β. Prove that the function $C(\cdot)$ as generated by this mechanism satisfies the functional acyclicity condition.

Let, on the contrary, $C(\cdot)$ contain a functional cycle, that is let there be a number r, options $x_1, \ldots, x_r$ and sets $X_1, \ldots, X_r$ such that $\forall i \in \{1, \ldots, r\}$ $x_i \in C(X_i)$, $x_{i+1} \in X_i \setminus C(X_i) (x_{r+1} = x_1)$. Then, $\forall i \in \{1, \ldots, r\}$ from $x_{i+1} \in X_i \setminus C(X_i)$ follows that $\exists z_i \in X_i : z_i \beta x_{i+1}$.

In what follows, denote by r the length of the shortest functional cycle, and consider two cases.

1. Let $r = 2$. From $z_1 \beta x_2$, $z_2 \beta x_1$ and the weak interval condition, obtain then $z_1 \beta x_1$ $z_2 \beta x_2$ $x_1 \beta x_2$ $x_2 \beta x_1$ $x_1 \beta x_1$ $x_2 \beta x_2$. All of these relations are impossible because $x_1 \in C(X_1)$, $x_2 \in C(X_2)$.

2. Let $r > 2$. Show that $\forall i \in \{1, \ldots, r\}$ $x_i \beta x_{i+1}$. Let, on the contrary, $\exists i : x_i \overline{\beta} x_{i+1}$. Without loss of generality, assume that $i = 1$, i.e., $x_1 \overline{\beta} x_2$.

Consider the set $X' = \{x_1, z_2, x_3\}$. The options $x_1, x_3, \ldots, x_r$ and sets $X_1', X_3, \ldots, X_r$ form a $(r-1)$-long cycle that contradicts the assumption of minimal r. To determine this fact, suffices it to demonstrate that $x_1 \in C(X_1'), x_3 \notin C(X_1')$. The second assertion follows from $z_2 \beta x_3$. Prove the first one. If, on the contrary, $x_1 \notin C(X_1')$, $x_3 \beta x_1$ or $z_2 \beta x_1$ If $x_3 \beta x_1$, obtain $x_1 \beta x_2$ $x_2 \beta x_1$ $x_1 \beta x_1$ $x_2 \beta x_2$ $z_1 \beta x_1$ $x_3 \beta x_2$ using $z_1 \beta x_2$ and the weak interval condition. All of these relations contradict the assumptions $x_1 \overline{\beta} x_2$, $x_1 \in C(X_1)$ or $x_2 \in C(X_2)$. Thus, $x_3 \overline{\beta} x_1$. The fact that $z_2 \overline{\beta} x_1$ is proved similarly.

The existence of $(r-1)$-long functional cycle is indicative of the fact that $x_i \beta x_{i+1}$ for all i. Then, $\exists i x_{i+1} \beta x_i$ $x_i \beta x_i$ according to the weak cyclicity condition that contradicts to the assumption $x_i \in C(X_i)$.

The contradictions in Cases 1) and 2) proves that the functional acyclicity conditions are satisfied.

Let now $C(\cdot)$ satisfy the functional acyclicity condition and be generated by some pair-dominant mechanism $< \beta, (2) >$. Show that β is a weak bi-order.

Let, on the contrary, there exist in β a cycle of length $r > 3$ without symmetrical pairs and loops, or the weak interval condition be violated.

Consider both cases in succession.

a) $x_1 \beta x_2 \beta \ldots \beta x_r \beta x_1$. Then, considering the sets $X_1 = \{x_1, x_2\}, X_2 = \{x_2, x_3\}, \ldots, X_r = \{x_r, x_1\}$ obtain that for $C(\cdot)$ there is a functional cycle.

b) If there exist x_1, x_2, x_3, and x_4 such that $x_1 \beta x_2$ and $x_3 \beta x_4$, but $x_1 \overline{\beta} x_4$, $x_3 \overline{\beta} x_2$, $x_2 \overline{\beta} x_4$, $x_4 \overline{\beta} x_2$, $x_2 \overline{\beta} x_2$, $x_4 \overline{\beta} x_4$ obtain through consideration of the sets $X_1 = \{x_1, x_2, x_4\}$ and $X_2 = \{x_2, x_3, x_4\}$ the following functional cycle: $x_4 \in C(X_1)$, $x_2 \notin C(X_1)$, $x_2 \in C(X_2)$, $x_4 \notin C(X_2)$.

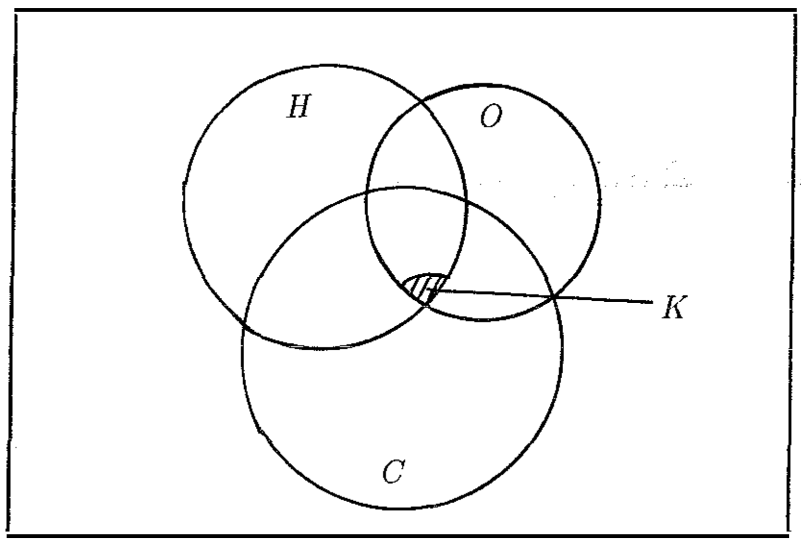

Figure 1.

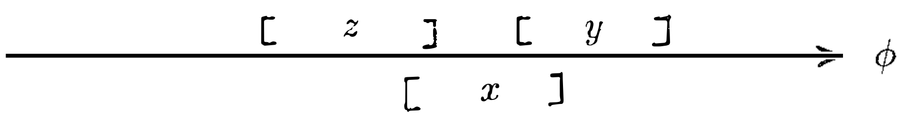

Figure 2.

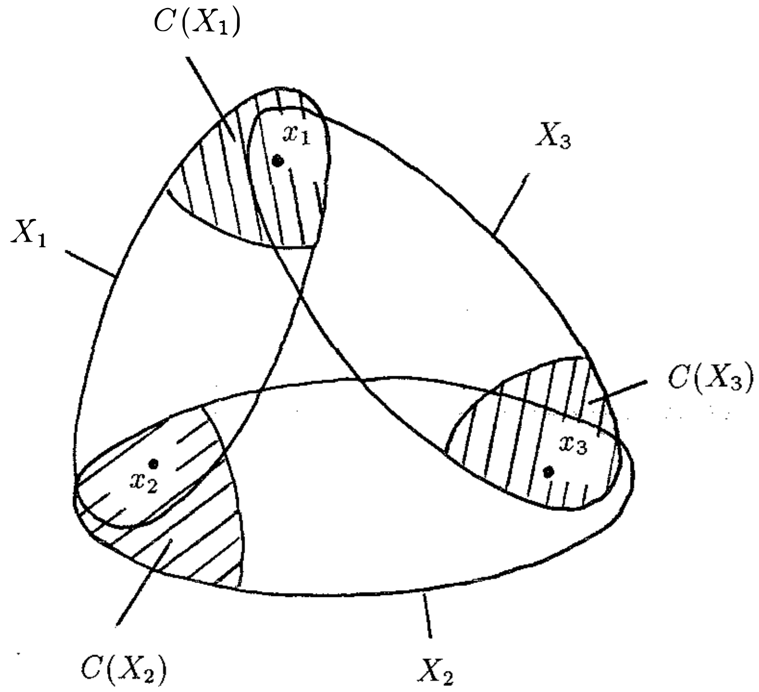

Figure 3.

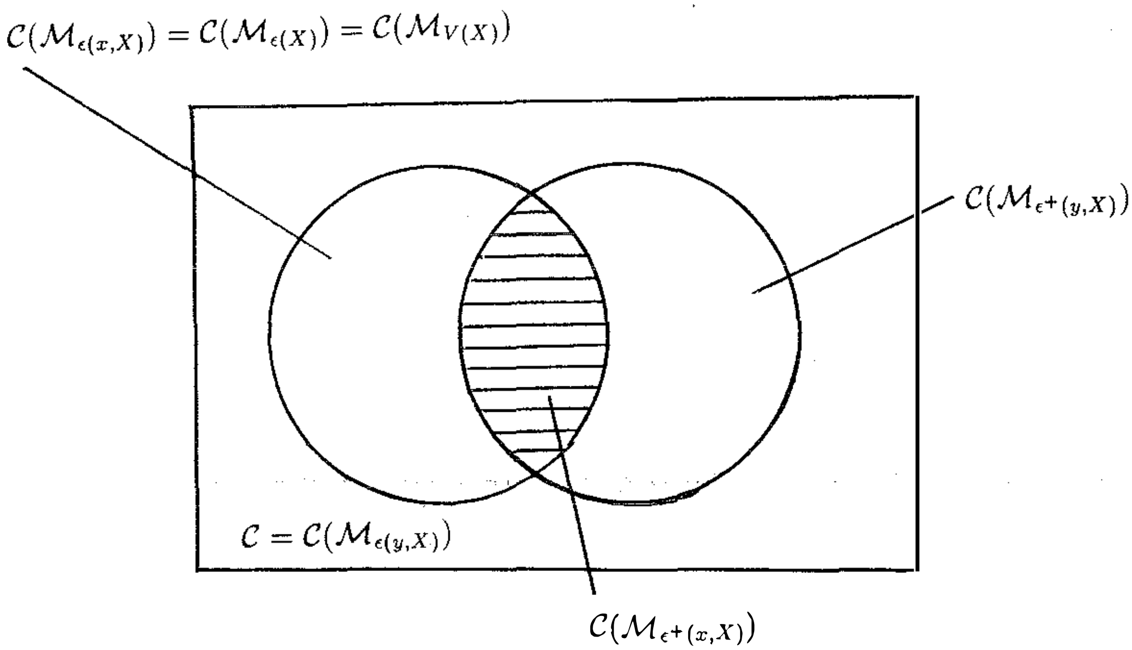

Figure 4.

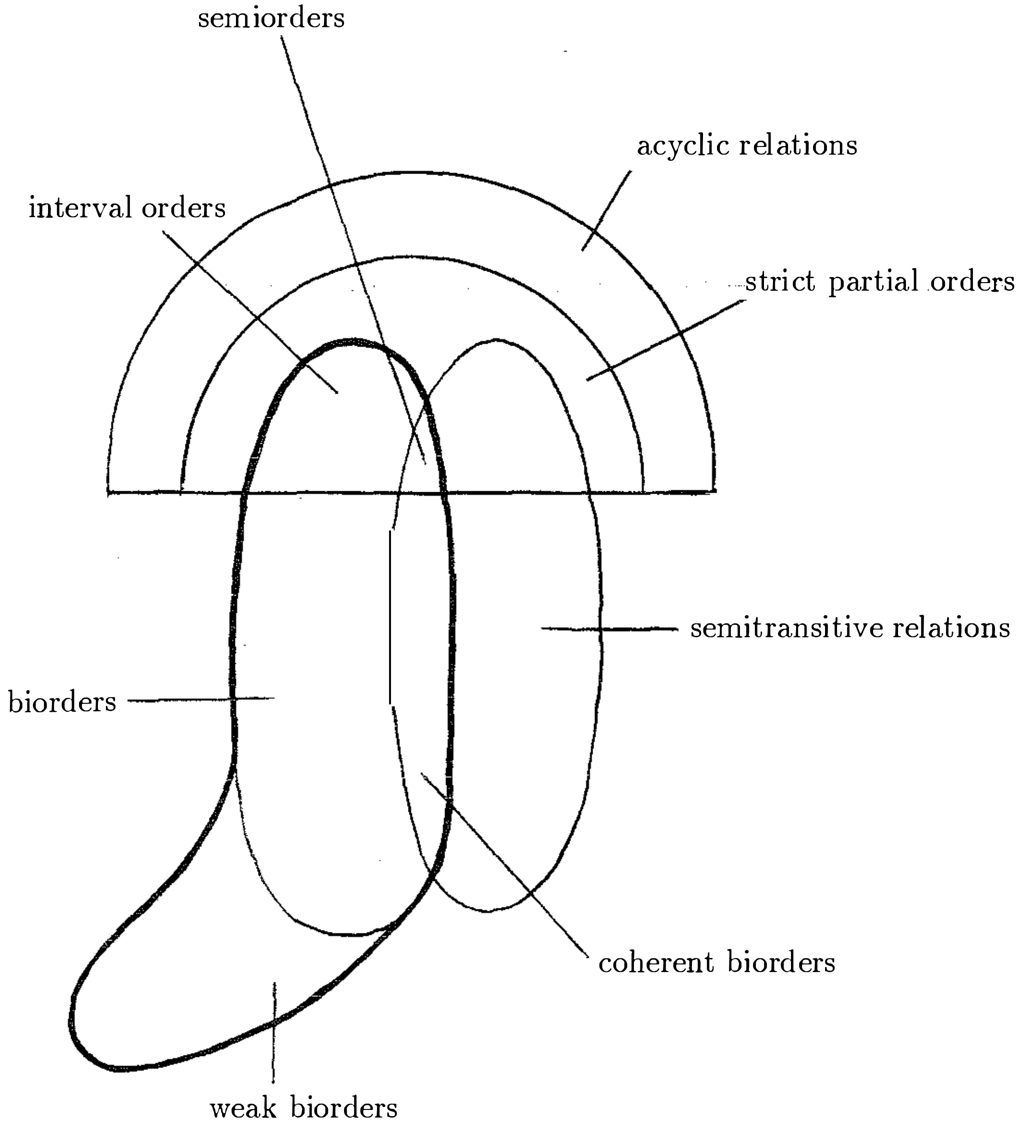
semiorders
acyclic relations
interval orders
strict partial orders
semitransitive relations
biorders
coherent biorders
weak biorders

Figure 5.

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